

7, K. & Benney, D. The propagation of nonlinear shear flow with a free surface, *Studies in Applied Math.*, 1976, 69-92.

ical simulation of the kinetic equations for surface waves, *Izv. Akad. Nauk SSSR. Fiz. Atmosf. Ok.*, 1990, 26, 118-123 (English translation: *Izvestia, Atmospheric and Oceanic Physics*, 1990, 26, 118-123)

waves on shear currents: Solution of the boundary-value problem, *Fluid Mech.*, 1993, 253, 565-584.

Formation of a narrow angular spectrum of wind-generated nonlinear interaction between wind and waves, *Izv. Akad. Nauk SSSR. Fiz. Atmosf. Ok.*, 1989, 25, no.4, 411-420 (English translation: *Atmospheric and Oceanic Physics*, 1989, 25, no.4, 411-420)

Linear and Nonlinear Waves, Wiley, New York, 1974, 1-100.

Stability of periodic waves of finite amplitude on the surface of a deep fluid, *J. Appl. Math. Techn. Phys.*, 1968, 9, 190-194.

Smilga, A.V. On the dynamics of water wave spectra in the presence of a current, *Zh. Eksp. Teor. Fiz.*, 1981, 81, 1318-1324 (English translation: *Sov. Phys. JETP*, 1981, 54, 700-706).

Spiridonov, V.I. 1990: On the formation of the directional spectrum of wind waves, *Zh. Eksp. Teor. Fiz.*, 1990, 98, no.6(12), 1941-1946 (English translation: *Sov. Phys. JETP*, 1990, 71, no.6, 1091-1100).

Solov'ev, S.V., Novikov, S.P. & Pitaevsky, L.P. *Solitons: Method of inverse scattering*, Nauka, Moscow, 1980, 1-288 (English translation: Plenum Press, N.Y., 1984, 288 pp.)

Stashenko, M.M. Generation and dissipation ranges of weak turbulence theory for wind waves, *Izv. Akad. Nauk SSSR. Fiz. Atmosf. Ok.*, 1983, 18, no.10, 1066-1076 (English translation: *Izvestia, Atmospheric and Oceanic Physics*, 1983, 18, no.10, 1066-1076)

The long-wave cut-off in the wind driven wave spectrum, *Izv. Akad. Nauk SSSR. Fiz. Atmosf. Ok.*, 1989, 25, 118-123 (English translation: *Izvestia, Atmospheric and Oceanic Physics*, 1989, 25, 118-123)

Chapter 7

Direct modeling of one-dimensional nonlinear potential waves

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Abstract

A method for numerical investigation of nonlinear wave dynamics based on direct hydrodynamical modeling of 1-D potential periodic surface waves is presented. By a nonstationary conformal mapping, the principal equations are rewritten in a surface-following coordinate system and reduced to two simple evolutionary equations for the elevation and the velocity potential of the surface. For stationary equations, the proposed approach coincides with the conventional complex variable method. For this case, numerical algorithms for solution of gravity (Stokes) and gravity-capillary wave equations are proposed, and examples of numerical solutions are given. The results imply that gravity-capillary waves do not approach Stokes waves as the capillarity coefficient decreases. Both stationary and nonstationary schemes use Fourier series representation for spatial approximation and the Fourier transform method to calculate nonlinearities. The nonstationary model was validated by simulation of propagating waves with initial conditions obtained as numerical (for gravity and gravity-capillary waves) or analytical (for pure capillary, or Crapper's waves) solutions of the stationary problem. The simulated progressive waves did not change their shape during long-term time integration, which indicates high accuracy of the scheme. Another criterion used for model validation was conservation of integral invariants of simulated multi-mode wave fields. A number of long-term model simulations of gravity, gravity-capillary, and pure capillary waves, with various initial conditions, were performed; for the simulated wave fields, distributions of energy and phase speed over full spectra were analyzed. It was found that the wavenumber-frequency spectra are

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l separated into patterns lying along regularly located curves, with most of the energy concentrated along the curves corresponding to free and bound waves. This set of curves can be described by the equation $D(\omega/n, k/n) = 0$ ($n = 1, 2, 3, \dots$), where $D(\omega, k) = 0$ is approximated by the linear dispersion relation but does not coincide with it, especially for large k where there is a tendency for the indicated relation to approximate a straight line. Some other properties of simulated wave fields were also analyzed; these included temporal evolution of the spectra and spatial distribution of the energy of perturbations. The method developed may be applied to a broad range of problems where the assumption of one-dimensionality is acceptable.

Introduction

Computational techniques for numerical solution of the Navier–Stokes equations have brought new developments to geophysical fluid dynam-

Using modern numerical models, the long-term evolution of several complicated dynamical phenomena in different fluids, including the atmosphere, can be successfully simulated. However, the long-term simulation of a nonlinear multi-mode wave field is difficult to perform, since the best numerical schemes for the Euler equations fail to provide sufficient accuracy for treating nonlinearities in wave motion. The main source of error is primarily due to the finite difference representation of the vertical structure of the flow when waves with different wavenumbers are present. In this case, theoretical and numerical investigations of surface gravity waves are usually based on the equations for potential flow with a free surface. In this case the flow is fully determined by the form of the surface and the velocity potential on the surface and in its vicinity. The potential motion assumption, of course, idealizes the phenomenon, since actual wave motion is both rotational and turbulent. Fortunately, potential theory gives many results which agree well with observations. For example, it is well-known that even linear theory yields phase velocity estimates with an accuracy of the order of 1%. A much more sophisticated theory, dealing with nonlinear wave–wave interactions (Hasselmann, 1962), which is based on the potential motion assumption, gives results which are confirmed by experimental data.

The main advantage of the potential motion approximation is that the system of Euler dynamical equations is reduced to the Laplace equation. However, the solution to the problem of surface wave motion is compli-

ated by boundary conditions which is not the case in the previous system (Cartesian coordinate system). This does not mean that the method is not applicable on a for coordinate Cartesian coordinate system. This does not mean that the method is not applicable on a for coordinate Cartesian coordinate system. This does not mean that the method is not applicable on a for coordinate Cartesian coordinate system.

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Our generalization of the 1-D potential motion approximation to three-dimensional scales much larger than the wavelength is based on

along regularly located curves, with most of the modes corresponding to free and bound waves. This is the equation $D(\omega/n, k/n) = 0$ ($n = 1, 2, 3, \dots$), defined by the linear dispersion relation but does not depend on k where there is a tendency for the indicated curve to be a straight line. Some other properties of simulated wave fields include the temporal evolution of the spectra and spatial distributions. The method developed may be applied to the simulation of one-dimensionality is acceptable.

The numerical solution of the Navier-Stokes equations has led to developments in geophysical fluid dynamics. Models of the long-term evolution of several phenomena in different fluids, including the atmosphere and ocean, are simulated. However, the long-term simulation of a wave field is difficult to perform, since the Euler equations fail to provide sufficient detail in wave motion. The main source of error is the finite difference representation of the vertical derivatives with different wavenumbers are present. Numerical investigations of surface gravity waves have been conducted for potential flow with a free surface, determined by the form of the surface and the velocity field in its vicinity. The potential model idealizes the phenomenon, since actual wave fields are turbulent. Fortunately, potential theory agrees well with observations. For example, it is a simple theory yields phase velocity estimates with good accuracy. A much more sophisticated theory, dealing with nonlinear interactions (Hasselmann, 1962), which is based on a nonlinear motion assumption, gives results which are in good agreement with observations.

The potential motion approximation is that the nonlinear interactions is reduced to the Laplace equation. The problem of surface wave motion is compli-

cated by the requirement of having to apply the kinematic and dynamic boundary conditions (both nonlinear) on the free surface, the location of which is unknown at any given moment. Some attempts have been made previously to reproduce the evolution of waves in a Cartesian coordinate system (e.g., Prosperetti and Jacobs, 1983), but such techniques are not applicable to long-term integrations. A more feasible approach is based on a formulation of the governing equations in a surface-following coordinate system; the simplest technique uses the difference between the Cartesian vertical coordinate and the surface height as the new vertical coordinate, along with Cartesian coordinates in the horizontal. However, this does not eliminate all of the problems, since the Laplace equation is transformed into a general elliptic equation, and an integral equation must be solved at each time step (Chalikov and Liberman, 1991) to calculate the vertical derivative of the velocity potential. Another approach is based on expanding the velocity potential in power series in the vicinity of the surface. Such a method, developed by Watson and West (1975), was applied to the solution of the two-dimensional potential wave equations (West *et al.*, 1987). Even though this model gave excellent results for a relatively small number of modes, the method is not universal since the convergence of the power series is slow for the case of multi-mode wave fields with typical spectral energy distributions.

In this study we consider only 1-D nonlinear waves. Such waves were simulated numerically with a quasi-Lagrangian technique (Longuet-Higgins and Cokelet, 1976), and with a Cauchy-type integral algorithm (Dold, 1991). The performance of neither scheme was limited by wave steepness, and both were capable of simulating the initial phase of wave breaking (a phenomenon whose later stages are rotational and remain extremely difficult to simulate directly). A method based on a Taylor expansion of the Dirichlet-Neumann operator was developed by Craig and Sulem (1993). The method was illustrated by computing evolution of modulated wave packets and a low order approximation of the Stokes wave for relatively short periods. However, the applicability of these methods to simulating longer time scales is uncertain.

Our goal is to construct a numerical scheme for direct modeling of 1-D potential waves so that the effects of nonlinear interactions on time scales much longer than the wave period may be analyzed. The approach is based on a nonstationary conformal mapping which allows us to rewrite

the equations of potential waves (which take into account the effects of capillarity and finite depth) in a surface-following coordinate system, where the Laplace equation retains its form, so that the original system can be represented by two relatively simple evolutionary equations (Section 2). These equations may be solved by using Fourier transform method with high accuracy and computational efficiency. Section 3 deals with stationary solutions of the system; a numerical method to obtain stationary gravity (Stokes) and gravity-capillary waves is presented and results of the computations are discussed. The numerical scheme for the nonstationary equations is described in Section 4. The results of Section 3, as well as mass, momentum and energy conservation criteria, are used for validation of the nonstationary model (Section 5). In Section 6, results of long-term model simulations are discussed, and spectral properties of the obtained wave fields are analyzed.

2. Equations

Consider the principal 2-D equations for potential waves written in Cartesian coordinates, i.e., the Laplace equation for the velocity potential Φ

$$\Phi_{xx} + \Phi_{zz} = 0, \quad (1)$$

and the two boundary conditions at the free surface $z = h(x, t)$: the kinematic condition

$$h_t + h_x \Phi_x - \Phi_z = 0, \quad (2)$$

and the Lagrange integral

$$\Phi_t + \frac{1}{2} (\Phi_x^2 + \Phi_z^2) + h + p - \sigma h_{xx} (1 + h_x^2)^{-3/2} = 0, \quad (3)$$

where p is the external surface pressure.¹

The equations are to be solved in the domain

$$-\infty < x < \infty, \quad -H \leq z \leq h(x, t). \quad (4)$$

The variables Φ and h are considered to be periodic with respect to x , the period being 2π , and a zero normal velocity condition at the bottom is assumed:

$$\Phi_z(x, z = -H, t) = 0. \quad (5)$$

¹Subscripts of independent variables denote partial differentiation with respect to this variable.

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Equations (1)–(3) are written in nondimensional form, with the fol-
lowing scales: length L , where $2\pi L$ is the (dimensional) period in the
horizontal, time $\mathcal{T} = L^{1/2} g^{-1/2}$, and the velocity potential $L^{3/2} g^{1/2}$ (g -
acceleration of gravity). The last term in eqn (3) describes the effect of
surface tension, and

$$\sigma = \frac{\Gamma}{g L^2} \tag{6}$$

is a nondimensional parameter ($\Gamma \simeq 8 \cdot 10^{-5} \text{m}^3 \text{s}^{-2}$ is the kinematic co-
efficient of surface tension for water).

System (1)–(3) is solved as an initial value problem for the unknown
functions Φ and h with given initial conditions $\Phi(x, z = h(x, t = 0), t = 0)$
and $h(x, t = 0)$. However, straightforward numerical integration of this
system is known to be computationally inefficient and, for time periods
much greater than the time scale \mathcal{T} , virtually impracticable. To make a
numerical solution feasible, we introduce a surface-following coordinate
system which conformally maps the original domain (4) onto the strip

$$-\infty < \xi < \infty, \quad -\tilde{H} \leq \zeta \leq 0, \tag{7}$$

with a periodicity condition given as

$$\begin{aligned} x(\xi, \zeta, \tau) &= x(\xi + 2\pi, \zeta, \tau) + 2\pi \\ z(\xi, \zeta, \tau) &= z(\xi + 2\pi, \zeta, \tau), \end{aligned} \tag{8}$$

where τ is the new time coordinate, $\tau = t$. (Note that the mapping is
time-dependent, since it involves the surface height h .)

It can readily be shown that the required conformal mapping exists
and, due to periodicity condition (8), can be represented through Fourier
series:

$$x = \xi + x_0(\tau) + \sum_{-M \leq k \leq M, k \neq 0} \eta_{-k}(\tau) \frac{\cosh k(\zeta + \tilde{H})}{\sinh k \tilde{H}} \vartheta_k(\xi), \tag{9}$$

$$z = \zeta + \eta_0(\tau) + \sum_{-M \leq k \leq M, k \neq 0} \eta_k(\tau) \frac{\sinh k(\zeta + \tilde{H})}{\sinh k \tilde{H}} \vartheta_k(\xi), \tag{10}$$

where η_k are the coefficients of Fourier expansion of the free surface $\eta(\xi, \tau)$
with respect to the new horizontal coordinate ξ :

$$\eta(\xi, \tau) = h(x(\xi, \zeta = 0, \tau), t = \tau) = \sum_{-M \leq k \leq M} \eta_k(\tau) \vartheta_k(\xi); \tag{11}$$

denotes the function

$$\vartheta_k(\xi) = \begin{cases} \cos k\xi & k \geq 0 \\ \sin k\xi & k < 0 \end{cases} \quad (12)$$

note that $(\vartheta_k)_\xi = k\vartheta_{-k}$, and $\sum(A_k\vartheta_k)_\xi = -\sum kA_{-k}\vartheta_k$; M is the truncation number to be used in numerical integration (so far $M = \infty$ is assumed); $x_0(\tau)$ can be chosen arbitrarily, though it is convenient to sume

$$x_0(\tau) = 0. \quad (13)$$

the lower boundary $\zeta = -\tilde{H}$ cannot be chosen arbitrarily, since the relation

$$z(\xi, \zeta = -\tilde{H}, \tau) = -H \quad (14)$$

must hold, which, after substituting expansion (10), yields:

$$\tilde{H} = H + \eta_0(\tau). \quad (15)$$

Since η_0 is determined by the Fourier expansion (11), and, generally, is an unknown function of time, \tilde{H} also depends on time.

Due to the conformity of the mapping, Laplace equation (1) retains its form in (ξ, ζ) coordinates. Standard derivations show that system (1)–(3) can be written in the new coordinates as follows:

$$\Phi_{\xi\xi} + \Phi_{\zeta\zeta} = 0 \quad (16)$$

$$-z_\xi x_\tau + x_\xi z_\tau = \Phi_\zeta \quad (17)$$

$$\begin{aligned} \Phi_\tau - J^{-1}(x_\xi x_\tau + z_\xi z_\tau)\Phi_\xi + \frac{1}{2}J^{-1}(\Phi_\xi^2 - \Phi_\zeta^2) \\ + z + p - \sigma J^{-3/2}(-x_{\xi\xi}z_\xi + z_{\xi\xi}x_\xi) = 0 \end{aligned} \quad (18)$$

where eqns (17) and (18) are written for the surface $\zeta = 0$ (so that $z = \eta$ is represented by expansion (11)), and

$$J = x_\xi^2 + z_\xi^2 = x_\zeta^2 + z_\zeta^2 \quad (19)$$

the Jacobian of the transformation. Boundary condition (5) readily yields:

$$\Phi_\zeta(\xi, \zeta = -\tilde{H}, \tau) = 0. \quad (20)$$

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The Laplace equation (16) with boundary condition (20) is solved via Fourier expansion (which reduces system (16) - (18) to a 1-D problem):

$$\begin{cases} \cos k\xi & k \geq 0 \\ \sin k\xi & k < 0 \end{cases} \quad (12)$$

$(\partial_k \vartheta_k)_\xi = -\sum k A_{-k} \vartheta_k$; M is the truncational integration (so far $M = \infty$ is arbitrarily, though it is convenient to $\tau) = 0$. (13)

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$$\Phi_{\zeta\zeta} = 0 \quad (16)$$

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$$\begin{aligned} & \Phi_\xi + \frac{1}{2} J^{-1} (\Phi_\xi^2 - \Phi_\zeta^2) \\ & (-x_{\xi\xi} z_\xi + z_{\xi\xi} x_\xi) = 0 \end{aligned} \quad (18)$$

en for the surface $\zeta = 0$ (so that $z = \eta$ and

$$z_\xi^2 = x_\zeta^2 + z_\zeta^2 \quad (19)$$

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$$(-\tilde{H}, \tau) = 0. \quad (20)$$

$$\Phi = \sum_{-M \leq k \leq M} \phi_k(\tau) \frac{\cosh k(\zeta + \tilde{H})}{\cosh k\tilde{H}} \vartheta_k(\xi), \quad (21)$$

where ϕ_k are Fourier coefficients of the surface potential $\Phi(\xi, \zeta = 0, \tau)$. Thus, eqns (17) and (18) constitute a closed system of prognostic equations for the surface functions $z(\xi, \zeta = 0, \tau) = \eta(\xi, \tau)$ and $\Phi(\xi, \zeta = 0, \tau)$. In principle, it can be written as a system of ordinary differential equations for the Fourier coefficients η_k, ϕ_k using (11) and the following formulae which are easily obtained from (9), (10), (12), (13), (15), (21):

$$\Phi(\xi, \zeta = 0, \tau) = \sum_{-M \leq k \leq M} \phi_k(\tau) \vartheta_k(\xi) \quad (22)$$

$$\Phi_\xi(\xi, \zeta = 0, \tau) = - \sum_{-M \leq k \leq M} k \phi_{-k}(\tau) \vartheta_k(\xi) \quad (23)$$

$$\Phi_\zeta(\xi, \zeta = 0, \tau) = \sum_{-M \leq k \leq M} k \phi_k(\tau) \tanh(k\tilde{H}) \vartheta_k(\xi) \quad (24)$$

$$x_\xi(\xi, \zeta = 0, \tau) = 1 + \sum_{-M \leq k \leq M, k \neq 0} k \eta_k(\tau) \coth(k\tilde{H}) \vartheta_k(\xi) \quad (25)$$

$$z_\xi(\xi, \zeta = 0, \tau) = - \sum_{-M \leq k \leq M} k \eta_{-k}(\tau) \vartheta_k(\xi) \quad (26)$$

$$x_{\xi\xi}(\xi, \zeta = 0, \tau) = - \sum_{-M \leq k \leq M, k \neq 0} k^2 \eta_{-k}(\tau) \coth(k\tilde{H}) \vartheta_k(\xi) \quad (27)$$

$$z_{\xi\xi}(\xi, \zeta = 0, \tau) = - \sum_{-M \leq k \leq M} k^2 \eta_k(\tau) \vartheta_k(\xi) \quad (28)$$

$$\Phi_\tau(\xi, \zeta = 0, \tau) = \sum_{-M \leq k \leq M} \dot{\phi}_k(\tau) \vartheta_k(\xi) \quad (29)$$

$$x_\tau(\xi, \zeta = 0, \tau) = \sum_{-M \leq k \leq M, k \neq 0} \left(\dot{\eta}_{-k}(\tau) \coth(k\tilde{H}) - \frac{k \eta_{-k}(\tau) \dot{\eta}_0(\tau)}{\sinh^2(k\tilde{H})} \right) \vartheta_k(\xi) \quad (30)$$

$$z_\tau(\xi, \zeta = 0, \tau) = \sum_{-M \leq k \leq M} \dot{\eta}_k(\tau) \vartheta_k(\xi). \quad (31)$$

