

# COE CST Tenth Annual Technical Meeting

**399-UCF**

## Efficient Uncertainty Quantification, Probability of Collision and Benchmarking

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Center of Excellence for  
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# Agenda

- Team Members
- Task Description
- Schedule
- Goals
- Results
- Conclusions and Future Work

# Team Members

- People

Principal Investigator



Tarek A. Elgohary

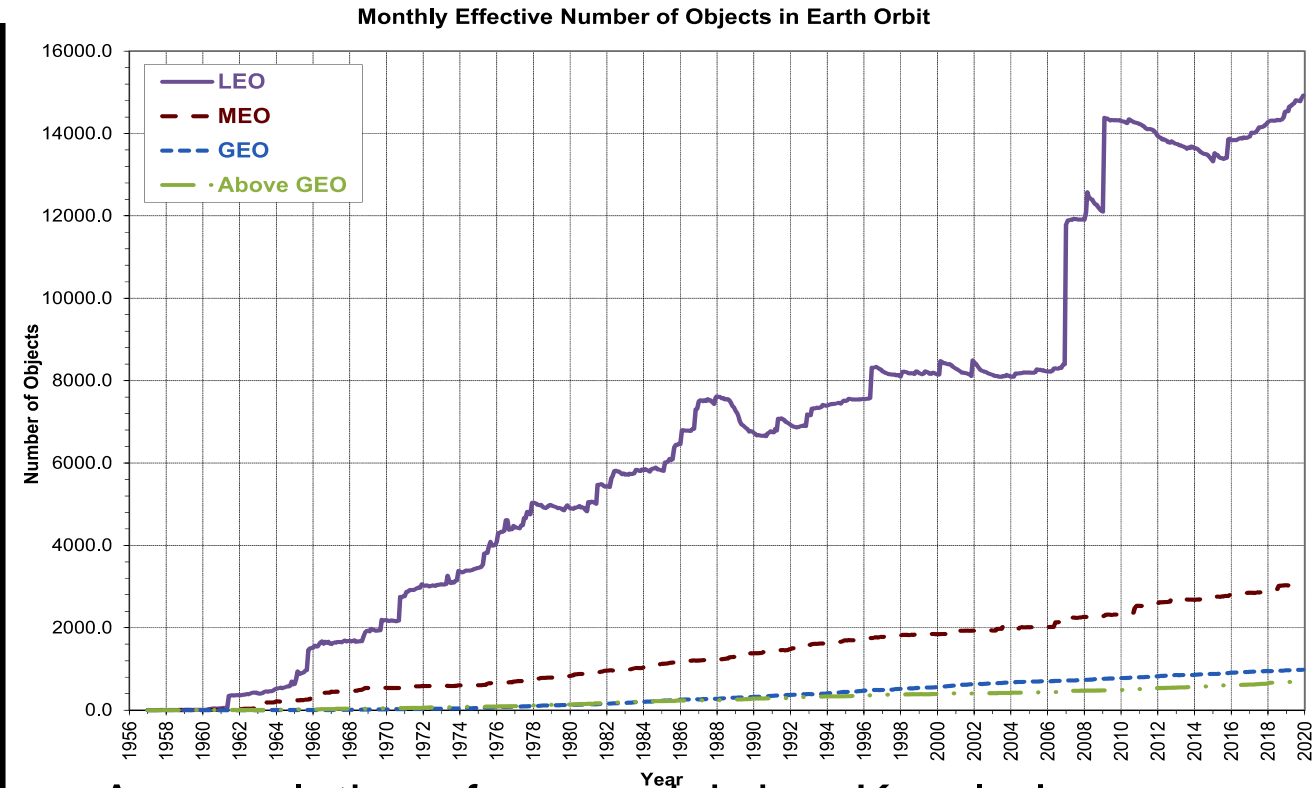
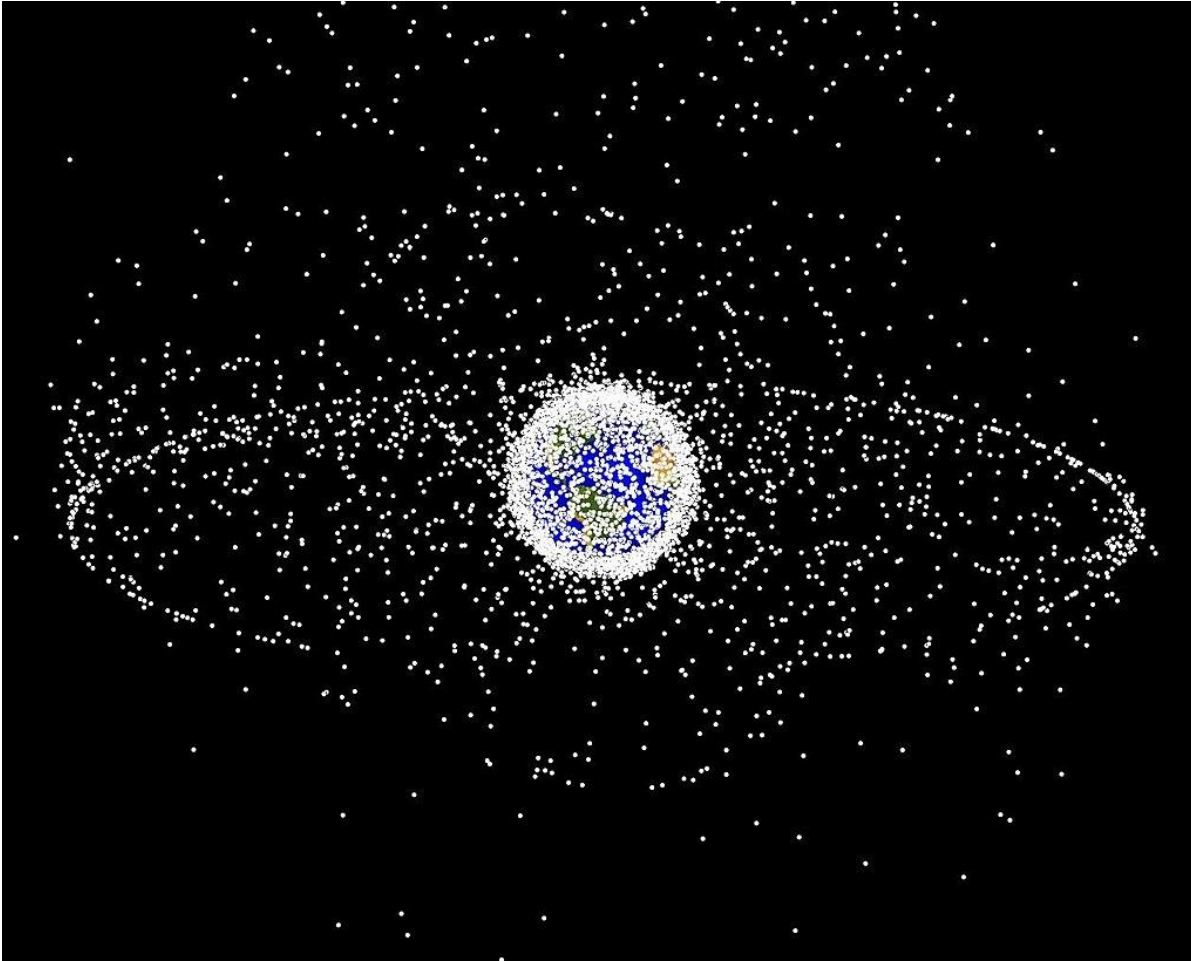
Ph.D. Student



