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form. Various examples of the use of these models will be shown in the conference presentation.

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ON THE TRANSFORMATION OF WAVE SPECTRA BY CURRENTS AND BATHYMETRY[1]

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Abstract

Application of the conservation principle for action spectral density along rays, frequently described in literature, is not sufficient to specify the refracted wave spectrum. In order to determine the change of wave spectrum by current-depth refraction correctly, the effect of ray separation which causes convergence or divergence of wave energy must be included. This effect can be derived from the divergence of the velocity field in the conservation equation for action spectral energy density.

Introduction

A number of papers which apply ray theory to calculate refracted wave spectra have appeared recently (e.g., Le Méhauté and Wang, 1982; Mathiesen, 1987; Liu et al., 1989). However, the theoretical bases upon which these calculations are made are not always clear. Some apply only the conservation principle for action spectral density while others include an additional transformation factor based on purely mathematical reasoning. The purpose here is to clarify the problem from a theoretical point of view and to provide physical insight to this transformation factor.

Basic Equations

The change of wave field due to the presence of varying currents and bathymetry can be specified based on conservation of wave action along characteristic curves or rays (Bretherton and Garrett, 1969; Phillips, 1977). The path of a ray is determined by simultaneous solution of the following set of equations:

$$\frac{dk_i}{dt} = -\frac{\partial \omega}{\partial \lambda}\frac{\partial \lambda}{\partial x_i} = -\frac{\partial \sigma}{\partial h}\frac{\partial h}{\partial x_i} - k_j\frac{\partial u_j}{\partial x_i}, \quad (1)$$

$$\frac{dx_j}{dt} = \frac{\partial \omega}{\partial k_j} = c_{g_j} + u_j, \quad (2)$$

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$$\frac{d\omega}{dt} = \frac{\partial\omega}{\partial\lambda}\frac{\partial\lambda}{\partial t} = \frac{\partial\sigma}{\partial h}\frac{\partial h}{\partial t} + k_i\frac{\partial u_i}{\partial t}, \tag{3}$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{\partial\omega}{\partial k_j}\frac{\partial}{\partial x_j}. \tag{4}$$

Tensor notation is used to make these relations more concise. Here we express the wave-number vector by $\mathbf{k} = (k_1, k_2)$, group velocity by $\mathbf{C_g} = (c_{g1}, c_{g2})$, flow velocity by $\mathbf{U} = (u_1, u_2)$, and the horizontal cartesian coordinates by $\mathbf{x} = (x_1, x_2)$. $\omega(\mathbf{k}, \lambda) = \sigma + k_i u_i$ represents the apparent frequency and σ, the intrinsic frequency of waves in a frame of reference moving with flow velocity $\mathbf{U}(\mathbf{x}, t)$, obeys the dispersion relationship, $\sigma = (gk \tanh kh)^{1/2}$, where g is the gravitaional acceleration, $k = |\ \mathbf{k}\ |$, and $h(\mathbf{x}, t)$ is water depth. $\lambda(\mathbf{x}, t)$ represents local properties of the medium, i.e., h and $\mathbf{U}$. Equation 3 indicates that if water depth and current velocity do not vary with time, ω remains constant along the rays.

Conservation of wave action for a slowly varying wavetrain of small amplitude can be expressed in terms of rays as

$$\frac{d}{dt}\left(\frac{E}{\sigma}\right) + (\nabla\cdot\mathbf{V})\left(\frac{E}{\sigma}\right) = 0. \tag{5}$$

