Contemporary Mathematics

Gauss Sum Combinatorics and Metaplectic Eisenstein Series

Ben Brubaker, Daniel Bump, and Solomon Friedberg

It is a great pleasure to dedicate this paper to Steve Gelbart, who was one of the first to work in this area.

1. Introduction

1.1. Steve Gelbart, Whittaker models and the metaplectic group. The metaplectic double cover of $\mathrm{Sp}(2r)$ and the Weil representation were introduced by Weil [**37**] in order to formulate results of Siegel on theta functions in the adelic setting. This was followed by two initially independent developments. First, Shimura [**32**], [**33**] gave two extremely important constructions involving modular forms of half-integral weight. Both Shimura integrals involved Rankin-Selberg convolutions with a theta function which, in the modern view, lives on the metaplectic double cover of SL_2 (or GL_2). Second, the mainstream modern context of automorphic representations of adele groups emerged with Jacquet and Langlands [**23**] and Godement and Jacquet [**22**]. (Gelbart's book [**11**] was important in making this modern point of view accessible to a generation of workers in the field.) It became clear that the metaplectic group and Shimura's constructions needed to be reworked in the modern language. Gelbart [**12**] was perhaps the first to talk about automorphic forms of half-integral weight, particularly theta functions, in completely modern terms. The Shimura constructions were carried out on the adele group by Gelbart and Piatetski-Shapiro [**17**], [**18**] for the Shimura correspondence, and by Gelbart and Jacquet [**14**], [**15**] for the symmetric square. Shimura's important constructions were thus extended, and as a particular important point the lifting from GL_2 to GL_3 was established.

Jacquet and Langlands [**23**] emphasized the uniqueness of Whittaker models for representations of GL_2. Shalika [**31**] and Piatetski-Shapiro [**29**] showed that uniqueness holds over nonarchimedean local fields. (See also Gelfand and Kazhdan [**21**] and Bernstein and Zelevinsky [**1**].)

For metaplectic covers of these groups, the Whittaker models may or may not be unique. Gelbart and Piatetski-Shapiro considered the representations of the double cover of SL_2 that have unique Whittaker models and found them to be associated with theta functions. On the other hand Gelbart, Howe and Piatetski-Shapiro [**13**] showed that representations of the double cover of GL_2 have Whittaker models (in a slightly modified sense) that are unique. The failure of uniqueness of Whittaker models might seem a defect, but Waldspurger [**36**] showed that precisely this lack of uniqueness is the source of an important phenomenon, in which the global Whittaker models of modular forms of half-integral weight encode L-values of the Shimura correspondent. Waldspurger's phenomenon was clarifed by Gelbart and Piatetski-Shapiro [**19**]. And Gelbart and Soudry [**20**] gave examples of automorphic forms on the double cover of SL_2 that have local Whittaker models at all places but no global Whittaker models.

Higher metaplectic covers were defined by Kubota [**26**] (for SL_2) and Matsumoto [**28**] (for general simply-connected groups), and in general Whittaker models are not unique. Kazhdan and Patterson [**24**] showed that on the n-fold cover of GL_r, the Whittaker coefficients of theta functions (residues of Eisenstein series) have unique Whittaker models if $r = n$ or $n-1$. However for general automorphic forms, including Eisenstein series, Whittaker models are not unique.

At first sight, Whittaker models of Eisenstein series on metaplectic groups seem to be difficult to compute. When uniqueness of Whittaker models fails, the coefficients can sometimes be computed, but they at first appear chaotic, and this may account for the fact that their properties remained hidden for a very long time. Nevertheless in [**3**]–[**10**] a theory of *Weyl group multiple Dirichlet series* has been developed by Brubaker, Bump, Chinta, Friedberg, Gunnells and Hoffstein. These Dirichlet series are conjectured to be (global) Whittaker coefficients of metaplectic Eisenstein series, though recent progress in developing their properties has been made possible by not emphasizing this connection in favor of other methods.

1.2. Main results and outline of the paper. This paper continues to develop the theory of Weyl group multiple Dirichlet series from the perspective of [**6**]. There, we presented a definition of multiple Dirichlet series attached to the root system A_r, $r \geq 1$. We recall the basic set-up.