Tahsinul Haque Tasif



# Task Description



- Accumulation of space debris – Kessler’s Syndrome – Sustainability of the space environment
- More and more constellations in Earth orbit – SpaceX, OneWeb, India, China, etc.

# Two Approaches for UQ

## • Probability Density Function (PDF) via Higher Order State Transition Tensors

- Evolution of uncertainties

$$\delta x = \phi_1 \delta x_0 + \phi_2 \delta x_0 \delta x_0 + \dots$$

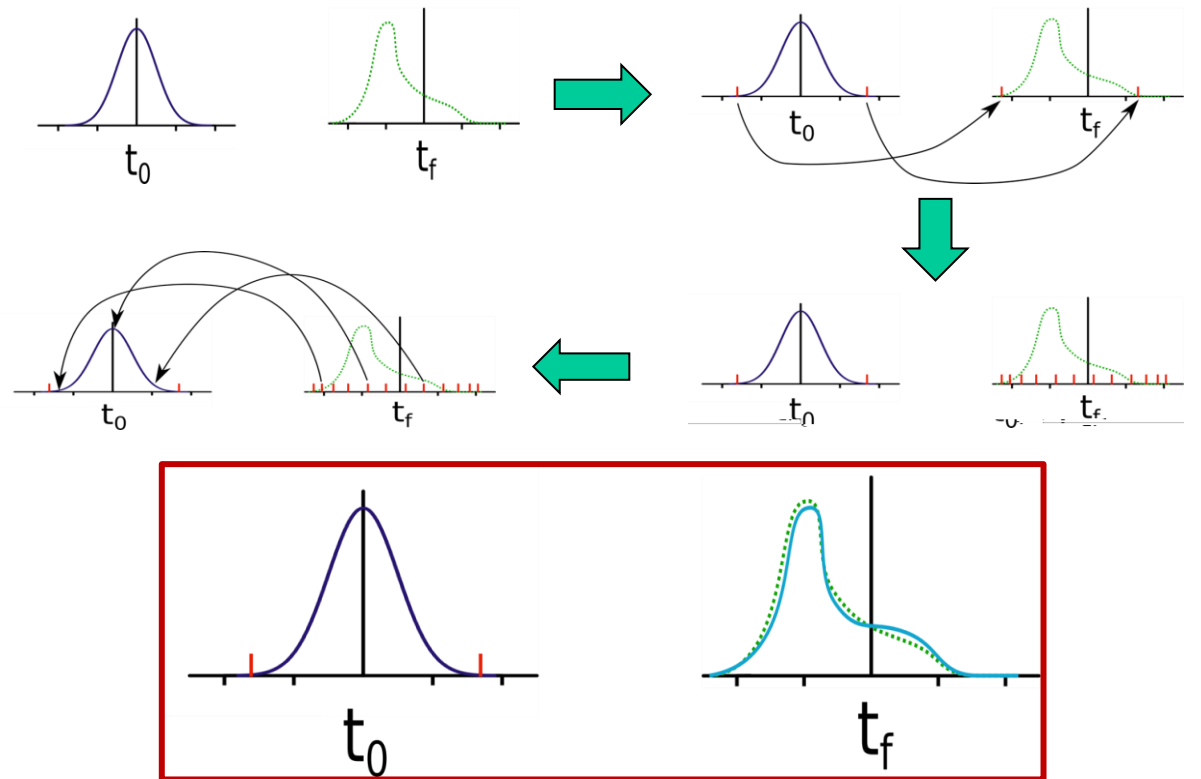
- Knowing the probability distribution of  $\delta x_0$ , the posterior PDF is given by,

$$P_{\delta x}(\delta x) = P_{\delta x_0}(\delta x_0) \left| \det \left( \frac{\partial g^{-1}(\delta x)}{\partial(\delta x)} \right) \right|$$

- Where,  $g^{-1}(\delta x)$  is the Taylor series reversion.

Focus of  
Today's Talk

## • Probability Density Function (PDF) Approximation Techniques



# Analytic Continuation Technique

- Analytic Continuation is an integration method applied to solve fundamental problems in Astrodynamics.
- This method has been proven to be highly precise and computationally efficient in orbit propagation.
- The full spherical harmonics gravity model and atmospheric drag model were also incorporated with Analytic Continuation method.

$$f = \mathbf{r} \cdot \mathbf{r} \text{ and } g_p = f^{-\frac{p}{2}}$$
$$\mathbf{r}_0^{(2)} = -\mu \frac{\mathbf{r}_0}{(\mathbf{r}_0 \cdot \mathbf{r}_0)^{3/2}} = -\mu \mathbf{r}_0 f^{-\frac{3}{2}} = -\mu \mathbf{r}_0 g_3$$

# Analytic Continuation - State Variables

- Taylor series expansion to obtain position and velocity:

$$\mathbf{r}(t_0 + dT) = \mathbf{r}_0 + \sum_{m=1}^n \mathbf{r}_0^{(m)} \frac{dT^{(m)}}{m!}$$

$$\mathbf{r}^{(1)}(t_0 + dT) = \mathbf{r}_0^{(1)} + \sum_{m=2}^n \mathbf{r}_0^{(m)} \frac{dT^{(m-1)}}{(m-1)!}$$

- The recursive equations to calculate  $\mathbf{r}_0^{(n)}$ ,  $f^{(n)}$  and  $g_p^{(n)}$  :

$$\mathbf{r}_0^{(n+2)} = -\mu \sum_{m=0}^n \binom{n}{m} \mathbf{r}_0^{(m)} g_3^{(n-m)} \text{ and } f^{(n)} = \sum_{m=0}^n \binom{n}{m} \mathbf{r}_0^{(m)} \cdot \mathbf{r}_0^{(n-m)}$$

$$g_p^{(n+1)} = -\frac{1}{f} \left\{ \frac{p}{2} f^{(1)} g_p^{(n)} + \sum_{m=1}^n \binom{n}{m} \left( \frac{p}{2} f^{(m+1)} g_p^{(n-m)} + f^{(m)} g_p^{(n-m+1)} \right) \right\}$$



# Analytic Continuation – State Transition Tensors

- Index based First and Second order State Transition Tensors:

$$\Phi_{ij}^1 = \frac{\partial \chi_i}{\partial \chi_{0j}} \text{ and } \Phi_{ijk}^2 = \frac{\partial^2 \chi_i}{\partial \chi_{0j} \partial \chi_{0k}}$$

where,  $\chi_i$  is the  $i$ -th element of the state vector,  $\chi = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T$ .

