A Proposed Definition for Vector Correlation in Geophysics: Theory and Application

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ABSTRACT

A universally accepted definition for vector correlation in oceanography and meteorology does not presently exist. To address this need, a generalized correlation coefficient, originally proposed by Hooper and later expanded upon by Jupp and Mardia, is explored. A short history of previous definitions is presented. Then the definition originally proposed by Hooper is presented together with supporting theory and associated properties. The most significant properties of this vector correlation coefficient are that it is a generalization of the square of the simple one-dimensional correlation coefficient, and when the vectors are independent, its asymptotic distribution is known; hence, it can be used for hypothesis testing. Because the asymptotic results hold only for large samples, and in practical situations only small samples are often available, modified sampling distributions are derived using simulation techniques for samples as small as eight. It is symmetric with respect to its arguments and has a simple interpretation in terms of canonical correlation. It is invariant under transformations of the coordinate axes, including rotations and changes of scale.

Finally, to assist in interpreting this vector correlation coefficient, several cases that lead to perfect correlation and zero correlation are examined, and the technique is applied to surface marine winds at two locations in the northwest Atlantic.

1. Introduction

The problem of correlating vector quantities has been of interest to meteorologists and oceanographers for at least the past 75 years (e.g., Detzius 1916; Sverdrup 1917; Charles 1959; Buell 1971; Breckling 1989). The problem arises because a vector requires both magnitude and direction, and is further complicated because direction is a circular function. When a vector is decomposed into its scalar components, standard correlation techniques can be applied such as the Pearson product-moment correlation coefficient. This approach has been used extensively in the fields of meteorology and oceanography. It is important to recognize, however, that when the scalar components of a vector are correlated, the results are not unique since they depend on the choice of the coordinate system used for the decomposition. For example, if the scalar components of a vector are correlated using a spherical, earth-oriented coordinate system, one will generally obtain different results than if a natural coordinate system were used.

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When vectors are correlated directly without decomposition, the effects of both magnitude and direction are included, yielding a simple scalar value that is taken as a measure of the degree of association between the vectors of interest. A number of definitions for vector correlation have appeared in the literature that take into account both speed and direction (e.g., Durst 1957; Court 1958; Breckling 1989), but no single definition is in general use at this time. Crosby et al. (1991) recently presented another definition for vector correlation that was originally proposed by Hooper (1959) and later refined by Jupp and Mardia (1980). This definition is a generalization of the standard definition for scalar correlation. Both the population parameter and its sample statistic have a rather complete set of useful statistical properties. It is the purpose of this study to present this definition with the supporting theoretical background along with an application of this vector correlation technique to a practical problem in marine meteorology.

The text is presented in the following order. First, a short review of the basic properties of vector quantities is followed by some of the properties of the standard product-moment correlation coefficient for scalar variables. Next, a number of the previous definitions of vector correlation are presented. Then the definition for vector correlation originally proposed by Hooper (1959) is presented. The properties of this vector correlation coefficient follow as well as some simulation results related to its statistical properties. Finally, several cases are presented that lead to perfect correlation, followed by a practical example in the application and interpretation of this vector correlation coefficient. Proofs of most of the properties are contained in the Appendix.

a. Basic properties

The following basic definitions for vector quantities are given for a two-dimensional vector **W** with components u and v. We follow the standard convention used in meteorology and oceanography by adopting an earth-oriented Cartesian coordinate system. Thus, u increases to the right along the x axis and v increases upward along the positive y axis.

The mean vector of a sample of vectors $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_n$ is given by

$$\bar{\mathbf{W}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{W}_i. \quad (1.1)$$

If we have a sample of vectors $\mathbf{W}_i$ ($i = 1$ to n), let $\mathbf{S}_w$ be the sample covariance matrix. That is,

$$\mathbf{S}_w = \frac{1}{n-1}\sum_{i=1}^{n} (\mathbf{W}_i - \bar{\mathbf{W}})(\mathbf{W}_i - \bar{\mathbf{W}})^{\mathrm{T}}$$

$$= \begin{pmatrix} s_u^2 & s_{uv} \\ s_{vu} & s_v^2 \end{pmatrix}, \quad (1.2)$$

where the sum is taken over a sample of size n. The sample variance of a vector can be defined as the trace (Tr) of the covariance matrix. That is, one definition of the variance is

$$\mathrm{Tr}(\mathbf{S}_W) = s_u^2 + s_v^2. \quad (1.3)$$

This, of course, is the sum of the variances of the individual u and v components. A more conventional definition is the generalized variance given by $|\mathbf{S}_W|$, the determinant of the covariance matrix.

b. Properties of the product-moment correlation coefficient

Most of the previous definitions of vector correlation have attempted to generalize the definition of the standard, one-dimensional, linear correlation coefficient. In order to present the history and further motivate the discussion related to vector correlation, we review some of the properties of the population standard product-moment correlation coefficient ρ.

Given two random variables u and v, with standard deviations σ_u, σ_v, and covariance σ_{uv}, the correlation coefficient is defined as

$$\rho = \frac{\sigma_{uv}}{\sigma_u \sigma_v}.$$

Given a sample of u and v, the sample correlation coefficient r is defined as

$$r = \frac{s_{uv}}{s_u s_v}.$$

The population correlation coefficient ρ has the following properties: $-1 \leqslant \rho \leqslant 1$, and if $x = a + bu$ and $y = c + dv$, where a, b, c, and d are any scalars such that b and d are not equal to zero, then

$$\rho_{UV} = \rho_{XY}.$$

Thus, ρ is invariant under linear transformations of u and v. If u and v are independent, then $\rho = 0$.

Further, $\rho = 1$ if and only if $u = a + bv$ for some a and b; thus, it is reasonable to expect that a vector correlation coefficient should have similar properties.

Similar properties hold for the sample correlation coefficient, except for property 3. That is, even if u and v are independent, the sample correlation coefficient will not be equal to zero. It will, of course, approach zero as the sample size approaches infinity.

The sample correlation coefficient is related to least-squares regression. If, from a sample of u and v, the least-squares regression of u on v is found so that the predicted value of u given v is

$$\hat{u}_i = a + bv_i,$$

then it can be shown that

$$b = r\left(\frac{s_u}{s_v}\right),$$

and

$$r^2 = \frac{\sum_{i=1}^{n} (\hat{u} - \bar{u})^2}{\sum_{i=1}^{n} (u - \bar{u})^2}.$$

The quantity r^2 represents the proportion of explained variance and is often referred to as the coefficient of determination.

2. History

a. Mathematical background

To examine vector correlation in greater detail, certain additional concepts are defined. Let $\mathbf{W}_1$ and $\mathbf{W}_2$ be a pair of two-dimensional random vectors. Then

$$\mathbf{X} = \begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}$$

is a four-dimensional vector. Further, let

$$\Sigma_X = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} = \begin{bmatrix} \sigma(u_1, u_1) & \sigma(u_1, v_1) & \sigma(u_1, u_2) & \sigma(u_1, v_2) \\ \sigma(v_1, u_1) & \sigma(v_1, v_1) & \sigma(v_1, u_2) & \sigma(v_1, v_2) \\ \sigma(u_2, u_1) & \sigma(u_2, v_1) & \sigma(u_2, u_2) & \sigma(u_2, v_2) \\ \sigma(v_2, u_1) & \sigma(v_2, v_1) & \sigma(v_2, u_2) & \sigma(v_2, v_2) \end{bmatrix} \quad (2.1)$$

be the 4×4 population covariance matrix of the vector $\mathbf{X}$. In Eq. (2.1), Σ_{11} is the covariance matrix of $\mathbf{W}_1$, Σ_{22} the covariance matrix of $\mathbf{W}_2$, Σ_{12} the cross-covariance matrix of $\mathbf{W}_1$ and $\mathbf{W}_2$, and Σ_{21} the cross-covariance matrix of $\mathbf{W}_2$ and $\mathbf{W}_1$.

b. Early definitions

Early definitions of vector correlation are summarized in Court (1958). Here, we express each of the definitions in terms of the population parameters and in matrix notation,[1] whereas previous studies have usually given the definitions in scalar notation and in terms of the sample parameters. An early definition was given by Detzius (1916) as

$$\rho_D^2 = \frac{\mathrm{Tr}(\Sigma_{12})^2}{\mathrm{Tr}\,(\Sigma_{11})\,\mathrm{Tr}(\Sigma_{22})}.$$

This definition has sometimes been referred to as the "stretch" correlation coefficient.

