ACTA ARITHMETICA
LI (1988)

On the equation $f(1)1^k+f(2)2^k+ \dots +f(x)x^k+R(x) = by^z$

by

JERZY URBANOWICZ (Warszawa)

To
Dr Zbigniew Religa
and his collaborators
as a token of gratitude

1. Introduction. Let $R \in \boldsymbol{Z}[x]$ be a polynomial, and let $b \neq 0$ and $k \geqslant 1$ be integral numbers. Let $N_0 = N \cup \{0\}$.

In this paper we deal with the equation

$$\sum_{i=0}^{x} f(i)i^k + R(x) = by^z \tag{1.1}$$

for periodic functions $f\colon N_0 \to \boldsymbol{Z}$.

We find some natural subclass of the class of all periodic functions $f\colon N_0 \to \boldsymbol{Z}$ such that: the number of solutions of the equation (1.1) in integers $x \geqslant 1$, $y, z > 1$ for f from this subclass and for any $R \in \boldsymbol{Z}[x]$ is finite. Here we exclude the cases $k \leqslant 3$ and $k = 5$. For example, all periodic functions $f\colon N_0 \to \{\pm 1\}$ with the period not divisible by 4 belong to the above considered subclass.

We give also examples of periodic functions $F\colon N_0 \to \{\pm 1\}$ such that for some $R \in \boldsymbol{Z}[x]$, $b \in \boldsymbol{Z}$, $b \neq 0$, and for large k in comparison with above excluded $k \leqslant 3$, $k = 5$, the equation (1.1) has infinitely many solutions in integers $x \geqslant 1$, $y, z > 1$. For example, it suffices to take a periodic function f with the period of length 4 satisfying $f(0) = f(3) = 1$, $f(1) = f(2) = -1$ and $k = 2^1, 2^2, 2^4, 2^8$ or 2^{16}. In general, we may take in the last example $k = 2^r$, where $r \geqslant 1$ and 2^r+1 is a prime number.

We conjecture that there exist a periodic function $f\colon N_0 \to \{\pm 1\}$ and infinitely many k such that the equation (1.1) (for this f and each k) has infinitely many solutions in integers $x \geqslant 1$, $y, z > 1$ for some $R \in \boldsymbol{Z}[x]$ and $b \in \boldsymbol{Z}$, $b \neq 0$ (dependent on f and k).

The results in the present paper are generalizations of results of [10]. We follow ideas of this paper. Similar problems were dealt in papers [8], [5] (here as in [10] $f=1$), and [3], [4] (here f was a quadratic character). All results in the paper are consequences of papers of Schinzel and Tijdeman [9], of LeVeque [7], and of Brindza [1], [2].

I wish to express my thanks to J. Browkin and A. Schinzel for their advice and encouragement.

2. Generalized Bernoulli polynomials. We use the notation from Chapter 13 in [6].

Let $x \geqslant 1$ be a natural number and let f be a function defined on a set containing $\{0, 1, \ldots, x-1\}$.

The polynomials $B_{k,f}^{(x)}(T)$ defined by

$$\sum_{a=0}^{x-1} f(a)\frac{te^{(a+T)t}}{e^{xt}-1} = \sum_{k=0}^{\infty} B_{k,f}^{(x)}(T)\frac{t^k}{k!}$$

are called *generalized Bernoulli polynomials belonging to f and x*. The *generalized Bernoulli numbers* are defined by

$$B_{k,f}^{(x)} = B_{k,f}^{(x)}(0).$$

Of course $B_{k,f}^{(x)}$ belong to a field generated by $f(a)$, $a \in \{0, 1, \ldots, x-1\}$ over $\boldsymbol{Q}$ and

$$B_{k,f}^{(x)}(T) = \sum_{i=0}^{k} \binom{k}{i} B_{i,f}^{(x)} T^{k-i}. \tag{2.1}$$

If $f=1$ then $B_{k,f}^{(x)}(T) = B_k(T)$ and $B_{k,f}^{(x)} = B_k$, where $B_k(T)$ and B_k are ordinary Bernoulli polynomials and numbers respectively.

It is known that the following formulas hold for $k \geqslant 0$ (see [6]):

$$B_{k,f}^{(x)}(T) = x^{k-1}\sum_{a=0}^{x-1} f(a) B_k\left(\frac{T+a}{x}\right), \tag{2.2}$$

$$\sum_{i=0}^{x-1} f(i)i^k = \frac{1}{k+1}[B_{k+1,f}^{(x)}(x) - B_{k+1,f}^{(x)}]. \tag{2.3}$$

Let $A \subset N_0$ and let f be defined on A. We say that f is *periodic* and x_0 is its *period*, if x_0 is a minimal natural number satisfying

$$f(i+x_0) = f(i) \quad \text{for every } i,\ i+x_0 \in A.$$

LEMMA 2.1 (see [6]). *Let x_0 and x be natural numbers and let x be divisible by x_0. If f is a periodic function on a set containing $\{0, 1, \ldots, x-1\}$ with the period x_0 then for $k \geqslant 0$:*

$$B_{k,f}^{(x)}(T) = B_{k,f}^{(x_0)}(T) \quad (\textit{and consequently } B_{k,f}^{(x)} = B_{k,f}^{(x_0)}).$$

3. Formulas for $B_{k,f}^{(x)}$. Let x be a natural number and let f be a function on a set containing $\{0, 1, \ldots, x-1\}$. We use following formulas for generalized Bernoulli numbers:

$$B_{k,f}^{(x)} = x^{k-1}B_k(s_0+f(0)) + x^{k-2}2B_{k-1}ks_1 + 2\sum_{l=0}^{k-2}\binom{k}{l}(2B_l)x^{l-1}2^{k-l-2}s_{k-l}, \tag{3.1}$$

where

$$s_r = \sum_{i=1}^{(x-1)/2} [f(2i) + (-1)^k f(x-2i)]i^r,$$

for $2 \nmid x$ and $k \geqslant 2$; and

$$B_{k,f}^{(x)} = \frac{1}{x}t_k - \frac{k}{2}t_{k-1} + \sum_{l=2}^{k}\binom{k}{l}B_l x^{l-1} t_{k-l}, \tag{3.2}$$

where

$$t_r = \sum_{i=0}^{x-1} f(i)i^r,$$

for any x and $k \geqslant 2$.

We prove that (3.1) ((3.2) is an obvious corollary from (2.2) with $T=0$ and from (2.1)). From (2.2) and from

$$B_k(1-T) = (-1)^k B_k(T)$$

we get for $2 \nmid x$

$$\begin{aligned} B_{k,f}^{(x)} &= x^{k-1}\sum_{\substack{0\leqslant i\leqslant x-1\\ 2\nmid i}} f(i)B_k(i/x) + x^{k-1}\sum_{\substack{0\leqslant i\leqslant x-1\\ 2\mid i}} f(i)B_k(i/x)\\ &= (-1)^k x^{k-1}\sum_{\substack{1\leqslant i\leqslant x-1\\ 2\mid i}} f(x-i)B_k(i/x) + x^{k-1}\sum_{\substack{0\leqslant i\leqslant x-1\\ 2\mid i}} f(i)B_k(i/x). \end{aligned}$$

Hence and from (2.1), (3.1) follows.

4. $B_{k,f}^{(x)}$ modulo powers of 2. Let $x \geqslant 1$ be a natural number and let f be a function of a set containing $\{0, 1, \ldots, x-1\}$ into $\boldsymbol{Z}$.

