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In many of the problems encountered in the heliospheric transport of energetic particles, particles are scattered effectively in pitch-angle during timescales of interest. The scattering is due to the irregular electromagnetic fluctuations in the plasma that have a secular effect on the particle velocity. Under these circumstances the particle distribution functions can be assumed to be nearly isotropic, and the appropriate transport equation is the energetic particle transport equation first derived by Parker (*Planet. Space Sci.*, *13*, 9, 1965). Applications of this transport equation have had a huge impact on this area of research from the solar modulation of galactic cosmic rays, to the transport of solar energetic particles and the mechanism of diffusive shock acceleration. It is therefore essential for a student of energetic particle transport to gain familiarity with the equation, the physics behind it, and illustrative applications of the equation to many of the important energetic particle populations in the heliosphere. The problems that follow are rather diverse and only ordered by their difficulty with the easiest problems presented first.

1. Particle Conservation

Consider the Parker transport equation

$$\frac{\partial f}{\partial t} + (\mathbf{V} + \mathbf{V}_D) \cdot \nabla f - \nabla \cdot \mathbf{K} \cdot \nabla f - \frac{1}{3}(\nabla \cdot \mathbf{V})\, p \frac{\partial f}{\partial p} = 0$$

with no source of particles on the right hand side, where the drift velocity

$$\mathbf{V}_D = \frac{pvc}{3e} \nabla \times \frac{\mathbf{B}}{B^2}$$

Show explicitly that the total number of particles in phase-space is conserved as long as $f(|\mathbf{x}| \to \infty)$ and $f(p \to \infty)$ vanish. It is helpful to write the equation in conservation (continuity) form.

2. Interplanetary Propagation of Solar Energetic Particles (SEPs)

High-energy particles are accelerated close to the Sun in association with flares and coronal mass ejections (CMEs). They occur either as discrete impulsive events or gradual events. The former events are thought to be accelerated as a byproduct of magnetic reconnection at the flare site, while the latter events are thought to be accelerated at the shocks driven by fast CMEs near the Sun. In both cases these particles propagate into interplanetary space after their release at the Sun. The particles that arrive first at an observing spacecraft propagate nearly scatter-free through the ambient electromagnetic fields. However, those that arrive later have been scattered by electromagnetic fluctuations, have nearly isotropic velocity distributions, and may be described very approximately by the Parker transport equation.

The simplest possible model neglects particle drift, advection with the solar wind and adiabatic deceleration in the diverging wind. If N particles of a specific momentum magnitude p_0 are released impulsively at the Sun with spherical symmetry, they then satisfy

$$\frac{\partial f}{\partial t} = K(p_0)\nabla^2 f + \frac{N}{4\pi p^2}\delta(p - p_0)\delta(\mathbf{r})\delta(t)$$

where we have assumed that the diffusion tensor is isotropic and homogeneous. Find $f(r,t,p_0)$. For an observer at heliocentric radius r, at what time is the maximum particle intensity observed?

3. The Solar Modulation of Galactic Cosmic Rays

Consider a simple model for the solar modulation of galactic cosmic rays, which nevertheless includes many of the important features of the process. Take the stationary spherically-symmetric Parker transport equation for constant solar wind speed V and $K_{rr} = Vr/2$ (independent of energy)

$$V\frac{\partial f}{\partial r} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{V}{2} r \frac{\partial f}{\partial r}\right) - \frac{1}{3}\frac{2V}{r} p \frac{\partial f}{\partial p} = 0$$

where drift transport is neglected. Find $f(r < r_0, p)$ subject to the boundary condition $f(r_0, p) = p_0\delta(p - p_0)$. The solution represents the modulation of a monoenergetic population of galactic cosmic rays. A more general energy spectrum of cosmic rays in interstellar space may be obtained by convolution. Hint: a more convenient choice of independent variables is $x = \ln(r/r_0)$ and $y = \ln(p/p_0)$. Describe the essential features of the solution. Find p_m, the momentum at which f has its maximum value, as a function of r.