System (17), (18) is not resolved with respect to the time derivative of the surface elevation $\eta(\xi, \tau)$. During numerical integration of the initial value problem, the values of the time derivative can be obtained with a

simple iterative algorithm making use of eqn (17) and Cauchy–Riemann relations $x_{\tau\xi} = z_{\tau\zeta}$, $x_{\tau\zeta} = -z_{\tau\xi}$. However, a more efficient approach may be applied (V. Zakharov, private communication; see also Kuznetsov *et al.*, 1994). Introducing complex variables $\rho = \xi + i\zeta$ and $r(\rho, \tau) = x(\xi, \zeta, \tau) + iz(\xi, \zeta, \tau)$, we can rewrite the left-hand side of eqn (17) as follows

$$\operatorname{Im} \left(\frac{r_\tau}{r_\rho} \right)_{\zeta=0} = (J^{-1}\Phi_\zeta)_{\zeta=0}. \quad (32)$$

Note that due to conformity of the transformation, $r(\rho, \tau)$ is an analytic function of ρ , and so are $r_\tau = x_\tau + iz_\tau$, $r_\rho = x_\xi + iz_\xi$, and their ratio in the left-hand side of relation (32). Therefore, if we denote

$$\frac{r_\tau}{r_\rho} = F(\xi, \zeta, \tau) + iG(\xi, \zeta, \tau), \quad (33)$$

functions F and G are bound by the Cauchy–Riemann relations:

$$F_\xi = G_\zeta, \quad F_\zeta = -G_\xi. \quad (34)$$

Considering that G is a harmonic function of ξ and ζ , and that it becomes zero at the lower boundary $\zeta = -\tilde{H}$ (because at that boundary $z = -H$, $z_\tau = z_\xi = 0$), so that

$$(F + iG)_{\zeta=-\tilde{H}} = \left(\frac{x_\tau}{x_\xi} \right)_{\zeta=-\tilde{H}} = (F)_{\zeta=-\tilde{H}}, \quad (35)$$

we can write the following expansion:

$$G(\xi, \zeta, \tau) = \sum_{-M \leq k \leq M, k \neq 0} g_k(\tau) \frac{\sinh k(\zeta + \tilde{H})}{\sinh k\tilde{H}} \vartheta_k(\xi), \quad (36)$$

and relations (34) yield:

$$F(\xi, \zeta, \tau) = f_0(\tau) + \sum_{-M \leq k \leq M, k \neq 0} g_{-k}(\tau) \frac{\cosh k(\zeta + \tilde{H})}{\sinh k\tilde{H}} \vartheta_k(\xi). \quad (37)$$

Function $f_0(\tau)$ can be found using assumption (13), which together with (33) yields (for any ζ and τ):

$$0 = \int_0^{2\pi} x_\tau d\xi = \int_0^{2\pi} (Fx_\xi - Gz_\xi) d\xi;$$

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$$r(\rho, \tau) = (J^{-1}\Phi_\zeta)_{\zeta=0}. \tag{32}$$

the transformation, $r(\rho, \tau)$ is an analytic function of $\xi + iz_\tau$, $r_\rho = x_\xi + iz_\xi$, and their ratio in (2). Therefore, if we denote

$$r(\rho, \tau) = G(\xi, \zeta, \tau) + iG(\xi, \zeta, \tau), \tag{33}$$

by the Cauchy–Riemann relations:

$$r_\zeta = F_\zeta, \quad F_\zeta = -G_\xi. \tag{34}$$

is a function of ξ and ζ , and that it becomes $-i\tilde{H}$ (because at that boundary $z = -H$,

$$r(\rho, \tau) = \left(\frac{x_\tau}{x_\xi} \right)_{\zeta=-\tilde{H}} = (F)_{\zeta=-\tilde{H}}, \tag{35}$$

in expansion:

$$r(\rho, \tau) = \sum_{k \neq 0} g_k(\tau) \frac{\sinh k(\zeta + \tilde{H})}{\sinh k\tilde{H}} \vartheta_k(\xi), \tag{36}$$

$$r(\rho, \tau) = \sum_{M, k \neq 0} g_{-k}(\tau) \frac{\cosh k(\zeta + \tilde{H})}{\sinh k\tilde{H}} \vartheta_k(\xi). \tag{37}$$

Using assumption (13), which together with

$$r(\rho, \tau) = \int_0^{2\pi} (F x_\xi - G z_\xi) d\xi;$$

substituting expansions (36), (37), (9), (10), and integrating the products of the Fourier series, we obtain:

$$f_0 = \frac{1}{2} \sum_{-M \leq k \leq M, k \neq 0} k \eta_{-k} g_k \sinh^{-2}(k\tilde{H}). \tag{38}$$

Then if

$$g(\xi, \tau) = G(\xi, \zeta = 0, \tau) = \sum_{-M \leq k \leq M, k \neq 0} g_k(\tau) \vartheta_k(\xi) \tag{39}$$

is known,

$$f(\xi, \tau) = F(\xi, \zeta = 0, \tau) = f_0(\tau) + \sum_{-M \leq k \leq M, k \neq 0} g_{-k}(\tau) \coth(k\tilde{H}) \vartheta_k(\xi) \tag{40}$$

is also known: f is a generalization of the Hilbert transform of g , which, for $k \neq 0$, may be defined in Fourier space as

$$f_k = g_{-k} \coth(k\tilde{H}), \quad g_k = -f_{-k} \tanh(k\tilde{H}), \tag{41}$$

whereas $g_0 = 0$ and f_0 is defined by (38). Thus, we can replace eqn (17) by explicit expressions for the time derivatives x_τ and z_τ which follow from (33). Finally, eqns (17) and (18) can be rewritten as a system which is resolved with respect to the time derivatives (here $\zeta = 0$):

$$z_\tau = x_\xi g + z_\xi f \tag{42}$$

$$\Phi_\tau = f \Phi_\xi - \frac{1}{2} J^{-1} (\Phi_\xi^2 - \Phi_\zeta^2) - z - p + \sigma J^{-3/2} (-x_{\xi\xi} z_\xi + z_{\xi\xi} x_\xi), \tag{43}$$

where according to (32),

$$g = (J^{-1}\Phi_\zeta)_{\zeta=0}, \tag{44}$$

f is obtained from g according to (38)–(40), and the derivatives can be expressed through Fourier series (22)–(29), (31) (expansion (30) is no longer needed, since x_τ has been eliminated from the system).

Thus, the original system of equations is transformed into two evolutionary equations (42), (43) which can be effectively solved using the Fourier transform method (see Section 4).

For deep water ($H = \infty$), the coefficients in expansions (24), (25), (30), (38), (40), (41) become simpler, since $\tanh(k\tilde{H})$ and $\coth(k\tilde{H})$ replaced by $\text{sign}(k)$, and the terms with $\sinh^{-2}(k\tilde{H})$ vanish. In particular, operator (41) becomes a conventional Hilbert transform, and (38) reduced to $f_0 = 0$.

To include the case of pure capillary waves, it is convenient to use different scaling: $L^{3/2}\Gamma^{-1/2}$ for time and $L^{1/2}\Gamma^{1/2}$ for velocity potential. With the new nondimensional variables, eqns (43), (42) acquire the following form:

$$z_\tau = x_\xi g + z_\xi f \tag{45}$$

$$\partial_\tau = f\Phi_\xi - \frac{1}{2}J^{-1}(\Phi_\xi^2 - \Phi_\zeta^2) - \alpha z - p + J^{-3/2}(-x_{\xi\xi}z_\xi + z_{\xi\xi}x_\xi), \tag{46}$$

where $\alpha = \sigma^{-1}$.

Stationary solutions

For the stationary problem, the method of conformal mapping is a well-known approach based on using the velocity potential Φ and the stream function Ψ as the independent variables (e.g. Crapper, 1984). It is easy to show that in this case

$$\Phi = -c\xi + \Phi_0, \quad \Psi = c\zeta + \Psi_0, \tag{47}$$

where $-c$ is the velocity of the mean flow, $\Phi_x = -\Psi_z$ and $\Phi_z = \Psi_x$ are the horizontal and the vertical Cartesian velocity components respectively, Φ_0 and Ψ_0 are constants.

For the stationary version of system (1)-(3) (or (16)-(18)) to describe progressive waves, the periodicity condition on Φ , which implies a zero mean flow velocity, must be replaced by the weaker condition of periodicity of the velocity components, i.e., of the spatial derivatives of Φ . In a coordinate system moving with the wave's phase velocity c , the mean flow velocity is equal to $-c$, and the velocity potential Φ is given by relations (47) where Φ_0 must be allowed to depend on time (since stationarity is assumed for the velocity field rather than the velocity potential). Consequently, with the external pressure $p = 0$, system (16)-(18) is reduced to the equation written for the surface $\zeta = 0$:

$$\frac{1}{2}c^2 J^{-1} + z - \sigma J^{-3/2}(-x_{\xi\xi}z_\xi + z_{\xi\xi}x_\xi) = a \tag{48}$$

where $a = \dots$
on time, a is
linear).

In this work
(9), (10) for
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is discussed

Note that
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had to develop
gravity-capillary
numerical solutions

Below we
generalization
straightforward

3.1 Pure gravity

With $\sigma = 0$,
on expansion
series of the
1880) who introduced
recent studies
oriented rectangular
coefficients;
relations up to 10th

Here, the
surface height
conformal mapping
by the coefficient

coefficients in expansions (24), (25), simpler, since $\tanh(k\tilde{H})$ and $\coth(k\tilde{H})$ terms with $\sinh^{-2}(k\tilde{H})$ vanish. In particular, the Hilbert transform, and (38)

primary waves, it is convenient to use Γ and $L^{1/2}\Gamma^{1/2}$ for velocity potential variables, eqns (43), (42) acquire the

$$\Gamma + z_\xi f \tag{45}$$

$$-p + J^{-3/2}(-x_{\xi\xi}z_\xi + z_{\xi\xi}x_\xi), \tag{46}$$

method of conformal mapping is a well-known technique for velocity potential Φ and the stream function Ψ (e.g. Crapper, 1984). It is easy

$$\Psi = c\zeta + \Psi_0, \tag{47}$$

where $\Phi_x = -\Psi_z$ and $\Phi_z = \Psi_x$ are the velocity components respectively,

in (1)–(3) (or (16)–(18)) to describe the motion on Φ , which implies a zero velocity by the weaker condition of periodicity of the spatial derivatives of Φ . In a frame moving with the wave's phase velocity c , the mean flow velocity potential Φ is given by relations (16)–(18) which depend on time (since stationarity is assumed in the velocity potential). Consequently, system (16)–(18) is reduced to $\zeta = 0$:

$$-x_{\xi\xi}z_\xi + z_{\xi\xi}x_\xi = a \tag{48}$$

where $a = -\frac{d\Phi_0}{d\tau}$, and since the left-hand side of (48) does not depend on time, a is a constant (so that the dependence of Φ_0 on τ may only be linear).

In this work, eqn (48) is solved numerically using Fourier expansions (9), (10) for x and z and (25)–(28) for the derivatives in (48) and (19). The nonlinearities are evaluated at gridpoints $\xi^{(j)} = 2\pi(j-1)/N$, N being the total number of gridpoints. This approach, developed by Orszag (1970) and Eliassen *et al.* (1970), is known as the transform method and is discussed in more detail in Section 4.

Note that because σ is a factor in a term having the highest differential order, we may face effects of singularity for small σ . Indeed, we had to develop two different schemes for the cases of pure gravity and gravity-capillary waves, and it will be seen that in the latter case the numerical solution does not approach a Stokes wave as σ decreases.

Below we consider only the case of deep water ($H = \infty$); however, generalization of the algorithms described in the following subsections is straightforward for the case of a finite depth.

3.1 Pure gravity waves

With $\sigma = 0$, the solutions of eqn (48) are Stokes waves. A method based on expansion of the Fourier coefficients of the surface height in power series of the wave amplitude was initially proposed by Stokes (1847, 1880) who in his latter work obtained a fifth-order approximation. In recent studies, the method has been further developed into a computer-oriented recursive scheme which produces consecutive power expansion coefficients; Drennan *et al.* (1992) carried out the power series calculations up to 170 terms.

Here, the solutions in the form of Fourier expansion coefficients for the surface height were sought numerically with an iterative algorithm. The conformal mapping with surface boundary condition (48) is determined by the coefficients η_k through the relationships

$$x(\xi, \zeta) = \xi + \sum_{-M \leq k \leq M, k \neq 0} \eta_{-k} \exp(k\zeta) \vartheta_k(\xi) \tag{49}$$

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$$z(\xi, \zeta) = \zeta + \sum_{-M \leq k \leq M} \eta_k \exp(k\zeta) \vartheta_k(\xi) \quad (50)$$

the stationary deep-water versions of expansions (9), (10).), formulae (49), (50) may serve as a parametric representation of the surface.

For gravity waves, eqn (48) can be rewritten in the form

$$\log\left(\frac{1}{2}c^2\right) - \log J = \log(a - z). \quad (51)$$

If $\log \frac{d\tau}{d\rho}$ with τ, ρ defined as in (32), it can be seen that

$$\log J = 2\text{Re}(w), \quad z_\xi = \text{Im}(\exp w). \quad (52)$$

If a Fourier expansion for $\log J$ is known, $\text{Im}(w)$ (also in the real space) can be found via the Hilbert transform as in the second of eqns (41), after which w and then $\exp w$ can be calculated at the gridpoints.

This yields z_ξ , and after finding the corresponding Fourier coefficients by direct Fourier transform, z can be obtained by integration in Fourier space. Thus, z can be easily found if $\log J$ is known. This reduces the differential relationship (51) to an equation with a difference operator, which may be solved by a simple iterative procedure.

Since z is an even function of ξ , it is convenient to choose

$$s = \frac{1}{4} (\log J(\xi = 0, \zeta = 0) - \log J(\xi = \pi, \zeta = 0)) \quad (53)$$

as a parameter determining the amplitude of the wave (in linear approximation, s is equal to the amplitude). With $\chi^{[n]}$ denoting the value of z at the gridpoint χ on n th iteration, the scheme can be written as follows:

For $n = 0$, $\log J^{[0]} = 2s e^\zeta \cos \xi$ (this is the solution of the linearized problem).

For $n > 0$, use Hilbert transform (41), complex exponentiation, and numerical integration in Fourier space to find $z^{[n]}$ as described above. If the maximum surface gridpoint value of $|z^{[n]} - z^{[n-1]}|$ is less than the prescribed accuracy ϵ , the iterations are completed, and $z^{[n]}$ is an approximate solution within the accuracy given.

G3. Let

$a =$

This will ensure real gridpoint values of

$$\log J^{[n+1]}$$

where $(\)_0$ denotes real part

G4. Find the Fourier coefficients J_n , $n = n + 1$ and return

The last term in iteration: in accordance with Equality (54) is zero, which follows from $\zeta \rightarrow -\infty$.

Figure 1 illustrates profiles for $N = 1728$, $\epsilon = 0.04$, $s = 0.04$ from 28 for $s = 0.04$.

$$A = \frac{1}{2} (\eta(\xi) + \eta(\xi + \pi))$$

is approximately 0.4. The maximum steepness (ab-

Theoretically, s is the wave whose crest coincides with $J(\xi = 0, \zeta = 0)$. For s much larger than the linear value, the accuracy of the solution drops down dramatically, and the accuracy of the solution for large s the numerical values of M, N and

$$\exp(k\zeta)\varphi_k(\xi) \tag{50}$$

ous of expansions (9), (10).
as a parametric representa-

rewritten in the form

$$; (a - z). \tag{51}$$

(32), it can be seen that

$$(\exp w). \tag{52}$$

known, $\text{Im}(w)$ (also in the
transform as in the second of
 w can be calculated at the
g the corresponding Fourier
n be obtained by integration
and if $\log J$ is known. This
ip (51) to an equation with
a simple iterative procedure.

it is convenient to choose

$$J(\xi = \pi, \zeta = 0) \tag{53}$$

of the wave (in linear ap-
With $\chi^{[n]}$ denoting the value
e can be written as follows:

is the solution of the lin-

(41), complex exponent cal-
find $z^{[n]}$ as described above.
 $|z^{[n]} - z^{[n-1]}|$ is less than
e completed, and $z^{[n]}$ is an
con.

G3. Let

$$a = a^{[n+1]} = \frac{e^{4s} z^{[n]}(\xi = 0) - z^{[n]}(\xi = \pi)}{e^{4s} - 1}.$$

This will ensure relation (53) for the next iteration. Calculate surface gridpoint values of

$$\log J^{[n+1]} = -\log(a^{[n+1]} - z^{[n]}) + (\log(a^{[n+1]} - z^{[n]}))_0 \tag{54}$$

where $(\)_0$ denotes mean over ξ , i.e. 0th Fourier coefficient.

G4. Find the Fourier expansion of $\log J^{[n+1]}$ by a Fourier transform; let $n = n + 1$ and return to step G2.

The last term in (54) allows us to find the phase velocity on $(n + 1)$ -st iteration: in accordance with eqn (51), this term is equal to $\log(\frac{1}{2}(c^{[n+1]})^2)$. Equality (54) is based on the fact that the mean value of $\log J$ over ξ is zero, which follows from the first of relations (52), as $w \rightarrow 0$ when $\zeta \rightarrow -\infty$.

Figure 1 illustrates some results of the described procedure for $M = 384$, $N = 1728$, $\epsilon = 10^{-11}$. The values of the parameter s for the given profiles are 0.04, 0.08, 0.16, 0.32, 1.06. The number of iterations varied from 28 for $s = 0.04$ to 44 for $s = 1.06$. For $s = 1.06$, the amplitude

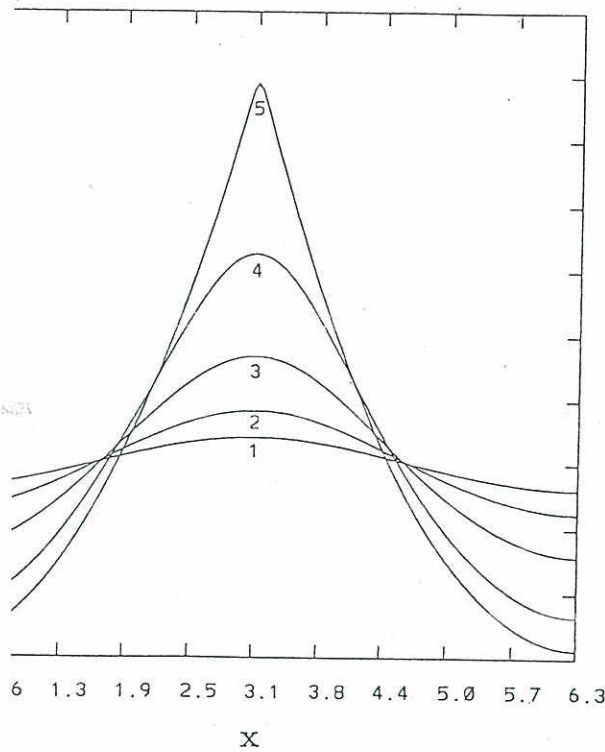
$$A = \frac{1}{2}(\eta(\xi = 0) - \eta(\xi = \pi)) = \frac{1}{2}(h(x = 0) - h(x = \pi)) \tag{55}$$

is approximately 0.4374 which is close to that of Stokes wave with maximum steepness (about 0.443 according to Longuet-Higgins, 1975).

Theoretically, s can be arbitrarily large; $s = \infty$ for the steepest Stokes wave whose crest constitutes an angle of 120° and thus is a singularity with $J(\xi = 0, \zeta = 0) = \infty$. The algorithm does converge for values of s much larger than those used in Fig. 1; however, when the profile becomes close to that of the steepest wave, convergence of the Fourier series slows down dramatically, and at the same time, due to strong nonlinearities, the accuracy of the transform method decreases sharply. As a result, for large s the numerical solution contains spurious oscillations; with the values of M, N indicated above, for $s > 1.06$, the maximum slope of the

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ned profile exceeds 30°, which is a theoretical maximum.



