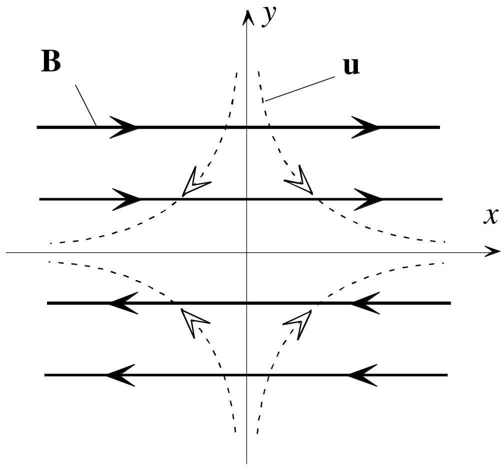
T. G. Forbes (2010)

Problem 1. The figure below shows a steady-state configuration where anti-parallel field lines (solid lines) merge and annihilate at the $y = 0$ plane. The annihilation is driven by an imposed stagnation-point flow (dashed lines).



Resistive MHD Equations

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\mathbf{u}\rho) = 0 ,$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \bullet \nabla)\mathbf{u}\right) = -\nabla p + \mathbf{j} \times \mathbf{B} ,$$

$$\nabla \times \mathbf{B} = -\mu_0 \mathbf{j} , \quad \mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta_e \mathbf{j} ,$$

$$-\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} .$$

The stagnation-point flow $\mathbf{u}$ is prescribed by:

$$\mathbf{u} = -ky\,\hat{\mathbf{y}} + kx\,\hat{\mathbf{x}} ,$$

and the magnetic field has the form

$$\mathbf{B}(x, y) = B_x(y)\,\hat{\mathbf{x}} ,$$

where k is a constant and $B_x(y) = -B_x(-y)$.

(a) Use the resistive MHD equations listed above to show that the mass continuity equation is satisfied if the density, ρ, is a constant.

(b) Verify that Faraday's equation is satisfied if the electric field is $\mathbf{E} = -E_o\,\hat{\mathbf{z}}$ where E_o is a constant and $\hat{\mathbf{z}}$ is the unit vector perpendicular to the x-y plane.

(c) Use the resistive MHD Ohm's Law above to determine $B_x(y)$ in terms of the electrical resistivity, η_e, the magnetic permeability, μ_0, and the constants, k and E_o. "Explicitly" means you should solve and integrate the different equation that comes from Ohm's Law. (Hint: You will need to use the Dawson Integral Function.)

(d) Plot the solution for B_x with B_x normalized to B_0 and y normalized to l. B_0 and l are constants defined by

$$B_0 = \frac{E_0 \mu_0 l_0}{\eta_e}, \quad \text{and} \quad l_0 = \sqrt{\frac{2\eta_e}{k\mu_0}}.$$

What is the physical significance of B_0 and l_0?

(e) Use the momentum equation to determine the gas pressure $p(x, y)$. What happens to the gas pressure as x or y tends to infinity?

Reconnection — problems (D.W. Longcope)

2. Two-dimensional current-free magnetic fields can be conveniently represented using complex functions. For example, the field from a pair of parallel wires, located at $x = \pm a$ and each carrying current I_0, satisfies the complex equation

$$B_y + iB_x \;=\; \frac{\mu_0 I_0/2\pi}{x+iy-a} + \frac{\mu_0 I_0/2\pi}{x+iy+a} \;=\; \hat{F}_0(x+iy) \quad . \tag{1}$$

This is a complex function of the complex variable $w = x+iy$. The magnetic field it describes is current-free ($\nabla \times \mathbf{B} = 0$) where ever the function is *analytic* in w. From eq. (1) we see that this is everywhere except at its two simple poles, $w = a$ and $w = -a$, which are the wires.

(a) Compute the total amount of magnetic flux (per length in the third direction) passing though a surface stretching along the x axis from the null point at the origin to the edge of a wire; take the radius $r \ll a$. We'll call this "private flux" since it separately encircles one wire, while the rest of the flux, "shared flux", encircles them both.

(b) With a current sheet (or tangential discontinuity like that in fig. 4.6 of Vol. I) of length $2L$ at the erstwhile null point, the magnetic field is given by the function

$$B_y + iB_x \;=\; \frac{a\mu_0 I_0/\pi}{\sqrt{a^2-L^2}}\,\frac{\sqrt{(x+iy)^2-L^2}}{[(x+iy)^2-a^2]} \quad . \tag{2}$$

This function is non-analytic (and thus has current) at two simple poles, $w = \pm a$ (the wires) and a branch cut running along $-L < x < L$; this magnetic discontinuity (tangential discontinuity) is the current sheet. Show that the magnetic field along the x axis is horizontal within the current sheet and vertical everywhere else. Expand about one of the branch points, $w = +L$, to determine which sense the field has on each side of the current sheet.