Let F be a totally complex algebraic number field containing the group μ_{2n} of $2n$-th roots of unity. Let S be a finite set of places of F containing the archimedean ones and those ramified over $\mathbb{Q}$ that is large enough that the ring $\mathfrak{o}_S$ of S-integers in F is a principal ideal domain.

Then to any r-tuple of $\mathfrak{o}_S$ integers $\mathbf{m} = (m_1, \cdots, m_r)$, we associate a multiple Dirichlet series of type A_r of form

$$Z_\Psi(\mathbf{s};\mathbf{m}) = Z_\Psi(s_1,\cdots,s_r;m_1,\cdots,m_r) \quad = \tag{1}$$
$$\sum H\Psi(c_1,\cdots,c_r;m_1,\cdots,m_r)\mathbb{N}c_1^{-2s_1}\cdots\mathbb{N}c_r^{-2s_r}$$

where the sum is over nonzero ideals c_i of $\mathfrak{o}_S$, and we are denoting $\mathbf{s} = (s_1, \cdots, s_r) \in \mathbb{C}^r$.

Remark 1.1. Here H and Ψ are defined when c_i are nonzero elements of $\mathfrak{o}_S$, but their product is well-defined over ideals, since H and Ψ behave in a coordinated way when c_i is multiplied by a unit. Thus the sum is essentially over ideals $c_i\mathfrak{o}_S$. However we will want to consider H independently of Ψ, so for each prime $\mathfrak{p}$ of $\mathfrak{o}_S$ we fix a generator p of $\mathfrak{p}$, and only consider c_i and m_i which are products of powers of these fixed p's.

The function Ψ is chosen from a finite-dimensional vector space $\mathcal{M}$ of functions on $(F_S^\times)^n$ that is well-understood and defined in [**5**] and [**4**], but the function H is more interesting. It has a twisted multiplicativity with respect to both the c_i, $i = 1, \ldots, r$ and $\mathbf{m}$. That is, H can be decomposed into relatively prime pieces up to an nth root of unity determined by the $\mathbf{m}$ and the c_i. While this implies that Z_Ψ

is not an Euler product, the specification of its coefficients is nevertheless reduced to the case where the c_i and m_i are powers of the same prime p. See [**3**], [**5**] or [**6**] for further details, including a precise definition of the twisted multiplicativity.

Thus to specify Z_Ψ we are reduced to defining $H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r})$ for a fixed prime p. These "local coefficients" are defined by weighted sums over strict Gelfand-Tsetlin patterns with a certain fixed top row. Recall that a *Gelfand-Tsetlin pattern of rank* r is an array of integers

$$\mathfrak{T} = \left\{ \begin{array}{ccccccccc} a_{00} & & a_{01} & & a_{02} & & \cdots & & a_{0r} \\ & a_{11} & & a_{12} & & & & a_{1r} & \\ & & \ddots & & & & \iddots & & \\ & & & & a_{rr} & & & & \end{array} \right\} \tag{2}$$

where the rows interleave; that is, $a_{i-1,j-1} \geqslant a_{i,j} \geqslant a_{i-1,j}$. We will say that the pattern is *strict* if each row is strictly decreasing. The weighting factors attached to each pattern are products of Gauss sums formed with nth power residue symbols. A precise definition is reviewed in the next section.

In [**6**] we made two conjectures concerning these multiple Dirichlet series. The first stated that the multiple Dirichlet series associated to A_r appear in the Fourier-Whittaker coefficients of certain minimal parabolic metaplectic Eisenstein series on n-fold covers of $\mathrm{GL}(r+1)$. The second conjectured that these series have analytic continuation to meromorphic functions on the whole complex space $\mathbb{C}^r$ and possess functional equations isomorphic to the Weyl group S_{r+1} of the corresponding root system A_r. This paper concerns the latter of these two conjectures.