- Taylor series expansion of the terms of the State Transition Tensors:

$$\Phi_{i=1,\dots,3,jk}^2(t + dT, t) = \frac{\partial^2 \chi_i(t+dT)}{\partial \chi_j(t) \partial \chi_k(t)} = \frac{\partial^2 \chi_i(t)}{\partial \chi_j(t) \partial \chi_k(t)} + \sum_{m=1}^n \frac{\partial^2 \chi_i^{(m)}(t)}{\partial \chi_j(t) \partial \chi_k(t)} \frac{dT^{(m)}}{(m)!}$$
$$\Phi_{i=4,\dots,6,jk}^2(t + dT, t) = \frac{\partial^2 \chi_i(t+dT)}{\partial \chi_j(t) \partial \chi_k(t)} = \frac{\partial^2 \chi_i(t)}{\partial \chi_j(t) \partial \chi_k(t)} + \sum_{m=2}^n \frac{\partial^2 \chi_i^{(m)}(t)}{\partial \chi_j(t) \partial \chi_k(t)} \frac{dT^{(m-1)}}{(m-1)!}$$



# Schedule

| Task  | Time Frame         |
|---|--------------------|
| Develop Analytic Continuation for arbitrary order perturbed state transition tensors for accurate error propagation | Fall 2020          |
| Posterior PDF approximation via high-order state transition tensors and computation of probability of collisions    | Spring/Summer 2021 |
| Orthogonal Probability Approximation for posterior PDF with parametric uncertainty                                  | Summer/Fall 2021   |
| Computing Probability of collisions of RSOs via two approaches + Benchmarking problems                              | Spring/Summer 2022 |

# Goals

- Accurate and efficient approaches to quantify uncertainty and compute probability of collision for RSOs
- Benchmarking platform for other methods to provide synthetic or real cases and compare results
- Sustainability of the space environment
- Tools to predict space debris trajectories and potential hazardous events to various operators
- Accurate orbit prediction for newly deployed constellations and their potential collisions with debris and/or other RSOs.