Stokes waves: Curve 1 - $A = 0.0399$, 2 - $A = 0.0793$, 3 - $A = 0.2806$, 5 - $A = 0.4374$.

tion convergence criterium ϵ is small enough, the error of the truncation error, which can be evaluated by comparing results obtained with different resolutions. Such a comparison is shown for three resolutions: $M = 384$, $N = 1728$; $M = 192$, $N = 864$; $M = 96$, $N = 432$. The results are illustrated in Table 1, where A is the value of the amplitude A (as defined in (55)) obtained with resolution number M ; $MAXD_{M_1, M_2}$ is the maximum gridpoint distance

$$= \left((x_{M_1} - x_{M_2})^2 + (z_{M_1} - z_{M_2})^2 \right)^{1/2}, \quad (56)$$

where $(x, z) = (x_M, z_M(\xi))$ is the numerical solution for x, z at

$\zeta = 0$ obtained with the t

$$RMSD_{M_1, M_2}$$

is the root mean square d

Table 1 shows that for all tested resolutions the error is very small for all tested resolutions. The convergence decelerates for the steepest wave. For all cases the error is small at the wave's crest.

Table 1: Comparison of Stokes wave solutions

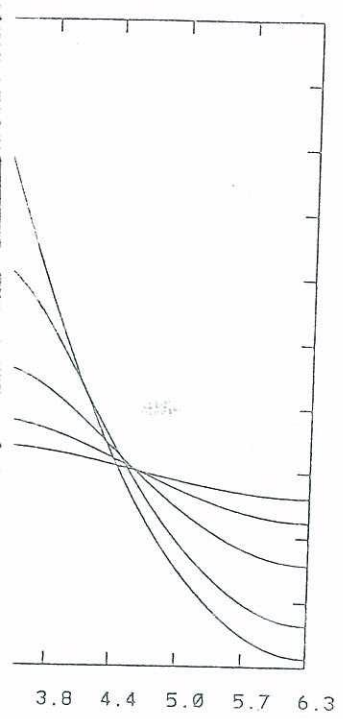
s	A_s
0.4	0.32
0.6	0.39
0.8	0.42
1.0	0.43
1.2	0.43
1.8	0.44
2.0	0.43

s	$RMSD_{96,384}$	$RMSD_{192,864}$
0.4	$1.3 \cdot 10^{-12}$	1.5
0.6	$3.9 \cdot 10^{-6}$	3.0
0.8	$4.8 \cdot 10^{-4}$	3.0
1.0	$3.0 \cdot 10^{-3}$	6.7
1.2	$6.8 \cdot 10^{-3}$	2.2
1.8	$2.1 \cdot 10^{-2}$	7.6
2.0	$2.6 \cdot 10^{-2}$	9.7

3.2 Gravity-capillary waves

An iterative algorithm similar to the one used for Stokes waves is used to obtain numerical solutions

which is a theoretical maximum.



$A = 0.0399, 2 - A = 0.0793, 3 -$

is small enough, the error can be evaluated by comparisons of the numerical solutions. Such a comparison is illustrated in Table 1, where the numerical solutions (as defined in (55)) obtained with M_2 is the maximum gridpoint

$$-z_{M_2})^2)^{1/2}, \tag{56}$$

numerical solution for x, z at

$\zeta = 0$ obtained with the truncation number M ;

$$RMSD_{M_1, M_2} = \left(\frac{1}{2\pi} \int_0^{2\pi} (R(\xi))^2 d(x(\xi)) \right)^{1/2} \tag{57}$$

is the root mean square difference over x coordinate.

Table 1 shows that for amplitudes $A < 0.4$ the truncation errors are very small for all tested resolutions and decrease rapidly with increasing M ; the convergence decelerates when the amplitude approaches that of the steepest wave. For all examples, the maximum error was located near the wave's crest.

Table 1: Comparison of Stokes wave profiles calculated with different spectral resolutions

s	A_{384}	$A_{96} - A_{384}$	$A_{192} - A_{384}$
0.4	0.3273	$4.2 \cdot 10^{-13}$	$1.6 \cdot 10^{-15}$
0.6	0.3986	$1.5 \cdot 10^{-6}$	$1.1 \cdot 10^{-9}$
0.8	0.4264	$8.3 \cdot 10^{-5}$	$3.2 \cdot 10^{-6}$
1.0	0.4360	$-1.4 \cdot 10^{-4}$	$-1.2 \cdot 10^{-4}$
1.2	0.4394	$-1.1 \cdot 10^{-3}$	$-1.3 \cdot 10^{-3}$
1.8	0.4405	$-5.5 \cdot 10^{-3}$	$-3.0 \cdot 10^{-3}$
2.0	0.4393	$-6.9 \cdot 10^{-3}$	$-3.9 \cdot 10^{-3}$

s	$RMSD_{96,384}$	$RMSD_{192,384}$	$MAXD_{96,384}$	$MAXD_{192,384}$
0.4	$1.3 \cdot 10^{-12}$	$1.5 \cdot 10^{-15}$	$3.7 \cdot 10^{-12}$	$2.2 \cdot 10^{-15}$
0.6	$3.9 \cdot 10^{-6}$	$3.0 \cdot 10^{-8}$	$1.7 \cdot 10^{-5}$	$1.3 \cdot 10^{-8}$
0.8	$4.8 \cdot 10^{-4}$	$3.0 \cdot 10^{-5}$	$2.3 \cdot 10^{-3}$	$1.7 \cdot 10^{-4}$
1.0	$3.0 \cdot 10^{-3}$	$6.7 \cdot 10^{-4}$	$1.1 \cdot 10^{-2}$	$3.3 \cdot 10^{-3}$
1.2	$6.8 \cdot 10^{-3}$	$2.2 \cdot 10^{-3}$	$2.1 \cdot 10^{-2}$	$9.3 \cdot 10^{-3}$
1.8	$2.1 \cdot 10^{-2}$	$7.6 \cdot 10^{-3}$	$4.9 \cdot 10^{-2}$	$2.6 \cdot 10^{-2}$
2.0	$2.6 \cdot 10^{-2}$	$9.7 \cdot 10^{-3}$	$5.9 \cdot 10^{-2}$	$3.1 \cdot 10^{-2}$

3.2 Gravity-capillary waves

An iterative algorithm similar to that described above has been worked out to obtain numerical solutions of eqn (48) with $\sigma > 0$. Here, we again

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The height is an even function of x , and hence of ξ .

For algorithm it is convenient to rewrite eqn (48) in the

$$\frac{x}{\alpha+1}z - a_* - \frac{1}{\alpha+1}J^{-1}(-x_{\xi\xi}z_{\xi} + z_{\xi\xi}x_{\xi}) = 0 \quad (58)$$

$a_* = c^2\alpha/(\alpha+1)$; $c_*^2 = c^2\alpha/(\alpha+1) = c^2/(1+\sigma)$. Note the actual phase velocity to the phase velocity of

$$= \left(\frac{1}{k} + \sigma k\right)^{1/2} = \left(\frac{1}{k} + \frac{k}{\alpha}\right)^{1/2} \quad (59)$$

1; the convenience of representing the results in that it does not depend on the choice of the time

at term on the left-hand side of eqn (58), it can be

$$z_{\xi\xi}x_{\xi} = \text{Im} \left(\frac{dw}{d\rho} \right) = \text{Im}(w_{\xi}) = (\text{Im } w)_{\xi} \quad (60)$$

the same meaning as in relations (52). Also, if we of the surface height over the x coordinate is zero, the following choice of the 0th Fourier coefficient:

$$\eta_0 = -\frac{1}{2} \sum_{1 \leq k < M} k\eta_k^2, \quad (61)$$

and c_* are bound by the relationship:

$$a_* = \frac{1}{2}c_*^2. \quad (62)$$

Thus, this property directly follows from results by (5); for the general case of gravity-capillary waves, as it can be deduced from the observation that any term in eqn (3) over x is zero.

Relations (60), (62) and (63)

$$-2(\alpha+1)a_* \sinh(R)$$

Choosing

$$S = -\frac{1}{2}(\text{Im } w_{\xi}(\xi) =$$

as the parameter determining S , it is equal to the amplitude of w . To formulate the iteration scheme as

GC1. Assume $n = 0$, $(\text{Im } w)_{\xi}^{[n]}$ linearized problem).

GC2. For given $(\text{Im } w)_{\xi}^{[n]}$, and a Hilbert transform (as in step G1) at each gridpoint, then find $z^{[n]}$ according to the second relation (58) terminated by relation (61). If $|z^{[n]} - z^{[n-1]}|$ is less than the tolerance, the iteration is completed, and $z^{[n]}$ is an approximation to the exact solution.

GC3. Calculate surface values w from eqn (63), by substituting $w = z^{[n]}$. Similarly to step G3, $a_* = a_*^{[n]}$ holds for $w = w^{[n+1]}$:

$$a_*^{[n+1]} = \frac{\alpha}{2(\alpha+1)} \frac{\eta^{[n]}(0) \text{erf}(\sqrt{\alpha} R)}{\text{si}(\sqrt{\alpha} R)}$$

where

$$\eta^{[n]}(\xi) = z^{[n]}(\xi, \zeta = 0)$$

GC4. Find the Fourier expansion of w for $n = n + 1$ and return to step G2.

Convergence of the algorithm is tested on the parameter S for different values of α . The wave profiles obtained for

$$\hat{A} = \frac{1}{2} (\text{Max}_{0 \leq j \leq N} \eta(\xi_j))$$

function of x , and hence of ξ .

it to rewrite eqn (48) in the

$$-x_{\xi\xi}z_{\xi} + z_{\xi\xi}x_{\xi} = 0 \quad (58)$$

$(\alpha + 1) = c^2/(1 + \alpha)$. Note
ity to the phase velocity of

$$\left(\frac{k}{\alpha}\right)^{1/2} \quad (59)$$

representing the results in
d on the choice of the time

d side of eqn (58), it can be

$$\text{Im}(w_{\xi}) = (\text{Im } w)_{\xi} \quad (60)$$

relations (52). Also, if we
ver the x coordinate is zero,
the 0th Fourier coefficient:

$$(61)$$

relationship:

$$(62)$$

tly follows from results by
of gravity-capillary waves,
from the observation that
r x is zero.

Relations (60), (62) and (52) allow us to rewrite eqn (58) as follows:

$$-2(\alpha + 1)a_* \sinh(\text{Re } w) + \alpha \exp(\text{Re } w)z = (\text{Im } w)_{\xi}. \quad (63)$$

Choosing

$$S = -\frac{1}{2}(\text{Im } w_{\xi}(\xi = 0, \zeta = 0) - \text{Im } w_{\xi}(\xi = \pi, \zeta = 0)) \quad (64)$$

as the parameter determining the wave amplitude (like s in relation (53), it is equal to the amplitude for the linearized problem), we can now formulate the iteration scheme as follows:

GC1. Assume $n = 0$, $(\text{Im } w)_{\xi}^{[0]} = -Se^{\zeta} \cos \xi$ (this is the solution of the linearized problem).

GC2. For given $(\text{Im } w)_{\xi}^{[n]}$, find $w^{[n]}$ in Fourier space by integration and a Hilbert transform (as in the first of equalities (41)); find $\exp(w^{[n]})$ at each gridpoint, then find $z^{[n]}$ by a Fourier transform and integration according to the second relation (52), with the integration constant determined by relation (61). If the maximum surface gridpoint value of $|z^{[n]} - z^{[n-1]}|$ is less than the prescribed accuracy ϵ , the iterations are completed, and $z^{[n]}$ is an approximate solution within the accuracy given.

GC3. Calculate surface values of $(\text{Im } w)_{\xi}^{[n+1]}$ as the right-hand side of eqn (63), by substituting $w = w^{[n]}$, $z = z^{[n]}$ into the left-hand side. Similarly to step G3, $a_* = a_*^{[n+1]}$ must be chosen so that relation (64) holds for $w = w^{[n+1]}$:

$$a_*^{[n+1]} = \frac{\alpha}{2(\alpha + 1)} \frac{\eta^{[n]}(0) \exp(R^{[n]}(0)) - \eta^{[n]}(\pi) \exp(R^{[n]}(\pi))}{\sinh(R^{[n]}(0)) - \sinh(R^{[n]}(\pi))} \quad (65)$$

where

$$\eta^{[n]}(\xi) = z^{[n]}(\xi, \zeta = 0), \quad R^{[n]}(\xi) = \text{Re } w^{[n]}(\xi, \zeta = 0).$$

GC4. Find the Fourier expansion of $(\text{Im } w)_{\xi}^{[n+1]}$ by a Fourier transform; let $n = n + 1$ and return to step GC2.

Convergence of the algorithm and the dependence of wave amplitude on the parameter S for different α are characterized by Table 2. Since the wave profiles obtained for large α have two maxima, values of

$$\hat{A} = \frac{1}{2} (\text{Max}_{0 \leq j \leq N} \eta(\xi = \xi^{(j)}) - \text{Min}_{0 \leq j \leq N} \eta(\xi = \xi^{(j)})) \quad (66)$$

ves

α and A (relation (55)). The calculations were performed with $\epsilon = 10^{-11}$.

Number of iterations N_{it} for gravity-capillary wave calculations. Blank entries mean that the scheme failed to converge.

	0.1	0.2	0.4	0.6	0.8	1
1000	0.2000	0.4000	0.6000	0.8000	1.0000	
1000	0.2000	0.4000	0.6000	0.8000	1.0000	
16	18	20	21	21	20	
3996	0.1969	0.3772	0.5315	0.6583	0.7606	
3996	0.1969	0.3772	0.5315	0.6583	0.7606	
41	41	39	39	43	43	
3971	0.1814	0.3109	0.4066	0.4815	0.5420	
3971	0.1814	0.3109	0.4066	0.4815	0.5420	
61	48	47	44	43	42	
3880	0.1497	0.2382	0.3050	0.3593	0.4049	
3880	0.1497	0.2382	0.3050	0.3593	0.4049	
93	58	48	43	41	39	
3655	0.1090	0.1760	0.2291	0.2738		
3711	0.1165	0.1844	0.2375	0.2819		
157	81	44	41	40		
3262	0.0512	0.0967	0.1361			
3895	0.1064	0.1417	0.1750			
106	88	62	54			
3162	0.0323	0.0636				
3853	0.0955	0.1182				
134	122	97				
3124	0.0247	0.0490				
3653	0.0742	0.0946				
178	157	138				
3109						
3247						
2051						

For $\alpha = 0.1$, samples of calculated wave profiles with Figs 2 and 3, respectively. These results were obtained using the algorithm above, which, more con-

veniently, uses \hat{A} instead of S . A major advantage of the modified version in the (α, \hat{A}) plane is notably larger than in the (α, S) plane; however, computationally it is less efficient than the nested calculations to numerical solution, while in step GC3 a_* is re-

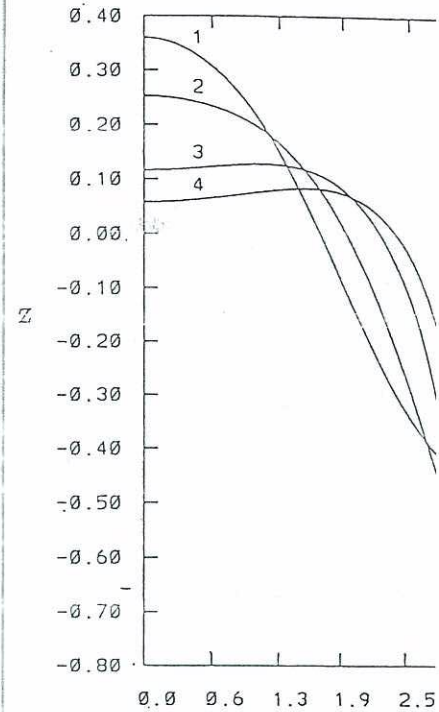


Figure 2: Profiles of gravity-capillary wave), 2 - $\alpha = 1$, 3 - $\alpha = 2$, 4 - $\alpha = 3$

From Table 2 and Figs 2 and 3, it is seen that the wave profiles do not approach those of pure gravity waves as α increases. For smaller scales where capillarity is dominant, the wave profiles with $\alpha = 2$, two maxima emerge, and for $\alpha = 3$, two maxima emerge, (the absolute minimum being al-

5)). The calculations were per-

s N_H for gravity-capillary waves calcu-
 ean that the scheme failed to converge

	0.4	0.6	0.8	1
0.4000	0.6000	0.8000	1.0000	
0.4000	0.6000	0.8000	1.0000	
20	21	21	20	
0.3772	0.5315	0.6583	0.7606	
0.3772	0.5315	0.6583	0.7606	
39	39	43	43	
0.3109	0.4066	0.4815	0.5420	
0.3109	0.4066	0.4815	0.5420	
47	44	43	42	
0.2382	0.3050	0.3593	0.4049	
0.2382	0.3050	0.3593	0.4049	
48	43	41	39	
0.1760	0.2291	0.2738		
0.1844	0.2375	0.2819		
44	41	40		
0.0967	0.1361			
0.1417	0.1750			
62	54			
0.0636				
0.1182				
97				
0.0490				
0.0946				
138				

veniently, uses \hat{A} instead of S as a wave parameter. An important advantage of the modified version is that its domain of convergence in the (α, \hat{A}) plane is notably larger than that of the original scheme GC1-GC4; however, computationally it is less efficient than the latter, as it requires nested calculations to numerically determine values of a_* for each iteration, while in step GC3 a_* is readily yielded by formula (65).

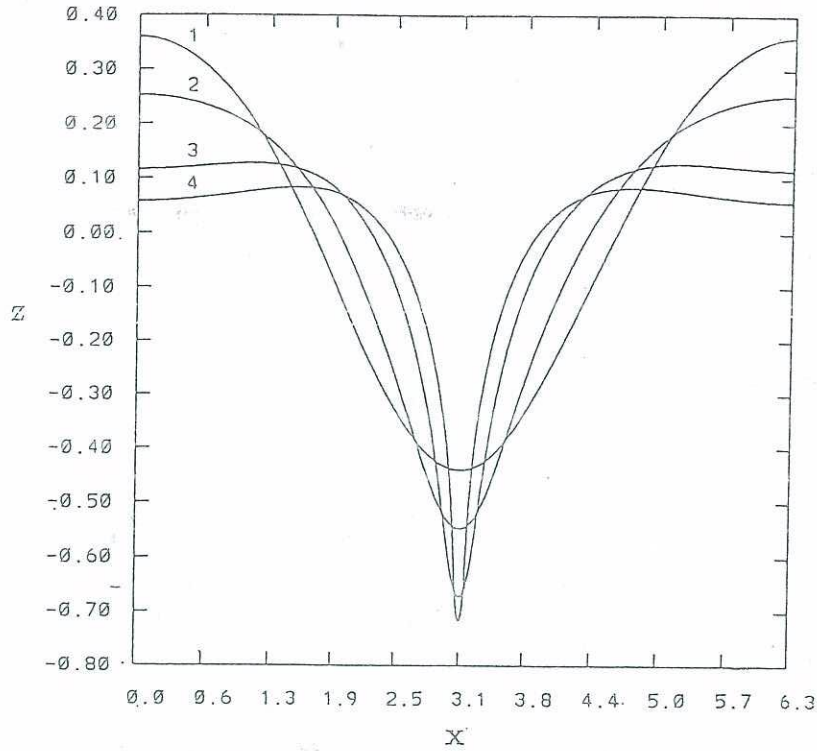


Figure 2: Profiles of gravity-capillary waves, $\hat{A} = 0.4$. Curve 1 - $\alpha = 0$ (Crapper's wave), 2 - $\alpha = 1$, 3 - $\alpha = 2$, 4 - $\alpha = 3$.

