ProductLog

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Notations

Traditional name

Lambert function

Traditional notation

$W(z)$

Mathematica StandardForm notation

`ProductLog[z]`

`LambertW[z]`

Primary definition

01.31.02.0001.01

$$W(z) = W_0(z)$$

Specific values

Values at fixed points

01.31.03.0001.01

$$W(0) = 0$$

01.31.03.0007.01

$$W\left(-\frac{\pi}{2}\right) = \frac{i\,\pi}{2}$$

01.31.03.0008.01

$$W\left(-\frac{1}{e}\right) = -1$$

Values at infinities

01.31.03.0002.01

$$W(\infty) = \infty$$

01.31.03.0003.01

$$W(-\infty) = \infty$$

01.31.03.0004.01

$$W(i\,\infty) = \infty$$

01.31.03.0005.01

$W(-i\,\infty) == \infty$

01.31.03.0006.01

$W(\tilde{\infty}) == \infty$

General characteristics

Domain and analyticity

$W(z)$ is an analytical function of z which is defined over the whole complex z-plane.

01.31.04.0001.01

$z \longrightarrow W(z) :: \mathbb{C} \longrightarrow \mathbb{C}$

Symmetries and periodicities

Mirror symmetry

01.31.04.0002.01

$W(\bar{z}) == \overline{W(z)} \;/;\; z \notin \left(-\infty, -\frac{1}{e}\right)$

Periodicity

No periodicity

Poles and essential singularities

The function $W(z)$ does not have poles and essential singularities.

01.31.04.0003.01

$\mathcal{S}ing_z(W(z)) == \{\}$

Branch points

The function $W(z)$ has two branch points: $z == -\frac{1}{e}$, $z == \tilde{\infty}$.

01.31.04.0004.01

$\mathcal{BP}_z(W(z)) == \left\{-\frac{1}{e}, \tilde{\infty}\right\}$

01.31.04.0005.01

$\mathcal{R}_z(W(z), 0) == \log$

01.31.04.0006.01

$\mathcal{R}_z\left(W(z), -\frac{1}{e}\right) == 2$

01.31.04.0007.01

$\mathcal{R}_z(W(z), \tilde{\infty}) == \log$

Branch cuts

The function $W(z)$ is a single-valued function on the z-plane cut along the interval $\left(-\infty, -\frac{1}{e}\right)$, where it is continuous from above.

01.31.04.0008.01

$$\mathcal{BC}_z(W(z)) = \left\{\left\{\left(-\infty, -\frac{1}{e}\right), -i\right\}\right\}$$

01.31.04.0009.01

$$\lim_{\epsilon \to +0} W(x + i\,\epsilon) = W(x) \;/;\; x < -\frac{1}{e}$$

01.31.04.0010.01

$$\lim_{\epsilon \to +0} \mathrm{W}(x - i\,\epsilon) = W_{-1}(x) \;/;\; x < -\frac{1}{e}$$

Series representations

Generalized power series

Expansions at $z = 0$

01.31.06.0001.02

$$W(z) \propto z - z^2 + \frac{3 z^3}{2} - \ldots \;/;\; (z \to 0)$$

01.31.06.0008.01

$$W(z) \propto z - z^2 + \frac{3 z^3}{2} - O\left(z^4\right)$$

01.31.06.0002.01

$$W(z) = \sum_{k=1}^{\infty} \frac{(-k)^{k-1} z^k}{k!} \;/;\; |z| < \frac{1}{e}$$

01.31.06.0003.02

$$W(z) \propto z + O\left(z^2\right)$$

Expansions at $z = -\frac{1}{e}$

01.31.06.0004.02

$$W(z) \propto -1 + \sqrt{2 e} \sqrt{z + \frac{1}{e}} - \frac{2 e}{3}\left(z + \frac{1}{e}\right) + \frac{11 e^{3/2}}{18\sqrt{2}}\left(z + \frac{1}{e}\right)^{3/2} + \ldots \;/;\; \left(z \to -\frac{1}{e}\right)$$

01.31.06.0009.01

$$W(z) \propto -1 + \sqrt{2 e} \sqrt{z + \frac{1}{e}} - \frac{2 e}{3}\left(z + \frac{1}{e}\right) + \frac{11 e^{3/2}}{18\sqrt{2}}\left(z + \frac{1}{e}\right)^{3/2} + O\left(\left(z + \frac{1}{e}\right)^2\right)$$

01.31.06.0005.01

$$W(z) == \sum_{k=0}^{\infty} c_k\, p^k \;/;\; p == \sqrt{2}\,\sqrt{e\, z+1} \bigwedge |e\, z+1| < 1 \bigwedge c_0 == -1 \bigwedge c_1 == 1 \bigwedge$$

$$a_0 == 2 \bigwedge a_1 == -1 \bigwedge a_k == \sum_{j=2}^{k-1} c_j\, c_{1-j+k} \bigwedge c_k == -\frac{a_k}{2} + \frac{k-1}{k+1}\left(\frac{a_{k-2}}{4} + \frac{c_{k-2}}{2}\right) - \frac{c_{k-1}}{k+1}$$