For integral r denote

$$a_r = \sum_{\substack{0\leqslant i\leqslant x-1\\ i\equiv r \pmod 2}} f(i) \quad (\text{so } a_0+a_1 = t_0), \qquad b_r = \sum_{\substack{0\leqslant i\leqslant x-1\\ i\equiv r \pmod 4}} f(i),$$

$$c_r = \sum_{\substack{0\leqslant i\leqslant x-1\\ i\equiv r \pmod 8}} f(i).$$

The symbol $a \| b$ for $a, b \in \boldsymbol{Z}$, $a \neq 0$ means that $a|b$ and a, b/a are co-prime.

LEMMA 4.1. *Let x and f be as above. We have:*

I. *If $2 \nmid x$ then $2B_{k,f}^{(x)}$ for $k \geqslant 1$ and $B_{0,f}^{(x)}$ are 2-integral and*

$$2B_{k,f}^{(x)} \equiv t_0 2B_k \pmod 2 \quad \textit{for } k \geqslant 1,$$

$$B_{0,f}^{(x)} \equiv t_0 \pmod 2.$$

II. *If* $2\|x$ *then* $2B_{k,f}^{(x)}$ *for* $k \geqslant 0$ *are* 2-*integral and*

$$2B_{k,f}^{(x)} \equiv 2B_k a_1 \pmod 2 \quad \text{for } k \geqslant 2,$$
$$2B_{1,f}^{(x)} \equiv a_0 \pmod 2,$$
$$2B_{0,f}^{(x)} \equiv t_0 \pmod 2.$$

III. *If* $2^\alpha\|x$ *and* $\alpha \geqslant 2$ *then* $2^\alpha B_{k,f}^{(x)}$ *for* $k \geqslant 0$ *are* 2-*integral and*

$$2^\alpha B_{k,f}^{(x)} \equiv a_1 \pmod 2 \quad \text{for } k \geqslant 1,$$
$$2^\alpha B_{0,f}^{(x)} \equiv t_0 \pmod 2.$$

Let a_1 *be even and let* $\varepsilon, \delta, \varrho = 0$ *or* 1. *If*

$$\left.\begin{array}{l} \left.\begin{array}{l} b_1 \equiv b_3 \equiv \varrho \pmod 2 \\ b_1 \equiv b_3 + 2\varepsilon \pmod 4 \\ \beta = \alpha - 1 \end{array}\right\} \quad \text{or} \quad \begin{array}{l} c_{-1} \equiv c_{-3} + \delta + \varrho + 1 \pmod 2 \\ b_1 \equiv b_3 \equiv 2\delta + 2 \pmod 4 \\ b_1 \equiv b_3 + 4(\varepsilon + \delta + \varrho + 1) \pmod 8 \\ \beta = \alpha - 2 \end{array} \end{array}\right\}$$

then for $k \geqslant 2$: $2^\beta B_{k,f}^{(x)}$ *are* 2-*integral and*

$$2^\beta B_{k,f}^{(x)} \equiv 2B_{\varrho k+\varepsilon} \pmod 2 \tag{4.1}$$

unless $k = 2$ *and* b_2 *is odd in the second case; then* (4.1) *changes into the congruence*

$$2^\beta B_{k,f}^{(x)} \equiv 2B_{\varrho k+\varepsilon+3} \pmod 2.$$

Remark. The congruence (4.1) states that for $k \geqslant 2$

$$2^\beta B_{k,f}^{(x)} \equiv 2B_k,\ 2B_{k+1},\ 1 \quad \text{or} \quad 0 \pmod 2$$

according as the 2-tuple $\{\varepsilon, \varrho\}$ equals

$$\{0, 1\},\ \{1, 1\},\ \{1, 0\} \quad \text{or} \quad \{0, 0\}.$$

Proof. We consider the case I. From (3.1) we have in this case for $k \geqslant 2$

$$B_{k,f}^{(x)} = x^{k-1} B_k(s_0 + f(0)) + 2\text{-integral}.$$

Therefore for $2|k$, $k \neq 0$ it suffices to use the von Staudt–Clausen theorem for 2, i.e., to use the congruence

$$2B_k \equiv 1 \pmod 2 \quad \text{for } 2|k,\ k \neq 0$$

and to observe that for $2|k$

$$s_0 + f(0) = t_0.$$

For $2\nmid k$, $k \neq 1$ the case I follows from $B_k = 0$. For $k \leqslant 1$ it is an immediate corollary from (2.2).

From (3.2) we have in the case $2|x$ for $k \geqslant 2$

$$B_{k,f}^{(x)} = \frac{1}{x} t_k - \frac{k}{2} t_{k-1} + 2\text{-integral}. \tag{4.2}$$

Hence we have for $2|k$, $k \neq 0$

$$B_{k,f}^{(x)} = \frac{1}{x} t_k + 2\text{-integral} \tag{4.3}$$

so in the case II, i.e., if $2\|x$

$$2B_{k,f}^{(x)} \equiv t_k \equiv a_1 \pmod 2.$$

But in the same case for $2\nmid k$, $k \neq 1$

$$2B_{k,f}^{(x)} \equiv t_k - t_{k-1} = \sum_{i=0}^{x-1} f(i) i^{k-1}(i-1) \equiv 0 \pmod 2.$$

Therefore to prove II it is sufficient to use the von Staudt–Clausen theorem for 2, again. For $k \leqslant 1$ the case II is an immediate corollary from (2.2).

We consider the case III. The first congruence of it is an immediate consequence of (4.2) because $\alpha > 1$. The second one is an obvious corollary from (2.2). Let a_1 be even. To prove (4.1) it suffices to observe that if $i \equiv r \pmod 4$ then for $2|k$, $k \geqslant 2$ and $2\nmid r$ we have $i^k \equiv 1 \pmod 8$, but for $2\nmid k$, $k \geqslant 3$ and for $2\nmid r$ we have $i^k \equiv i \pmod 8$. Hence

$$t_k = \sum_{r=0}^{3} \sum_{\substack{0 \leqslant i \leqslant x-1 \\ i \equiv r \pmod 4}} f(i) i^k \equiv \begin{cases} b_1 + b_3 \pmod 8 & \text{if } 2|k,\ k \geqslant 4 \text{ or } k = 2 \text{ and } 2|b_2, \\ b_1 + b_3 + 4 \pmod 8 & \text{if } k = 2 \text{ and } 2\nmid b_2, \\ b_1 + 3b_3 + 4(c_{-1} + c_{-3}) \pmod 8 & \text{if } 2\nmid k \text{ and } k \geqslant 3 \end{cases} \tag{4.4}$$

because for $2\nmid k$, $k \geqslant 3$ we have

$$\sum_{\substack{0 \leqslant i \leqslant x-1 \\ i \equiv r \pmod 4}} f(i) i^k \equiv \sum_{\substack{0 \leqslant i \leqslant x-1 \\ i \equiv r \pmod 4}} f(i) i \equiv r \sum_{\substack{0 \leqslant i \leqslant x-1 \\ i \equiv r \pmod 8}} f(i) + (r+4) \sum_{\substack{0 \leqslant i \leqslant x-1 \\ i \equiv r+4 \pmod 8}} f(i)$$
$$= r \sum_{\substack{0 \leqslant i \leqslant x-1 \\ i \equiv r \pmod 4}} f(i) + 4 \sum_{\substack{0 \leqslant i \leqslant x-1 \\ i \equiv r+4 \pmod 8}} f(i) = rb_r + 4c_{r+4} \pmod 8.$$

If $a_1 = b_1 + b_3 \equiv 0 \pmod 2$ then for $k \geqslant 2$, $t_{k-1} \equiv 0 \pmod 2$ and from (4.2) we get (4.3). Now, it is sufficient to use the tables:

A. Let $b_i \equiv r_i \pmod 4$ for $i = 1$ and 3, and $t_k \equiv s_k \pmod 4$, where $0 \leqslant r_i, s_k < 4$.

For $k \geqslant 2$ we get from (4.4)

r_1	r_3	s_k: $2\mid k$	s_k: $2\nmid k$
1	1	2	0
3	3		
1	3	0	2
3	1		
0	0	0	
2	2		
0	2	2	
2	0		

B. The case $r_1 = r_3 = 0$ or 2 we consider in more details. Let $b_i \equiv \bar{r}_i \pmod 8$ for $i = 1$ and 3, and $t_k \equiv \bar{s}_k \pmod 8$, where $0 \leqslant \bar{r}_i, \bar{s}_k < 8$.