From Table 2 and Figs 2 and 3, it can be seen that with increasing α (i.e., with the capillarity coefficient σ decreasing), the wave profiles do not approach those of pure gravity waves, but rather shift their energy to smaller scales where capillarity effects are more pronounced. Beginning with $\alpha = 2$, two maxima emerge, so that $x = 0$ becomes a local minimum (the absolute minimum being always at $x = \pi$); at the same time the

calculated wave profiles with
 respectively. These results were
 algorithm above, which, more con-

Nonlinear Ocean Waves

of convergence decreases rapidly, especially for small amplitudes or those values of α which ensure equal linear phase velocities (59) of neighboring wavenumbers $k, k + 1$, namely $\alpha = 2$ ($k = 1$) and $\alpha = 3$ ($k = 2$).

Neither the scheme GC1-GC4 nor its aforementioned modification converged with $\alpha > 6$; however, the modified scheme converged for \hat{A} considerably larger than those indicated in Table 2.

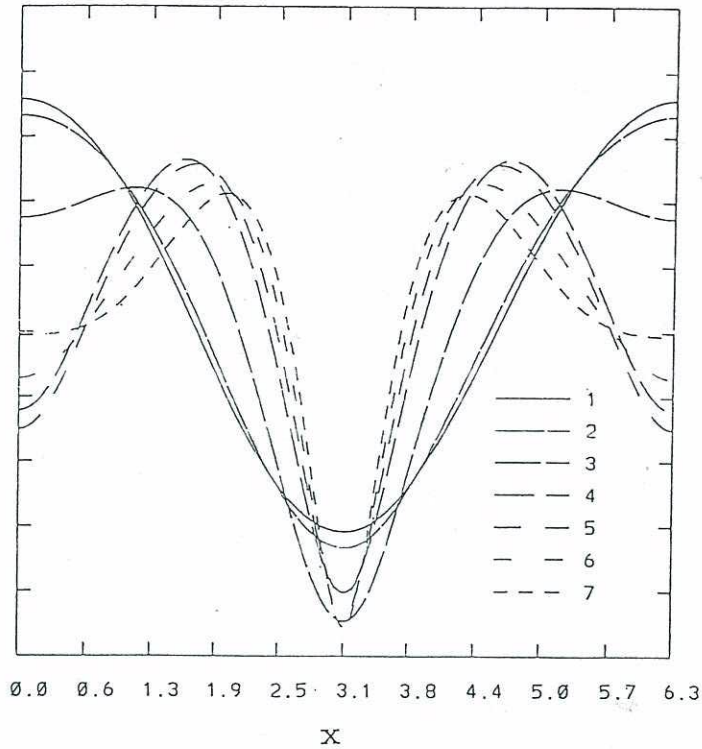


Figure 3: Profiles of gravity-capillary waves, $\hat{A} = 0.1$. Curve 1 - $\alpha = 0$ (Crapper's), 2 - $\alpha = 1$, 3 - $\alpha = 2$, 4 - $\alpha = 3$, 5 - $\alpha = 4$, 6 - $\alpha = 5$, 7 - $\alpha = 6$.

The dependence of the normalized phase velocity c_* on $\sigma = 1/\alpha$, as stated by Fig. 4, is consistent with the behavior of the corresponding profiles. When σ decreases but remains positive, c_* decreases and does not approach its value at $\sigma=0$. The latter is always greater than 1

and, except for increasing function of \hat{A} phase velocity minimum value

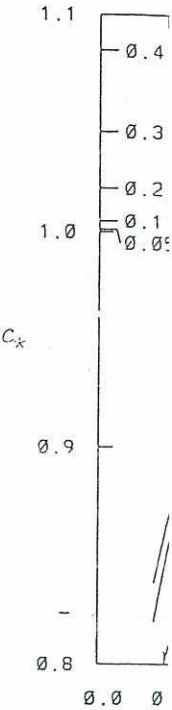
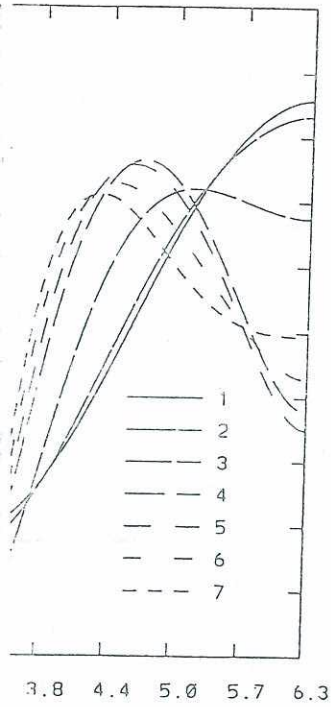


Figure 4: Normalized capillarity coefficient

For most cases with results of maximum difference exceeded $1.2 \cdot 10^{-4}$

y, especially for small amplitudes
 equal linear phase velocities (59)
 $k + 1$, namely $\alpha = 2$ ($k = 1$) and

puts aforementioned modification
 modified scheme converged for \hat{A}
 indicated in Table 2.



$\hat{A} = 0.1$. Curve 1 - $\alpha = 0$ (Crapper's
 = 4, 6 - $\alpha = 5$, 7 - $\alpha = 6$.

phase velocity c_* on $\sigma = 1/\alpha$, as
 the behavior of the corresponding
 is positive, c_* decreases and does
 the latter is always greater than 1

and, except for a small interval in the vicinity of the maximum \hat{A} , is an
 increasing function of \hat{A} , while for $\sigma \neq 0$ the calculated c_* is a decreasing
 function of \hat{A} and always less than 1. In fact, for small values of \hat{A} , the
 phase velocity corresponding to a given σ turns out to be close to the
 minimum value of c_l (relation (59)) over k .

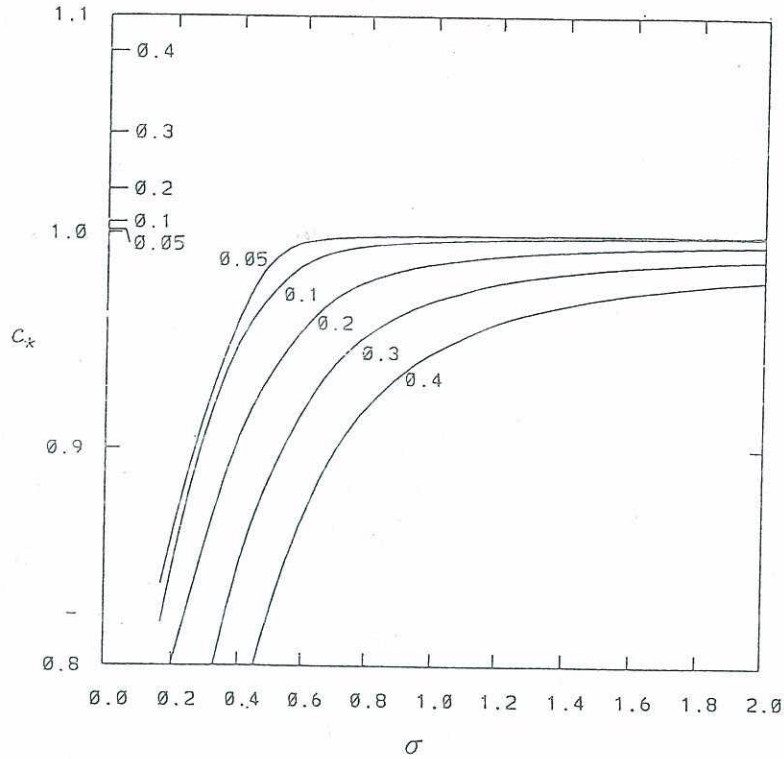


Figure 4: Normalized phase velocity $c_* = c/\sqrt{1+\sigma}$ as a function of nondimensional capillarity coefficient σ . Wave amplitudes \hat{A} are indicated at the respective curves.

For most cases represented in Table 2, a comparison was performed
 with results obtained for a higher resolution ($M = 384$, $N = 1728$). The
 maximum difference between surface profiles as defined in (56) never ex-
 ceeded $1.2 \cdot 10^{-11}$, which means that the truncation errors are negligible.

illary waves

ary waves, which are described by eqn (58) if we formally e solution is represented by a simple formula (Crapper, rotation, it can be written as

$$x(\xi, \zeta) = \xi + \frac{4q \sin \xi}{1 - 2q \cos \xi + q^2}, \tag{67}$$

$$z(\xi, \zeta) = \zeta + \frac{4q(\cos \xi - q)}{1 - 2q \cos \xi + q^2} - \frac{1}{2}A^2, \tag{68}$$

$$q = be^\zeta, \quad b = \frac{-2 + \sqrt{4 + A^2}}{A}, \tag{69}$$

b, c*, a* are bound by the following relationships:

$$= \hat{A} = S = \frac{4b}{1 - b^2}, \quad \frac{1}{2}c_*^2 = a_* = \frac{1}{\sqrt{4 + A^2}}. \tag{70}$$

n (68) may be any constant but here it is chosen to satisfy

ion (67)–(70) was used as another means to validate scheme h $\alpha = 0$. For all tested amplitudes, up to the maximum tude (e.g., Crapper, 1984), the maximum difference be erical ($M = 96$) and the exact solutions as defined in (56) $\cdot 10^{-12}$.

i of the nonstationary equations

proximation of system (42), (43) we use a Galerkin-type method based on Fourier expansion of the prognostic a finite truncation number M . The system is thus reduced ary differential equations for $4M + 2$ Fourier coefficients $\cdot M \leq k \leq M$:

$$= E_k(\eta_{-M}, \eta_{-M-1}, \dots, \eta_M, \phi_{-M}, \phi_{-M-1}, \dots, \phi_M) \tag{71}$$

$$= F_k(\eta_{-M}, \eta_{-M-1}, \dots, \eta_M, \phi_{-M}, \phi_{-M-1}, \dots, \phi_M) \tag{72}$$

where E_k, F_k are, respect right-hand sides of eqns (

To calculate E_k, F_k as mulae (23)–(28) are used ated exactly), and the transform method (Orsz evaluation on a spatial gr tion of its arguments whi gridpoint values $u(\xi^j), v$ Fourier transforms are per are evaluated at each gr the function Y are four $2\pi(j - 1)/N$, and N is th

For the method to be imum mean square appro need to be evaluated exa must choose

where ν is the maximum sides of eqns (42), (43) in is of infinite order so tha met. However, numerical N ensuring exact evaluat a further increase in N numerical solution. For n

However high the spo grations one must paran of the spectrum ($k > M$ large wavenumbers will c tion terms were added to this purpose:

where E_k, F_k are, respectively, the Fourier expansion coefficients for the right-hand sides of eqns (42) and (43) as functions of ξ .

To calculate E_k, F_k as functions of the prognostic variables η_k, ϕ_k , formulae (23)–(28) are used for the spatial derivatives (which are thus evaluated exactly), and the nonlinearities are calculated with the so-called transform method (Orszag, 1970; Eliassen *et al.*, 1970), i.e., by their evaluation on a spatial grid. If $Y(u(\xi), v(\xi), w(\xi), \dots)$ is a nonlinear function of its arguments which are represented by their Fourier expansions, gridpoint values $u(\xi^{(j)}), v(\xi^{(j)}), w(\xi^{(j)}), \dots$ are first calculated, i.e., inverse Fourier transforms are performed; then $Y^{(j)} = Y(u(\xi^{(j)}), v(\xi^{(j)}), w(\xi^{(j)}), \dots)$ are evaluated at each gridpoint; finally, the Fourier coefficients Y_k of the function Y are found by direct Fourier transform. Here $\xi^{(j)} = 2\pi(j - 1)/N$, and N is the number of gridpoints.

For the method to be a purely Galerkin one, i.e., to ensure the minimum mean square approximation error, the Fourier coefficients E_k, F_k need to be evaluated exactly for $-M \leq k \leq M$. For this purpose, we must choose

$$N > (\nu + 1)M \tag{73}$$

where ν is the maximum order of nonlinearities. Since the right-hand sides of eqns (42), (43) include division by the Jacobian, the nonlinearity is of infinite order so that, strictly, the above condition on N cannot be met. However, numerical integrations show that if we choose a value of N ensuring exact evaluation of the cubic nonlinearities ($\nu = 3$ in (73)), a further increase in N (with fixed M) virtually does not impact the numerical solution. For most runs, $M = 96$ and $N = 432$ were used.

However high the spectral resolution might be, for long-term integrations one must parameterize the energy flux into the severed part of the spectrum ($k > M$); otherwise, spurious energy accumulation at large wavenumbers will corrupt the numerical solution. Simple dissipation terms were added to the right-hand sides of eqns (71) and (72) for this purpose:

$$\dot{\eta}_k = E_k - \mu_k \eta_k \tag{74}$$

$$\dot{\phi}_k = F_k - \mu_k \phi_k \tag{75}$$

eqn (58) if we formally
 ple formula (Craper,

$$\sqrt{2}, \tag{67}$$

$$\frac{1}{2}A^2, \tag{68}$$

$$\sqrt{2}, \tag{69}$$

g relationships:

$$\frac{1}{\sqrt{4 + A^2}} \tag{70}$$

re it is chosen to satisfy

means to validate scheme
 3, up to the maximum
 maximum difference be-
 tions as defined in (56)

ations

re use a Galerkin-type
 sion of the prognostic
 system is thus reduced
 + 2 Fourier coefficients

$$-1, \dots, \phi_M) \tag{71}$$

$$-1, \dots, \phi_M) \tag{72}$$

$$\mu_k = \begin{cases} rM \left(\frac{|k| - k_d}{M - k_d} \right)^2 & \text{if } |k| > k_d \\ 0 & \text{otherwise} \end{cases} \quad (76)$$

and $r = 0.25$ were chosen for most runs; the sensitivity to reasonable variations of k_d and r was low. The model absorbs the energy at wavenumbers close to the cut-off M while leaving longer waves virtually intact (note that $-k_d \leq k \leq k_d$ are not affected).

For numerical integration, the fourth-order Runge-Kutta scheme was used.

Validation of the model

The numerical solutions dealt with in Section 3 were used for validation of the numerical model (system (42), (43)). In the model's coordinate system, the mean flow, progressive waves were simulated starting from initial conditions calculated as stationary solutions in a moving coordinate system. If such a wave is not affected by truncation errors, it should propagate with its speed without changes of shape. The model was validated against Stokes waves (experiments 1-3 in Table 3, respectively); pure gravity waves described analytically by (67)-(70) (Crapper, 1957); pure gravity waves; and gravity-capillary waves obtained numerically in Section 3.

The numerical solution of eqns (45), (46) for Crapper's waves ($\alpha = 0$) was performed for $A = 0.7$ up to $\tau = 100$, i.e., for about 16 periods with the time step $\Delta\tau = 0.001$. Stokes waves ($\sigma = 0$) and gravity-capillary waves ($\sigma = 0.4$) were simulated for $A = 0.3$ up to $\tau = 0.01$ and $\tau = 100$ (with $\Delta\tau = 0.001$), respectively. Snapshots of instantaneous wave profiles (not shown) obtained at various values of τ during the simulations showed that in all cases the wave propagated without any visible disturbances. To estimate "steady-state" solution quantitatively, we calculated the phase amplitudes of the Fourier components for consecutive intervals of $8\Delta\tau$ and obtained their temporal means and standard deviations over the period of integration (Table 4).

Table 3: List of numerical experiments: C - gravity-capillary waves; GC - gravity-capillary waves; power spectrum (83) with μ_k of k th Fourier component of the initial condition.

No.	Type
1	C
2	G
3	$GC, \sigma = 0.4$
4	$GC, \sigma = 0.05$
5	G
6	G
7	G
8	$GC, \sigma = 0.005$
9	C
10	G

The instantaneous phase $\eta(\xi, \tau)$ was calculated as follows:

$$\eta(\xi, \tau) = \dots$$

where η_k are the Fourier components calculated by expansion (11), and

If there is only one wave component corresponding to wavenumber k , the phase speed of the wave surface propagates with a constant value of C for each wave component. For their shapes, this value of C may be different. There may be many components which have the same phase speed due to nonlinearity of the waves there are, generally in the same direction with the same

Table 3: List of numerical experiments (*G* – pure gravity waves; *C* – pure capillary waves; *GC* – gravity-capillary waves; *A* – amplitude of stationary wave (55); *PS* – power spectrum (83) with parameters k_0, A_0 ; *RP* – random phases; a_k – amplitude of k th Fourier component of surface height. The last column indicates nonzero Fourier components of the initial conditions, in ξ -coordinate)

No.	Type	Initial conditions	Modes
1	<i>C</i>	Crapper's wave, $A = 0.7$	All
2	<i>G</i>	Stokes wave, $A = 0.3$	All
3	<i>GC</i> , $\sigma = 0.4$	Stationary <i>GC</i> wave, $A = 0.3$	All
4	<i>GC</i> , $\sigma = 0.05$	<i>PS</i> , <i>RP</i> , $k_0 = 1, A_0 = 0.1$	1 – 25
5	<i>G</i>	Lake & Yuen waves, $a_3 = a_5 = 0.04$	3, 5
6	<i>G</i>	White noise, $a_k = 0.001$	1 – 25
7	<i>G</i>	<i>PS</i> , <i>RP</i> , $k_0 = 5, A_0 = 0.01$	1 – 25
8	<i>GC</i> , $\sigma = 0.005$	<i>PS</i> , <i>RP</i> , $k_0 = 5, A_0 = 0.01$	1 – 25
9	<i>C</i>	<i>PS</i> , <i>RP</i> , $k_0 = 5, A_0 = 0.01$	1 – 25
10	<i>G</i>	Stokes wave, $A = 0.3$, + white noise, $a_k = 0.001$	All+ 15 – 39

The instantaneous phase velocity of the k th wave component may be calculated as follows:

$$C(k) = \dot{\lambda}_k = \frac{\eta_{-k}\dot{\eta}_k - \eta_k\dot{\eta}_{-k}}{k(\eta_k^2 + \eta_{-k}^2)}, \tag{77}$$

where η_k are the Fourier expansion coefficients of the surface as defined by expansion (11), and λ_k are the phases:

$$\eta(\xi, \tau) = \sum_{0 \leq k \leq M} \sqrt{\eta_k^2 + \eta_{-k}^2} \cos(k(\xi - \lambda_k(\tau))).$$

If there is only one wavenumber–frequency spectrum component corresponding to wavenumber k , i.e., if each Fourier component of the surface propagates with a single phase velocity, formula (77) yields a constant value of C for each k . In our case of progressive waves retaining their shapes, this value is the same for all k . For arbitrary wave fields, there may be many modes (wavenumber–frequency spectrum components) which have the same wavenumber but propagate with different phase speed due to nonlinear effects, and even in the case of linear waves there are, generally, two k th modes propagating in opposite directions with the same absolute speed. In such cases, formula (77) yields

Ocean Waves

age phase velocity of the modes, which generally vary in scribed runs, modes moving in the opposite direction were ng the integration because of truncation errors. The am- e modes were by several orders of magnitude smaller than :rue" (positively directed) components of the simulated r manifest themselves in small shifts and perturbations of us phase velocities, whose temporal means and standard given in Table 4 for the first 15 modes.

al means and standard deviations of the phase velocities for the Drapper's, Stokes and gravity-capillary progressive waves (C is the city, i.e. that obtained for the stationary solution)

waves	Stokes waves	GC waves
0.971524	$A = 0.3, C = 1.046040$	$A = 0.3, C = 1.160514$
$1 \cdot 10^{-5}$	1.045997 ($2 \cdot 10^{-5}$)	1.160514 ($2 \cdot 10^{-5}$)
$1 \cdot 10^{-5}$	1.045940 ($2 \cdot 10^{-5}$)	1.160513 ($2 \cdot 10^{-5}$)
$1 \cdot 10^{-5}$	1.045844 ($2 \cdot 10^{-5}$)	1.160512 ($2 \cdot 10^{-5}$)
$1 \cdot 10^{-5}$	1.045711 ($2 \cdot 10^{-5}$)	1.160510 ($2 \cdot 10^{-5}$)
$1 \cdot 10^{-5}$	1.045539 ($2 \cdot 10^{-5}$)	1.160507 ($2 \cdot 10^{-5}$)
$1 \cdot 10^{-5}$	1.045329 ($2 \cdot 10^{-5}$)	1.160505 ($2 \cdot 10^{-5}$)
$1 \cdot 10^{-5}$	1.045081 ($2 \cdot 10^{-5}$)	1.160501 ($2 \cdot 10^{-5}$)
$1 \cdot 10^{-5}$	1.044790 ($2 \cdot 10^{-5}$)	1.160497 ($2 \cdot 10^{-5}$)
$1 \cdot 10^{-5}$	1.044472 ($2 \cdot 10^{-5}$)	1.160493 ($2 \cdot 10^{-5}$)
$1 \cdot 10^{-5}$	1.044410 ($2 \cdot 10^{-5}$)	1.160488 ($2 \cdot 10^{-5}$)
$1 \cdot 10^{-5}$	1.043709 ($2 \cdot 10^{-5}$)	1.160482 ($2 \cdot 10^{-5}$)
$1 \cdot 10^{-5}$	1.043271 ($2 \cdot 10^{-5}$)	1.160476 ($2 \cdot 10^{-5}$)
$1 \cdot 10^{-5}$	1.042795 ($2 \cdot 10^{-5}$)	1.160470 ($2 \cdot 10^{-5}$)
$1 \cdot 10^{-5}$	1.042281 ($2 \cdot 10^{-5}$)	1.160463 ($2 \cdot 10^{-5}$)
$1 \cdot 10^{-4}$	1.041730 ($2 \cdot 10^{-5}$)	1.160456 ($2 \cdot 10^{-5}$)

d phase velocities are very close to their values obtained r solutions (also given in Table 4); although they slightly reasing the wavenumber, their mean errors and standard all for all the three types of waves. Since conservation s (not shown) was also very accurate (the deviations of ng the simulations from their initial values were always r the Stokes wave and less than 10^{-11} for the capillary

and gravity-capillary and remained consis not noticeably chang implies that these wa that the numerical so tions of the original

Another criterion invariants during the

horizontal momentur

and energy $E = E_K$

is the kinetic energy,

is the potential energ

is the potential ener, tained by transforme for the invariants into the horizontal axis.