A later definition that involved both "stretch" and "turn" was given by Sverdrup (1917) as

$$\rho_S^2 = \frac{\mathrm{Tr}(\Sigma_{12})^2 + [\sigma(u_1, v_2) - \sigma(u_2, v_1)]}{\mathrm{Tr}(\Sigma_{11})\,\mathrm{Tr}(\Sigma_{22})}.$$

This definition was often used by British meteorologists during the 1950s (e.g., Durst 1957). This definition is also related to the so-called complex correlation coefficient, which is defined as

$$\rho_T = \frac{\mathrm{Tr}(\Sigma_{12}) + i[\sigma(u_1, v_2) - \sigma(u_2, v_1)]}{[\mathrm{Tr}(\Sigma_{11})\,\mathrm{Tr}(\Sigma_{22})]^{0.5}},$$

where $i = (-1)^{0.5}$. See Kundu (1976) for a geometric interpretation of this coefficient. We note that the square of the absolute value of the complex vector correlation coefficient is the same as the definition given by Sverdrup.

Hotelling (1936) presented the following definition: $\rho_H^2 = |[(\Sigma_{11})^{-1}\Sigma_{12}(\Sigma_{22})^{-1}\Sigma_{21}]|$. Hotelling derived many of the statistical properties of the sample statistic for this parameter.

A definition proposed by Court (1958) and later elaborated on by Lenhard et al. (1963a), Buell (1963), and Lenhard et al. (1963b) was based on a generalization of the concept of explained variance. His definition was

$$\rho_c^2 = \frac{\mathrm{Tr}[\Sigma_{12}(\Sigma_{22})^{-1}\Sigma_{21}]}{\mathrm{Tr}(\Sigma_{11})}.$$

In this definition, $\mathbf{W}_1$ represents the dependent variable. This definition is not symmetric in $\mathbf{W}_1$ and $\mathbf{W}_2$ and is not invariant under changes in scale.

Since 1960, a series of papers on the correlation of directional data or angular association have appeared. For a history and discussion of these more recent studies see Breckling (1989).

c. The definition

Here, we consider the following definition:

$$\rho_v^2 = \mathrm{Tr}[(\Sigma_{11})^{-1}\Sigma_{12}(\Sigma_{22})^{-1}\Sigma_{21}]. \quad (2.2)$$

This is essentially the definition originally given by Hooper (1959) and further developed by Jupp and Mardia (1980). Many of the definitions of vector correlation have been normalized so that this parameter will fall between 0.0 and 1.0. Following Jupp and Mardia, we have not divided the right-hand side of Eq. (2.2) by a constant, which will depend on the dimensions of the vectors, to ensure that ρ_v^2 lies between 0.0 and 1.0. In the two-dimensional case discussed in this paper, ρ_v^2 will be between 0.0 and 2.0. This is the only case considered in this study.

In the next section, we show that this definition is a generalization of the standard scalar correlation coefficient. Unlike some of its predecessors, however, which have been restricted to the unit circle (e.g., Mardia and Puri 1978; Stephens 1979), this definition includes both direction and magnitude. For applied problems in oceanography and meteorology, this is a very important distinction. Also, it has a rather complete set of desirable properties.

3. Basic theory

As indicated in the Introduction, a number of definitions for vector correlation have been proposed. In this section we present the most important properties of the vector correlation coefficient defined by Jupp and Mardia (1980). The proofs of most of these properties are given in the Appendix. We give the results for two-dimensional vectors in ordinary Euclidean space. To generalize the results to three or four dimensions is conceptually straightforward.

a. Properties

The most significant properties of the vector correlation coefficient defined above follow. The vector correlation coefficient is a generalization of the square of

[1] Matrix notation does not necessarily simplify the expressions in the two-dimensional case. However, it is essential for higher dimensions.

the simple one-dimensional correlation coefficient, is symmetric in its arguments, and has a simple interpretation in terms of canonical correlation. It is invariant under transformations of the coordinate axes, including rotations and changes in scale. It is equal to zero when the vectors are independent and obtains its maximum value if and only if they are linearly dependent. In addition, the statistic that is used to estimate this parameter has a sampling distribution that is asymptotically robust. That is, if $\mathbf{W}_1$ and $\mathbf{W}_2$ are independent and if their distributions have all fourth-order moments, then the asymptotic distribution of the statistic does not depend on the distributions of $\mathbf{W}_1$ and $\mathbf{W}_2$ (Jupp and Mardia 1980). Since the distributions of the $\mathbf{W}_i$ may be unknown or difficult to express in closed form, this property is very significant for applications such as large-sample hypothesis testing. We consider this last property to be particularly important for its use and interpretation in practice.

b. Basic definition

Let $\mathbf{W}_1$ and $\mathbf{W}_2$ be a pair of two-dimensional random vectors. Then we define the vector $\mathbf{X}$, its covariance matrix Σ_X, and the submatrices of Σ_X exactly as in Eq. (2.1). Here, we always assume that Σ_{11} and Σ_{22} are nonsingular. Thus, the vectors $\mathbf{W}_1$ and $\mathbf{W}_2$ are nondegenerate; that is, $u_i \neq a_i + b_i v_i$ $(i = 1, 2)$ for some a_i and b_i. We also assume that all moments of the vector $\mathbf{X}$ exist.

Then the definition of the vector correlation coefficient between $\mathbf{W}_1$ and $\mathbf{W}_2$ is

$$\rho_v^2 = \mathrm{Tr}[(\Sigma_{11})^{-1}\Sigma_{12}(\Sigma_{22})^{-1}\Sigma_{21}], \qquad (3.1)$$

where the Σ_{ij} are as in Eq. (2.1).[2] In terms of the u's and v's, this definition is given by

$$\rho_v^2 = \frac{f}{g}, \quad \text{where} \qquad (3.2)$$

$$\begin{aligned} f = {} & \sigma(u_1, u_1)\{\sigma(u_2, u_2)[\sigma(v_1, v_2)]^2 + \sigma(v_2, v_2)[\sigma(v_1, u_2)]^2\} \\ & + \sigma(v_1, v_1)\{\sigma(u_2, u_2)[\sigma(u_1, v_2)]^2 + \sigma(v_2, v_2)[\sigma(u_1, u_2)]^2\} + 2[\sigma(u_1, v_1)\sigma(u_1, v_2)\sigma(v_1, u_2)\sigma(u_2, v_2)] \\ & + 2[\sigma(u_1, v_1)\sigma(u_1, u_2)\sigma(v_1, v_2)\sigma(u_2, v_2)] - 2[\sigma(u_1, u_1)\sigma(v_1, u_2)\sigma(v_1, v_2)\sigma(u_2, v_2)] \\ & - 2[\sigma(v_1, v_1)\sigma(u_1, u_2)\sigma(u_1, v_2)\sigma(u_2, v_2)] - 2[\sigma(u_2, u_2)\sigma(u_1, v_1)\sigma(u_1, v_2)\sigma(v_1, v_2)] \\ & - 2[\sigma(v_2, v_2)\sigma(u_1, v_1)\sigma(u_1, u_2)\sigma(v_1, u_2)], \end{aligned}$$

and

$$g = \{\sigma(u_1, u_1)\sigma(v_1, v_1) - [\sigma(u_1, v_1)]^2\}\{\sigma(u_2, u_2)\sigma(v_2, v_2) - [\sigma(u_2, v_2)]^2\}.$$