E is the local wave energy per unit area (proportional to the square of the wave amplitude) and $\mathbf{V} = \mathbf{C_g} + \mathbf{U}$. The wave action is defined as E/σ. For a continuous spectrum, E corresponds to the energy density of a group of waves whose wave-numbers lie in the element of area δA of the wave-number plane, specified by the vectors $\mathbf{k}$, $\mathbf{k} + \delta\mathbf{k}'$, and $\mathbf{k} + \delta\mathbf{k}''$ such that

$$\delta E(\mathbf{k}) = \rho g F(\mathbf{k})\,\delta A, \tag{6}$$

$$\delta A = |\delta\mathbf{k}' \times \delta\mathbf{k}''|. \tag{7}$$

$F(\mathbf{k})$ is the spectral density and ρ the water density. By applying the kinematic conservation principle, Phillips(1977) has shown that

$$\frac{d}{dt}\delta A + (\nabla\cdot\mathbf{V})\,\delta A = 0. \tag{8}$$

Therefore

$$\frac{d}{dt}\left(\frac{F(\mathbf{k})}{\sigma}\right) = 0. \tag{9}$$

Equation 9 expresses the conservation of action spectral density along the ray. In the absence of a current, $F(\mathbf{k})$ remains constant along the ray. This result was first demonstrated by Longuet-Higgins(1957).

Ray Separation Factor

Equation 8 cannot be directly integrated along a ray because knowledge of neighboring solutions is required to determine the divergence of the velocity. In order to solve this problem, we introduce the Jacobian $J(s,r)$ of the transformation from the ray coordinates (s,r) to $\mathbf{x} = (x,y)$,

$$J(s,r) = \frac{\partial(x,y)}{\partial(s,r)}, \tag{10}$$

where r is a parameter which is a constant along each ray and s is the arclength along the ray. Then differentiating Eq. 10 with respect to s we obtain (Chao and Bertucci, 1989)

$$\frac{dJ}{ds} = \frac{d}{ds}\left\{\frac{\partial x}{\partial s}\frac{\partial y}{\partial r} - \frac{\partial x}{\partial r}\frac{\partial y}{\partial s}\right\} = J\left[\mathbf{V}^{-1}\nabla\cdot\mathbf{V} + \mathbf{V}\cdot\nabla\mathbf{V}^{-1}\right]. \tag{11}$$

If the temporal variation of current and water depth is small compared to wave period

$$\frac{\partial}{\partial t}\mathbf{V}^{-1} = 0, \tag{12}$$

and we have from Eqs. 2 and 4 that

$$\mathbf{V}\cdot\nabla\mathbf{V}^{-1} = \frac{d}{dt}\mathbf{V}^{-1} = \mathbf{V}\frac{d}{ds}\mathbf{V}^{-1}. \tag{13}$$

By substituting Eq. 13 into Eq. 11 and rearranging, we obtain

$$\nabla\cdot\mathbf{V} = \mathbf{V}\left[\frac{1}{J}\frac{dJ}{ds} - \mathbf{V}\frac{d}{ds}\mathbf{V}^{-1}\right]. \tag{14}$$

Therefore Eq. 8 becomes

$$\frac{d}{dt}\ln(J\mathbf{V}\delta A) = 0, \tag{15}$$

which states that the quantity $J\mathbf{V}\,\delta A$ is conserved along the ray.

The Jacobian $J(s,r)$ can be given a geometric interpretation. We note that the Jacobian defined by Eq. 10 can also be expressed as

$$J(s,r) = (0,0,1)\cdot\frac{\partial\mathbf{x}}{\partial s}\times\frac{\partial\mathbf{x}}{\partial r}, \tag{16}$$

where $\partial\mathbf{x}/\partial s = \hat{s}$ is the unit tangent vector along a ray. Thus

$$\delta\nu \equiv J(s,r)\,\delta r \tag{17}$$

is the width between two neighboring rays associated with parameters r and $r + \delta r$. Since δr is a constant, we obtain from Eqs. 9, 15, and 17

$$F(\mathbf{k})\,\delta A = \frac{\sigma}{\sigma'}\left\{\frac{\delta\nu'}{\delta\nu}\frac{\mathbf{V}'}{\mathbf{V}}\right\}F'(\mathbf{k}')\,\delta A' = \frac{\sigma}{\sigma'}\left\{\frac{\delta A}{\delta A'}\right\}F'(\mathbf{k}')\delta A'. \tag{18}$$

Here we have used prime ($\prime$) to designate parameters in the source area. Equation 18 shows that the wave action flux is conserved between two neighboring rays.