An example of th resented in Fig. 5 fo The initial surface wa linear modes with an

$$a_k = \langle$$

and gravity-capillary waves), the modes retained their initial energies and remained consistent in phase; consequently, the simulated waves did not noticeably change their shapes during the integration. This result implies that these waves are stable with respect to truncation errors, and that the numerical solutions yielded by the model approximate the solutions of the original differential equations with high accuracy.

Another criterion of model accuracy is conservation of the integral invariants during the integration, i.e. of volume

$$V = (2\pi)^{-1} \int_0^{2\pi} z x_\xi d\xi, \tag{78}$$

horizontal momentum

$$I = (2\pi)^{-1} \int_0^{2\pi} \phi z_\xi d\xi, \tag{79}$$

and energy $E = E_K + E_{PG} + E_{PT}$, where

$$E_K = (2\pi)^{-1} \int_0^{2\pi} \phi \phi_\zeta d\xi \tag{80}$$

is the kinetic energy,

$$E_{PG} = (2\pi)^{-1} \int_0^{2\pi} z^2 x_\xi d\xi \tag{81}$$

is the potential energy of gravity, and

$$E_{PT} = (2\pi)^{-1} \sigma \int_0^{2\pi} (J^{-1/2} - 1) d\xi \tag{82}$$

is the potential energy of surface tension. Formulae (78)–(82) are obtained by transformation of standard Cartesian-coordinate expressions for the invariants into (ξ, ζ) coordinates and refer to a unit length along the horizontal axis.

An example of the temporal evolution of E_K , E_P , E_T , and E is represented in Fig. 5 for the case of gravity-capillary waves with $\sigma = 0.05$. The initial surface was chosen in the form of a superposition of $M_m = 25$ linear modes with amplitudes a_k assigned according to

$$a_k = \begin{cases} A_0 \left(\frac{k}{k_0}\right)^P & k_0 \leq k \leq k_0 + M_m - 1 \\ 0 & \text{otherwise} \end{cases} \tag{83}$$

which generally vary in opposite direction were truncation errors. The amplitudes were smaller than the amplitudes of the simulated waves and perturbations of the initial means and standard deviations.

the phase velocities for the progressive waves (C is the group velocity solution)

GC waves

$A = 0.3, C = 1.160514$

- 1.160514 ($2 \cdot 10^{-5}$)
- 1.160513 ($2 \cdot 10^{-5}$)
- 1.160512 ($2 \cdot 10^{-5}$)
- 1.160510 ($2 \cdot 10^{-5}$)
- 1.160507 ($2 \cdot 10^{-5}$)
- 1.160505 ($2 \cdot 10^{-5}$)
- 1.160501 ($2 \cdot 10^{-5}$)
- 1.160497 ($2 \cdot 10^{-5}$)
- 1.160493 ($2 \cdot 10^{-5}$)
- 1.160488 ($2 \cdot 10^{-5}$)
- 1.160482 ($2 \cdot 10^{-5}$)
- 1.160476 ($2 \cdot 10^{-5}$)
- 1.160470 ($2 \cdot 10^{-5}$)
- 1.160463 ($2 \cdot 10^{-5}$)
- 1.160456 ($2 \cdot 10^{-5}$)

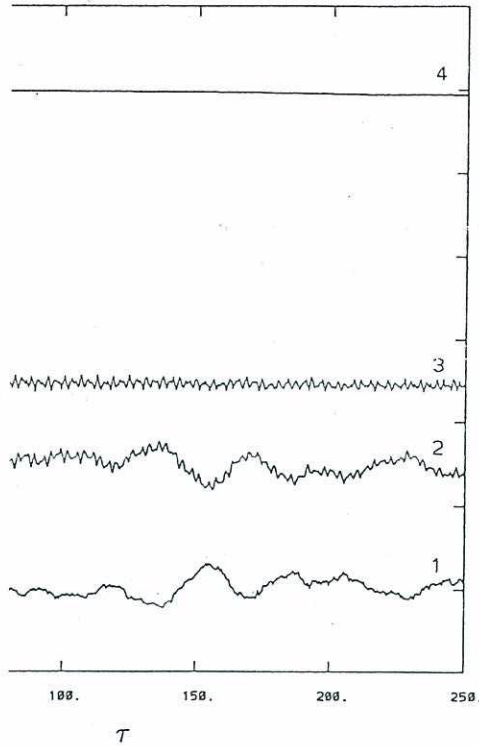
their values obtained although they slightly differ from errors and standard deviations. Since conservation of energy (the deviations of the values were always small) for the capillary

es

plitude of the k_0 th mode, and power $P < 0$ is
The Fourier coefficients were calculated as

$$= a_k \begin{cases} \sin \lambda_k & k \leq 0 \\ \cos \lambda_k & k > 0 \end{cases} \quad (84)$$

phases.



potential energy of surface tension E_{PT} (curve 1), gravi-
(2), kinetic energy E_K (3), and their sum E (4) (Exp.
($\sigma = 0.05$).

s for the initial surface velocity potential Φ were
r approximation, all wave components propa-
gation. It can be easily derived from the linear

theory that this is ensured by

$$\phi_k = b_k \eta_{-k}, \quad b_k$$

For the calculations repre-
 $P = -\frac{3}{2}$, $k_0 = 1$, $M_m = 25$,
tal momentum I and the volt-
margins of the order of 10^{-13}
can be seen that while the ene-
significant fluctuations, their s
due to damping at high waven-
a number of other test simulat-

The results described in th
uccessfully simulate evolution of

6. Results of the simu

The progressive gravity, capilla-
in Sections 3 and 5 represent a
they consist of Fourier modes
sion relation (59), propagate w
evident that similar effects ma-
tions: due to the impact of non-
not be represented as a superp-
their linear phase speeds; mor-
not associated with any single
manifestation of nonlinearity i-
phase speeds close to that of lo-
"bound") components was clea-
observational studies (Yuen ar-
been proposed for this phenom-
current coupling, but Lake a-
mainly due to nonlinearity of t-
contain both types of waves, f-
and the "observed" phase vel-
Partitioning of the energy bet-
wave spectrum, whose shape
oretical explanation of this ef-

theory that this is ensured by the relations:

$$\phi_k = b_k \eta_{-k}, \quad b_k = \text{sign}(k) \sqrt{\frac{1 + \sigma k^2}{k \tanh(kH)}} \quad (85)$$

For the calculations represented in Fig. 5, $\sigma = 0.05$, $A_0 = 0.1$, $P = -\frac{3}{2}$, $k_0 = 1$, $M_m = 25$, and $H = \infty$ (deep water). The horizontal momentum I and the volume V were conserved with relative error margins of the order of 10^{-13} and 10^{-11} respectively. From Fig. 5 it can be seen that while the energy components E_K , E_{PG} , and E_{PT} show significant fluctuations, their sum nearly conserves; its slow decrease is due to damping at high wavenumbers. Similar results were obtained for a number of other test simulations, including ones with a finite depth H .

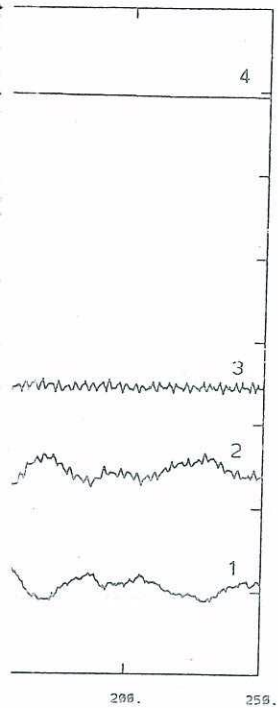
The results described in this section suggest that the model can successfully simulate evolution of multi-component wave fields.

6. Results of the simulations

The progressive gravity, capillary, and gravity-capillary waves dealt with in Sections 3 and 5 represent a very special case of nonlinear interactions; they consist of Fourier modes which, rather than obey the linear dispersion relation (59), propagate with one and the same phase speed. It is evident that similar effects may also be observed in more general situations: due to the impact of nonlinearity, a multi-mode wave motion cannot be represented as a superposition of Fourier modes propagating with their linear phase speeds; moreover, a specific wavenumber, generally, is not associated with any single phase speed. Perhaps the most striking manifestation of nonlinearity is that some shorter waves propagate with phase speeds close to that of long waves. The existence of such forced (or "bound") components was clearly demonstrated in many laboratory and observational studies (Yuen and Lake, 1982). Various explanations have been proposed for this phenomenon, including wind-wave and wave-current coupling, but Lake and Yuen (1978) found that this effect is mainly due to nonlinearity of the waves themselves. Realistic wave fields contain both types of waves, free and bound, for the same wavenumber, and the "observed" phase velocities reflect a combined effect of the two. Partitioning of the energy between these types of waves depends on the wave spectrum, whose shape is influenced by external forcing. A theoretical explanation of this effect based on Zakharov's (1968) equation

th mode, and power $P < 0$ is
 coefficients were calculated as

$$\begin{cases} \leq 0 \\ > 0 \end{cases} \quad (84)$$



face tension E_{PT} (curve 1), gravi-
 κ (3), and their sum E (4) (Exp.

face velocity potential Φ were
 all wave components propa-
 gately derived from the linear

Yuen and Lake (1982). The phenomenon was reproduced by a hydrodynamical potential wave model by Chalikov and Yuen (1982) who pointed out that, with phase velocities calculated from formula (77), each Fourier component turns out to have a phase velocity much greater than that predicted by the linear theory. In fact, the phase velocity varies in time, and that its standard deviation increases with the wavenumber. However, their model was based on long-term simulations with high spectral resolution, greater computational efficiency and accuracy. The experimental model described above has shown that, for analysis of the frequency spectra to be virtually unaffected by further spectral resolution and the length of simulations, it is essential that the model be run for several hundred periods of the longest Fourier component. In the present work, $M = 96$ and $N = 432$ were used.

In this section we consider the results of a model simulation (Exp. 5) of the interaction of two gravity waves with wavenumbers close to each other. The phase velocities of different modes by the interference of the surface elevation values in two sections (Fig. 8 of Lake and Yuen, 1978), and found out that some of the waves not produced by the wave maker were secondary waves. It is hardly possible to exactly reproduce the interaction in a model simulation, since the amplitudes of the waves are not reported, and there are uncertainties as to modeling the model, to obtain a flow qualitatively similar to that of the experiment, we used a superposition of the 3rd and the 5th Fourier components with amplitudes of 0.04 as the initial conditions for the surface elevation. The forcing was imposed. In this and all other runs (except for the progressive waves described in Section 3), the initial phase velocities on the surface were prescribed according to the linear theory for unidirectional waves (formula (85)). The simulation was terminated with $\Delta\tau = 0.01$ up to $\tau = 1000$.

The frequency spectra (calculated from Fourier expansions of the surface elevation with respect to x coordinate) averaged for 6 consecutive runs (167 nondimensional time units each) are given in Figure 6. The energy decreases slowly in time because of dissipation at

high wavenumbers. The energy of the 5th mode is considerably less than that of the 3rd mode)

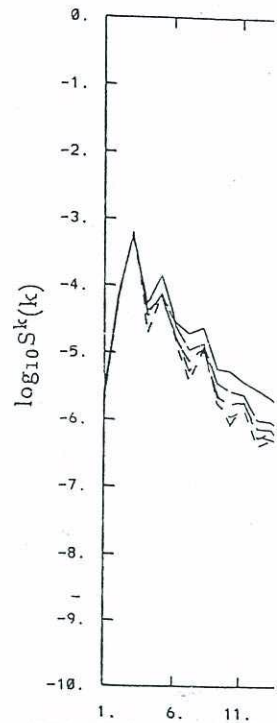


Figure 6: Time-averaged wave energy spectrum of length $\delta\tau = 167$, Exp. 5.

In Fig. 7, temporal variations of the phase velocities (7) are shown. A linear relation is observed only for the primary waves propagate significantly faster than the secondary waves. This is a natural means of the phase velocity dispersion pointed out in Chalikov and Yuen (1982). The standard deviations for the phase velocities are shown by the presence of both

phenomenon was reproduced by Chalikov and phase velocities calculated component turns out to have a predicted by the linear theory. in time, and that its standard. However, their model high spectral resolution, accuracy. Experiments shown that, for analysis unaffected by further of simulations, it is expected periods of the longest of the order of 100. In the re used.

simulation (Exp. 5) of (78) who investigated the wavenumbers close to of different modes by values in two sections (73), and found out that by the wave maker were possible to exactly reproduce, since the amplitudes of uncertainties as to modeling qualitatively similar to of the 3rd and the 5th conditions for the surrounding all other runs (except in Section 3), the initial were prescribed according to (85). The $\tau = 1000$.

Fourier expansions of the averaged for 6 consecutive intervals each) are given in because of dissipation at

high wavenumbers. The energy of the 3rd mode is nearly conserved but the energy of the 5th mode (which was initially equal to that of the 3rd mode) is considerably less.

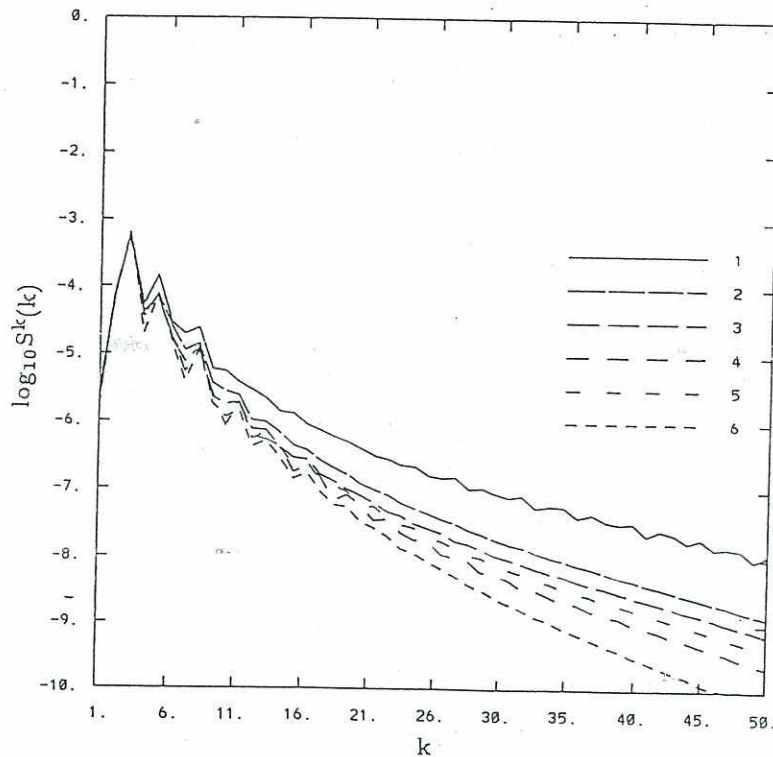
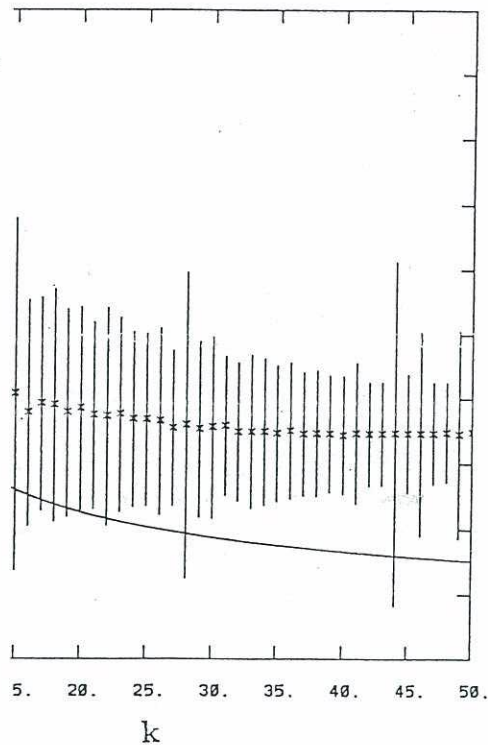


Figure 6: Time-averaged wavenumber spectra $S^k(k)$ for 6 consecutive time intervals of length $\delta\tau = 167$, Exp. 5.