This expression is found by expanding the matrix products in Eq. (3.1) and then simplifying the results. This expanded form is useful for computational applications (Crosby et al. 1990).

c. Generalizing the basic definition

It is easily seen that the definition given in Eq. (3.1) is a generalization of the square of the standard Pearson correlation coefficient for a pair of one-dimensional random variables.

d. Relationship to canonical correlation

An intuitive justification for this definition of vector correlation is based on canonical correlation. Let $\mathbf{W}_1$ and $\mathbf{W}_2$ be a pair of two-dimensional random vectors. The canonical correlations are defined in the following manner: linear combinations of u_1 and v_1 and u_2 and v_2 are formed where

$$z_{11} = a_{11}u_1 + b_{11}v_1,$$

$$z_{12} = a_{12}u_2 + b_{12}v_2,$$

such that for all linear combinations the standard one-dimensional correlation $\rho_1 = \mathrm{corr}(z_{11}, z_{12})$ between z_{11} and z_{12} is a maximum. The variables z_{11} and z_{12} are called the first canonical variables. Note that $\rho_1 \geq 0$. The parameter ρ_1 is the first canonical correlation. Then a second set of variables

$$z_{21} = a_{21}u_1 + b_{21}v_1, \quad \text{and}$$

$$z_{22} = a_{22}u_2 + b_{22}v_2,$$

is found such that

$$\mathrm{corr}(z_{11}, z_{21}) = \mathrm{corr}(z_{11}, z_{22}) = \mathrm{corr}(z_{12}, z_{21}) = \mathrm{corr}(z_{12}, z_{22}) = 0, \quad \text{and}$$

$$\rho_2 = \mathrm{corr}(z_{21}, z_{22})$$

is a maximum. The parameter ρ_2 is the second canonical correlation and is nonnegative. The vector correlation coefficient given by Eq. (3.1) is the sum of the squares of the two canonical correlations. That is, $\rho_v^2 = \rho_1^2 + \rho_2^2$. The remaining properties and proofs are given in the Appendix.

[2] Although not explicitly shown, this definition is also an extension of the coefficient of determination in multiple regression.

4. Sampling distribution

a. The asymptotic behavior

If the covariance matrix Σ_X of $\mathbf{X}$ is estimated in the usual way from a sample of size n by

$$\mathbf{S}_X = \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{pmatrix} = \frac{1}{n-1} \sum_{i=1}^{n} (\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})^{\mathrm{T}};$$

then ρ_v^2 is estimated by

$$\hat{\rho}_v^2 = \mathrm{Tr}(\mathbf{S}_{11}^{-1}\mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21}). \tag{4.1}$$

The statistic defined in Eq. (4.1) will have several of the properties of ρ_v^2. Namely, it is symmetric in $\mathbf{W}_1$ and $\mathbf{W}_2$. It is invariant under transformations of the coordinate axes. It is equal to the sum of the squares of the sample canonical correlations. If $\mathbf{W}_1$ and $\mathbf{W}_2$ are linearly dependent, then $\hat{\rho}_v^2 = 2.0$.

If $\mathbf{W}_1$ and $\mathbf{W}_2$ are independent, it does not follow that

$$\mathbf{S}_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

hence, $\hat{\rho}_v^2 \neq 0$. However, as the sample size increases, $\mathbf{S}_{12}$ will approach the zero matrix and $\hat{\rho}_v^2$ will approach 0.

If $\mathbf{W}_1$ and $\mathbf{W}_2$ are two-dimensional and independent, then $n\hat{\rho}_v^2$ is distributed asymptotically as a chi-square variable with four degrees of freedom (Jupp and Mardia 1980). This asymptotic distribution is valid for most realistic marginal distributions of the $\mathbf{W}_1$ and $\mathbf{W}_2$ vectors. Hence, for large samples, the statistic $\hat{\rho}_v^2$ can be used to test for independence even when the distributions of the W's are not known. This situation is rather common in the geophysical sciences. A more general development of the asymptotic distributions for statistics of this type is given in Anderson (1984).

b. Small sample distribution of $\hat{\rho}_v^2$

In practice, sample sizes that are too small to use the asymptotic result given above (i.e., $n \ll 64$) are frequently encountered. Thus, we seek to extend the results of Jupp and Mardia to small sample sizes by estimating the small sample distributions, using Monte Carlo techniques. In particular, a random number generator (Press et al. 1986) was used to generate normally distributed (0, 1) u and v vector components for two-component vectors for sample sizes of 8, 12, 32, and 64. Values of $\hat{\rho}_v^2$ were calculated for 1 000 000 cases for each sample size. One million runs were required for each sample size in order to achieve reasonable accuracy (i.e., to the second decimal place) over the tails of the distributions that were generated.

The resulting empirical cumulative frequency distributions are shown together with the theoretical chi-square distribution with four degrees of freedom in Fig. 1. There is a significant departure from the theoretical chi-square distribution for small sample sizes. The reason for the crossover of the curves at approximately constant values of $n\hat{\rho}_v^2$ is not known. For sample sizes greater than 64, the form of the distribution closely approximates chi square with four degrees of freedom; for samples smaller than eight, the general form of the distribution breaks down, no longer resembling chi square.

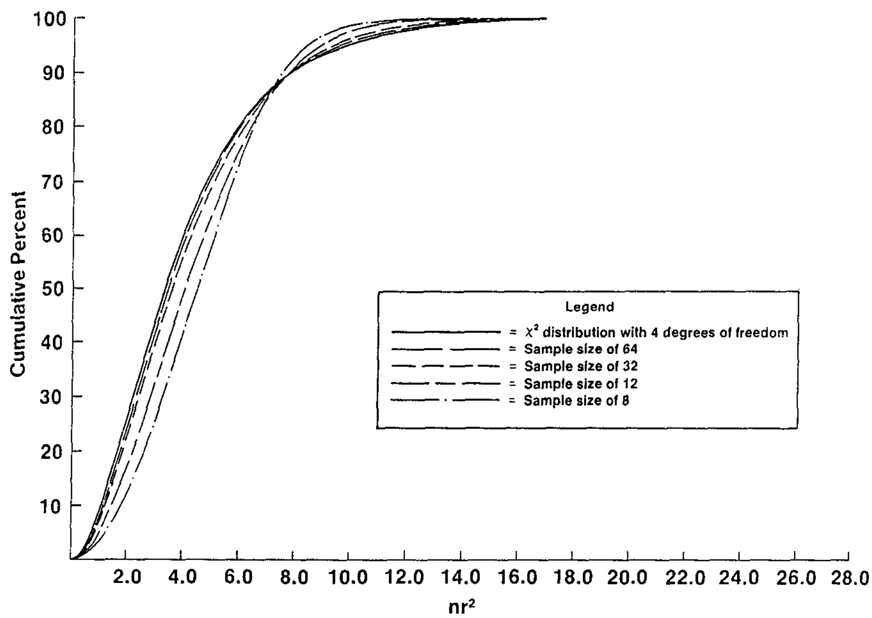

FIG. 1. Cumulative frequency distributions for two-dimensional vectors for sample sizes of 8, 12, 32, 64, and for the theoretical chi-square distribution with four degrees of freedom.