(c) Assume $L \ll a$ to simplify the field at and around the current sheet. Find the peak magnetic field at the edge of the current sheet. Use this value to show that the field along the y axis agrees with (5.21) of Vol. I provided $|y| \ll a$. (Please note that there may be a small error when comparing to fig. 5.4.)

(d) Use the approximate version from above to compute the total current, $I_{\rm cs}$, carried by the current sheet. Is the sense of current the same or opposite to the wires? Show that your result agrees with the net current inferred from the large-distance limit of eq. (2).

(e) Write down a definite integral representing the "private flux" now encircling one wire. Rather than performing this integral we can find an approximate expression by replacing the current sheet with a wire at the origin carrying the same current as the sheet. Compute the flux encircling either of the original wires explicitly in terms of $I_{\rm cs}$. Show that result differs from that of part (a) by a term $\Delta\psi(I_{\rm cs}) \sim I_{\rm cs}\ln(I_{\rm cs})$.

(f) Using the approximation of $\psi(I_{\rm cs})$ above compute the electrodynamic work (per length in the third direction), $dW = -I_{\rm cs}d\psi$, required to raise the current from zero to its final value. Express the result purely in terms of $I_{\rm cs}$, ultimately scaling as $I_{\rm cs}^2\ln(I_{\rm cs})$.

(g) Begin with a potential field ($I_{\rm cs} = 0$) between wires with current I_0 separated by $2a_0$. Assume that a small change to the separation results in a current sheet described by eq. (2) without changing the private flux. (Assume also that I_0 is held fixed throughout this move). Which direction are the wires moved? Compute the mechanical work done on the wires in changing their separation. Compare this to the electrodynamic work from part (f).

3. Consider a simplified version of the current sheet created between parallel wires in the previous problem, or in fig 4.6 of Vol. I. The sheet has width $2L$ and a very small thickness $2\delta > 0$ in place of a genuine discontinuity. The field within this sheet can be written using a flux function, $\mathbf{B} = \nabla A \times \hat{\mathbf{z}}$ where

$$A(x,y) \;=\; -\frac{\mu_0 I_{\rm cs} y^2}{2\pi\delta L^2}\sqrt{L^2 - x^2} \quad , \quad |x| \le L \quad , \quad |y| \le \delta \ . \tag{3}$$

(a) Show that the field strength on each side of the sheet (except very near the tips, $|x| \simeq L$) has the Green-Syrovatskii form

$$|B| \sim B_{\rm pk}\sqrt{1 - x^2/L^2} \ , \tag{4}$$

and that the current density within is $\sim |B|/\delta$. Show also that the net current carried by the sheet is $I_{\rm cs}$.

(b) Using the approximations from above find the Lorentz force density within the sheet. Are the forces directed toward increasing or decreasing the current density? Would flow generated by this force increase or decrease the magnetic energy?

(c) Define the Lunquist number (see eq. [5.16] of Vol. I) of the current sheet, $L_u = \mu_0 L v_{\rm A,pk}/\eta_e$, where $v_{\rm A,pk}$ is the Alfvén speed using $B_{\rm pk}$. Show that L_u is proportional to $I_{\rm cs}$ with a constant of proportionality having nothing to do with the current sheet or magnetic field. What current (in Amps) produces $L_u = 1$ in the Earth's magnetosphere? in the solar coronal? (You may use the typical values from table 5.1 of Vol. 1).

(d) Assume a resistivity which is η_e within a central portion of the sheet $|y| < \delta$, $|x| < \Delta \le L$ and vanishes elsewhere. Compute the electric field at the center of the current sheet. Use this electric field to compute the rate of electrodynamic work (per length) done on the current sheet, $I_{\rm cs}E_z$. Compare this with the total Ohmic dissipation equal to the integral of $\eta_e|\mathbf{J}|^2$. Compare the two for (i) Sweet-Parker reconnection $\Delta = L$ and (ii) Petschek reconnection $\Delta \ll L$. How do you account for any significant discrepancy?

(e) Assume the electric field is uniform and matches the value found above for the center of the sheet. Use this to compute the net Poynting flux into the resistive region. Take the limits $\Delta \ll L$ and then $\Delta \to L$. How does this compare to the Ohmically dissipated power? the rate of electrodynamic work?