Several special cases of the conjecture, depending on the initial data r, $\mathbf{m}$ and n, have been proved. For example, in [**4**] we proved this when n is sufficiently large (depending on r and $\mathbf{m}$). In this case, there are fewer non-zero local coefficients, so the proofs of functional equations are much simpler. The conjecture for general n remains open.

The present paper reduces this conjecture to a single combinatorial identity, and then outlines an approach to proving this resulting identity. More detailed pieces of this proof will appear in subsequent papers, but we feel that it is particularly illuminating to see several examples of the sorts of combinatorial objects involved.

The key idea is that while the prime power contributions to H are in bijection with Gelfand-Tsetlin patterns of fixed top row, there is no canonical way of identifying the two sets. In fact, given one such map from patterns to H, we can compose this with any involution of Gelfand-Tsetlin patterns preserving the top row to get another parametrization of the local coefficients H. A distinguished role is played by an involution, originally defined by Schützenberger [**30**] for Young tableaux, which has been translated into the language of Gelfand-Tsetlin patterns by Berenstein and Kirillov [**25**].

In Theorem 2.2, we explain how our initial parametrization of H in terms of patterns from [**6**] can be used to inductively prove all but one functional equation associated to simple reflections in the Weyl group. A second parametrization, obtained by composing the coordinate map of [**6**] with the Schützenberger involution, can be used to inductively prove a different collection of all but one of these functional equations. Thus, we reduce our conjecture to proving that the two parametrizations of H by Gelfand-Tsetlin patterns are equal.

In Theorem 3.5, we further reduce the conjecture by decomposing the Schützenberger involution into involutions t_i corresponding to simple reflections σ_i, $i = 1, \ldots, r$, and prove that one only needs the equivalence of two new parametrizations of H by Gelfand-Tsetlin patterns which differ by a single t_r. This allows us to restrict our attention, within Gelfand-Tsetlin patterns, to the top three rows, as t_r does not affect the remaining rows.

In the final section, we illustrate some of the combinatorial techniques used to prove the reduction resulting from Theorem 3.5. These techniques prove the required identity for all but one class of degenerate patterns, and we give an example in the degenerate case to give a flavor of the remaining complexity.

We would like to thank Gautam Chinta, Paul Gunnells and the referee for helpful comments. This work was supported by NSF FRG grants DMS-0354662 and DMS-0353964 and by NSA grant H98230-07-1-0015.

2. Definitions and a first reduction of the FE Conjecture

As noted above, in [**6**] we defined a *Weyl group multiple Dirichlet series* of type A_r in terms of Gauss sums indexed by strict Gelfand-Tsetlin patterns. We will repeat the definition now, with slightly different notation and give a second dual definition. The two definitions are not known to be equivalent, and the remainder of this paper is devoted to progress toward this claim. To conclude the section, we will explain how the equivalence of these two definitions would imply Conjecture 1 of [**6**], which asserts that the resulting multiple Dirichlet series possesses functional equations isomorphic to the Weyl group S_{r+1} of A_r. The conjecture will henceforth be referred to as the **FE conjecture**. In the middle of the section, we also digress to discuss the connections between these formulas and the representation theory of $SL_{r+1}(\mathbb{C})$.

2.1. Definitions for coefficients of Weyl group multiple Dirichlet series.

Given a number field F and a set of places S, as described in the introduction, let ψ be an additive character of $F_S = \prod_{v \in S} F_v$ that is trivial on $\mathfrak{o}_S$ but no larger fractional ideal. If $m, c \in \mathfrak{o}_S$ with $c \neq 0$ let

$$\mathfrak{g}(m, c) = \sum_{a \bmod c} \left(\frac{a}{c}\right) \psi\left(\frac{am}{c}\right), \tag{3}$$

where $\left(\frac{a}{c}\right)$ is the n-th power residue symbol. In much of the following discussion, we fix a prime p, and use the following abbreviated notation when no confusion can arise:

$$\mathfrak{g}(a) = \mathfrak{g}(p^{a-1}, p^a), \qquad \mathfrak{h}(a) = \mathfrak{g}(p^a, p^a), \qquad q = |\mathfrak{o}/p\mathfrak{o}|. \tag{4}$$