<table>
<tr><th rowspan="3"></th><th rowspan="3">$\bar{r}_1$</th><th rowspan="3">$\bar{r}_3$</th><th colspan="5">$\bar{s}_k$</th></tr>
<tr><th>$2\mid k$</th><th colspan="2">$k = 2$</th><th colspan="2">$2\nmid k$</th></tr>
<tr><th>$k \geqslant 4$</th><th>$2\mid b_2$</th><th>$2\nmid b_2$</th><th>$2\mid c_{-1}+c_{-3}$</th><th>$2\nmid c_{-1}+c_{-3}$</th></tr>
<tr><td rowspan="4">$r_1 = r_3 = 0$</td><td>0</td><td>0</td><td colspan="2" rowspan="2">0</td><td rowspan="2">4</td><td rowspan="2">0</td><td rowspan="2">4</td></tr>
<tr><td>4</td><td>4</td></tr>
<tr><td>0</td><td>4</td><td colspan="2" rowspan="2">4</td><td rowspan="2">0</td><td rowspan="2">4</td><td rowspan="2">0</td></tr>
<tr><td>4</td><td>0</td></tr>
<tr><td rowspan="4">$r_1 = r_3 = 2$</td><td>2</td><td>2</td><td colspan="2" rowspan="2">4</td><td rowspan="2">0</td><td rowspan="2">0</td><td rowspan="2">4</td></tr>
<tr><td>6</td><td>6</td></tr>
<tr><td>2</td><td>6</td><td colspan="2" rowspan="2">0</td><td rowspan="2">4</td><td rowspan="2">4</td><td rowspan="2">0</td></tr>
<tr><td>6</td><td>2</td></tr>
</table>

We investigate the first subcase of III using the table A, the second one using the table B. The congruence (4.1) follows from the von Staudt–Clausen theorem for 2. Lemma 4.1 is proved.

COROLLARY 4.2. *Let $x \geqslant 1$ be an integer, and let f be a function defined on a set containing $\{0, 1, \ldots, x-1\}$ with values in $\mathbf{Z}$. We take the notation from*

Lemma 4.1. Put $\beta = 1$ for $4 \nmid x$. Let

(a) $2 \nmid x$ and $2 \nmid t_0$, or

(b) $2 \| x$ and $2 \nmid a_0, a_1$, or

(c) $4 \mid x$, $2 \nmid b_i$ and $b_i \equiv b_{i+2} \pmod 4$ for $i = 0, 1$.

Then for $k \geqslant 0$, $2^\beta B_{k,f}^{(x)}$ are 2-integral and the following congruences hold:

$$2^\beta B_{k,f}^{(x)} \equiv 2B_k \pmod 2.$$

Proof. The corollary in the cases (a) and (b) follows immediately from the lemma. The case (c) for $k \geqslant 2$ follows from the first subcase of III for $\varepsilon = 0$ and $\varrho = 1$, but for $k \leqslant 1$ from (2.2) for $T = 0$.

Remark. The corollary is also true in the second subcase of III of Lemma 4.1 with $\varepsilon = 0$ and $\varrho = 1$, and with some additional conditions.

5. The facts from the theory of diophantine equations. We use the following theorems from the theory of diophantine equations:

LEMMA 5.1 (see [9]). *Let b be a nonzero integer, and let P be a polynomial with rational coefficients with least two distinct zeros. Then the equality*

$$P(x) = by^z, \quad |y| > 1$$

in integers implies that $z < C$, where C is an effectively computable constant depending only on P and b.

LEMMA 5.2 (see [7] and [1], [2]). *Let $P \in \mathbf{Q}[x]$,*

$$P(x) = a_0 x^N + a_1 x^{N-1} + \ldots + a_N = a_0 \prod_{i=1}^{n} (x - \alpha_i)^{r_i},$$

with $a_0 \neq 0$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. Let $0 \neq b \in \mathbf{Z}$, $m \in \mathbf{N}$ and define $v_i = m/(m, r_i)$. Then the equation

$$P(x) = by^m$$

has only finitely many solutions $x, y \in \mathbf{Z}$ unless $\{v_1, \ldots, v_n\}$ is a permutation of one of the n-tuples $\{v, 1, \ldots, 1\}$, $v \geqslant 1$ or $\{2, 2, 1, \ldots, 1\}$. These solutions can be effectively determined.

6. The equation (1.1). We extend Lemma 4 in [10].

LEMMA 6.1. *Let $x_0 \geqslant 1$ be an integer and let f be a function of a set containing $\{0, 1, \ldots, x_0-1\}$ into $\mathbf{Z}$ such that for some $\beta \geqslant 1$ and for any $k \geqslant 0$, $2^\beta B_{k,f}^{(x_0)}$ are 2-integral, and the following congruences hold:*

$$2^\beta B_{k,f}^{(x_0)} \equiv 2B_k \pmod 2. \tag{6.1}$$

Let $R^* \in \mathbf{Z}[x]$ be a polynomial. Set for $n \geqslant 3$

$$P(x) = B_{n,f}^{(x_0)}(x) - B_{n,f}^{(x_0)} + nR^*(x).$$

Then:

(i) $P(x)$ *has at least three zeros of odd multiplicities unless* $n = 3$, 4 *and* 6.

(ii) *For any odd prime* p, *at least two zeros of* $P(x)$ *have multiplicities prime to* p *unless* $n = 4$.

Proof. Since $B_{i,j}^{(x_0)} \in \boldsymbol{Q}$ by (2.1) we can choose $d \in \boldsymbol{N}$ such that

$$dP(x) \in \boldsymbol{Z}[x].$$

Let $d \in \boldsymbol{N}$ be minimal satisfying this condition. Using the same arguments as in the proof of Lemma 4 in [10] we get by the congruence (6.1): $2^{\beta} \| d$ unless $n = 2^r$ for some $r \geqslant 1$. In this case $2^{\beta-1} \| d$.

We distinguish two cases:

(a) Let $n \geqslant 3$ be odd. To prove the lemma in this case for $n > 3$ it suffices to repeat the part A of the proof of Lemma 4 in [10] with d as above. In this case the polynomial $P(x)$ has at least three simple zeros so it satisfies (i) and (ii). Similarly, if $n = 3$ then $P(x)$ has at least two simple zeros, so it satisfies (ii).