# Results

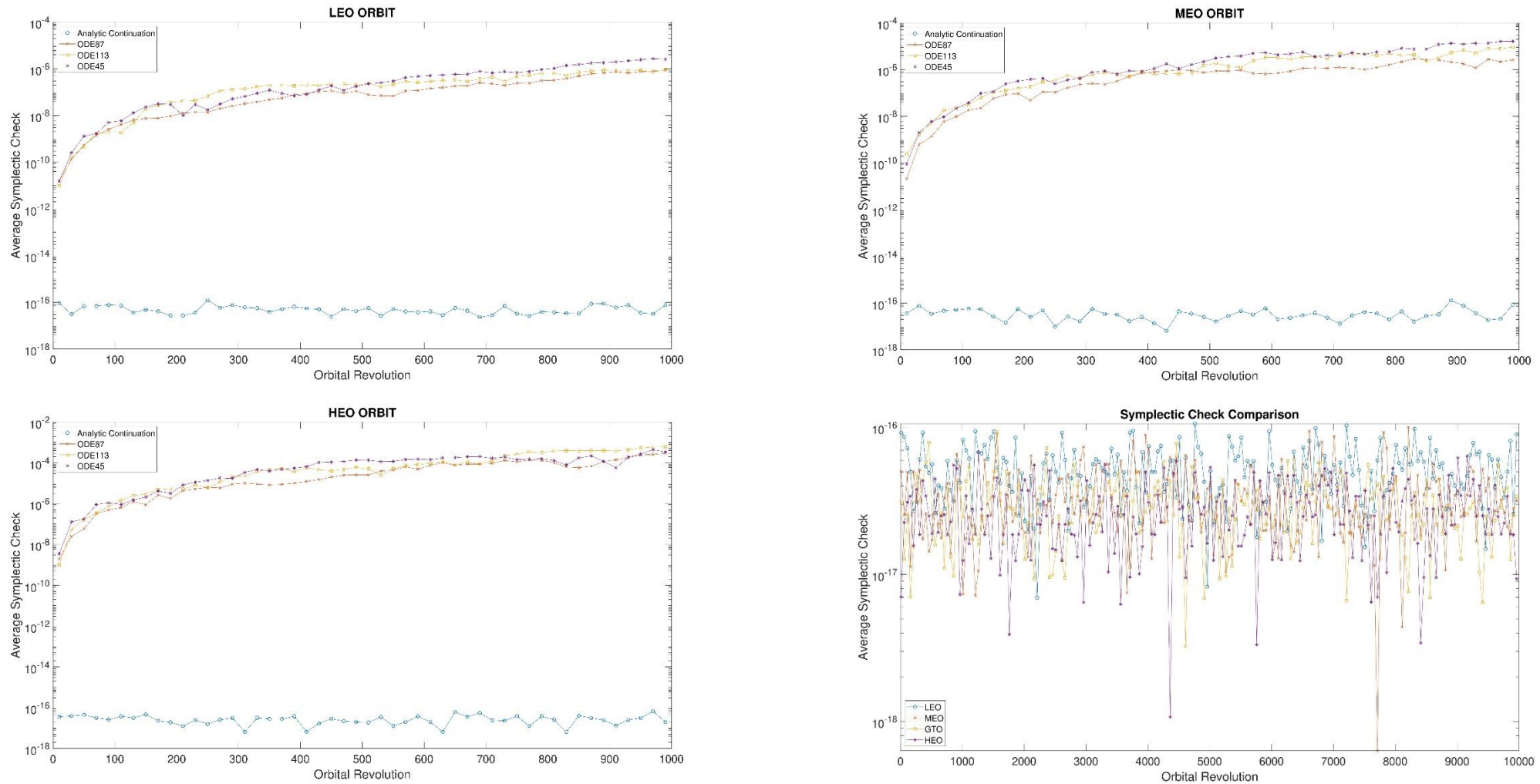


Fig: Symplectic Error in  $J_2 - J_6$  gravity perturbed orbits and comparison with MATLAB ODE suite