If $\mathfrak{T}$ is the Gelfand-Tsetlin pattern (2), define $G_R(\mathfrak{T}) = G_L(\mathfrak{T}) = 0$ if $\mathfrak{T}$ is not strict; assuming strictness, we define them as follows. Let $1 \leqslant i \leqslant j \leqslant r$ and

$$R_{i,j} = R_{i,j}(\mathfrak{T}) = \sum_{k=j}^{r} (a_{i,k} - a_{i-1,k}), \qquad L_{i,j} = L_{i,j}(\mathfrak{T}) = \sum_{k=i}^{j} (a_{i-1,k-1} - a_{i,k}). \tag{5}$$

We say that $\mathfrak{T}$ is *left-leaning at* (i,j) if $a_{i,j} = a_{i-1,j-1}$ and that $\mathfrak{T}$ is *right-leaning at* (i,j) if $a_{i,j} = a_{i-1,j}$. We define

$$G_R(R_{i,j}) = G_R(R_{i,j}(\mathfrak{T})) = \begin{cases} q^{R_{i,j}} & \text{if } \mathfrak{T} \text{ is right-leaning at } (i,j); \\ \mathfrak{g}(R_{i,j}) & \text{if } \mathfrak{T} \text{ is left-leaning at } (i,j); \\ \mathfrak{h}(R_{i,j}) & \text{otherwise.} \end{cases} \tag{6}$$

Similarly, we define

$$G_L(L_{i,j}) = G_L(L_{i,j}(\mathfrak{T})) = \begin{cases} q^{L_{i,j}} & \text{if } \mathfrak{T} \text{ is left-leaning at } (i,j); \\ \mathfrak{g}(L_{i,j}) & \text{if } \mathfrak{T} \text{ is right-leaning at } (i,j); \\ \mathfrak{h}(L_{i,j}) & \text{otherwise.} \end{cases} \tag{7}$$

Thus we have attached one of the number-theoretic quantities in (4) to each entry of the strict Gelfand-Tsetlin pattern.

Let

$$G_R(\mathfrak{T}) = \prod_{1 \leqslant i \leqslant j \leqslant r} G_R(R_{i,j}), \qquad G_L(\mathfrak{T}) = \prod_{1 \leqslant i \leqslant j \leqslant r} G_L(L_{i,j}).$$

If the dependence on p is to be emphasized, we will sometimes denote these by $G_R(\mathfrak{T};p)$ and $G_L(\mathfrak{T};p)$. We further define two sets of "local coordinates"

$$k_R(\mathfrak{T}) = (k_1^R, \cdots, k_r^R), \qquad k_L(\mathfrak{T}) = (k_1^L, \cdots, k_r^L),$$

by

$$k_i^R = k_i^R(\mathfrak{T}) = \sum_{j=i}^{r} (a_{i,j} - a_{0,j}) \tag{8}$$

and

$$k_i^L = k_i^L(\mathfrak{T}) = \sum_{j=r+1-i}^{r} (a_{0,j-r-1+i} - a_{r+1-i,j}). \tag{9}$$

Finally, we define

$$H_R(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) = \sum_{k_R(\mathfrak{T}) = (k_1, \cdots, k_r)} G_R(\mathfrak{T}) \tag{10}$$

and similarly

$$H_L(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) = \sum_{k_L(\mathfrak{T}) = (k_1, \cdots, k_r)} G_L(\mathfrak{T}), \tag{11}$$

where, in each case, the sum runs over all Gelfand-Tsetlin patterns with top row

$$\lambda + \rho = \{l_1 + \ldots + l_r + r, l_2 + \ldots + l_r + r - 1, \cdots, l_r + 1, 0\}, \tag{12}$$

satisfying the indicated condition beneath each sum. Here

$$\lambda = \{l_1 + \ldots + l_r, l_2 + \ldots + l_r, \cdots, l_r, 0\},$$

and $\rho = (r, r-1, \cdots, 0)$. As noted in the introduction, because we know how to decompose the multiple Dirichlet series in (1), via twisted multiplicativity in $\mathbf{m}$ and the c_i, $i = 1, \ldots, r.$, into prime power pieces, then either description (10) or (11) suffices to complete the definition of the series. For a precise description of the twisted multiplicativity, see [**6**].