01.31.06.0006.02

$$W(z) \propto -1 + O\left(\sqrt{z+\frac{1}{e}}\right)$$

Expansions at $z == \infty$

01.31.06.0007.02

$$W(z) == \log(z) - \log(\log(z)) - \sum_{k=0}^{\infty} \frac{(-1)^k}{\log^k(z)} \sum_{j=1}^{k} \frac{S_k^{(k-j+1)}}{j!} \log^j(\log(z)) \;/;\; (|z| \to \infty)$$

Integral representations

On the real axis

Of the direct function

01.31.07.0001.01

$$W(z) == 1 + (\log(z) - 1)\exp\left(\frac{i}{2\pi}\int_0^{\infty} \frac{1}{t+1} \log\left(\frac{\log(z) + t - \log(t) - i\pi}{\log(z) + t - \log(t) + i\pi}\right) dt\right) \;/;\; z \notin \left(-\frac{1}{e}, 0\right)$$

Dispersion relation

01.31.07.0002.01

$$W(z) == \int_{-\infty}^{-\frac{1}{e}} -\frac{1}{\pi}\, \mathrm{Im}\left(\frac{\partial W(x)}{\partial x}\right) \log\left(1 - \frac{z}{x}\right) dx \;/;\; z \notin \left(-\infty, -\frac{1}{e}\right)$$

Limit representations

01.31.09.0001.01

$$W(x) == \lim_{n\to\infty} \underbrace{x \exp(-x \exp(-\ldots\, x \exp(-x)))}_{n-\text{times}} \;/;\; 0 < x < e$$

01.31.09.0002.01

$$W(-\log(x)) == -\log(x) \lim_{n\to\infty} \left.x^{x^{\cdot^{\cdot^{x}}}}\right\} n-\text{times} \;/;\; e^{-e} < x < e^{1/e}$$

Differential equations

Ordinary nonlinear differential equations

01.31.13.0001.01

$$w'(z)\, z\, (w(z) + 1) - w(z) == 0 \;/;\; w(z) == W(c_1\, z)$$

Transformations

Transformations and argument simplifications

01.31.16.0001.01

$$W(x \log(x)) = \log(x) \text{ /; } x > \frac{1}{e}$$

01.31.16.0002.01

$$W\left(-\frac{\log(x)}{x}\right) = -\log(x) \text{ /; } 0 \leq x \leq e$$

01.31.16.0003.01

$$W\left(-\frac{\log(x+1)}{x\,(x+1)^{1/x}}\right) = -\frac{x+1}{x}\log(x+1) \text{ /; } -1 < x < 0$$

Products, sums, and powers of the direct function

Sums of the direct function

01.31.16.0004.01

$$W(x) + W(y) = W\left(x\,y\left(\frac{1}{W(x)} + \frac{1}{W(y)}\right)\right) \text{ /; } x > 0 \lor y > 0$$

01.31.16.0005.01

$$w^w = z \text{ /; } w = \frac{\log(z)}{W(\log(z))}$$

Related transformations

01.31.16.0006.01

$$e^{\alpha W(z)} = z^\alpha\, W(z)^{-\alpha}$$

Identities

Functional identities

01.31.17.0001.01

$$e^{n\,w(z)} = z^n\, w(z)^{-n} \text{ /; } w(z) = W(z) \land n \in \mathbb{Z}$$

Complex characteristics

Real part

01.31.19.0001.01

$$\operatorname{Re}(W(x + i\,y)) = \sum_{k=1}^{\infty} \frac{(-k)^{k-1}}{k!} \left(x^2+y^2\right)^{k/2} \cos\left(k \tan^{-1}(x, y)\right) \text{ /; } x^2 + y^2 < \frac{1}{e^2}$$

Imaginary part

01.31.19.0002.01

$$\operatorname{Im}(W(x+i\,y)) = \sum_{k=1}^{\infty} \frac{(-k)^{k-1}}{k!} \left(x^2+y^2\right)^{k/2} \sin\left(k \tan^{-1}(x, y)\right) /;\ x^2+y^2 < \frac{1}{e^2}$$

Absolute value

01.31.19.0003.01

$$|W(x+i\,y)| = W\left(\sqrt{x^2+y^2}\right)$$

Argument

01.31.19.0004.01

$$\arg(W(x+i\,y)) = \sum_{k=1}^{\infty} \frac{(-k)^{k-1}}{k!} \tan^{-1}\left(\cos\left(k \tan^{-1}(x, y)\right), \sin\left(k \tan^{-1}(x, y)\right)\right) /;\ x^2+y^2 < \frac{1}{e^2}$$

Conjugate value

01.31.19.0005.01

$$\overline{W(x+i\,y)} = \sum_{k=1}^{\infty} \frac{(-k)^{k-1}}{k!} \left(\cos\left(k \tan^{-1}(x, y)\right) - i \sin\left(k \tan^{-1}(x, y)\right)\right) \left(x^2+y^2\right)^{k/2} /;\ x^2+y^2 < \frac{1}{e^2}$$