# Results (continued)

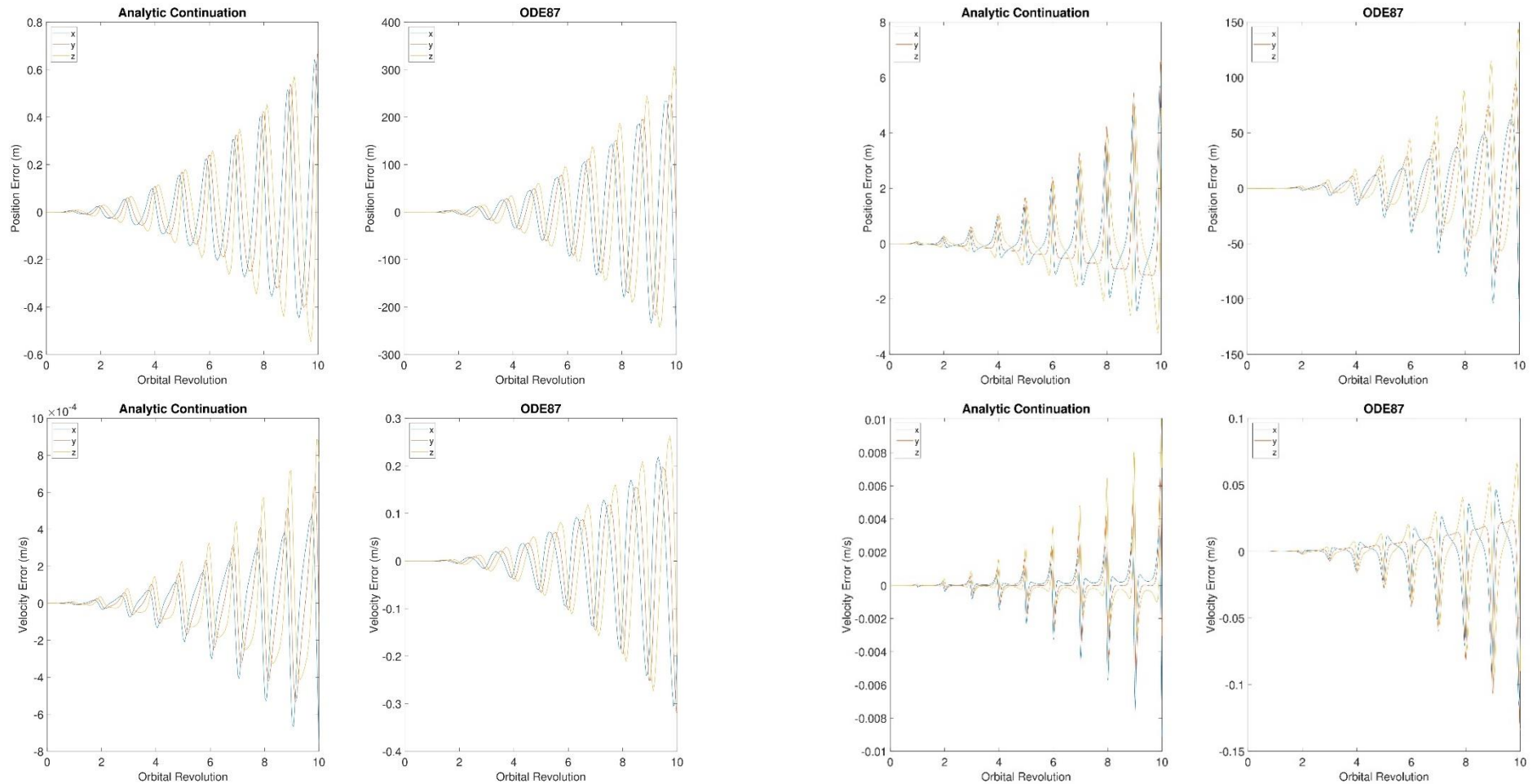


Fig: Linear prediction error of states of  $J_2 - J_6$  gravity and drag perturbed LEO and MEO orbit using Analytic Continuation and comparison with ODE87