2.2. Gelfand-Tsetlin patterns and representation theory. We now explain how the above data defined from Gelfand-Tsetlin patterns relates to the representation theory of $SL_{r+1}(\mathbb{C})$.

First we review the case where $n = 1$, which is explained more fully in [**6**]. When $n = 1$, the Weyl group multiple Dirichlet series are just Whittaker coefficients of Eisenstein series of minimal parabolic type on PGL_{r+1}. According to the formula of Shintani [**34**] and of Casselman-Shalika [**7**], these are Schur polynomials, represented by the Weyl character formula. Specifically, the Satake isomorphism associates with $\mathbf{s}$ and the prime p a conjugacy class A in the L-group $SL_{r+1}(\mathbb{C})$, and the Shintani-Casselman-Shalika formula identifies the p-part of a Whittaker coefficient of the Eisenstein series with the character of an irreducible representation of $SL_{r+1}(\mathbb{C})$ applied to A. Here the Whittaker coefficient is with respect to a character of the maximal unipotent whose p-part is determined by a partition λ associated to a highest weight vector in the representation.

The relevance of Gelfand-Tsetlin patterns can be seen from this – one may give a description of the irreducible representations of $SL_{r+1}(\mathbb{C})$ based on Gelfand-Tsetlin patterns in which the patterns parametrize vectors in irreducible modules of $SL_{r+1}(\mathbb{C})$; in this parametrization, the top row of the pattern is the highest weight vector of the representation, and the row sums are data determining the weight space in which the vector lies. We will not use this description – rather we will use a variant due to Tokuyama in which the top row of the pattern is not λ but $\lambda + \rho$ as in (12), and only strict Gelfand-Tsetlin patterns are used. But we mention it in order to inculcate the idea that the top row of the pattern corresponds to the highest weight vector, and the row sums correspond to weights.

We return to the fact that the p-part of the multiple Dirichlet series when $n = 1$ is the character of the irreducible representation with highest weight vector λ applied to the conjugacy class A in $SL_{r+1}(\mathbb{C})$. The Weyl character formula expresses this character value as a ratio of two quantities. The numerator is an alternating sum of $|W| = (r + 1)!$ monomials (where W is the Weyl group). The denominator in the Weyl character formula can be expressed as either an alternating sum or as a product. Writing it as a product, it *resembles* the p-part of the normalizing factor of the Eisenstein series, which is a product of $\frac{1}{2}r(r + 1)$ zeta functions. However, the zeta functions are at the wrong values; they are shifted.

But Tokuyama [**35**] gave a deformation of the Weyl character formula in which both the numerator and the denominator are altered. The denominator remains a product of $\frac{1}{2}r(r + 1)$ factors, and these can be brought into agreement with the p-part of the normalizing factor. The numerator becomes a sum over Gelfand-Tsetlin patterns. This deformation of the Weyl character formula can now be recognized as the p-part of the multiple Dirichlet series as we have defined it. Specifically, H_R as we have defined it in (10) is the contribution of all terms in a particular weight space of the irreducible representation to Tokuyama's formula. The row sums of the pattern, that is, the numbers k_R, correspond to the weight to which the term contributes in the formula for Tokuyama's numerator. Summing over all k_R gives the p-part of the multiple Dirichlet series, which is in exact agreement with the numerator in Tokuyama's formula. The denominator in Tokuyama's formula is then the p-part in the normalizing factor of the Eisenstein series.

Thus when $n = 1$, there is a well-understood connection with the representation theory of $SL_{r+1}(\mathbb{C})$. When $n > 1$, we are presented with a further "deformation" of

Tokuyama's formula involving Gauss sums. The connection apparently persists but it is no longer with the representation theory of $SL_{r+1}(\mathbb{C})$ but with its quantum analog.