(b) Let $n \geqslant 4$ be even. First, we prove (i). In the case $2|n$ we consider two subcases as in [10]. First, let $n = 2^r$ for some $r \geqslant 1$. Then $2^{\beta-1} \| d$ and to prove (i) it suffices to repeat the part B of the proof of Lemma 4 in [10]. Thus using the same arguments as in this proof we get for $r \geqslant 3$

$$dP(x) \equiv d'x^{4s} + 2x^{3s} \pm x^{2s} + 2x^s \pmod 4, \tag{6.2}$$

where $s = n/4$ and

$$dB_{0,j}^{(x_0)} \equiv d' \pmod 4.$$

Let

$$dP(x) = T^2(x)Q(x), \tag{6.3}$$

where $T, Q \in \boldsymbol{Z}[x]$ and Q contains each factor of odd multiplicity of dP exactly once. Assume $\deg Q \leqslant 2$. From (6.2) we get

$$T^2(x)Q(x) \equiv x^{2s}(d'x^{2s} + 1) \pmod 2. \tag{6.4}$$

Therefore $T^2(x)$ must be divisible by x^{2s-2} modulo 2. The rest of the proof goes like the part B of the proof of Lemma 4 in [10]. So

$$T(x) = x^{s-1}T_1(x) + 2T_2(x),$$

$$T^2(x) = x^{2s-2}T_1^2(x) + 4T_3(x),$$

where $T_1, T_2, T_3 \in \boldsymbol{Z}[x]$ and the last identity for $n > 8$ (i.e. $s > 2$) is incompatible with (6.2) because of the term $2x^s$. So we have proved (i) for $n = 2^r$, $r > 3$. If $n = 8$ then the congruence (6.2) holds with $s = 2$. We get

$$dP(x) \equiv d'x^8 + 2x^6 \pm x^4 + 2x^2 \pmod 4 \tag{6.5}$$

and

$$T^2(x) \equiv x^2 T_1^2(x) \pmod 4. \tag{6.6}$$

Since $T_1(x) \not\equiv 0 \pmod 2$ the polynomial $T_1^2(x)$ modulo 4 is monic. Moreover we have

$$\deg(T_1^2(x) \bmod 4) = 2\deg(T_1(x) \bmod 2).$$

Hence and from (6.3) and (6.6) we obtain

$$\deg(dP(x) \bmod 4) = 2 + 2\deg(T_1(x) \bmod 2) + \deg(Q(x) \bmod 4),$$

so, from (6.5),

$$2\deg(T_1(x) \bmod 2) + \deg(Q(x) \bmod 4) = \begin{cases} 6 & \text{if } d' \not\equiv 0 \pmod 4, \\ 4 & \text{if } d' \equiv 0 \pmod 4). \end{cases}$$

Therefore the 2-tuple $\{\deg(T_1(x) \bmod 2), \deg(Q(x) \bmod 4)\}$ equals

1° $\{3, 0\}$ or 2° $\{2, 2\}$ if $d' \not\equiv 0 \pmod 4$; and

3° $\{2, 0\}$ or 4° $\{1, 2\}$ if $d' \equiv 0 \pmod 4$.

We prove that the case 1° is impossible. Let

$$T_1(x) \equiv x^3 + \ldots + c \pmod 2, \quad \text{where } c = 0 \text{ or } 1,$$

and let

$$Q(x) \equiv q \pmod 4, \quad \text{where } q = 1, 2 \text{ or } 3.$$

Here $T_1^2(x) \equiv (x^3 + \ldots + c)^2 \pmod 4$. Hence and from (6.3), (6.5) and (6.6) we get by comparing the coefficients of x^8

$$q \equiv d' \pmod 4.$$

Therefore if $d' \equiv \pm 1 \pmod 4$ then by comparing the coefficients of x^2 we find that

$$c^2 q \equiv 2 \pmod 4 \quad \text{so} \quad c^2 \equiv 2 \pmod 4.$$

We obtain a contradiction. Let $d' \equiv 2 \pmod 4$. Then we have $q \equiv 2 \pmod 4$ so $Q(x) \equiv 0 \pmod 2$. It is incompatible by (6.3) with (6.4) for $s = 2$. We consider the case 2°. Let

$$T_1(x) \equiv x^2 + ax + b \pmod 2, \quad \text{where } a, b = 0 \text{ or } 1,$$

and let

$$Q(x) \equiv px^2 + qx + r \pmod 4, \quad \text{where } p, q, r = 0, 1, 2 \text{ or } 3 \text{ and } p \neq 0.$$

Then $T_1^2(x) \equiv (x^2 + ax + b)^2 \pmod 4$. Therefore we get from (6.3), (6.5) and (6.6) by comparing the coefficients of x^8

$$p \equiv d' \pmod 4$$

and by comparing the coefficients of x^2

$$b^2 r \equiv 2 \pmod 4 \quad \text{so} \quad b = 1 \text{ and } r \equiv 2 \pmod 4.$$

Hence by comparing the coefficients of x^3 we get

$$q+4a \equiv 0 \pmod 4 \quad \text{so} \quad q \equiv 0 \pmod 4.$$

Let $d' \equiv 2 \pmod 4$. Then $p \equiv 2 \pmod 4$ and $Q(x) \equiv 0 \pmod 2$. It is incompatible with (6.4) for $s = 2$. If $d \equiv \pm 1 \pmod 4$ then by comparing the coefficients of x^6 in both sides of (6.5) by (6.3) and (6.6) we find that

$$2+d'(2+a^2) \equiv 2 \pmod 4.$$

Therefore

$$d'(2+a^2) \equiv 0 \pmod 4 \quad \text{and} \quad a^2 \equiv 2 \pmod 4.$$

We get a contradiction, too.

Now, let $d' \equiv 0 \pmod 4$. We consider the case 3°. Let

$$T_1(x) \equiv x^2+\ldots \pmod 2,$$

and let

$$Q(x) \equiv q \pmod 4, \quad \text{where } q = 1, 2 \text{ or } 3.$$

Here $T_1^2(x) \equiv (x^2+\ldots)^2 \pmod 4$ and we get from (6.3) and (6.6) by comparing the coefficients of x^6 in both sides of (6.5)

$$q \equiv 2 \pmod 4, \quad \text{i.e.,} \quad Q(x) \equiv 0 \pmod 2.$$

Therefore we obtain a contradiction with (6.4) for $s = 2$, again. We prove that the case 4° is impossible. Let

$$T_1(x) \equiv x+a \pmod 2, \quad \text{where } a = 0 \text{ or } 1$$

and let

$$Q(x) \equiv px^2+qx+r \pmod 4, \quad \text{where } p, q, r = 0, 1, 2, \text{ or } 3 \text{ and } p \neq 0.$$

Here $T_1^2(x) \equiv (x+a)^2 \pmod 4$. Therefore we get from (6.8), (6.6) and (6.5) by comparing the coefficients of x^6

$$p \equiv 2 \pmod 4$$

and by comparing the coefficients of x^2

$$a^2 r \equiv 2 \pmod 4 \text{ so } a^2 = 1, \quad \text{i.e.,} \quad a = 1 \text{ and } r \equiv 2 \pmod 4.$$

Therefore it is sufficient to compare the coefficients of x^3 and we obtain

$$2ar+a^2 q \equiv 0 \pmod 4, \quad \text{i.e.,} \quad q \equiv 0 \pmod 4.$$

Hence $Q(x) \equiv 0 \pmod 2$ and we get a contradiction with (6.4) for $s = 2$, again. We have proved (i) for $n = 8$, too.