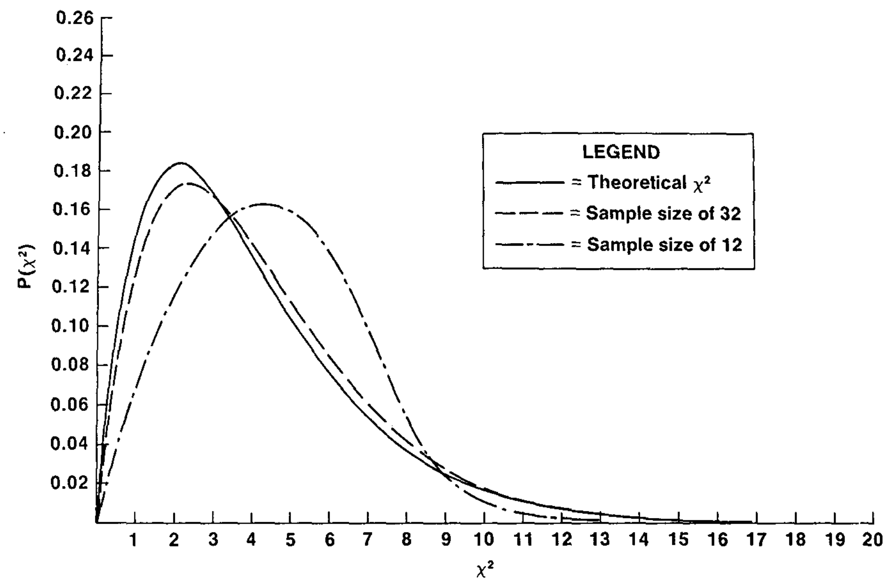

FIG. 2. Probability density functions (PDFs) corresponding to the cumulative frequency distributions shown in Fig. 1 for sample sizes of 12 and 32. The PDF for the theoretical chi-square distribution is also included.

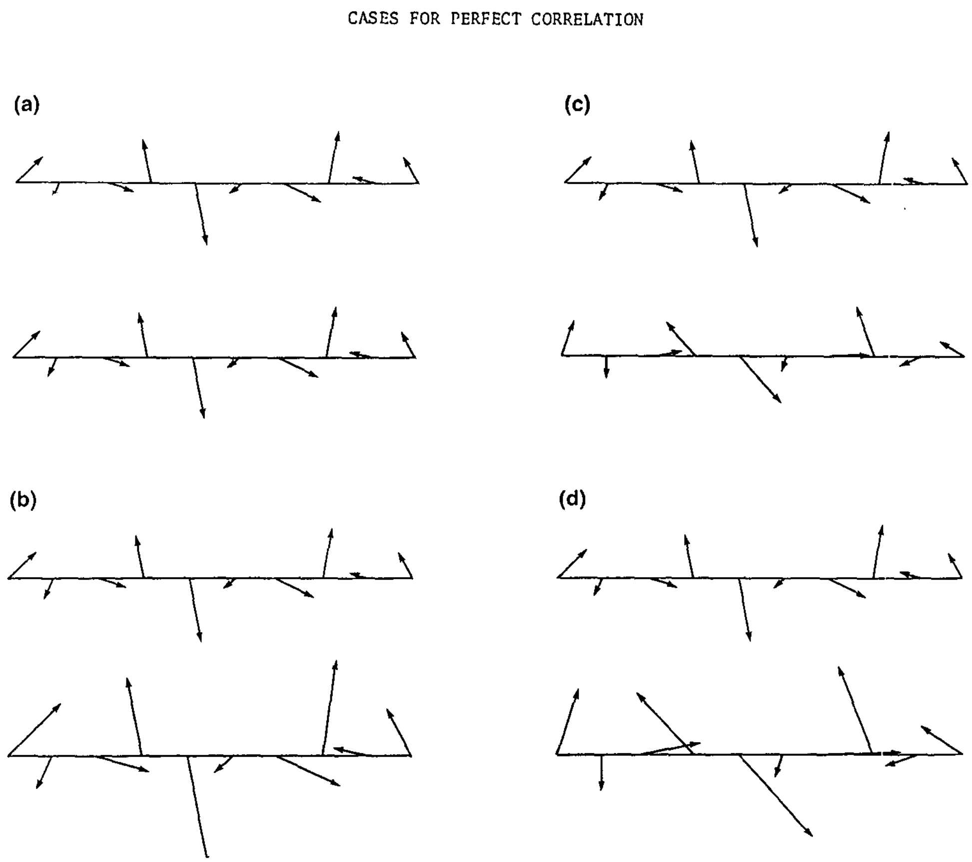

FIG. 3. Examples of vector sequences that produce perfect correlation (i.e., $\rho_v^2 = 2.0$). In (a) the vectors are identical; in (b) the magnitudes of the vectors in the second sequence are multiplied by a constant; in (c) the directions of the vectors in the second sequence are each rotated by a constant angle; and in (d) the second sequence is both multiplied by a constant and rotated by a constant angle.

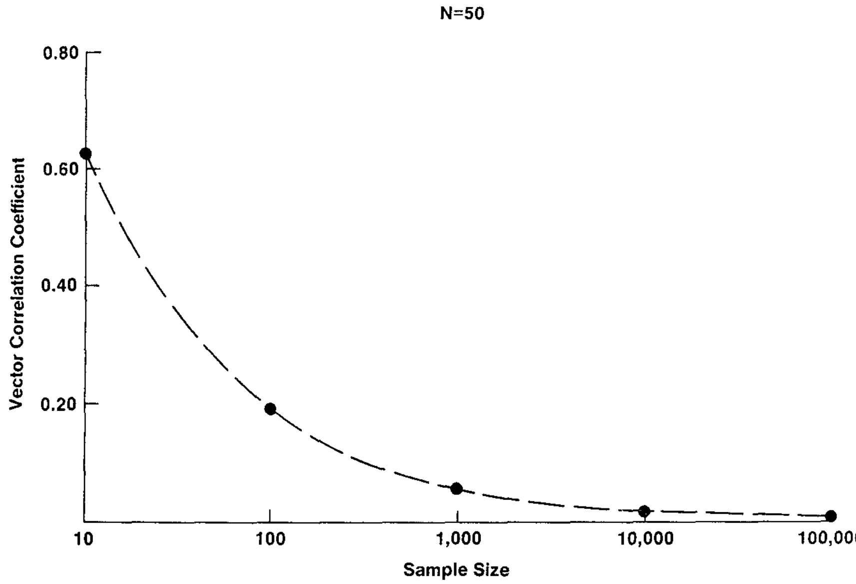

FIG. 4. The case leading to zero correlation between two vector sequences. In this case, the vectors in each sequence are generated randomly and the results averaged over 50 realizations and then plotted for sample sizes of 10, 100, 1000, 10 000, and 100 000.

The corresponding density functions for the cumulative distributions given in Fig. 1 are shown in Fig. 2. The results are given for sample sizes of 12 and 32. As sample size increases, the distributions become more positively skewed and more peaked.

In cases where the variables are normally distributed, the theory for testing for the independence of sets of variables is well developed. For a discussion of this theory, see Morrison (1990).

5. Interpreting ρ_v^2

The simple linear correlation coefficient is a measure of linear interdependence and as such it does not measure other types of dependence. Even in the simple case of two random variables, any expression of their joint variability is far too complex to be summarized in a single parameter. Thus, any correlation coefficient should be used only as an indicator of interdependence rather than as a precise measure. These comments should apply even more strongly to a single parameter that attempts to measure the association between vectors.