# Results (continued)

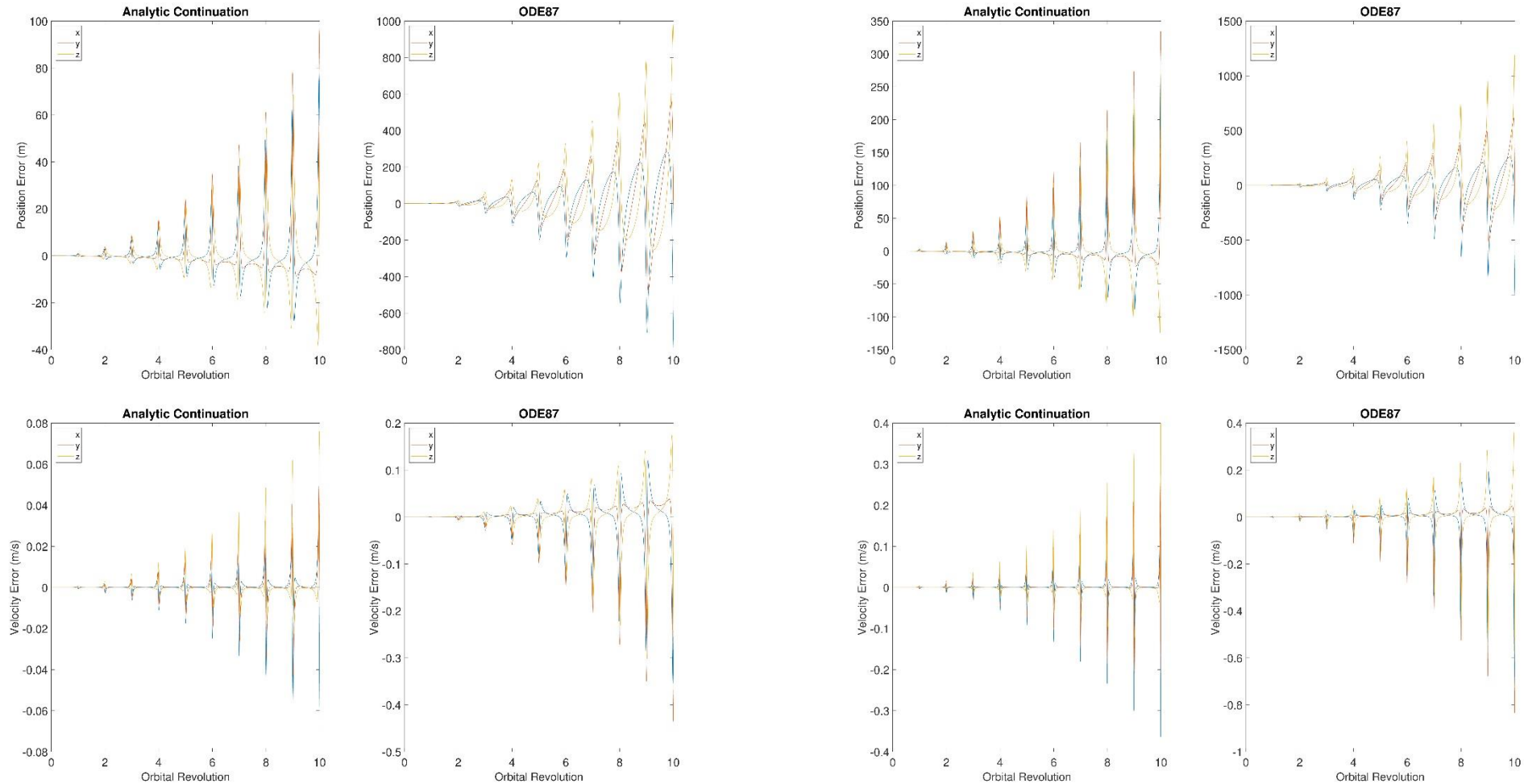


Fig: Linear prediction error of states of  $J_2 - J_6$  gravity and drag perturbed GTO and HEO orbit using Analytic Continuation and comparison with ODE87