4. A Simple Model for the Production and Evolution of Interstellar Pickup Ions in the Solar Wind

Interstellar gas enters the heliosphere under the influence of solar gravity, radiation pressure, and ionization losses. The resulting neutral atom density is $n(r,\theta)$, where r is heliocentric radial distance and θ is the angle of the heliocentric position vector relative to the bulk inflow velocity of the atoms. We may assume that the ionization rate per atom is $\beta_0(r_0/r)^2$. When an atom is ionized it has a speed approximately equal to the solar wind speed V in the frame of the solar wind. We assume that these ions are immediately picked up by the solar wind via gyration and pitch-angle scattering to form an isotropic shell of speed V in the solar wind frame.

(a) Assuming that the pitch-angle scattering rate is so large that the spatial diffusion tensor is negligible, write down the Parker equation for the evolution of the pickup ion omnidirectional distribution function $f(r,\theta,v)$

with an appropriate source term. We assume that the configuration is stationary and that the solar wind has constant speed and spherical symmetry.

(b) Solve the Parker equation for $f(r,\theta,v)$.

(c) Approximate $f(r,\theta,v)$ for large *r*.

Draw a schematic plot of $f(r,\theta,v)$ versus *v*.

5. Diffusive Acceleration at a Planar Stationary Shock

Consider particle acceleration and transport at a planar stationary shock at $x = 0$, for which the Parker transport equation in the shock frame is

$$V_x \frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\left(K \frac{\partial f}{\partial x}\right) - \frac{1}{3}\frac{dV_x}{dx} p \frac{\partial f}{\partial p} = 0 \qquad (1)$$

The upstream fluid flow is $V_x(x<0) = V_u > 0$ and the downstream fluid flow is $V_x(x>0) = V_d > 0$, where both V_u and V_d are constants. The diffusion coefficients are $K(x<0) = K_u$ and $K(x>0) = K_d$, where K_u and K_d are functions only of *p*. The boundary conditions are that $f(x \to \infty)$ is finite and $f(x \to -\infty) = f_\infty(p)$, where $f_\infty(p)$ represents the ambient population of energetic particles. The objective of this problem is to calculate $f(x,p)$.

(a) Solve equation (1) separately upstream ($x < 0$) and downstream ($x > 0$) of the shock. Each solution should involve two undetermined functions of *p*.

(b) Impose the boundary conditions at $x \to -\infty$ and $x \to \infty$.

(c) Impose the condition $f(x=-\varepsilon) = f(x=\varepsilon)$ at the shock as $\varepsilon \to 0$. Why is this condition appropriate?

(d) The final undetermined function of *p* is determined by integrating equation (1) from $x = -\varepsilon$ to $x = \varepsilon$ and allowing ε to approach zero. This “jump condition” yields a first-order differential equation for the remaining unknown function. What is the physical meaning of this jump condition? Solve the differential equation to determine the function.

(e) Write out $f(x<0,p)$ and $f(x>0,p)$ explicitly.

(f) Evaluate $f(x,p)$ for the specific case $f_\infty(p) = f_0\delta(p-p_0)$.

In this case write the power-law index in terms of the shock compression ratio $\rho_d/\rho_u = V_u/V_d \equiv X$, where ρ is the fluid mass density.

6. A Simple Example of a Shock Modified by Energetic Particle Pressure

Consider a fluid with mass density ρ, velocity **V**, and negligible pressure. It transports nonrelativistic energetic particles, which are coupled to it by a constant diffusion coefficient *K*. The relevant equations are the hydrodynamic equations for the fluid and the Parker equation for the energetic particles (ignoring the magnetic field):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \quad (1)$$

$$\rho \frac{\partial \mathbf{V}}{\partial t} + \rho(\mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla P, \quad (2)$$

$$\frac{\partial f}{\partial t} + \mathbf{V} \cdot \nabla f - K\nabla^2 f - \frac{1}{3}\nabla \cdot \mathbf{V} v \frac{\partial f}{\partial v} = 0, \quad (3)$$

where P is the energetic particle pressure $\left(P = (4\pi/3)\int dv m v^4 f\right)$

(a) Take the pressure moment of equation (3) to derive an equation for $P(\mathbf{x},t)$. You should get a factor 5/3; set $\gamma = 5/3$.

(b) Now consider a stationary planar system with variations in the x-direction only. Rewrite equations (1) and (2), and the equation for $\partial P/\partial t$ derived in (a) specifically for this system.