Now, let $n \neq 2^r$ for any r. Put $n = 2^r u$, where $r \geqslant 1$ and $u > 1$ is odd. Then $\binom{n}{2^r}$ is odd and $2^\beta \| d$. To prove (i) it is sufficient to repeat the part C of the proof of Lemma 4 in [10]. In the case $2^{\beta-1} B_{0,f}^{(x_0)} \equiv 1 \pmod 2$ and $n \geqslant 10$ the proof is the same as the part C of that proof.

Let

$$2^{\beta-1} B_{0,f}^{(x_0)} \equiv 0 \pmod 2, \quad \text{i.e.,} \quad dB_{0,f}^{(x_0)} \equiv 0 \pmod 4.$$

Here $4 | 2^\beta n$ and by (2.1) and (6.1) we get

$$\text{(6.7)} \quad dP(x) \equiv \pm nx^{n-1} \pm \binom{n}{2} x^{n-2} + \ldots + \binom{n}{i} dB_{i,f}^{(x_0)} x^{n-i} + \ldots \pm \binom{n}{2} x^2 \pmod 4.$$

Put (6.3) and let

$$T(x) \equiv x^{l_1}+x^{l_2}+\ldots x^{l_m} \pmod 2,$$

where $l_1 > l_2 > \ldots > l_m \geqslant 0$. Then

$$\text{(6.8)} \qquad T^2(x) \equiv x^{2l_1}+x^{2l_2}+\ldots+x^{2l_m}+2\sum_t p_t x^t \pmod 4,$$

where p_t is the number of solutions of $l_i+l_j = t$, $l_i < l_j$, $1 \leqslant i, j \leqslant m$.

Assume $\deg Q \leqslant 2$ and let

$$Q(x) = ax^2+bx+c,$$

where $a, b, c \in \mathbf{Z}$.

We consider two cases: $2 \| n$ $(r = 1)$ and $4 | n$ $(r > 1)$. In the first case, comparing

$$\text{(6.9)} \qquad T^2(x)Q(x) \equiv ax^{2l_1+2}+bx^{2l_1+1}+\ldots \pmod 4$$

and (6.7), we get $a \equiv 0 \pmod 4$ and $b \equiv 2 \pmod 4$. Therefore c must be odd. In this case $l_1 = (n-2)/2$.

In the second case we note that $4 | \binom{n}{i}$ for $1 \leqslant i \leqslant 2^r$ unless $i = 2^{r-1}$ or 2^r. In these cases $2 \| \binom{n}{2^{r-1}}$ and $2 \nmid \binom{n}{2^r}$. Therefore, comparing (6.7) and (6.9), we get by (6.1) $a \equiv 2 \pmod 4$ and $b \equiv 0 \pmod 4$. Therefore c must be odd, again. In this case $l_1 = (n-2^{r-1}-2)/2$.

In both cases $Q(x) \equiv 1 \pmod 2$ and

$$\text{(6.10)} \qquad dP(x) \equiv T^2(x) \equiv x^{2l_1}+x^{2l_2}+\ldots+x^{2l_m} \pmod 2.$$

Denote as in [10]

$$L = \{l_1, l_2, \ldots, l_m\}.$$

By (6.7) we have from (6.10)

$$(6.11)\qquad l \in L \Leftrightarrow 2 \leqslant 2l \leqslant n-2 \quad \text{and} \quad \binom{n}{2l} \text{ is odd.}$$

On the other hand from (6.3) and (6.8) in the case $2\|n$

$$dP(x) \equiv \sum_{l\in L} (2x^{2l+1}+cx^{2l})+2\sum_t p_t x^t (\text{mod } 4).$$

Therefore by (6.7)

$$\left.\begin{matrix} l \in L \\ l < l_1 \end{matrix}\right\} \Rightarrow p_{2l+1} \text{ is odd.}$$

Now it is sufficient in this case to repeat the part C of the proof of Lemma 4 in [10]. Here it must be $n \geqslant 10$ (i.e., $n \neq 6$). Now, let $4|n$. From (6.11) we conclude that $\binom{n}{2l_1}$ is odd so $\binom{n}{2^{r-1}+2}$ is odd. We get a contradiction for $r \geqslant 3$ because, for $1 \leqslant i < 2^r$, $\binom{n}{i}$ is even.

Let $r = 2$. Then $0, 1 \notin L$ and $2 \in L$. On the other hand, in the case $4|n$ we have from (6.3) and (6.8)

$$dP(x) \equiv \sum_{l\in L} (2x^{2l+2}+cx^{2l})+2\sum_t p_t x^t (\text{mod } 4).$$

In the case $r = 2$ we have

$$\sum_{l\in L} (2x^{2l+2}+cx^{2l}) \equiv \ldots + cx^4 (\text{mod } 4).$$

Moreover, from the definition of p_t, if $2\sum_t p_t x^t \not\equiv 0 \,(\text{mod } 4)$ then

$$\deg\bigl(2\sum_t p_t x^t (\text{mod } 4)\bigr) > 2.$$

Thus we get a contradiction with (6.7) because of the term $\pm\binom{n}{2}x^2$.

The proof of (i) is complete.

Now, we prove (ii). If $n \neq 2^r$ for any r then to prove (ii) it is sufficient to repeat the beginning of the part C of the proof of Lemma 4 in [10].

Let $n = 2^r$ for some $r \geqslant 3$. From (6.6) if d' is odd then

$$dP(x) \equiv x^{\mu}(x+1)^{\mu} (\text{mod } 2),$$

where $\mu = n/2$. Since μ is prime to p for any odd prime p, the polynomial $P(x)$ has at least two zeros of multiplicities prime to p and (ii) is proved in this case.

Let d' be even and let p be an odd prime number. Assume that

$$(6.12)\qquad dP(x) = cT^p(x)(ax+b)^k,$$

where $T \in \mathbf{Z}[x]$, $a, b, c \in \mathbf{Z}$, c is odd and $0 \leqslant k \leqslant p-1$. If $a \equiv 0 \,(\text{mod } 2)$ or $k = 0$ then we get from the congruence (6.4)

$$2^{r-1} = 2s = p\deg(T(x) \text{mod } 2).$$

It is impossible so $a \equiv 1 \,(\text{mod } 2)$ and $k \geqslant 1$. Moreover it follows from (6.4) that $b \equiv 0 \,(\text{mod } 2)$ and

$$2^{r-1} = 2s = p\deg(T(x) \text{ mod } 2)+k.$$

Therefore $2s-k \geqslant 0$ must be divisible by p and

$$T(x) \equiv x^{(2s-k)/p} (\text{mod } 2).$$

Hence

$$T^{p-1}(x) \equiv x^{\frac{p-1}{p}(2s-k)} (\text{mod } 4)$$

and

$$T^p(x) \equiv x^{\frac{p-1}{p}(2s-k)}T(x) (\text{mod } 4).$$

Therefore (6.12) is incompatible with (6.2) because of the term $2x^s$ unless $r = 3$, $k = 1$, $p = 3$ and $b \equiv 2 \,(\text{mod } 4)$. Precisely, we have $T(x) \not\equiv 0 \,(\text{mod } 4)$, i.e., $\deg(T(x) \text{mod } 4) \geqslant 0$ and $\frac{p-1}{p}(2s-k)+k_1 > s$, where $k_1 = k$ if $b \equiv 0 \,(\text{mod } 4)$ or $b \equiv 2 \,(\text{mod } 4)$ and $k \equiv 0 \,(\text{mod } 2)$, and $k_1 = k-1$ if $b \equiv 2 \,(\text{mod } 4)$ and $k \equiv 1 \,(\text{mod } 2)$. In the case $b \equiv 2 \,(\text{mod } 4)$ we have used the congruence

$$(ax+2)^k \equiv (ax)^k + 2k(ax)^{k-1} (\text{mod } 4).$$

For $r = 3$ (i.e., $n = 8$), $k = 1$, $p = 3$ and $b \equiv 2 \,(\text{mod } 4)$ we have

$$T^3(x) \equiv x^2 T(x) (\text{mod } 4).$$

Hence and from (6.12) we get

$$dP(x) \equiv x^2 T(x)(\pm x+2) (\text{mod } 4).$$

Let $T(x) \equiv \ldots + qx + r (\text{mod } 4)$, where $q, r = 0, 1, 2$ or 3. We obtain by (6.5) that $r = 1$ or 3 and

$$\pm r + 2q \equiv 0 \,(\text{mod } 4).$$

It is impossible. We have proved (ii) in the case $n = 2^r$ for some $r \geqslant 3$. The proof of Lemma 6.1 is complete.