To provide more insight into the types of vectors (i.e., vector sequences) that may lead to relatively high sample values for this parameter, we consider situations that lead to perfect correlation (i.e., $\rho_v^2 = 2$) and zero correlation (i.e., $\rho_v^2 = 0$). Then we present an application using marine surface winds from the northwest Atlantic.

a. Situations that lead to perfect and zero correlation

Four cases that lead to perfect correlation are shown in Fig. 3. The first, or trivial case, arises when the vector pairs are identical (Fig. 3a). The second case that produces perfect correlation arises when the magnitudes of the vectors in the second sequence are multiplied

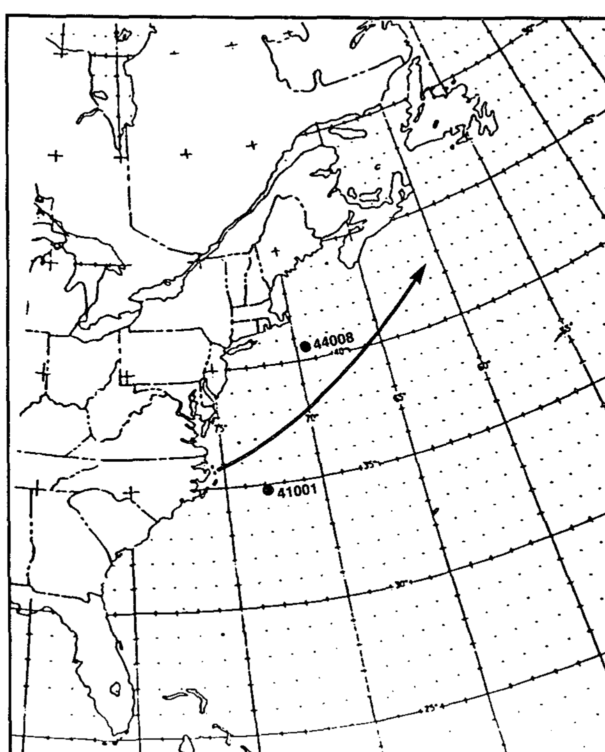

FIG. 5. Locations of the two NDBC environmental data buoys from which time-series surface winds were extracted. Period covers 1 December 1987–4 February 1988. A typical winter storm track has been included (Klein 1957).

by a constant (i.e., magnification; Fig. 3b). A third case of perfect correlation arises when the directions of the vectors in the second sequence are each rotated by a constant angle (Fig. 3c). The fourth case arises when the second sequence is derived from the first by combining both magnification and rotation (Fig. 3d).

From the above, we generalize these results to include any situation where one vector sequence can be expressed as a linear combination of the other (i.e., any case where two vector sequences are linearly dependent). In vector notation, if there exists a nonsingular matrix $\mathbf{A}$ and a vector $\mathbf{B}$ such that $\mathbf{W}_{1i} = \mathbf{A}\mathbf{W}_{2i} + \mathbf{B}$ for all i, then the vector correlation between $\mathbf{W}_{1i}$ and $\mathbf{W}_{2i}$ will be perfect. This is a restatement of property 7 of the Appendix.

Next, we consider the situation when there is zero correlation between two vector sequences. It is true that if two vectors are independent, then their vector correlation will be zero. In cases where the vectors are jointly normally distributed, their vector correlation will be zero if and only if they are independent. Using the random number generator described above, we generated independent vector sequences with normally distributed vector components for sample sizes of 10, 100, 1000, 10 000, and 100 000, and computed $\hat{\rho}_v^2$ for each sample size. This experiment was repeated 50 times for each sample size and the results averaged. As theory predicts, $\hat{\rho}_v^2$ clearly approaches zero for increasing sample size (Fig. 4). For a sample size of 100 000, for example, the averaged value of $\hat{\rho}_v^2$ is approximately 0.006. These results also demonstrate that relatively high correlations (e.g., >0.6) can be obtained solely by chance for small sample sizes (e.g., of order 10).

In the interpretation of $\hat{\rho}_v^2$, it is also important to consider the proper choice of sample size. The optimum choice will depend in part on the length of time it takes for the vectors to vary significantly. For example, if a sample size is chosen that is too small to encompass significant variability within the vector sequences, the resulting values of $\hat{\rho}_v^2$ may not be meaningful. Help in identifying this problem may be obtained by examining the trace of the sample covariance matrix from which $\hat{\rho}_v^2$ is calculated. The trace of the covariance matrix may provide a measure of the "signal-like" character for the calculated values of $\hat{\rho}_v^2$. Thus, values of the trace that exceed some threshold may have corresponding vector correlations that are meaningful.

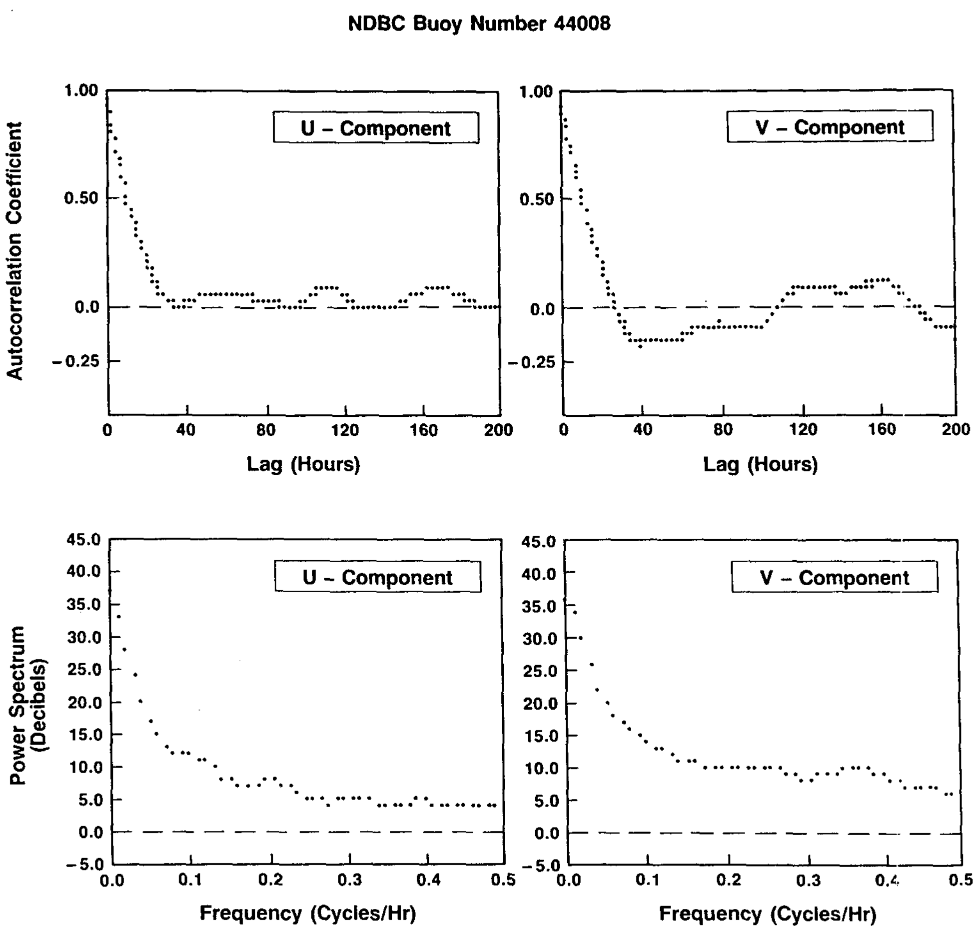

FIG. 6. Autocorrelation functions and power spectra for u and v components of winds from NDBC buoy 44008.

b. Application to marine surface winds

In the following application, we calculate vector correlations between surface winds at two locations in the northwestern Atlantic. The main purpose of this analysis was to construct a time series of sample parameters that will give a measure of the linear association of these winds at the two locations. In order to obtain similar information from the individual scalar correlations it would be necessary to examine four time series. The wind observations were acquired from NOAA Data Buoy Center (NDBC) environmental data buoys located at 40.5°N, 69.5°W (buoy number 44008) and at 34.9°N, 72.9°W (buoy number 41001). These buoys, whose locations are shown in Fig. 5, are approximately 700 km apart. The observations, taken hourly, extend from 1 December 1987 to 4 February 1988, a period of 65 days. An expected winter storm track for this region has been included (Klein 1957). As winter low pressure systems leave the east coast of the United States, they often deepen over the Gulf Stream and expand as they propagate to the northeast. Thus, the winds at both buoys are expected to be strongly influenced by the passage of these low pressure systems that pass through the area during the winter months. The stick diagram shown in the upper two panels of Fig. 8 depict the time series of wind vectors at each location.