In Fig. 7, temporal means and standard deviations of the instantaneous phase velocities (77) are shown. It is seen that the linear dispersion relation is observed only for wavenumbers $2 \leq k \leq 5$. For $k > 5$, the waves propagate significantly faster than the linear waves but slower than the primary waves. This effect is clearly pronounced only for the temporal means of the phase velocities; the instantaneous values vary highly (as pointed out in Chalikov and Liberman, 1991), which is reflected by large standard deviations for wavenumbers $k > 5$. This scattering is caused by the presence of both bound and free waves.



the velocity (77) (*) and its standard deviation (vertical bars) versus wavenumber k , Exp. 5 (simulation of the nonwind experiment). The smooth curve corresponds to the linear dispersion relation.

free and bound waves and their phase velocity. Figure 8 along with the logarithms of the time-averaged wavenumber-frequency spectrum $S(k, \omega)$ and frequency $S^\omega(\omega)$ spectra. The picture consists of patches; this effect is caused by the irregularities in the contour lines. To calculate S for each k , instantaneous values of the velocity with respect to the x coordinate were stored. The time intervals of integration $0 \leq \tau \leq 1000$ with time intervals of 100 units were used. In this case, the number of simulation ensured sufficient frequency resolution (0.001), which is essential for the analysis of the

spectra, and the maximum resolution far exceeded any possible "phase transforms" aliasing error negli

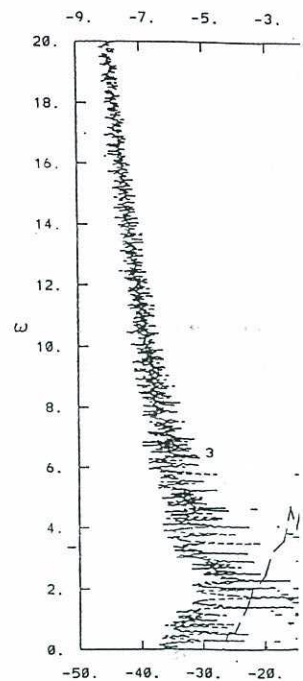
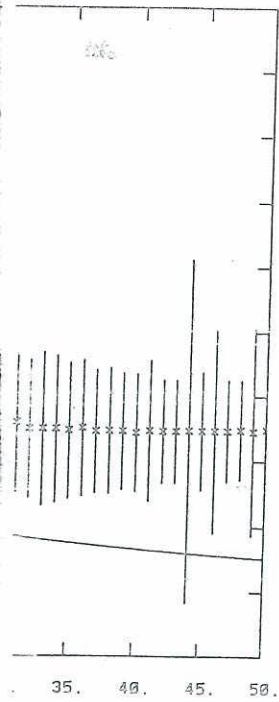


Figure 8: Time-averaged spectral plots. Curve 1 - linear dispersion relation, curves 2, ..., 9. The contour lines of constant values are seen as concatenated in patches. Curve 3 is $\log_{10} S^\omega(\omega)$ (frequency scale).

The most remarkable feature is that it is split into a set of well pronounced for the waves originally by using relations (1) and (2). Noticeable for the waves moving in the present in the initial condition is determined by the sign of the

spectra, and the maximum resolved frequency ($\omega = \pi/0.08$ for this run) far exceeded any possible "physical" value of ω and thus rendered the transforms' aliasing error negligible.



its standard deviation (vertical axis) of the nonwind experiment compared with the linear dispersion relation.

waves and their phase velocity spectrum $S(k, \omega)$ whose contours are the logarithms of the time-averaged spectra ($S^\omega(\omega)$). The picture shows this effect is caused by the discrete nature of S for each k , instantaneous values of x coordinate were stored at $\Delta x \leq 1000$ with time intervals Δt to time were used. In this run sufficient frequency resolution was achieved for the analysis of the

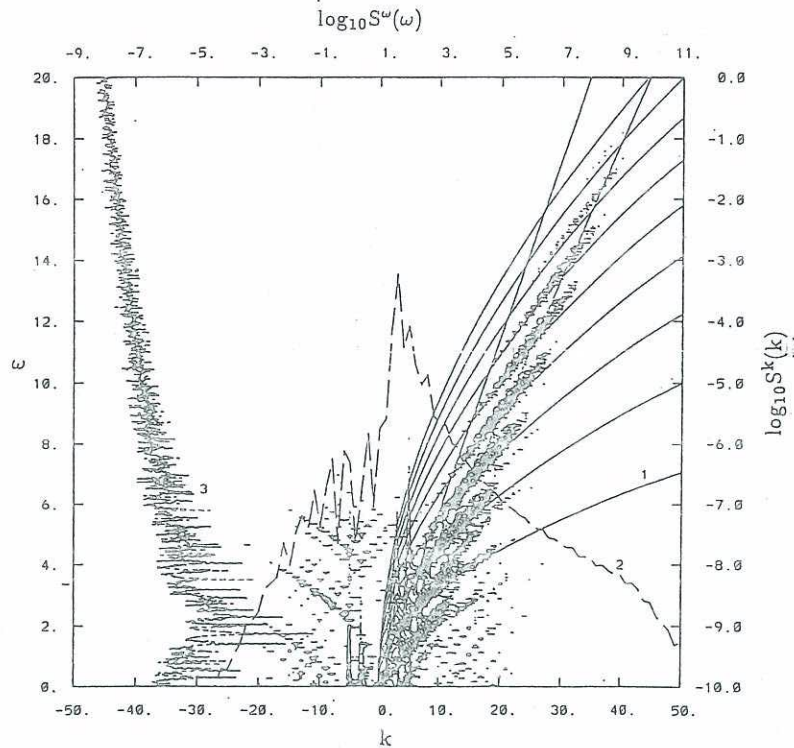


Figure 8: Time-averaged spectral characteristics for the period of $\delta\tau = 1000$, Exp. 5. Curve 1 - linear dispersion relation $\omega^2 = k$, the other parabolas - $\omega^2 = nk$, $k = 1, 2, \dots, 9$. The contour lines of $\log_{10} S(k, \omega)$ (wavenumber-frequency spectrum) are seen as concatenated in patches. Curve 2 is $\log_{10} S^k(k)$ (wavenumber spectrum, right scale), Curve 3 is $\log_{10} S^\omega(\omega)$ (frequency spectrum, top scale).

The most remarkable feature of the wavenumber-frequency spectrum is that it is split into a set of branches in a regular way. This effect is well pronounced for the waves propagating in the direction prescribed originally by using relations (85) ($k > 0$ in Fig. 8), but it is also noticeable for the waves moving in the opposite direction, which were not present in the initial conditions ($k < 0$ in Fig. 8 where the sign of k is determined by the sign of the component's phase velocity, while ω is

1 Waves

A considerable part of energy is borne by the component obeying the linear dispersion relation $\omega^2 = |k|$ (Curve 1 energy mostly belongs to what may be, with some interpreted as bound components, which propagate with their carrying waves from near Curve 1 and lie on imated by the curves

$$\omega^2 = n |k|, \tag{86}$$

where n (the order of the branch) is a positive integer. (Strictly, not all curves (86) with $n > 1$ may be called bound, since those with n a multiple of n have no "carrier". However, they are bound to a free wave with wavenumber k/n).

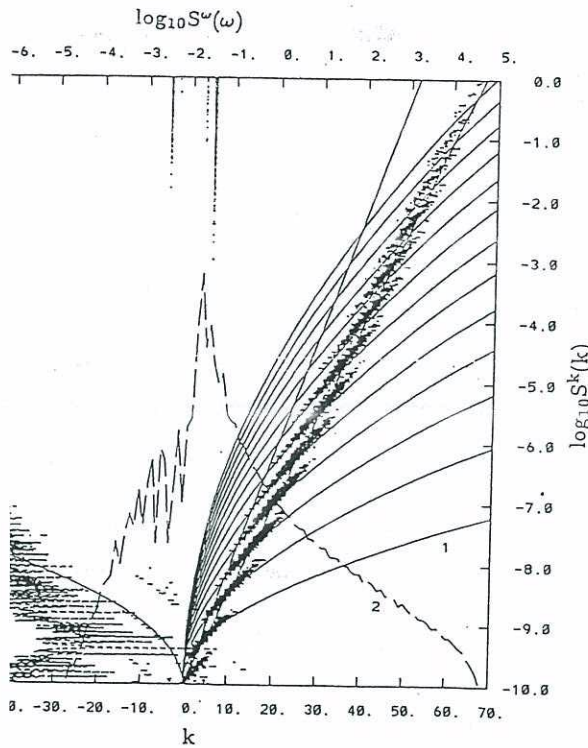


Figure 8, but instead of $\log_{10} S$ the contours of normalized spectrum are plotted.

Each of the branches with wavenumber k and tends to straighten out as k increases, the group velocity is the same for all the branches. There are also other patterns with notably regular discrete

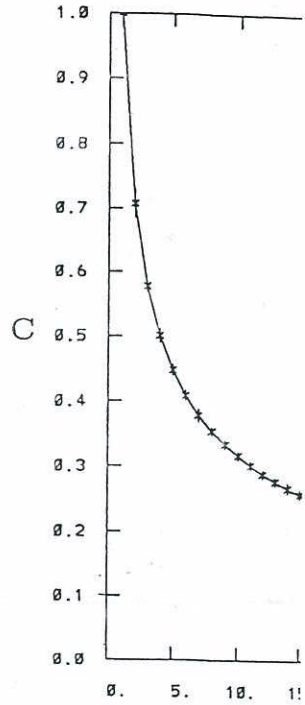


Figure 10: Same as in Fig. 7 (with noise).

Another representation can be obtained through normalization for a given k is the spectral density spectrum (Fig. 9), which is plotted over frequencies for each type described above. A

of energy is borne by the component dispersion relation $\omega^2 = |k|$ (Curve 1 belongs to what may be, with some components, which propagate with waves from near Curve 1 and lie on curves

$$|k|, \quad (86)$$

is a positive integer. (Strictly, not may be called bound, since those have no "carrier". However, they force wave with wavenumber k/n).

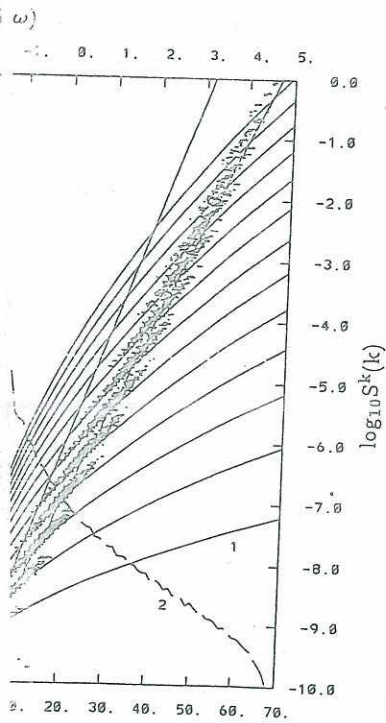


Figure 9: The contours of normalized spec-

Each of the branches follows relation (86) closely at lower wavenumbers k and tends to straighten at higher wavenumbers so that, with k increasing, the group velocity appears to tend to a constant whose value is the same for all the branches. Along with this set of branches, there are also other patterns which, despite their relatively low energy, show a notably regular discrete structure.

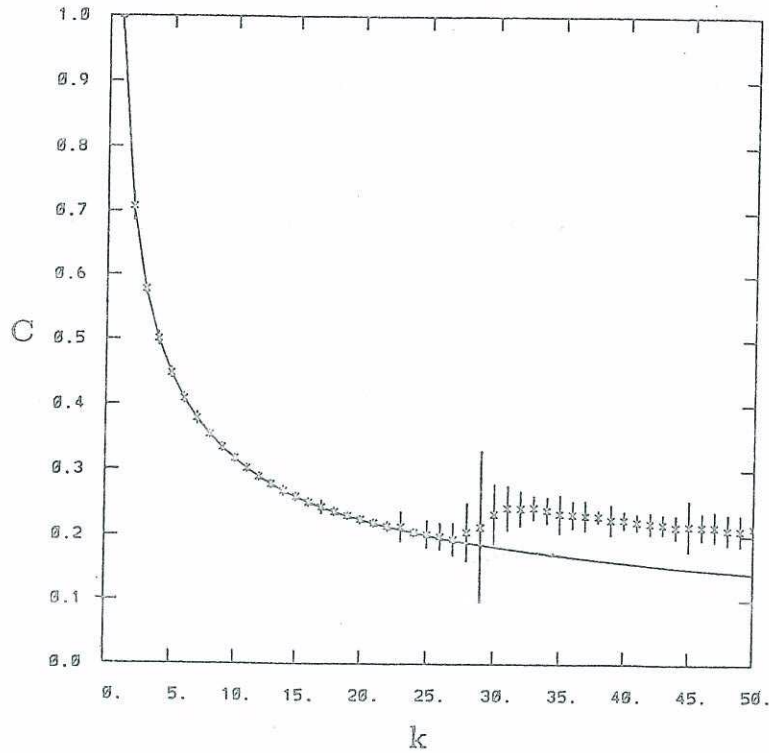
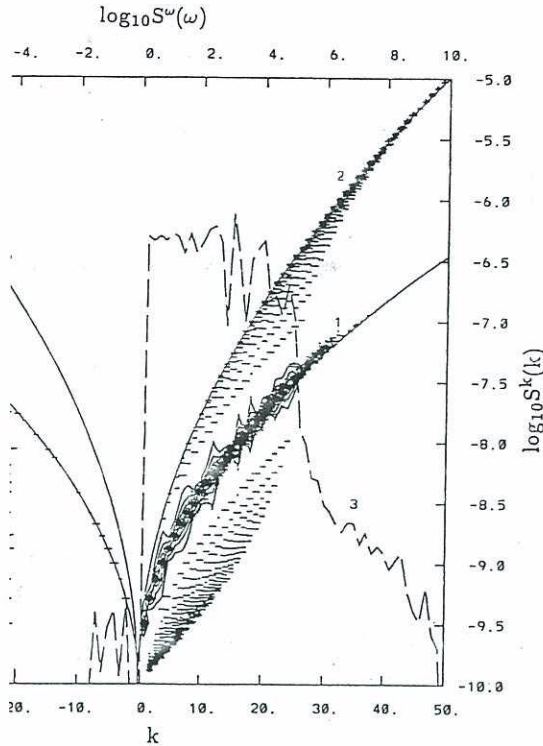


Figure 10: Same as in Fig. 7 but for Exp. 6 (initial conditions approximating white noise).

Another representation of the wavenumber–frequency spectrum can be obtained through normalizing each value of $S(k, \omega)$ by $S^k(k)$ (which for a given k is the spectral density integrated over ω). The normalized spectrum (Fig. 9), which characterizes the fractional energy distribution over frequencies for each wavenumber, exhibits up to 13 branches of the type described above. A most peculiar feature of the spectrum, clearly

es

), is a pattern which may be approximated by rough the origin of the coordinates; the corre- d to propagate with roughly the same phase be equal to the aforementioned apparent limit



acteristics as in Fig. 8 but for "white noise" simulation dispersion relation $\omega^2 = nk$ ($n = 1, 2$), 3 - $S^k(k)$, 4 -

xperiment (Exp. 6) simulated a pure gravity ditions approximating white noise; other fea- Exp. 5. For the first 25 wavenumbers k , the signed the same value of 0.001, with random mplitudes were set to zero. The phase veloci- rd deviations are shown in Fig. 10. Because

the amplitudes of the initial nearly obey the linear theory ear ones and have small sta the nonlinear interactions (waves. In the wavenumber- two ($n = 1$, where most of the "main branches" approxima the energy is small, and, aga line passing through the orig other regularly located curv planation of these features.

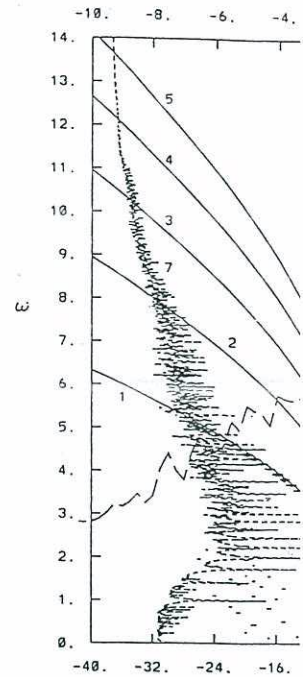
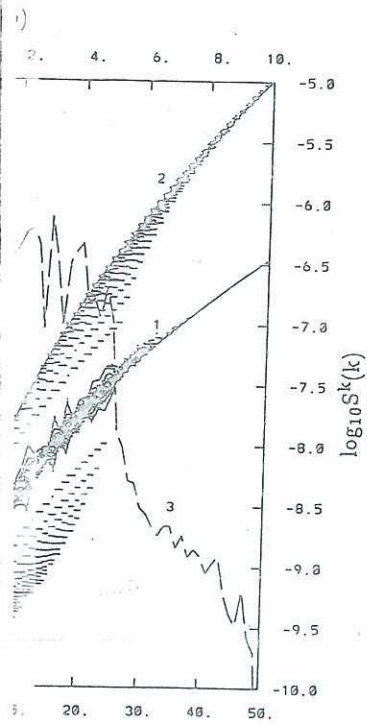


Figure 12: Same spectral charact with spectrum (83) approximati

Another run (Exp. 7),

which may be approximated by
of the coordinates; the corre-
with roughly the same phase
aforementioned apparent limit



6) but for "white noise" simulation
 $\omega^2 = nk$ ($n = 1, 2$), 3 - $S^k(k)$, 4 -

6) simulated a pure gravity
ating white noise; other fea-
e first 25 wavenumbers k , the
value of 0.001, with random
er to zero. The phase veloci-
e shown in Fig. 10. Because

the amplitudes of the initially assigned waves were small, these waves nearly obey the linear theory: their phase velocities are close to the linear ones and have small standard deviations. The waves produced by the nonlinear interactions ($k > 25$) again propagate faster than linear waves. In the wavenumber-frequency spectrum (Fig. 11), only the first two ($n = 1$, where most of the energy is concentrated, and $n = 2$) of the "main branches" approximated by (86) are seen. The remaining part of the energy is small, and, again, most of it is concentrated near a straight line passing through the origin, while the remainder is distributed along other regularly located curves. Further investigations are needed for explanation of these features.

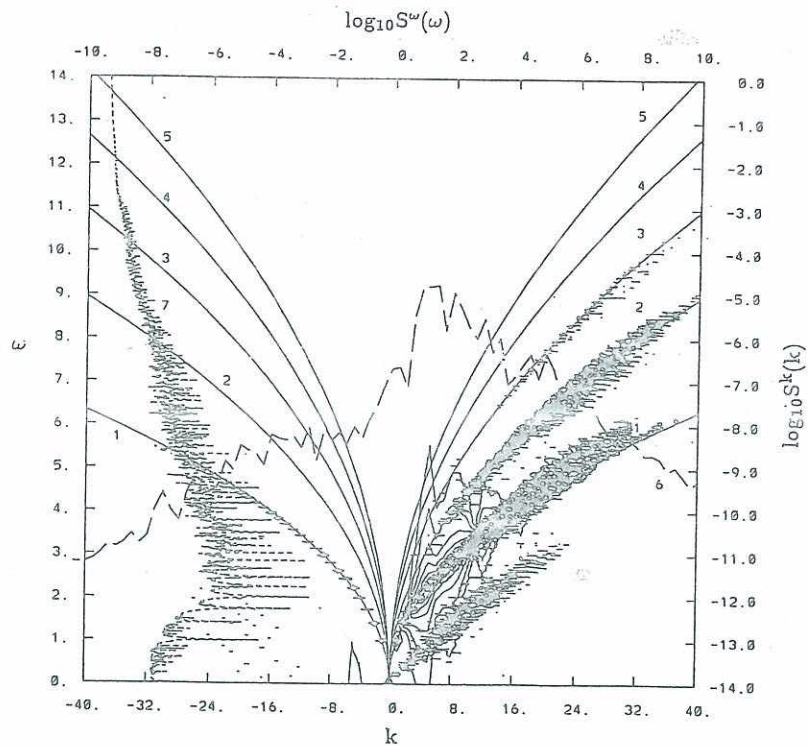
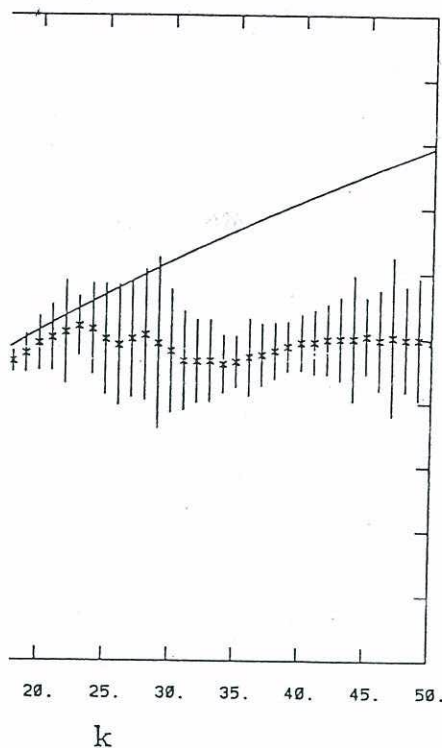


Figure 12: Same spectral characteristics as in Fig. 8 but for Exp. 7 (initial conditions with spectrum (83) approximating real waves).