Remarks.

1 (On the case $n = 3$). If $n = 3$ then by (2.1)

$$P(x) = B_{0,f}^{(x_0)}x^3 + 3B_{1,f}^{(x_0)}x^2 + 3B_{2,f}^{(x_0)}x + 3R^*(x).$$

Moreover, we get from (2.2) that

$$B_{0,f}^{(x_0)} = \frac{t_0}{x_0}, \quad B_{1,f}^{(x_0)} = \frac{t_1}{x_0} - \frac{t_0}{2} \quad \text{and} \quad B_{2,f}^{(x_0)} = \frac{t_2}{x_0} - t_1 + \frac{x_0 t_0}{6},$$

where

$$t_r = \sum_{i=0}^{x_0-1} f(i) i^r.$$

Therefore

$$2x_0 P(x) \equiv x(2x^2 + x_0^2) t_0 \pmod 3.$$

Hence if $3 \nmid t_0 x_0$ then

$$P(x) \equiv \pm x(x+1)(x-1) \pmod 3$$

so $P(x)$ has at least three simple zeros in this case. If $3x_0 | t_0$ then we can choose a polynomial $R^* \in \mathbf{Z}[x]$ such that $\deg P \leqslant 2$. Therefore (i) for $n = 3$ need not be satisfied.

2 (On the functions f not satisfying (6.1)). In Lemma 6.1 we have proved that if f and x_0 satisfy (6.1) (i.e., (4.1) with $\varepsilon = 0$ and $\varrho = 1$) then the polynomial $P(x)$, for any polynomial $R^* \in \mathbf{Z}[x]$, satisfies (i) and (ii). For example, it holds in the cases (a), (b) and (c) of Corollary 4.2. In the remaining cases, i.e., if $\varepsilon \neq 0$ or $\varrho \neq 1$, the situation is more complicated. For example, consider the first subcase of III in Lemma 4.1 for $\varepsilon = 1$ and $\varrho = 1$. Let f: $\{0, 1, 2, 3\} \to \mathbf{Z}$ be a function such that:

(6.13) $\quad t_0 = 0, \quad 2 \nmid b_i$ for $i = 0, 1, 2, 3, \quad$ and $\quad b_1 \not\equiv b_3 \pmod 4$.

Here $\beta = 1$ in the notation of Lemma 4.1. We have in this case that $B_{1,f}^{(4)} \in \mathbf{Z}$. The condition $t_0 = 0$ implies that $B_{0,f}^{(4)} = 0$, of course. Take a prime number $n = p = 2^r + 1$, $r \geqslant 1$. In general, for $n = 2^r + 1$, $r \geqslant 1$, we have for $0 \leqslant i \leqslant n$, $2 | \binom{n}{i}$ unless $i = 0, 1, n-1$ or n. Since by (4.1) in this case

$$2B_{p-1,f}^{(4)} \equiv 2B_p \equiv 0 \pmod 2,$$

we get for $0 \leqslant i \leqslant p-1$

$$p | \binom{p}{i} B_{i,f}^{(4)} \in \mathbf{Z}.$$

Therefore all the coefficients of the polynomial $B_{p,f}^{(4)}(x) - B_{p,f}^{(4)}$ are integral and divisible by p. Consequently we can find a polynomial $R^* \in \mathbf{Z}[x]$ such that $P(x) = B_{p,f}^{(4)}(x) - B_{p,f}^{(4)} + pR^*(x)$ has zeros of any prescribed multiplicities.

Examples. 1. Let f: $\{0, 1\} \to \mathbf{Z}$ be a function satisfying $f(0) = -1$, $f(1) = 1$. Then

$$B_{n,f}^{(2)}(x) = B_n(x) - 2^n B_n(x/2).$$

It is an immediate consequence of (2.2) for above f and of

$$2^{n-1} B_n((x+1)/2) = B_n(x) - 2^{n-1} B_n(x/2).$$

The last equality is also a consequence of (2.2) for $f = 1$. In this case the polynomial $P(x)$ from Lemma 6.1 satisfies (i) unless $n \leqslant 4$. If $n = 6$ then

$$P(x) = 3x^5 + \tfrac{15}{2}x^4 + \tfrac{15}{2}x^2 + 6R^*(x).$$

Hence

(6.14) $$2P(x) \equiv 2x^5 + x^4 + 3x^2 \pmod 4.$$

Put (6.3) with $d = 2$ and assume $\deg Q \leqslant 2$. The polynomial $T^2(x)$ is monic mod 4 because by (6.14) $T(x) \not\equiv 0 \pmod 2$. Moreover we have

$$\deg(T^2(x) \bmod 4) = 2\deg(T(x) \bmod 2).$$

Hence and from (6.14) we find that

$$2\deg(T(x) \bmod 2) + \deg(Q(x) \bmod 4) = 5.$$

Therefore $\deg(T(x) \bmod 2) = 2$ and $\deg(Q(x) \bmod 4) = 1$. Let

$$T(x) \equiv x^2 + ax + b \pmod 2, \quad \text{where } a, b = 0 \text{ or } 1$$

and

$$Q(x) \equiv px + q \pmod 4, \quad \text{where } p, q = 0, 1 \text{ or } 3 \text{ and } p \neq 0.$$

Here

$$T^2(x) \equiv (x^2 + ax + b)^2 \pmod 4.$$

Therefore we get from (6.3) (with $d = 2$) and (6.14) by comparing the coefficients of x^5

$$p \equiv 2 \pmod 4.$$

Moreover

$$b^2 q \equiv 0 \pmod 4.$$

If $b = 0$ then by comparing the coefficients of x^2 we have

$$a^2 q \equiv 3 \pmod 4 \quad \text{so} \quad a = 1 \text{ and } q \equiv 3 \pmod 4.$$

Here $T^2(x) \equiv x^4 + 2x^3 + x^2 \pmod 4$ and $Q(x) \equiv 2x + 3 \pmod 4$ so

$$T^2(x)Q(x) \equiv 2x^5 + 3x^4 + 3x^2 \pmod 4.$$

It is incompatible by (6.3) with (6.14). If $b = 1$ then $q \equiv 0 \pmod 4$ so $Q(x) \equiv 0 \pmod 2$. Hence by (6.3) $2P(x) \equiv 0 \pmod 2$. It is incompatible with (6.14), again. Therefore $P(x)$ for $n = 6$ satisfies (i).