Autocorrelation and spectral analyses were initially conducted to estimate the time scales of persistence and to identify any major periodic components in the data for each buoy (Figs. 6 and 7). Autocorrelation analysis of the u and v wind components indicate correlation time scales that are on the order of half a day; consequently, the original data have been subsampled every 12th point to produce series with observations that are approximately independent. The plots of power spectral density indicate monotonically decreasing spectral variance with increasing frequency (i.e., the so-called "red noise" spectrum), in each case revealing no major peaks (i.e., periodic components).

Vector correlations have been calculated for four sample sizes—8, 16, 24, and the entire series (i.e., 130)—corresponding to periods of 96, 192, 288, and 1560 h, respectively. A sliding sample window was employed that was stepped forward one data interval at a

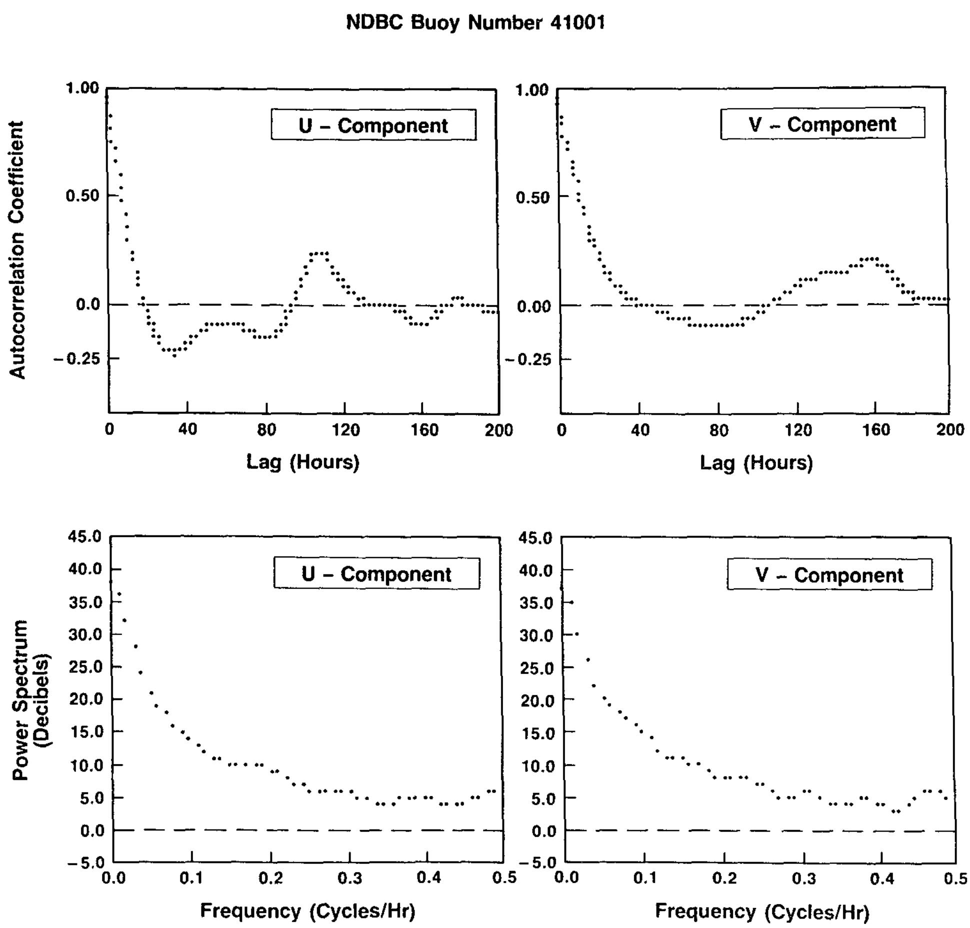

FIG. 7. Autocorrelation functions and power spectra for u and v components of winds from NDBC buoy 41001.

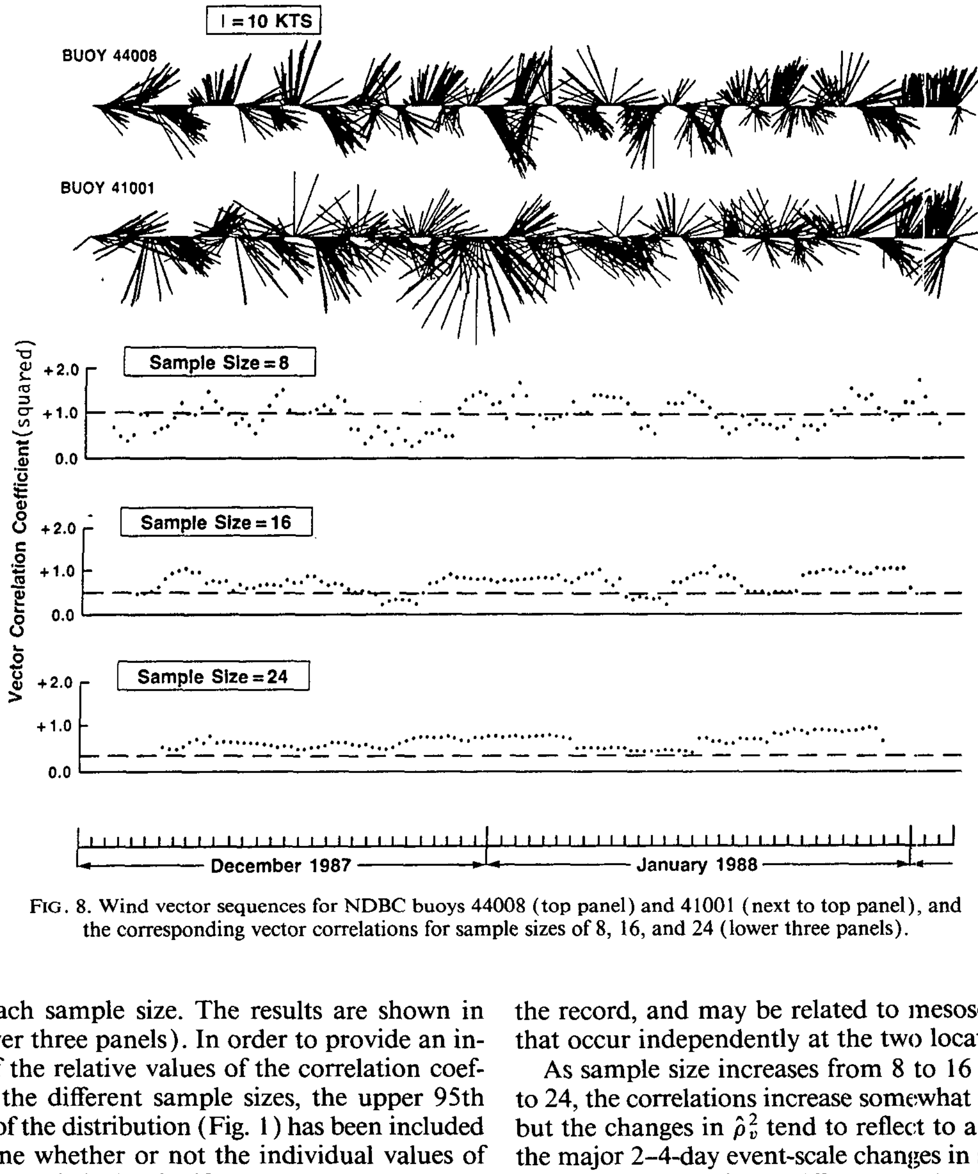

FIG. 8. Wind vector sequences for NDBC buoys 44008 (top panel) and 41001 (next to top panel), and the corresponding vector correlations for sample sizes of 8, 16, and 24 (lower three panels).

time for each sample size. The results are shown in Fig. 8 (lower three panels). In order to provide an indication of the relative values of the correlation coefficient for the different sample sizes, the upper 95th percentile of the distribution (Fig. 1) has been included to determine whether or not the individual values of $\hat{\rho}_v^2$ would be statistically significant at the 5% level. Use of this distribution is based on the assumption that the points within the series are independent.