Differentiation

Low-order differentiation

01.31.20.0001.01

$$\frac{\partial W(z)}{\partial z} = \frac{W(z)}{z\,(W(z)+1)}$$

01.31.20.0002.01

$$\frac{\partial^2 W(z)}{\partial z^2} = -\frac{W(z)^2\,(W(z)+2)}{z^2\,(W(z)+1)^3}$$

Symbolic differentiation

01.31.20.0003.01

$$\frac{\partial^n W(z)}{\partial z^n} = \sum_{k=0}^{\infty} \frac{(-k-n)^{k+n-1}\,(k+1)_n\,z^k}{(k+n)!} /;\ |z| < \frac{1}{e} \bigwedge n \in \mathbb{N}^+$$

01.31.20.0006.01

$$\frac{\partial^n W(z)}{\partial z^n} = \operatorname{boole}(n = 0,\ W(z)) + \frac{n!\,W(z)^n}{z^n\,(W(z)+1)^{2n-1}} \sum_{q=0}^{n} \sum_{m=0}^{q} \sum_{k=0}^{q-m} \sum_{j=0}^{q-k-m} \frac{(-1)^{q-j+n-1}\,(k+n)^{j+k+n-1} \binom{k+n}{n} (-2\,n)_{q-j-k-m}}{j!\,(k+n)!\,(q-j-k-m)!} W(z)^q /;$$

$$n \in \mathbb{N}^+$$

Eric Weisstein and Oleg Marichev

01.31.20.0004.01

$$\frac{\partial^n W(z)}{\partial z^n} == \frac{e^{-n\,W(z)}\,\mathrm{Pol}(n,\,W(z))}{(W(z)+1)^{2n-1}} \;/;\; \mathrm{Pol}(n+1,\,w) == (w+1)\,\mathrm{Pol}^{(0,1)}(n,\,w) - (w\,n+3\,n-1)\,\mathrm{Pol}(n,\,w) \bigwedge \mathrm{Pol}(1,\,w) == 1 \bigwedge n \in \mathbb{N}^+$$

Fractional integro-differentiation

01.31.20.0005.01

$$\frac{\partial^\alpha W(z)}{\partial z^\alpha} == \sum_{k=1}^{\infty} \frac{(-k)^{k-1}\, z^{k-\alpha}}{\Gamma(k-\alpha+1)} \;/;\; |z| < \frac{1}{e}$$

Integration

Indefinite integration

Involving only one direct function

01.31.21.0001.01

$$\int W(z)\,dz == \frac{z\left(W(z)^2 - W(z) + 1\right)}{W(z)}$$

Involving one direct function and elementary functions

Involving power function

01.31.21.0002.01

$$\int z^{\alpha-1}\,W(z)\,dz == \frac{1}{\alpha^2}\left(e^{-(\alpha-1)\,W(z)-W(z)}\,z^\alpha\,(\alpha\,\Gamma(\alpha+1,\,-\alpha\,W(z)) - \Gamma(\alpha+2,\,-\alpha\,W(z)))\,(-\alpha\,W(z))^{-\alpha}\right)$$

Definite integration

For the direct function itself

01.31.21.0003.01

$$\int_0^1 W(t)\,dt == \frac{(2\,W(1)-1)\left(2\,W(1)^2+1\right)}{8\,W(1)^2} + \frac{1}{8}$$

01.31.21.0004.01

$$\int_0^1 -\frac{\log\left(e^{W[x\log(x)]}\right)}{x}\,dx == \frac{\pi^2}{6}$$

Operations

Limit operation

01.31.25.0001.01

$$\lim_{n\to\infty}\left(\frac{m_*(n)}{n} - W(n)\right) == 0 \;/;\; \mathcal{S}_n^{(m_*(n))} \geq \mathcal{S}_n^{(m)} \bigwedge 0 \leq m \leq n$$

Representations through more general functions

Through other functions

01.31.26.0001.01

$W(z) = W_0(z)$

Representations through equivalent functions

With inverse function

01.31.27.0001.01

$e^{W(z)} W(z) == z$

With related functions

01.31.27.0002.01

$e^{\alpha W(z)} == z^{\alpha} W(z)^{-\alpha}$

Theorems

Periodic solutions of the two species Lotka-Volterra model

A closed form solution for periodic solutions of the two species Lotka-Volterra model $x'(t) == \alpha\, x(t)\,(1 - y(t)),\;\; y'(t) == \beta\, y(t)\,(1 - x(t))$ is given by $y(t) == -W\left(-c_1\, x(t)^{\frac{\beta}{\alpha}}\, e^{-\frac{\beta}{\alpha} x(t)}\right)$.

Solution of the bootstrap equation

A solution of the bootstrap equation $z == 2\,\Omega(z) - e^{\Omega(z)} + 1$, appearing for instance in the renormalization group equations, is given as $\Omega(z) == \left(z - 2\, W\left(-e^{\frac{z-1}{2}} \middle/ 2\right) - 1\right) \middle/ 2$.

The number of rooted trees

The number of rooted trees on n labeled points is $[z^n]\, W(z)$.

History

– J. H. Lambert (1758)
– L. Euler (1764, 1779)

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