# Results (continued)

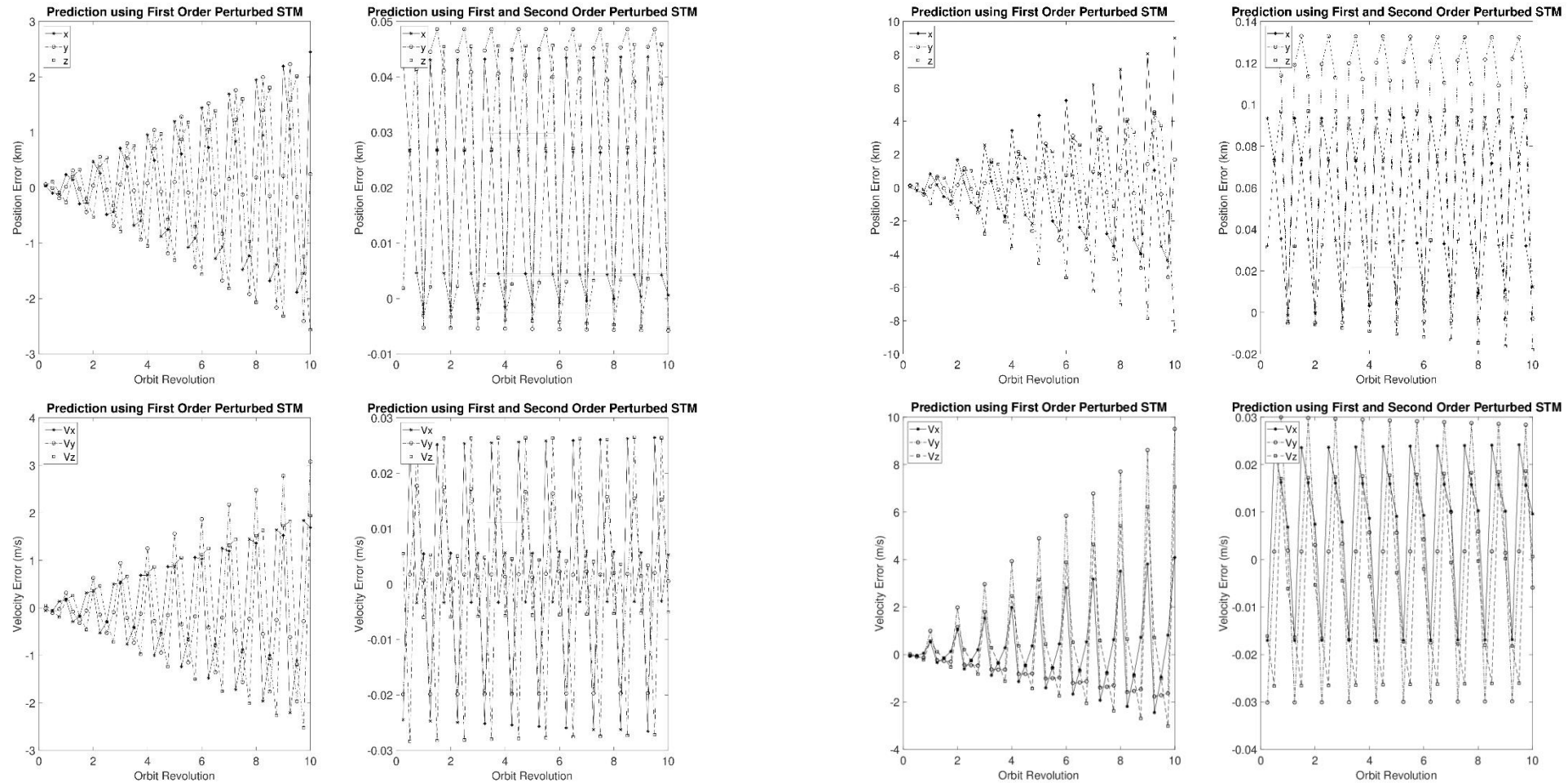


Fig: Linear prediction error improvement of states of  $J_2$  perturbed LEO and MEO orbit using Second Order State Transition Tensor derived using Analytic Continuation technique