Our choices of sample size are based primarily on the synoptic time scales of variation in the surface wind fields. The winds shown in Fig. 8 indicate time scales of variation (i.e., "event" time scales) on the order of 2–4 days. Sample sizes of 8 (4 days), 16 (8 days), and 24 (12 days) clearly encompass these time scales. It is important to recognize that the sample size must be sufficient to include significant variation in the vector sequences being correlated.

The results for a sample size of eight indicate that large variations in $\hat{\rho}_v^2$ occur over the length of the series. Relatively high values ($\hat{\rho}_v^2$ = 1.5 or greater) tend to occur where major changes in surface wind (particularly noticeable in wind direction) are similar at both locations (i.e., synoptic scale). Relatively low values of $\hat{\rho}_v^2$ (less than about 0.4) tend to occur throughout the record, and may be related to mesoscale motions that occur independently at the two locations.

As sample size increases from 8 to 16 and from 16 to 24, the correlations increase somewhat in most cases but the changes in $\hat{\rho}_v^2$ tend to reflect to a lesser extent the major 2–4-day event-scale changes in surface wind. It becomes increasingly difficult to relate the values of $\hat{\rho}_v^2$ to individual events in the wind field. In the limit, when N equals 130, we obtain a single value for $\hat{\rho}_v^2$ that represents the correlation between the surface wind fields at the two locations over the entire record. In this case $\hat{\rho}_v^2$ is equal to 0.54, a value that is statistically significant at the 5% level.

6. Discussion and conclusions

The vector correlation presented here has a number of desirable properties. It is a generalization of the square of the standard scalar correlation coefficient. Its properties 1) give an intuitive meaning to the definition, 2) relate the definition to the multivariate coefficient of determination, and 3) allow it to be used in a variety of practical situations, especially those where the distributions of the **W** vectors are unknown or nonstandard. Because of the simple form of the definition presented in this paper, its intuitive meaning, and its desirable statistical properties, we believe that the above definition of vector correlation should be considered in future applications.

Based on the results above, it becomes clear that care must be exercised in selecting the "proper" sample size for calculating vector correlations for time series data. At one extreme, choosing a sample size that is too small may lead to quite variable correlations that will not be amenable to interpretation. At the other extreme, when vector correlations for the entire series are calculated, a single value is obtained that will be meaningful, but the opportunity to examine time variations in correlation within the series will be lost. In cases where the sample sizes are small enough to reveal correlations related to individual events within the series, it may be possible to interpret $\hat{\rho}_v^2$ in terms of these events. We have not attempted to do so here, because these vector correlations may well depend on additional information to which we did not have access.

Our primary purpose has been to present Hooper's original definition of vector correlation with preliminary guidance on its use and interpretation. There are still many open questions about its application. For example, the distribution of this statistic is known for large samples when the correlation is zero and the sample points are independent. However, little is known about its distribution when the sample points are not independent, a situation often encountered in time series data. Consequently, considerably more effort should be devoted to the application of this technique to the practical problems that frequently arise in comparing vector quantities. In future studies we will apply the method of vector correlation presented here 1) to vector time series at several locations, 2) to datasets that are not time series, and 3) to the problem of forecasting vector sequences using regression techniques.

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APPENDIX

Proofs of the Properties of ρ_v^2

In order to develop the properties of ρ_v^2 we will need the following results. If $\mathbf{M}$ and $\mathbf{N}$ are square matrices, then

$$\mathrm{Tr}(\mathbf{MN}) = \mathrm{Tr}(\mathbf{NM}). \quad (A.1)$$

If $\mathbf{M}$ and $\mathbf{N}$ are nonsingular, then

$$(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}, \quad (A.2)$$

(Graybill 1969).

We now consider the properties of ρ_v^2.

Property 1. The coefficient ρ_v^2 is symmetric in $\mathbf{W}_1$ and $\mathbf{W}_2$. Using (A.1) twice, it is seen that

$$\begin{aligned}\rho_v^2(\mathbf{W}_1, \mathbf{W}_2) &= \mathrm{Tr}(\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})\\ &= \mathrm{Tr}(\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})\\ &= \rho_v^2(\mathbf{W}_2, \mathbf{W}_1).\end{aligned}$$

As discussed earlier, this is not true for some of the alternate definitions of vector correlation.

Property 2. The parameter ρ_v^2 is invariant under nonsingular linear transformations of the coordinate axes, including rotation and changes in scale.

For translations, this property is obvious since the covariance matrix is unchanged by such transformations. The second part of this property can be restated as the following theorem.

Theorem 1. The vector correlation ρ_v is invariant under linear transformations of $\mathbf{W}_1$ and $\mathbf{W}_2$ if the transformations are of rank 2. That is, if a linear transformation of the form

$$\mathbf{L} = \begin{pmatrix} a_{11} & a_{12} & 0 & 0\\ a_{21} & a_{22} & 0 & 0\\ 0 & 0 & b_{11} & b_{12}\\ 0 & 0 & b_{21} & b_{22}\end{pmatrix} = \begin{pmatrix}\mathbf{A} & 0\\ 0 & \mathbf{B}\end{pmatrix},$$

where $\mathbf{A}$ and $\mathbf{B}$ are nonsingular, is applied to the four-dimensional vector $\mathbf{X}$, then ρ_v^2 is unchanged. To see this, let

$$\mathbf{X}^* = \mathbf{LX} = \begin{pmatrix}\mathbf{W}_1^*\\ \mathbf{W}_2^*\end{pmatrix} = \begin{pmatrix}\mathbf{AW}_1\\ \mathbf{BW}_2\end{pmatrix}.$$

The covariance matrix of $\mathbf{X}^*$ is then given by

$$\begin{pmatrix}\mathbf{A} & 0\\ 0 & \mathbf{B}\end{pmatrix}\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}\begin{pmatrix}\mathbf{A}^{\mathrm{T}} & 0\\ 0 & \mathbf{B}^{\mathrm{T}}\end{pmatrix}.$$

The covariance of $\mathbf{X}^*$ is equal to

$$\left(\begin{array}{c|c}\mathbf{A}\Sigma_{11}\mathbf{A}^{\mathrm{T}} & \mathbf{A}\Sigma_{12}\mathbf{B}^{\mathrm{T}}\\ \mathbf{B}\Sigma_{21}\mathbf{A}^{\mathrm{T}} & \mathbf{B}\Sigma_{22}\mathbf{B}^{\mathrm{T}}\end{array}\right).$$

Then for the new vectors $\mathbf{W}_1^*$ and $\mathbf{W}_2^*$,

$$\begin{aligned}&\rho_v^2(\mathbf{W}_1^*, \mathbf{W}_2^*)\\ &= \mathrm{Tr}[(\mathbf{A}\Sigma_{11}\mathbf{A}^{\mathrm{T}})^{-1}(\mathbf{A}\Sigma_{12}\mathbf{B}^{\mathrm{T}})(\mathbf{B}\Sigma_{22}\mathbf{B}^{\mathrm{T}})^{-1}(\mathbf{B}\Sigma_{21}\mathbf{A}^{\mathrm{T}})].\end{aligned} \quad (A.3)$$

Using Eqs. (A.1) and (A.2), the right-hand side of Eq. (A.3) becomes

$$\mathrm{Tr}(\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}) = \rho_v^2(\mathbf{W}_1, \mathbf{W}_2).$$

Property 3. The parameter ρ_v^2 is the sum of the squares of the canonical correlations. This can be shown using property 2. Computing the canonical correlations is equivalent to finding an $\mathbf{A}$ and $\mathbf{B}$ of Eq.