(c) Find three integrals of the system and identify them as mass flux, momentum flux and energy flux conservation. Identifying the integral associated with the P equation is somewhat tricky. Rewrite the factor PdV/dx appearing in one term as $d/dx(PV) - VdP/dx$. Then in the terms involving the derivative dP/dx use the simplified version of equation (2) to replace dP/dx by the term in equation (2) involving V and dV/dx. The resulting equation may be integrated easily.

(d) Determine the three constants by setting $V = V_0 > 0$, $\rho = \rho_0$ and $P = 0$ as $x \to -\infty$.

(e) Derive the following equation for $V(x)$ alone by eliminating P in the energy flux integral:

7. Stochastic Acceleration of Particles in a Homogeneous Plasma

Stochastic acceleration of particles is a classical acceleration mechanism. The original version of the mechanism, second-order Fermi acceleration, was developed by Fermi to account for the acceleration of galactic cosmic rays by "collisions" with interstellar "clouds." Although the original application of the mechanism is no longer viable, subsequent versions describe the acceleration of particles by a spectrum of Alfvén waves, by a spectrum of magnetosonic waves, by stochastic compressions and expansions in a plasma, and by multiple shock waves. The basic mechanism may be understood by considering the elastic scattering of particles off a homogeneous isotropic ensemble of massive spheres with random velocities **V**, radius R, and density N. The appropriate transport equation is

$$\frac{\partial f}{\partial t} = \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2 D(p)\frac{\partial f}{\partial p}\right]$$

where f is the omnidirectional distribution function, p is momentum magnitude, $D(p) = (1/3)\langle V^2\rangle \lambda^{-1} p^2/v$, v is particle speed, and $\lambda[= (\pi R^2 N)^{-1}]$ is the scattering mean free path. Calculate $f(t,p)$ if $f(0,p) = n_0\left(4\pi p_0^2\right)^{-1}\delta\left(p - p_0\right)$ and the particles are nonrelativistic. It is helpful to choose variables $P = p/p_0$ and τ, an appropriate dimensionless time. Find limiting forms for $f(P,\tau)$ for (a) $\tau << 1$ and P arbitrary and (b) $\tau >> 1$ and P finite.

8. Particle Scattering by a Magnetic Irregularity

Consider the motion of a proton in a magnetic field given by

$$\mathbf{B} = B_0\left(\frac{dF}{dz}\mathbf{i} + \frac{dG}{dz}\mathbf{j} + \mathbf{k}\right)$$

where $F = F(z)$ and $G = G(z)$.

(a) Give the equations for $x(z)$ and $y(z)$ describing the magnetic field lines.

(b) Write down the three components of the equation of motion, $md^2\mathbf{r}/dt^2 = (e/c)(d\mathbf{r}/dt)\times\mathbf{B}$, involving d^2x/dt^2, d^2y/dt^2 and d^2z/dt^2.

(c) Show explicitly that the proton speed v is a constant.

(d) Integrate and manipulate the equations for d^2x/dt^2 and d^2y/dt^2 to show that if $F(z\to\pm\infty) = F_\pm$ and $G(z\to\pm\infty) = G_\pm$, where $F_\pm$ and $G_\pm$ are all constants, a proton that traverses the configuration from $z = -\infty$ to $z = +\infty$ encircles the same field line at $z = +\infty$ as it encircled at $z = -\infty$.

This means that in this configuration the particle precisely follows the field line.

(e) Now take

$$F(z) = \varepsilon\sin(2\pi z/L)\exp(-z^2/l^2)$$

$$G(z) = \varepsilon\cos(2\pi z/L)\exp(-z^2/l^2)$$

where $\varepsilon << 1$. Sketch a field line as carefully as you can.

(f) To zeroth order in ε, the proton trajectory satisfies $z = v_{z0}t$ and $\mathbf{v} = v_{\perp 0}\sin(\Omega t+\phi)\mathbf{i} + v_{\perp 0}\cos(\Omega t+\phi)\mathbf{j} + v_{z0}\mathbf{k}$, where $\Omega = e_pB_0/(m_pc)$. Integrate the equation for d^2z/dt^2 to calculate to order ε the change in v_z, Δv_z, as the proton moves from $z\to-\infty$ to $z\to+\infty$. You may wish to integrate by parts.

Interpret your answer. Do you see evidence for the cyclotron resonance condition? What determines the sign of Δv_z?