Before we can address this point, it will be useful explain the roles of k_R and k_L as defined in (8) and (9). The patterns $\mathfrak{T}$ with fixed top row and fixed value of either k_R or k_L parametrize basis vectors for a weight space given by the values $(k_1, \dots, k_r)$. (Typically, the coordinates for weight spaces are expressed in terms of differences of row sums in the Gelfand-Tsetlin pattern, and so our choice of k_R or k_L can be seen as a composition of these usual coordinates with an affine linear map. The affine linear map has been chosen so that the support of the coordinates has k_i non-negative for all $i = 1, \dots, r$.) The two coordinate choices k_R and k_L are related by the Schützenberger involution on Gelfand-Tsetlin patterns, an involution originally defined on semistandard Young tableaux by Schützenberger [**30**] in the context of *jeu de taquin*, which was translated into the language of Gelfand-Tsetlin patterns by Kirillov and Berenstein [**25**].

Following [**25**], the Schützenberger involution is defined in terms of simpler involutions on Gelfand-Tsetlin patterns labelled $t_1, \dots, t_r$. For any i with $1 \le i \le r$, t_i affects *only* the entries in the $(i+1)$st row of the pattern. Using the indexing as in (2), we observe that $a_{i,j}$ is constrained (by the definition of Gelfand-Tsetlin patterns) to lie between $\max(a_{i-1,j}, a_{i+1,j+1})$ and $\min(a_{i-1,j-1}, a_{i+1,j})$. The involution t_i reflects it in this range, so that

$$t_i(a_{i,j}) = \min(a_{i-1,j-1}, a_{i+1,j}) + \max(a_{i-1,j}, a_{i+1,j+1}) - a_{i,j} \tag{13}$$

for $j \in [i+1, r-1]$. At the ends of the $(i+1)$st row, we must modify this slightly:

$$\begin{aligned} t_i(a_{i,i}) &= a_{i+1,i} + \max(a_{i-1,i}, a_{i+1,i+1}) - a_{i,i}, \\ t_i(a_{i,r}) &= \min(a_{i-1,r-1}, a_{i+1,r}) + a_{i-1,r} - a_{i,r} \end{aligned} \tag{14}$$

From these t_i, we can build a collection of involutions as follows.

Let q_0 be the identity map, and define recursively

$$q_i = t_1 t_2 \cdots t_i q_{i-1}. \tag{15}$$

In particular, q_r is the desired Schützenberger involution. (Note the operations t_i obviously have order 2. They do not satisfy the braid relations, so $t_i t_{i+1} t_i \neq t_{i+1} t_i t_{i+1}$. However they do satisfy $t_i t_j = t_j t_i$ if $|i-j| > 1$, which implies that q_i has order 2.)

The Schützenberger involution interchanges the two weights k_R and k_L – see (29) below. It would be nice if we could assert that $G_R(\mathfrak{T}) = G_L(q_r \mathfrak{T})$, and indeed, this is true if $\mathfrak{T}$ is in some sense "in general position." However, there are exceptions to this, and careful bookkeeping is required. What appears to be actually true is that

$$\sum_{k_R(\mathfrak{T}) = (k_1, \cdots, k_r)} G_R(\mathfrak{T}) = \sum_{k_R(\mathfrak{T}) = (k_1, \cdots, k_r)} G_L(q_r \mathfrak{T}),$$

and the thrust of the proof of our later Theorem 3.5 is to reduce this statement to a simpler combinatorial one using the involution and its components t_i. Although only strict patterns have a nonzero contribution to (10) and (11), the Schützenberger involution does not perserve the property of strictness. This is the reason that in (10) and (11) we defined the terms to be zero and summed over all patterns.

Gautam Chinta and Paul Gunnells have called our attention to the fact that the R_{ij} and L_{ij} defined in (5) are the numbers appearing in what we might call *Littelmann patterns*. Littelmann [**27**] associated a sequence of integers with the following data: first, a Gelfand-Tsetlin pattern (or, more generally, a vertex in a crystal graph); and second, a reduced word representing the long element in the Weyl group as a product of simple reflections. The sequence of integers is obtained, as Littelmann explains in the introduction and in Section 5 of [**27**], by applying rasing operators in an order determined by the reduced expression of the long Weyl group element to the vertices of the crystal graph, and tabulating the number of times each raising operator can be applied. For one reduced word these numbers, put into an array, coincide with our R_{ij} (later defined as a "Γ array" in (21)). For another reduced word, they coincide with our L_{ij} (later defined as a "Δ array" in (22)). The observation of Chinta and Gunnells regarding this connection will be of doubtless importance in the further development of this theory.