2. Let $f\colon \{0, 1, 2, 3\} \to \mathbf{Z}$ be a function satisfying

$$f(0) = f(3) = 1 \quad \text{and} \quad f(1) = f(2) = -1.$$

This function satisfies the conditions (6.13). Let $n = 7$. Of course 7 is not of the form $2^r + 1$. Here

$$B_{7,f}^{(4)}(x) - B_{7,f}^{(4)} = 21x^5 - \tfrac{105}{2}x^4 - 140x^3 + \tfrac{525}{2}x^2 + 336x$$

so there exists a polynomial $R^* \in \mathbf{Z}[x]$ such that

$$P(x) = -\tfrac{105}{2}x^2(x^2-5).$$

This $P(x)$ satisfies (ii) but it does not satisfy (i), of course.

Now, we generalize Theorem of [10].

THEOREM 6.2. *Let $x_0 \geqslant 1$ be an integer and let*

$$f\colon N_0 \to \mathbf{Z}$$

be a periodic function with the period x_0 such that for some $\beta \geqslant 1$ and for any $k \geqslant 0$, $2^\beta B_{k,f}^{(x_0)}$ are 2-integral and the congruences

$$2^\beta B_{k,f}^{(x_0)} \equiv 2B_k \pmod 2$$

hold.

If $R \in \mathbf{Z}[x]$ is a fixed polynomial and $b \neq 0$ and $k \geqslant 4$, $k \neq 5$ are fixed integers then the equation

$$\sum_{i=0}^{x} f(i)i^k + R(x) = by^z$$

has only finitely many solutions in integers $x \geqslant 1$, $y, z > 1$. These solutions can be effectively determined.

Proof. Note that for every $x_1 < x_0$ the equation

$$\sum_{i=0}^{x_1} f(i)i^k + R(x_1) = by^z$$

has finitely many solutions in integers $y, z > 1$.

Let $x \geqslant x_0$ and let $x \equiv r \pmod{x_0}$, where $0 \leqslant r \leqslant x_0 - 1$. We can rewrite the equation (1.1) in the form

$$\sum_{i=0}^{x_0x'-1} f(i)i^k + f(x_0x')(x_0x')^k + f(x_0x'+1)(x_0x'+1)^k + \ldots + f(x)x^k + R(x) = by^z,$$

where $x = x_0x' + r$.

Then we get from (2.3) and Lemma 2.1

$$\frac{1}{k+1}\left[B_{k+1,f}^{(x_0)}(x-r) - B_{k+1,f}^{(x_0)}\right] + \sum_{i=0}^{r} f(x_0x'+i)(x_0x'+i)^k + R(x) = by^z.$$

Therefore we can rewrite the equation (1.1) in the form

$$\frac{1}{k+1}\left[B_{k+1,f}^{(x_0)}(x) - B_{k+1,f}^{(x_0)}\right] + R^*(x) = by^z, \tag{6.15}$$

where

$$R^*(x) = R(x+r) + \sum_{i=0}^{r} f(i)(x+i)^k \in \mathbf{Z}[x].$$

Note that to prove the theorem it suffices to prove that the equation (6.15) has finitely many integer solutions $x \geqslant x_0$ and $y, z > 1$ for any polynomial $R^* \in \mathbf{Z}[x]$ (not necessarily for R^* of the above form). Let $R^* \in \mathbf{Z}[x]$ be any polynomial. From Lemma 6.1 putting $n = k+1$ we conclude that

$$P(x) = \frac{1}{n}\left[B_{n,f}^{(x_0)}(x) - B_{n,f}^{(x_0)}\right] + R^*(x)$$

satisfies (i) and (ii). Thus it is sufficient to use Lemmas 5.1 and 5.2 and similarly as in [10] the theorem follows.

COROLLARY 6.3. *Let $f\colon N_0 \to \mathbf{Z}$ be a periodic function with the period x_0. Let $R \in \mathbf{Z}[x]$ be any fixed polynomial and let $b \neq 0$ and $k \geqslant 4$, $k \neq 5$ be fixed integers. If f and x_0 satisfy the conditions* (a), (b) *or* (c) *of Corollary* 4.2 *then the equation* (1.1) *has finitely many solutions in integers $x \geqslant 1$, $y, z > 1$. These solutions can be effectively determined.*

Proof. This is an immediate corollary from Theorem 6.2 and Lemma 4.2.

Hence we have

COROLLARY 6.4. *Let $f\colon N_0 \to \{\pm 1\}$ be a periodic function with the period x_0 and let*

$$4 \nmid x_0 \quad \text{or} \quad \begin{cases} 4 \| x_0, \\ b_i \equiv b_{i+2} \pmod 4 \quad \text{for } i = 0, 1. \end{cases}$$

If $R \in \mathbf{Z}[x]$ is any fixed polynomial, and $b \neq 0$ and $k \geqslant 4$, $k \neq 5$ are fixed integers then the equation (1.1) *has finitely many solutions in integers $x \geqslant 1$, $y, z > 1$. These solutions can be effectively determined.*

Remark. The last corollary is also true for $8 | x_0$ with some additional conditions (see the remark after Corollary 4.2).

EXAMPLES. 1. Let $f\colon N_0 \to \mathbf{Z}$ be a function defined by

$$f(i) = (-1)^{i+1} \quad \text{for } i \in N_0.$$

It is a periodic function with the period of length 2. We get from Theorem 6.2 and Example 1 after Lemma 6.1 that the equation

$$1^k - 2^k + \ldots + (-1)^{x+1} x^k + R(x) = by^z \tag{6.16}$$

for any $R \in \mathbf{Z}[x]$, $b \in \mathbf{Z}$, $b \neq 0$ and $k \geqslant 4$ has only finitely many solutions in integers $x \geqslant 1$, $y, z > 1$.

We consider the equation (6.16) for $k = 2$ and 3 and fixed $z = m > 1$. Put in Lemma 6.1 $n = k+1$. If $n = 3$ then $2P(x) \equiv x(x-1) \pmod 2$. Therefore by Lemma 5.2 the equation (6.16) for $k = 2$ has finitely many integer solutions unless $m = 2$. Let $S \in \mathbf{Z}[x]$ be a polynomial. Put

$$R^*(x) = x(x-1)(2S^2(x) + 2S(x)).$$

The equation

$$\tfrac{3}{2} x(x-1)(2S(x)+1)^2 = by^2 \tag{6.17}$$

reduces to Pell's equation so it has infinitely many integer solutions $x \geqslant 1$, $y > 1$ for infinitely many choices of b. Thus it has infinitely many solutions such that x is even or it has infinitely many solutions such that x is odd. Put in the equation (6.16)

$$R(x) = R^*(x) + x^k \quad \text{or} \quad R^*(x+1) \tag{6.18}$$

according as the equation (6.17) has infinitely many solutions with even or odd x. Then the equation (6.16) for this $R(x)$ has infinitely many solutions with even or odd x because it reduces to (6.17). Therefore the equation (6.16) for $k = 2$ has infinitely many solutions for suitably chosen b and R.