Another run (Exp. 7), which also differed from the previously de-

amplitudes only, was designed to approximate the initial spectrum of the surface elevation was (1) , with $A_0 = 0.01$, $k_0 = 5$, $P = -1.5$. The 12. In the wavenumber-frequency spectrum, energy concentrated along the "main" branches near" branch is again quite distinct.



for Exp. 8 (gravity-capillary waves, $\sigma = 0.005$).

gravity-capillary waves were simulated, with viscosity coefficient $\sigma = 0.005$. The initial conditions were (83) and (84) with $A_0 = 0.01$, $k_0 = 5$, $P = -1.5$. The waves bearing most of the energy are of the gravity-capillary type. The time integration was with the time step $\Delta\tau = 0.01$. The dependence of the energy on the wavenumber and its temporal standard deviation on the

wavenumber is shown in Fig. 13.

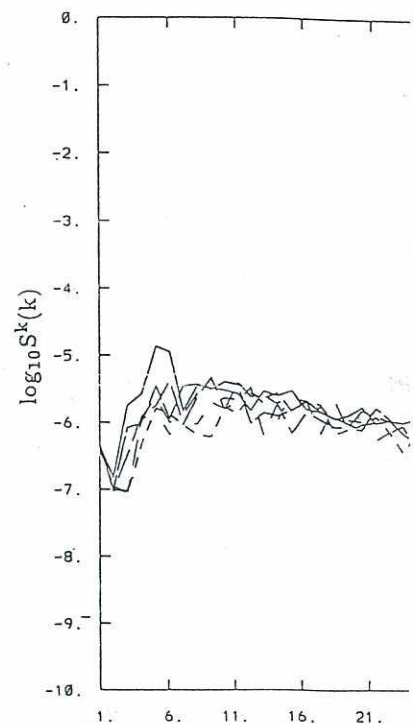
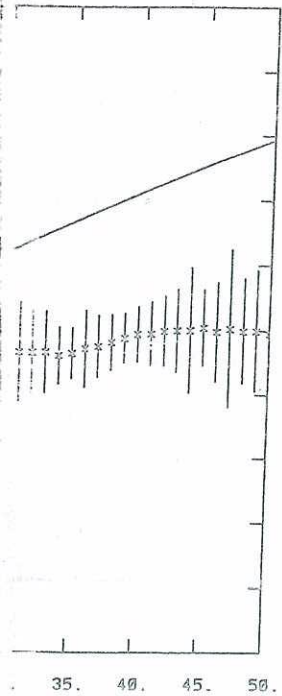


Figure 14: Same as Fig. 6 but for Exp. 6.

As in the runs with pure gravity waves, the longer waves tend to follow the linear dispersion relation, while the shorter waves show significant nonlinear interactions, which tend to scatter energy away from the linear theory. The wavenumber-frequency spectrum is shown in Fig. 14 in the case of gravity-capillary waves, whose mean over the period of integration is shown in Fig. 15. The spectrum shows a greater nonlinear character than in the case of pure gravity waves, with a less steep slope of the curve, compared to the previous run (Fig. 12, Curve 6).

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wavenumber is shown in Fig. 13.

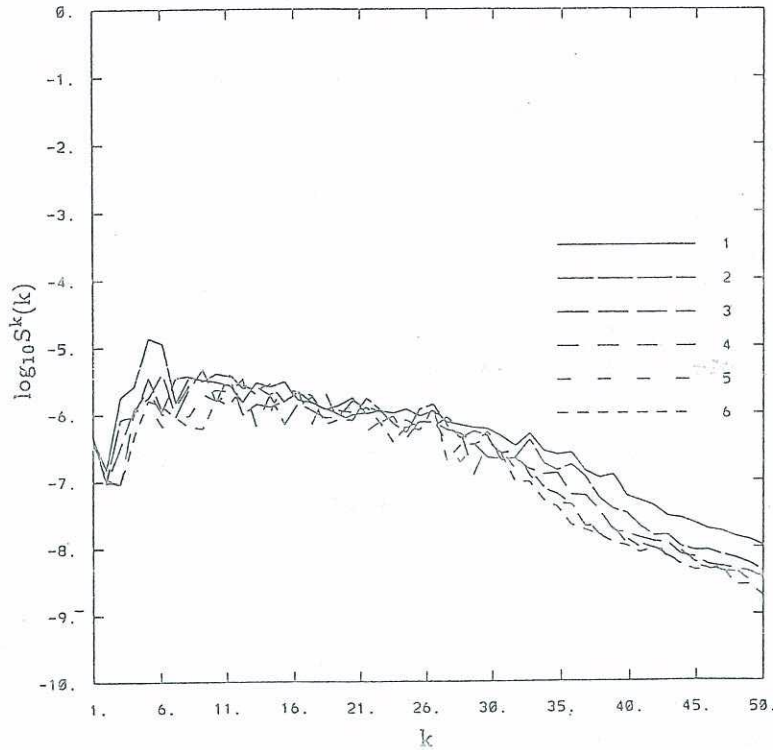
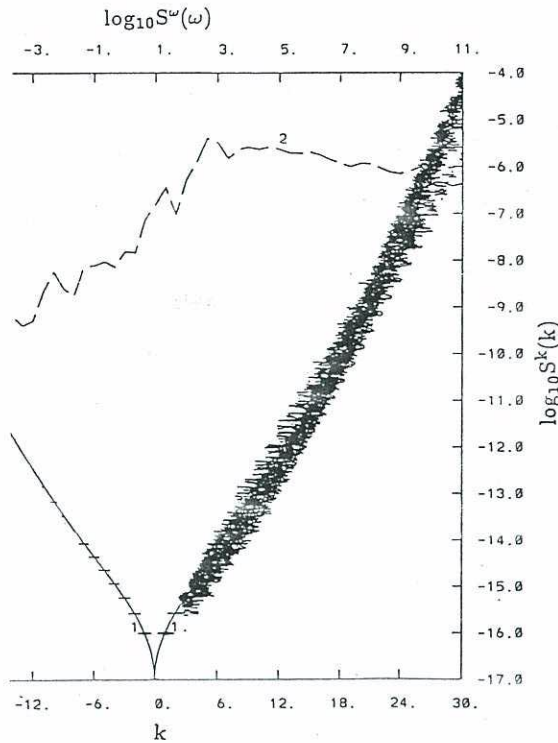


Figure 14: Same as Fig. 6 but for Exp. 8.

As in the runs with pure gravity waves, the lower wavenumbers tend to follow the linear dispersion relation (77), though with considerable scattering, while the shorter waves, which are mostly generated by the nonlinear interactions, tend to assume the phase velocities which are characteristic for longer waves and are in this case lower than those given by the linear theory. The wavenumber spectrum, whose temporal evolution is shown in Fig. 14 in the same way as in Fig. 6 for Exp. 5 and whose mean over the period of integration is represented by Curve 2 in Fig. 15, shows a greater nonlinear energy flux to higher wavenumbers than in the case of pure gravity waves. This is manifested in a much less steep slope of the curve, compared to the corresponding results of the previous run (Fig. 12, Curve 6), which was performed with the same

aves

, and in a relatively fast decrease of energy with on high wavenumbers. The same observation can ncy spectra (compare Fig. 15, Curve 3 and Fig.



haracteristics as in Fig. 8 but for Exp. 8. Curve 1 - linear in relation (87)), 2 - $\log_{10} S^k(k)$; 3 - $\log_{10} S(k, \omega)$.

requency spectrum (Fig. 15), it is hard to distin-free and bound waves, since Curve 1 which cor-dispersion relation (59) is too close to a straight gest waves whose energy is negligibly small. Here, d regarding that a bound wave propagates with y ω/k as its carrying wave and has a wavenumber multiple of the latter, we can write the following "main branches":

$$\left(\frac{\omega}{n}\right)^2$$

where $n = 1$ for the free (ca waves. However, the branch

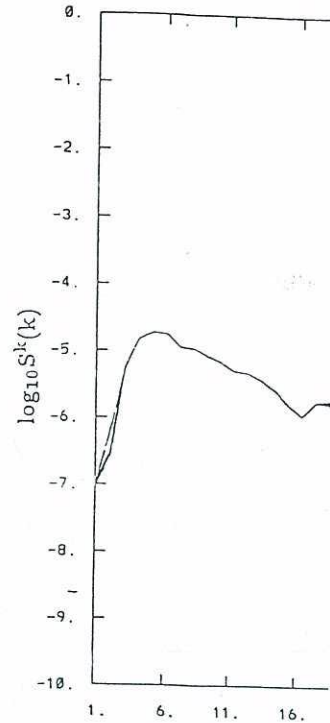


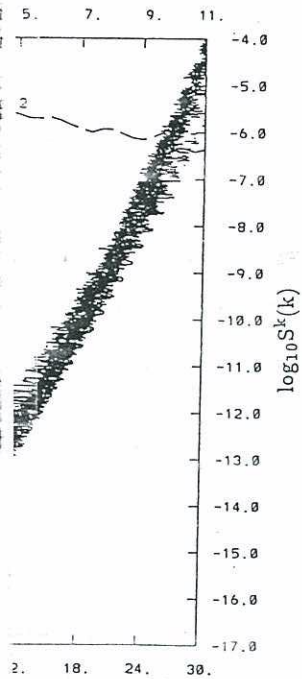
Figure 16: Same as in Fig. 6 but

Results of simulation of I (46)) are shown in Figs 16 a in Figs 14 and 15 respective were the same as in Exps 7 a and the time step $\Delta\tau = 0.0$ wavenumber spectrum is mu waves (Exp. 7) than to that wavenumber-frequency spect

$$\left(\frac{\omega}{n}\right)^2 = \frac{|k|}{n} + \sigma \left(\frac{|k|}{n}\right)^3, \tag{87}$$

where $n = 1$ for the free (carrying) waves and $n = 2, 3, \dots$ for the bound waves. However, the branches appear to merge with each other.

st decrease of energy with
The same observation can
Fig. 15, Curve 3 and Fig.



ut for Exp. 8. Curve 1 - linear
 $k(k)$; 3 - $\log_{10} S(k, \omega)$.

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since Curve 1 which cor-
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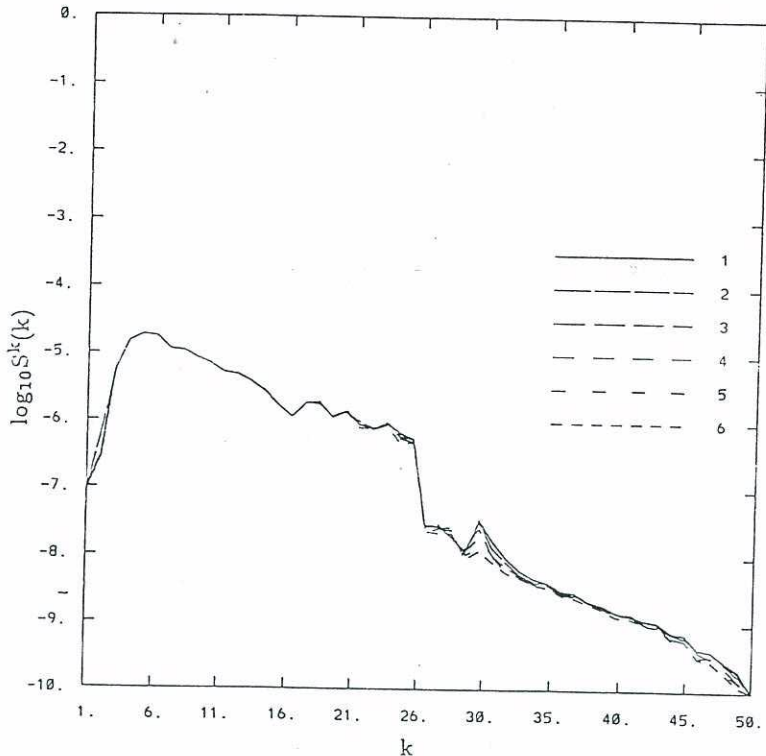


Figure 16: Same as in Fig. 6 but for Exp. 9 (capillary waves), $\delta\tau = 16.7$.

Results of simulation of pure capillary waves (Exp. 9, $\alpha = 0$ in eqn (46)) are shown in Figs 16 and 17 (the same spectral characteristics as in Figs 14 and 15 respectively). The initial values of surface elevation were the same as in Exps 7 and 8; the period of integration was $\tau = 100$, and the time step $\Delta\tau = 0.001$. It is remarkable that the slope of the wavenumber spectrum is much closer to that obtained for pure gravity waves (Exp. 7) than to that of gravity-capillary waves (Exp. 8). The wavenumber-frequency spectrum agrees well with the linear dispersion

Waves

Curve 1); however, below the curve is what appears as "main branches" corresponding to bound waves and

$$\omega^2 = \frac{|k|^3}{n} \tag{88}$$

ing from (87) with the gravity term omitted and an additional set of waves above the curve which "rectilinear" patterns observed in the pure gravity cases 8, 9, 11, 12 and 17 may suggest the hypothesis correspond to waves which, through nonlinear interaction shorter waves, are forced to propagate with the latter; further extensive simulations are needed to

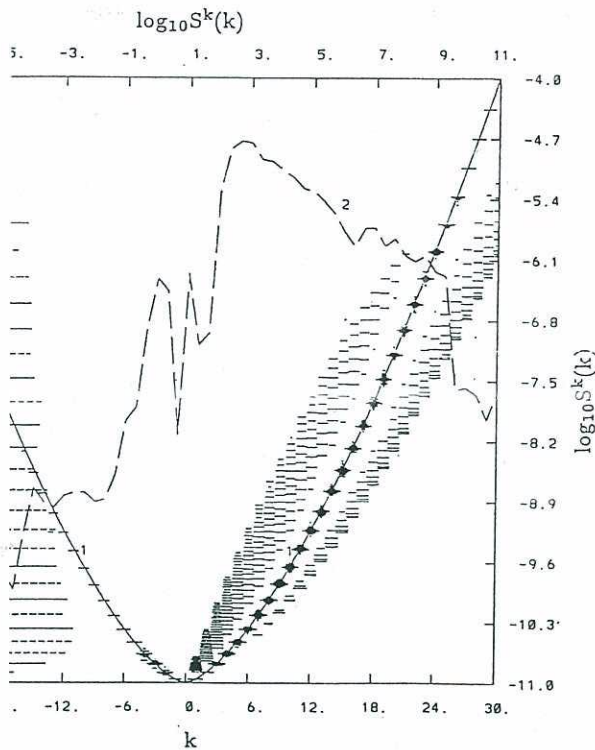


Fig. 8 but for Exp. 9 (capillary waves), with Curve 1 being relation ($n = 1$ in relation (88))

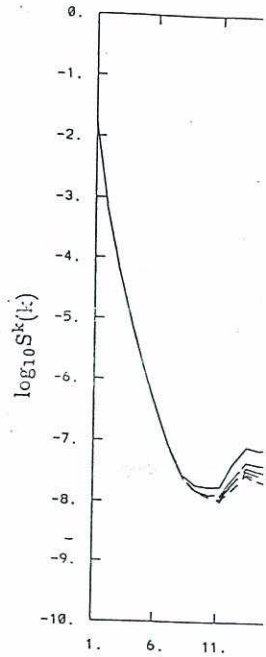


Figure 18: Same as in Fig. 8 but for Exp. 9 (capillary waves), with Curve 1 being relation ($n = 1$ in relation (88)) superimposed short gravity waves

Besides the simulation experiments with different and dissipation parameters and external forcing (not discussed) extra observed in all runs has follows. Most of the energy is called "main branches" and bound ($n = 2, 3, \dots$) wave

and, for not too large k , pure capillary waves).

the curve is what appears
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(88)

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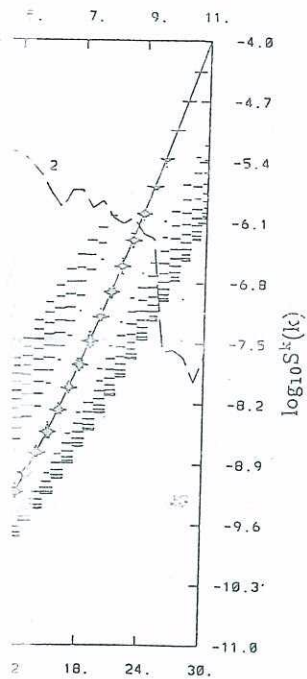


Figure 18: Same as in Fig. 6 but for Exp. 10 (initial conditions: Stokes wave with superimposed short gravity waves).

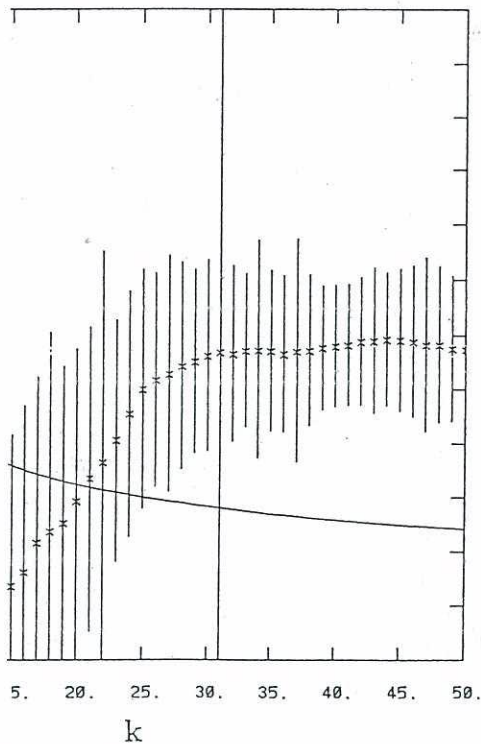
Besides the simulations described above, we ran a number of other experiments with different initial conditions, resolution in the horizontal, and dissipation parameters; some of the runs included various types of external forcing (not discussed here). The wavenumber-frequency spectra observed in all runs had similar features which may be summarized as follows. Most of the energy is concentrated along the curves which were called "main branches" and which consist of free ($n = 1$) and generalized bound ($n = 2, 3, \dots$) waves. The general formula for this set of curves is

$$D\left(\frac{\omega}{n}, \frac{k}{n}\right) = 0 \tag{89}$$

and, for not too large k , is approximated by relation (87) (or (88) for pure capillary waves).

y waves), with Curve 1 being

ves



' but for Exp. 10.

in (89), nonlinearity is manifested in two ways: branches instead of one curve corresponding to linear dispersion. The specific form of D depends on the energy spectrum and is determined by external forcing conditions. For example, in the very special case of a

$$D(\omega', k') = \omega'^2 - |k'| - \sigma |k'|^3 \quad (90)$$

Stokes wave. The specific form of D depends on the energy spectrum and is determined by external forcing conditions. For example, in the very special case of a

$$D(\omega', k') = \omega' - ck' \quad (91)$$

Stokes wave. The specific form of D depends on the energy spectrum and is determined by external forcing conditions. For example, in the very special case of a

except that c can be approximated by increasing k , there is a tendency for the "main branches" (the corresponding modes) to need further investigation.

The last model run to be shown is for gravity waves and illustrates short waves. A set of white-noise waves in the range $15 \leq k < 40$ with the amplitude $A = 0.1$ and $\tau = 1000$ with $\Delta\tau = 0.01$.