# Results (continued)

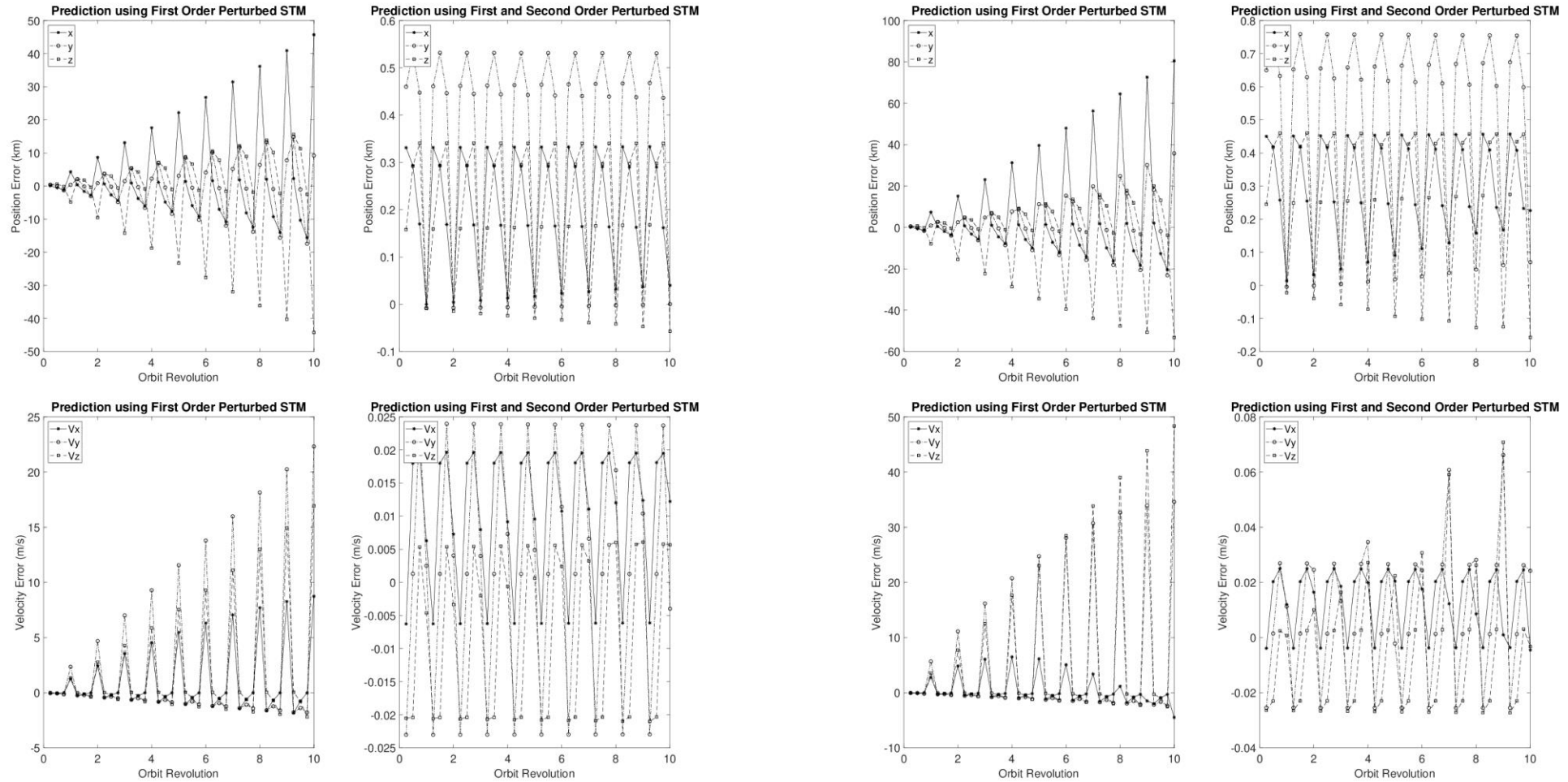


Fig: Linear prediction error improvement of states of  $J_2$  perturbed GTO and HEO orbit using Second Order State Transition Tensor derived using Analytic Continuation technique



# Publications

- Tasif, T.H., Elgohary, T.A.: A high order analytic continuation technique for the perturbed two-body problem state transition matrix, Advances in Astronautical Sciences: **AAS/AIAA Space Flight Mechanics Meeting** (2019)
- Tasif, T.H., Elgohary, T.A.: An adaptive analytic continuation technique for the computation of the higher order state transition tensors for the perturbed two-body problem, **AIAA Scitech 2020** Forum, p. 0958 (2020)
- Tasif, T.H., Elgohary, T.A.: An adaptive analytic continuation method for computing the perturbed two-body problem state transition matrix, **The Journal of the Astronautical Sciences** (2020), **In Press.**

# Conclusions and Future Work

- Implementation of Spherical Harmonics Gravity model on State Transition Matrix and Higher Order State Transition Tensor is under development now.
- Atmospheric drag model will be implemented on Higher Order State Transition Tensors.
- The results of the current research work will be extended to solve uncertainty quantification of states over time and perturbed Multi Revolution Lambert's Problem.