(A.3) such that the covariance matrix of $\mathbf{X}^*$ is equal to

$$\begin{pmatrix} 1 & 0 & \rho_1 & 0 \\ 0 & 1 & 0 & \rho_2 \\ \rho_1 & 0 & 1 & 0 \\ 0 & \rho_2 & 0 & 1 \end{pmatrix},$$

where $\rho_1 \geqslant \rho_2 \geqslant 0.0$ and ρ_1 and ρ_2 are maximized. That is, ρ_1 is the first canonical correlation and ρ_2 is the second (Anderson 1984). Next, let

$$\mathbf{D} = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}.$$

Then for $\mathbf{W}_1^*$ and $\mathbf{W}_2^*$,

$$\rho_v^2 = \mathrm{Tr}(\mathbf{I}^{-1}\mathbf{D}\mathbf{I}^{-1}\mathbf{D}),$$
$$= \rho_1^2 + \rho_2^2, \tag{A.4}$$

which by property 2 is equal to $\rho_v^2(\mathbf{W}_1, \mathbf{W}_2)$ and where $\mathbf{I}$ is the 2×2 identity matrix.

Property 4. If the covariance matrix takes on the special form below, then ρ_v^2 is a simple function of the squares of ordinary correlations. Let the covariance matrix of $\mathbf{W}_1$ and $\mathbf{W}_2$ be of the following form:

$$\begin{pmatrix} \mathbf{I} & \mathbf{R} \\ \mathbf{R}^T & \mathbf{I} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \sigma_{11} & \sigma_{12} \\ 0 & 1 & \sigma_{21} & \sigma_{22} \\ \sigma_{11} & \sigma_{21} & 1 & 0 \\ \sigma_{12} & \sigma_{22} & 0 & 1 \end{pmatrix}.$$

Then,

$$\rho_v^2 = \sigma_{11}^2 + \sigma_{12}^2 + \sigma_{21}^2 + \sigma_{22}^2. \tag{A.5}$$

The proof of this result is similar to that of property 3. Note that the covariances are correlations in this case.

Property 5. If $\mathbf{W}_1$ and $\mathbf{W}_2$ are independent, then $\rho_v^2 = 0$. Proof: if $\mathbf{W}_1$ and $\mathbf{W}_2$ are independent, then

$$\Sigma_{12} = \Sigma_{21} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0},$$

and

$$\rho_v^2 = \mathrm{Tr}(\Sigma_{11}^{-1}\mathbf{0}\Sigma_{22}^{-1}\mathbf{0}) = 0.$$

Property 6. If corr(u_1, u_2), corr(u_1, v_2), corr(v_1, u_2), and corr(v_1, v_2) are not all 0, then $\rho_v^2 > 0$.

To show this property, we note the following set of inequalities:

$$\rho_v^2 \geqslant \rho_1^2 \geqslant \max[|\mathrm{corr}(u_1, u_2)|,$$
$$|\mathrm{corr}(u_1, v_2)|, |\mathrm{corr}(v_1, u_2)|, |\mathrm{corr}(v_1, v_2)|].$$

Property 7. The random vectors $\mathbf{W}_1$ and $\mathbf{W}_2$ are linearly dependent in the two-dimensional case if and only if $\rho_v^2 = 2$.

Assume $\mathbf{W}_1$ and $\mathbf{W}_2$ are linearly dependent. Then there are nonsingular matrices $\mathbf{C}$ and $\mathbf{D}$ and a vector $\mathbf{A}$ such that

$$\mathbf{C}\mathbf{W}_1 + \mathbf{D}\mathbf{W}_2 + \mathbf{A} = \mathbf{0}.$$

Here, $\mathbf{0}$ represents a 2×1 null vector; hence,

$$\mathbf{W}_1 = -\mathbf{C}^{-1}\mathbf{D}\mathbf{W}_2 - \mathbf{C}^{-1}\mathbf{A}. \tag{A.6}$$

This relationship can be written as

$$\begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & -\mathbf{C}^{-1}\mathbf{D} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{0} \\ \mathbf{W}_2 \end{pmatrix} + \begin{pmatrix} -\mathbf{C}^{-1}\mathbf{A} \\ \mathbf{0} \end{pmatrix}.$$

It follows that the covariance matrix of $\mathbf{X}$ can be written as

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{C}^{-1}\mathbf{D}\Sigma_{22}\mathbf{D}^T(\mathbf{C}^{-1})^T & -\mathbf{C}^{-1}\mathbf{D}\Sigma_{22} \\ -\Sigma_{22}\mathbf{D}^T(\mathbf{C}^{-1})^T & \Sigma_{22} \end{bmatrix}. \tag{A.7}$$

Then from Eq. (3.1) we have

$$\rho_v^2 = \mathrm{Tr}\{[\mathbf{C}^{-1}\mathbf{D}\Sigma_{22}\mathbf{D}^T(\mathbf{C}^{-1})^T]^{-1}\mathbf{C}^{-1}\mathbf{D}\Sigma_{22}(\Sigma_{22})^{-1} \times \Sigma_{22}\mathbf{D}^T(\mathbf{C}^{-1})^T\} = \mathrm{Tr}(\mathbf{I}) = 2.$$

Next, assume

$$\rho_v^2 = 2.$$

Then the canonical correlations ρ_1 and ρ_2 are both equal to 1. This follows because

$$\rho_v^2 = \rho_1^2 + \rho_2^2,$$

and

$$0 \leqslant \rho_2 \leqslant \rho_1 \leqslant 1.$$

Then as in the proof of property 3, there are nonsingular matrices $\mathbf{A}$ and $\mathbf{B}$ such that

$$\mathbf{X}^* = \begin{pmatrix} \mathbf{W}_1^* \\ \mathbf{W}_2^* \end{pmatrix} = \begin{pmatrix} u_1^* \\ v_1^* \\ u_2^* \\ v_2^* \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}\begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{pmatrix}, \tag{A.8}$$

where the covariance matrix of $\mathbf{X}^*$ is of the form

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Hence, the correlation between u_1^* and u_2^* is 1.0 and the correlation between v_1^* and v_2^* is 1.0. Since these are ordinary correlation coefficients, this implies that

$$u_1^* = c_0 + c_1 u_2^*,$$

and

$$v_1^* = d_0 + d_1 v_2^*.$$

Hence, it follows that

$$\mathbf{W}_1^* = \begin{pmatrix} c_1 & 0 \\ 0 & d_1 \end{pmatrix}\mathbf{W}_2^* + \begin{pmatrix} c_0 \\ d_0 \end{pmatrix}. \tag{A.9}$$

From Eq. (A.8) we have

$$\begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{-1} \end{pmatrix}\begin{pmatrix} \mathbf{W}_1^* \\ \mathbf{W}_2^* \end{pmatrix}. \tag{A.10}$$

From Eqs. (A.9) and (A.10) it follows that

$$\mathbf{W}_1 = \mathbf{A}^{-1}\mathbf{W}_1^* = \mathbf{A}^{-1}\begin{pmatrix} c_1 & 0 \\ 0 & d_1 \end{pmatrix}\mathbf{W}_2^* + \mathbf{A}^{-1}\begin{pmatrix} c_0 \\ d_0 \end{pmatrix}$$

$$= \mathbf{A}^{-1}\begin{pmatrix} c_1 & 0 \\ 0 & d_1 \end{pmatrix}\mathbf{B}\mathbf{W}_2^* + \mathbf{A}\begin{pmatrix} c_0 \\ d_0 \end{pmatrix},$$

which proves the assertion.

Property 8. If $\mathbf{W}_1$ and $\mathbf{W}_2$ are a pair of independent two-dimensional vectors with finite fourth moments, then $n\hat{\rho}$ is distributed asymptotically as a chi-square variable with four degrees of freedom. This asymptotic distribution is valid for any form of the marginal distribution of the $\mathbf{W}$ vectors. The proof of this property is difficult and lengthy and is not included here. For details, see Anderson (1984) or Jupp and Mardia (1980).

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