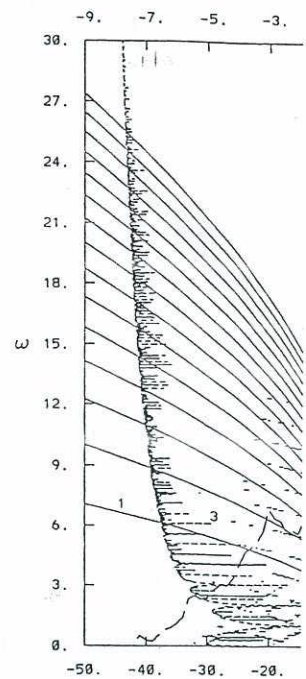
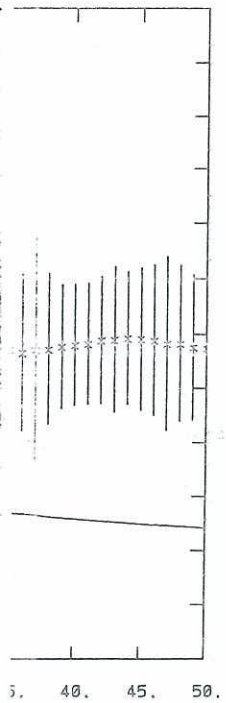


Figure 20: Same spectral characteristics as Figure 19, but for the linear dispersion relation $\omega^2 = k^3$ and the Stokes wave ($\omega = ck$).

except that c can be approximated by the linear phase velocity (59). With increasing k , there is a tendency for the branches to straighten. Along with the "main branches" (89), there are other patterns in the spectra; the corresponding modes bear relatively low energy. These structures need further investigation.

The last model run to be described (Exp. 10) again deals with pure gravity waves and illustrates interaction of a large long wave with small short waves. A set of white-noise-like waves with the amplitude of 0.001 in the range $15 \leq k < 40$ was superimposed on a 2π -periodic Stokes wave with the amplitude $A = 0.3$. The integration was performed up to $\tau = 1000$ with $\Delta\tau = 0.01$.



manifested in two ways: one corresponding to linear waves and another from its approximation

$$k^3 \quad (90)$$

where D depends on the energy input by external forcing. In the very special case of a

$$(91)$$

each branch consists of one mode with approximation (90)

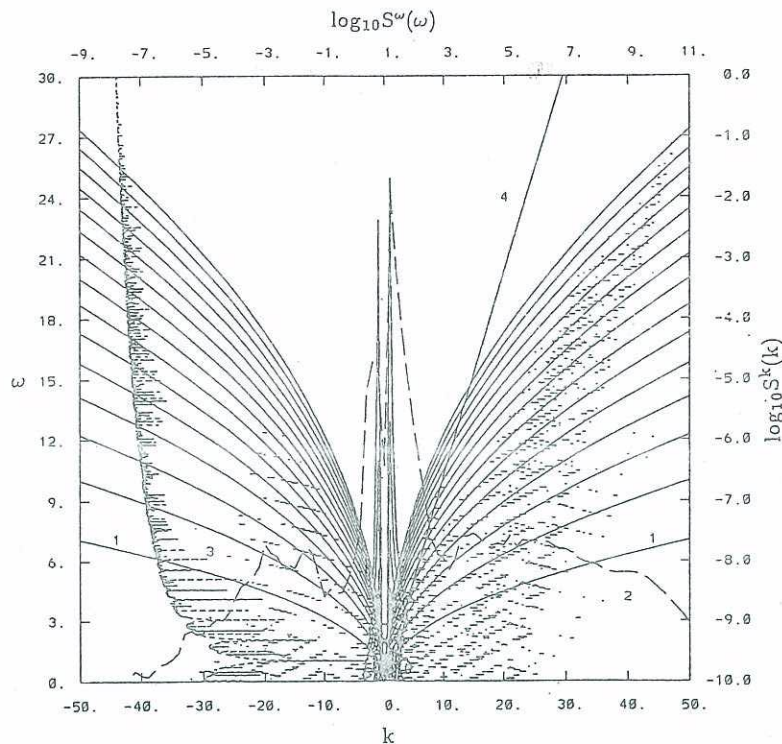
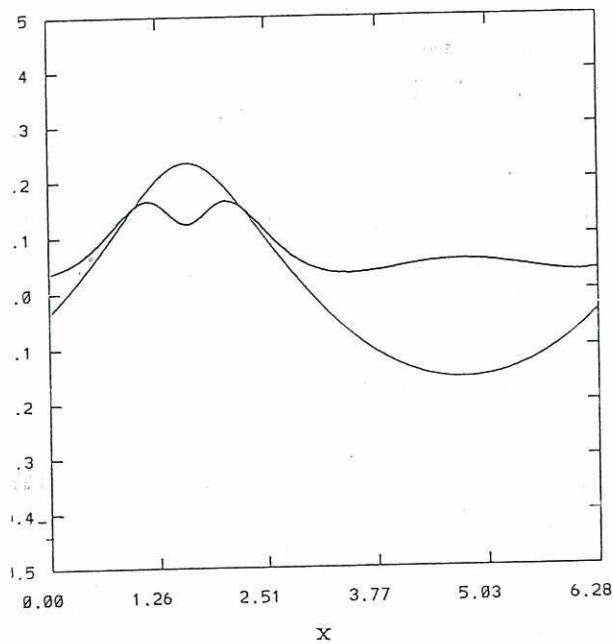


Figure 20: Same spectral characteristics as in Fig. 8 but for Exp. 10. Curve 1 - linear dispersion relation $\omega^2 = k$, 2 - $\log_{10} S^k(k)$; 3 - $\log_{10} S^\omega(\omega)$, 4 - components of the Stokes wave ($\omega = ck$).

near Ocean Waves



Distribution of the time-averaged potential energy of perturbations (multiplied by 100, thick curve) over the phase of the running Stokes-like wave (thin curve, by time averaging of the surface profile) (Exp. 10). The averaging is performed in a coordinate system moving with the component $k = 1$.

The evolution of the wavenumber spectrum is given in Fig. 18 which shows that the energy of the main Fourier components of Stokes wave is almost virtually unchanged while the energy of the short waves decreases considerably during the integration. The dependence of phase (77) on wavenumber differs substantially from the corresponding results of the previously described runs: it bears little resemblance to the linear-theory dependence (59) (the curve in Fig. 19). Correspondingly, in the wavenumber-frequency spectra (Fig. 20) the "main peaks" (89) appear to be represented only by the Fourier components of the Stokes wave and so are described by (91), while the superimposed

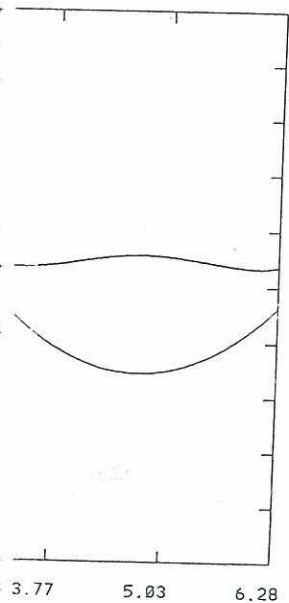
waves belong to the component from the Stokes wave. This was used as a check on the behavior of the numerical solution with the Stokes wave. It is clear that the distribution of the phase of the Stokes wave shows clear

7. Conclusions

The behavior of the Stokes wave. Even for the case of a single isolated component of a primary pure gravity wave for consecutive runs of a truncated Fourier series, which the accuracy of calculation number is investigated. This investigation may lead to the development of a numerical algorithm for simulation of the Stokes wave. It has been proven that the development of a high-order potential wave theory of wave phenomena is a theoretical

The main problem of the stationary potential flow is a numerical problem. It can be eliminated by the surface-following method. The most effective method of long-term

The results



potential energy of perturbations (multiple) superimposed on the running Stokes-like wave (thin curve, profile) (Exp. 10). The averaging is per the component $k = 1$.

spectrum is given in Fig. 18 which shows the Fourier components of Stokes wave and the energy of the short waves due to the perturbation. The dependence of phase on frequency is substantially from the corresponding runs: it bears little resemblance to the curve in Fig. 19). Correspondence spectra (Fig. 20) the "main" component is only by the Fourier components defined by (91), while the superimposed

waves belong to additional regular patterns which are completely different from those observed in Exp. 6 (Fig. 11) where "pure" white noise was used as the initial conditions. The difference suggests that the behavior of the superimposed waves is strongly controlled by interactions with the Stokes wave. The same conclusion can be drawn from the fact that the distribution of the energy density of the short waves over the phase of the Stokes wave (Fig. 21) is far from uniform or chaotic, and shows clear maxima at the points of maximum slope of the Stokes wave.

7. Conclusions

The behavior of nonlinear waves is difficult to investigate analytically. Even for the stationary equations, exact solutions are known only in the isolated case of pure capillary (Crapper's) waves. In the case of stationary pure gravity (Stokes) waves, construction of analytical expansions for consecutive Fourier coefficients provides only an approximation for truncated Fourier series and is thus, actually, a numerical procedure in which the amount of calculations increases sharply with increasing truncation number. As for general nonstationary wave fields, their analytical investigation is clearly impossible without drastic simplifications which may lead to unpredictable consequences. In the development of numerical algorithms for 1-D potential waves, considerable progress has been made during the last 15 years, but the applicability of these techniques for simulation of multi-mode wave fields over a long time period has not been proven. One possible approach to circumvent these problems is to develop a highly accurate numerical scheme for the principal equations of potential waves. With such a scheme, direct hydrodynamical modeling of wave phenomena may be expected to provide improvements in many theoretical and applied aspects of wave studies.

The main difficulty in constructing numerical methods for nonstationary potential waves is how to deal with the vertical dimension. For a numerical scheme to be really efficient, the vertical dimension must be eliminated from the model prognostic equations. The nonstationary surface-following conformal mapping used in our scheme is, indeed, a most effective way to resolve this problem and make the model capable of long-term multi-mode simulations.

The results obtained in this work may be divided into three groups.

al mapping developed by Stokes (1848) for stationary case extended to the nonstationary case, where the surface elevation becomes time dependent, and the stream function is no longer the velocity potential and the stream function is no longer effective because the original system contains one and two nonlinear boundary conditions on the surface and bottom, without any simplification, to a system of nonstationary equations. As in the Cartesian coordinate system the variables are the elevation and the velocity components and their dependence on the Cartesian horizontal coordinates is expressed parametrically via the new horizontal coordinate system. The system may be solved by numerical integration, and by calculating nonlinearities via Fourier transforms. The method may also be used for analytical investigations and for the development of the original system, with the problem of extrapolating the velocity potential not existing.

The solution of the stationary equations for gravity-capillary waves has been developed. The method allows us to solve the problem with computer accuracy and is based on representation of the surface height, written in the new coordinate system, as a sum of modes whose amplitudes are proportional to the velocity potential and the stream function via operators of integration and (generalized) differentiation calculated in Fourier space. Again, the use of the method to calculate nonlinearities allows a highly effective method. It should be noted that we had to use different algorithms for pure gravity and gravity-capillary waves. In the latter case our algorithm fails to converge when the capillarity coefficient becomes small. This problem is investigated. It was shown that, with decreasing capillarity coefficient, the phase velocities of gravity-capillary waves approach the values of the Stokes phase velocities. It does not appear to be an asymptotic form of the solution as the capillarity goes to zero. This, together with the algorithm (and its various modifications) for the stationary case, is an indication that stationary gravity-capillary waves are not affected by capillary forces are unstable or

While properties of stationary solutions are known, we used these results in the nonstationary model. The validation was performed using condition 3 as initial conditions for the stationary coordinate system of the latter wave profile, the model simulates the evolution of the wave profile. It should be emphasized that the nonstationary model is more accurate than the stationary ones and it "does not know" that the surface is not a straight line. Nevertheless, they differ in the amplitudes. This suggests that the solutions are stable with respect to truncation errors and (2) these errors are small.

3. We used the nonstationary linear wave fields. The cases discussed are chosen arbitrarily, as our aim was to demonstrate the applications of the technique developed. The most clearly seen in the simulation is the laboratory experiment by Yuen and Yeh of multiple wave fields with discrete frequency spectra into regular patterns of energy concentrated along with discrete branches satisfies a dispersion relation. The number n of the branch is discrete and is greater than 1 for branches. In this structure, nonlinear effects are observed of multiple branches and in deviation from the linear dispersion relation for the stationary cases the deviation clearly appears as a straight line and so the growth of this effect needs further analysis. It is concluded that other regular branches exist. These additional branches were observed as a remarkable exception of a peculiar pattern at least for not too small wave numbers. It is by a straight line passing through the origin was perhaps most strongly marked.

ed by Stokes (1848) for stationary case, where dependent, and the surface potential and the stream use the original system coordinate boundary conditions on simplification, to a system ns. As in the Cartesian elevation and the velocity on the Cartesian horizontal the new horizontal coordinate by numerical integration, nearities via Fourier transform for analytical investigations the original system, with the interpolating the velocity po-

ionary equations for gravity and. The method allows us to and is based on representative height, written in the new coordinate to the velocity potential integration and (generalized) space. Again, the use of nearities allows a highly efficient could be noted that we had gravity and gravity-capillary them fails to converge when becomes small. This problem shown that, with decreasing velocities of gravity-capillary uses of the Stokes phase velocity to be an asymptotic form of goes to zero. This, together with various modifications) for stationary gravity-capillary forces are unstable or

While properties of stationary solutions imply many intriguing problems, we used these results mainly as a tool to validate the nonstationary model. The validation was performed by using solutions obtained in Section 3 as initial conditions for the nonstationary problem. Since the coordinate system of the latter was bound to the mean flow rather than to the wave profile, the model simulates running Stokes and gravity-capillary waves. It should be emphasized that the validation was far from trivial, as the nonstationary model is based on equations much more complicated than the stationary ones and on a numerical procedure of its own, which "does not know" that the simulated waves are supposed to retain their shape. Nevertheless, they did retain it surprisingly well even for large amplitudes. This suggests that (1) Stokes and gravity-capillary waves are stable with respect to truncation errors of the nonstationary model, and (2) these errors are small.

3. We used the nonstationary model for case studies of evolution of nonlinear wave fields. The cases described in Section 6 were chosen somewhat arbitrarily, as our aim was to provide a possibly broader variety of applications of the technique developed. The effects of bound waves were most clearly seen in the simulation designed to approximate the laboratory experiment by Yuen and Lake (1982). A most surprising feature of multi-mode wave fields was a clear separation of the wavenumber-frequency spectra into regular curvilinear branches, with most of the energy concentrated along what we called "main branches". This set of branches satisfies a dispersion relation whose form is given by (89) where the number n of the branch is 1 for the branch consisting of free waves and is greater than 1 for branches consisting of generalized bound waves. In this structure, nonlinear effects were manifested both in the existence of multiple branches and in deviation of the "parent" curve ($n = 1$) from the linear dispersion relation for relatively large wavenumbers. In most cases the deviation clearly appeared to be such that the curve approaches a straight line and so the group velocity tends to a constant; however, this effect needs further analysis. The nonlinear interactions also produced other regular branches. The energy of the modes belonging to these additional branches was usually very small, sometimes with the remarkable exception of a peculiar pattern (or group of patterns) which, at least for not too small wavenumbers, could be roughly approximated by a straight line passing through the origin. The nonlinear behavior was perhaps most strongly manifested in the case of a long Stokes wave

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mulation of wind-wave interaction. Jour-
1 - 582.

simulation of the boundary layer above

waves. *Bound Layer Met*, 1986, 3.

3. Chalikov, D.V. & Liberman Yu
for potential waves. *Izv. Sov. Atm*

4. Craig, W. & Sulem, C. Numerical
Journal of Comp Phys, 1993, 108, 73 -

5. Crapper, G.D. An exact solution
arbitrary amplitude. *Journal of FL*

6. Crapper, G.D. Introduction to V
1984, 224 pp.

7. Dold, J.W. An Efficient Surface
steady Gravity Waves. *Journal of (*

8. Donelan, M.A. & Hui, W.H. *Me-
face Waves and Fluxes* eds Geerna
demic, Dordrecht, 1990, vol. 1, 209

9. Drennan, W.M., Hui, W.N. Ten-
water waves of large amplitude. *Z.*
380.

10. Eliassen, E.B., Machenhauer, I
method for integration of the hydro-
representation of the horizontal field
Meteorologi, Københavens Universit

11. Hasselmann, K. On the nonlinear
spectrum, part 1: General theory. *J*
- 500.

12. Kuznetsov, E.A., Spector, M.D.
singularities on the free surface of an ic
No. 2, 1283 - 1290.

13. Lake, B.M. & Yuen, H.C. A ne

- where the free wave branch is short waves is largely controlled. On the other hand, the nonlinear effects are remarkably larger in the case of other runs, which included pure solutions with the same initial surface elevations, as the structure of the solution is partially obscured by apparent effects caused to study a variety of problems in fields in a wide range of wavenumbers.
- These results are obtained by use of the developed hydrodynamic model for wave surfaces; the model can be studied by coupling of the developed model (e. g. Chalikov, 1978, 1980) for helpful discussions and to resolve the model equations with D. B. Rao, Dr W. Perrie, and Dr S. M. M. and making many valuable comments.
- of wind-wave interaction. *Journal of Fluid Mech*, 1986, 150, 1-15.
- of the boundary layer above
3. Chalikov, D.V. & Liberman Yu.M. Integration of primitive equations for potential waves. *Izv. Sov. Atm. Ocean Phys.*, 1991, 27, 42 - 47.
 4. Craig, W. & Sulem, C. Numerical Simulation of Gravity Waves. *Journal of Comp Phys*, 1993, 106, 73 - 83.
 5. Crapper, G.D. An exact solution for progressive capillary waves of arbitrary amplitude. *Journal of Fluid Mech*, 1957, 96, 417 - 445.
 6. Crapper, G.D. Introduction to Water Waves, John Wiley, Chichester, 1984, 224 pp.
 7. Dold, J.W. An Efficient Surface-Integral Algorithm Applied to Unsteady Gravity Waves. *Journal of Comp Phys*, 1992, 103, 90 - 115.
 8. Donelan, M.A. & Hui, W.H. Mechanics of ocean surface waves. *Surface Waves and Fluxes* eds Geernaert, G.L., Plant, W.J., Kluwer Academic, Dordrecht, 1990, vol. 1, 209 - 246.
 9. Drennan, W.M., Hui, W.N. Tenti G. Accurate calculations of Stokes water waves of large amplitude. *Z. angew Math Phys*, 1992, 43, 367 - 380.
 10. Eliassen, E.B., MACHENHAUER, B. & Rasmussen, E. On a numerical method for integration of the hydro-dynamical equations with a spectral representation of the horizontal fields. Report 2, Institute for Teoretisk. Meteorologi, Københavens Universitet, Copenhagen, Denmark, 1970.
 11. Hasselmann, K. On the nonlinear energy transfer in a gravity-wave spectrum, part 1: General theory. *Journal of Fluid Mech*, 1962, 12, 481 - 500.
 12. Kuznetsov, E.A., Spector, M.D. & Zakharov, V.E. Formation of singularities on the free surface of an ideal fluid. *Physical review*, 1994, 49, No. 2, 1283 - 1290.
 13. Lake, B.M. & Yuen, H.C. A new model for nonlinear wind waves.

Ocean Waves

al model and experimental evidence. *Journal of Fluid*
33 - 62.

gins, M.S. Integral properties of periodic gravity waves
le. *Proc. R. Soc. Lond.*, 1975, A 342, 157 - 174.

gins, M.S. & Cokelet, E.D. The deformation of steep
water. I. A numerical method of computation *Proc. R.*
, 350, 1 - 26.

. Transform method for calculation of vector coupled
n to the spectral form of vorticity equation, *Journal of*
27, 890 - 895.

A. & Jacobs, J.W. A Numerical Method for Potential
Surface. *Journal of Comp Phys*, 1983, 51, 365 - 386.

On the theory of oscillatory waves. *Trans. Cambridge*
41-445. *Math. Phys. Pap.*, 1847, 1, 197 - 229.

1880 Supplement to a paper on the theory of oscillatory
ys. *Pap.*, 1880, 1, 314 - 326.

. & West B.J. A transport-equation description of non-
ce wave interactions. *Journal of Fluid Mech*, 1975, 70,

rueckner K.A. & Janda R.S. A New Numerical Method
odynamics. *Journal of Geophys. Res.*, 1987, 92, No.
124.

Lake B.M. Nonlinear Dynamics of Deep-Water Gravity
pl Mech, 1982, 22, 67 - 229.



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