Munk and Arthur(1952) derived a second order differential equation based on physical grounds to determine the change in the ray separation factor $\delta\nu/\delta\nu'$ over variable depth. Skovgaard and Jonsson(1976) extended the approach to include the effect of currents. Their equations for computing the ray separation are essentially expansions of Eq. 16. Alternately, the ray separation factor can be determined without recourse to solving differential equations as shown below.

Under steady state assumption, we take advantage of the constancy of apparent wave frequency along the rays and consider the wave spectrum in the frequency-direction domain rather than wave-number space. Since the total energy of a given sea state at a location of interest should be the same either in the wave- number space or frequency-direction space,

$$\iint F(k_1, k_2)\, dk_1\, dk_2 = \iint F(\omega, \theta)\, d\omega\, d\theta \tag{19}$$

the differential elements $\delta A = \delta k_1 \delta k_2$ and $\delta\omega\delta\theta$ are related by the Jacobian as follows:

$$\frac{\delta A}{\delta\omega\delta\theta} = \frac{\partial(k_1, k_2)}{\partial(\omega, \theta)} = \frac{k}{c_g + u\cos\theta + v\sin\theta}, \tag{20}$$

where θ is the angle of wave-number vector measured from the $x-$ axis. Since $\delta\omega = \delta\omega'$, we have

$$\frac{\delta A}{\delta A'} = \left\{\frac{c_g' + u'\cos\theta' + v'\sin\theta'}{c_g + u\cos\theta + v\sin\theta}\right\}\left\{\frac{k}{k'}\frac{\delta\theta}{\delta\theta'}\right\}, \tag{21}$$

and Eq. 18 expressed in the frequency-direction domain as

$$F(\omega, \theta)\,\delta\omega\delta\theta = \frac{\sigma}{\sigma'}\left\{\frac{\delta A}{\delta A'}\right\} F'(\omega, \theta')\,\delta\omega\delta\theta'. \tag{22}$$

It is easy to show that if bottom contours are straight and parallel to the shoreline and $\mathbf{U} = \mathbf{U}' = 0$, then $\sigma = \sigma'$, and

$$\frac{\delta A}{\delta A'} = \left\{\frac{c_g'}{c_g}\right\}\left\{\frac{\cos\theta'}{\cos\theta}\right\}. \tag{23}$$

The quantities shown in the first and second curly brackets are the well-known shoaling and refraction coefficients, respectively, for calculating the wave height change over a sloping beach.

Concluding Remarks

One may easily fall into error by simply applying the conservation principle for action spectral density along rays (Eq. 9) to calculate refracted wave spectra in the frequency-direction domain through Eq. 20 as

$$F(\omega, \theta) = \frac{\sigma}{\sigma'}\left\{\frac{c_g' + u'\cos\theta' + v'\sin\theta'}{c_g + u\cos\theta + v\sin\theta}\right\}\frac{k}{k'}F'(\omega, \theta'), \tag{24}$$

and then integrate it to obtain the total energy density. A mathematically correct approach is through the use of

$$F(\omega, \theta)\, d\omega\, d\theta = \frac{\sigma}{\sigma'}\left\{\frac{c_g' + u'\cos\theta' + v'\sin\theta'}{c_g + u\cos\theta + v\sin\theta}\right\}\frac{k}{k'}\frac{\partial\theta}{\partial\theta'}F'(\omega, \theta')\, d\omega\, d\theta'. \tag{25}$$

We may conclude that in order to calculate refracted spectra correctly, the transformation factor $\partial\theta/\partial\theta'$ must be included. In view of Eqs. 18 and 21, the quantity $(k/k')(\partial\theta/\partial\theta')$ is equivalent to the ray separation factor, which measures the focusing or defocusing of wave energy due to refraction.

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