Similarly, if $k = 3$ (i.e., $n = 4$) then we have $P(x) = 2x^3 - 3x^2 + 4R^*(x)$. Let p be a prime number and let $S \in \mathbf{Z}[x]$ be a polynomial. Put

$$R^*(x) = \begin{cases} x^2 \sum_{i=2}^{p} \binom{p}{i} 2^{i-2} x^i (2S(x)+1)^i + x\left(pS(x) + \dfrac{p-1}{2}\right) + 1 & \text{if } p \geqslant 3, \\ x^2(2x-3)(S^2(x) + S(x)) & \text{if } p = 2. \end{cases}$$

The equations

$$x^2[2x(2S(x)+1)+1]^p = by^p, \tag{6.19}$$

$$\tfrac{3}{2} x(2x-3)(2S(x)+1)^2 = by^2 \tag{6.20}$$

have infinitely many integer solutions $x \geqslant 1$, $y > 1$ for infinitely many choices of b. Thus each of them has infinitely many solutions such that x is even or each of them has infinitely many solutions such that x is odd. Put in the equation (6.16) $z = p$ and $R(x)$ as in (6.18). Then the equation (6.16) for this $R(x)$ has infinitely many solutions with even or odd x because it reduces to (6.19) if $p \geqslant 3$ and to (6.20) if $p = 2$. Therefore the equation (6.16) for $k = 3$ and $z = p$ can have infinitely many solutions for suitably chosen b and R.

Before two next examples, we have the following remark. Take the notation of Theorem 6.2. Let $R^* \in \mathbf{Z}[x]$ be a polynomial. Put for $r \geqslant 0$

$$R(x) = R^*(x-r) - \sum_{i=0}^{r} f(i)(x-r+i)^k.$$

Note that if the equation

$$P(x) = by^z, \tag{6.21}$$

where

$$P(x) = \frac{1}{n}[B_{n,f}^{(x_0)}(x) - B_{n,f}^{(x_0)}] + R^*(x)$$

has infinitely many integer solutions $x \geqslant 1$, $y, z > 1$ then this equation has infinitely many integer solutions $x \geqslant 1$, $y, z > 1$ such that $x \equiv r \pmod{x_0}$ for some $0 \leqslant r \leqslant x_0 - 1$. Therefore the equation (1.1) for $k = n-1$ with $R \in \mathbf{Z}[x]$ defined above has infinitely many integer solutions $x \geqslant 1$, $y, z > 1$ such that $x \equiv r \pmod{x_0}$ so it has infinitely many solutions, in general.

2. Let $x_0 = 4$ and let $f\colon N_0 \to \{\pm 1\}$ be a periodic function with the period x_0 defined by means of $f(0) = f(3) = 1$ and $f(1) = f(2) = -1$. Then for these f and x_0: $b_0 = b_3 = 1$ and $b_1 = b_2 = -1$ and they satisfy the conditions (6.13). We use Remark 2 after Lemma 6.1 and Lemma 2.1. Take $k = 2^r$, $r \geqslant 1$ such that $2^r + 1$ is a prime number. Let $n = k+1$. Then we can find a polynomial $R^* \in \mathbf{Z}[x]$ such that the equation (6.12) has infinitely many integer solutions. Therefore the equation (1.1) with $R \in \mathbf{Z}[x]$ defined above has infinitely many integer solutions, too.

3. Let $x_0 = 4$ and let f be as above. Consider the equation (1.1) for $k = 6$ and for fixed $z = m > 1$. We use Example 2 after Lemma 6.1 and Lemma 2.1. Here

$$2P(x) \equiv x^2(x-1)(x+1) \pmod 2$$

so in view of Lemma 5.2 the equation (1.1) has only finitely many integer solutions x, y for any $R \in \mathbf{Z}[x]$ unless $m = 2$.

Let $S \in \mathbf{Z}[x]$ be any polynomial. Put

$$R^*(x) = -15x^2(x^2-5)(2S^2(x) + 2S(x)) - 3x^5 + 20x^3 - 48x.$$

Then for $m = 2$ the equation (6.21) takes the form

$$-\tfrac{15}{2} x^2(x^2-5)(2S(x)+1)^2 = by^2.$$

It amounts to Pell's equation having infinitely many integer solutions x, y for infinitely many choices of b. Therefore the equation (1.1) with above defined $R \in \mathbf{Z}[x]$ has also infinitely many integer solutions.

Remark. The cases $k = 1, 3, 5$ for $f = 1$ are discussed in [10].

References

[1] B. Brindza, *On some generalizations of the diophantine equation* $1^k+2^k+\ldots+x^k=y^z$, Acta Arith. 44 (1984), 99–107.
[2] — *On S-integral solutions of the equation* $f(x)=y^m$, Acta Math. Acad. Sci. Hungar. 44 (1984), 133–139.
[3] K. Dilcher, *Irreducibility and zeros of generalized Bernoulli polynomials*, C. R. Math. Rep. Acad. Sci. Canada 6 (1984), no. 5, 273–278.
[4] — *On a diophantine equation involving quadratic characters*, Comp. Math. 57 (1986), 383–403.
[5] K. Győry, R. Tijdeman and M. Voorhoeve, *On the equation* $1^k+2^k+\ldots+x^k=y^z$, Acta Arith. 37 (1980), 233–240.
[6] S. Lang, *Introduction to modular forms*, Springer-Verlag, Berlin, Heidelberg, New York 1976.
[7] W. J. LeVeque, *On the equation* $y^m=f(x)$, Acta Arith. 9 (1964), 209–219.
[8] J. J. Schäffer, *The equation* $1^p+2^p+\ldots+n^p=m^q$, Acta Math. 95 (1956), 155–189.
[9] A. Schinzel and R. Tijdeman, *On the equation* $y^m=P(x)$, Acta Arith. 31 (1976), 199–204.
[10] M. Voorhoeve, K. Győry and R. Tijdeman, *On the diophantine equation* $1^k+2^k+\ldots+x^k+R(x)=y^z$, Acta Math. 143 (1979), 1–8 and *Corrigendum*, Acta Math., to appear.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
Warsaw, Poland

Received on 17.10.1986
and in revised form on 7.7.1987 (1682)

ACTA ARITHMETICA
LI (1988)

On the average number of direct factors of a finite Abelian group

by

EKKEHARD KRÄTZEL (Jena)

1. Introduction. Let G be a finite Abelian group. Let $\tau(G)$ denote the number of direct factors of G and

$$T(x)=\sum\tau(G),$$

where the summation is extended over all Abelian groups of order not exceeding x. E. Cohen [1] proved the representation

$$T(x)=\gamma_{1,1}x(\log x+2C-1)+\gamma_{1,2}x+\Delta(x),$$

where $\Delta(x)$ is estimated by

$$\Delta(x)\ll\sqrt{x}\log^2 x.$$

In this paper we improve this result by

$$\Delta(x)=\gamma_{2,1}\sqrt{x}(\tfrac{1}{2}\log x+2C-1)+\gamma_{2,2}\sqrt{x}+O(x^{5/12}\log^4 x).$$

In these formulas C denotes Euler's constant, and $\gamma_{1,1},\ldots,\gamma_{2,2}$ are given by (22)–(25).

A similar situation takes place when we consider the unitary factors of G, that is, the total number of direct decompositions of G into 2 relatively prime factors. Let $t(G)$ denote the number of unitary factors of G and

$$T^*(x)=\sum t(G),$$

where again the summation is extended over all the Abelian groups of order not exceeding x. Here E. Cohen [1] proved that

$$T^*(x)=c_{1,1}x(\log x+2C-1)+c_{1,2}x+\Delta^*(x),\qquad \Delta^*(x)\ll\sqrt{x}\log x.$$

In this paper we prove

$$\Delta^*(x)=c_2\sqrt{x}+O(x^{11/29}\log^2 x),$$

where $c_{1,1}$, $c_{1,2}$, c_2 are defined by (13), (14).

It is not hard to prove this estimate for $\Delta^*(x)$. Therefore, the main point of