2.3. First Reduction of the FE Conjecture. The coefficients H_R defined in (10) agree with the coefficients H defined in [**6**], though the notation differs slightly. Their equivalence is easy to see if one bears in mind that, with notation as defined in this section, $\mathfrak{g}(p^a, p^b) = \mathfrak{h}(b)$ whenever $a \geqslant b$. The coefficients H_L defined in (11) are introduced here for the first time.

Conjecture 2.1. *With definitions as above, $H_R = H_L$.*

Before exploring the proof of this conjecture in subsequent sections, we note an important consequence.

Theorem 2.2. *Conjecture 2.1 implies the FE conjecture (Conjecture 1 in* [**6**]*).*

Proof. (Sketch) We will not make use of Conjecture 2.1 until quite late in the proof, and so will take Z_Ψ to be defined by the sum (1) with $H = H_R$ until further notice.

The functional equations that must be satisfied are formulated in [**4**]. In the case of A_r, the dependence on $m_1, \cdots, m_r$ can be made more explicit as follows. Let

$$\tilde{Z}_\Psi(\mathbf{s}; \mathbf{m}) = \tilde{Z}_\Psi(\mathbf{s}; \mathbf{m}; A_r) = \prod_{i,j} \mathbb{N}m_i^{\frac{2}{r+1} b_{ij} s_j} Z_\Psi(\mathbf{s}; \mathbf{m}),$$

where

$$b_{ij} = \begin{cases} i(r+1-j) & \text{if } i \leqslant j, \\ (r+1-i)j & \text{if } i \geqslant j, \end{cases}$$

and let $\tilde{Z}^*$ be $\tilde{Z}$ multiplied by certain Gamma factors, and a product of $\frac{1}{2}r(r+1)$ Dedekind zeta functions, which are given explicitly in [**3**]–[**5**].

Let $\mathcal{A}$ be the ring of (Dirichlet) polynomials in $q_v^{\pm 2s_1}, \dots, q_v^{\pm 2s_r}$ where v runs through the finite set of nonarchimedean places in S, and q_v denotes the cardinality of the residue field. Let $\mathfrak{M} = \mathcal{A} \otimes \mathcal{M}$. As in [**4**] we regard elements of $\mathfrak{M}$ as functions $\Psi : \mathbb{C}^r \times (F_S^\times)^r \longrightarrow \mathbb{C}$ such that for any fixed $(s_1, \dots, s_r) \in \mathbb{C}^r$ the function

$$(C_1, C_2, \dots, C_r) \longmapsto \Psi(s_1, \dots, s_r; C_1, \dots, C_r)$$

defines an element of $\mathcal{M}$, while for any $(C_1, \dots, C_r) \in (F_S^\times)^r$, the function

$$\mathbf{s} = (s_1, \dots, s_r) \longmapsto \Psi(s_1, \dots, s_r; C_1, \dots, C_r)$$

is an element of $\mathcal{A}$. Now the functional equation should have the following form: there is an action of the Weyl group $W \cong S_{r+1}$ on $\mathfrak{M}$ such that

$$\tilde{Z}^*_{w\Psi}(w\mathbf{s};\mathbf{m}) = \tilde{Z}^*_{\Psi}(\mathbf{s};\mathbf{m}).$$

As noted in [**4**] and [**5**], it suffices to prove one functional equation for each simple reflection in the Weyl group as these generate the entire group. Denoting the simple reflections by $\sigma_1, \cdots, \sigma_r$, we may argue inductively that the functional equations for A_{r-1} imply functional equations for $\sigma_2, \cdots, \sigma_r$, leaving us only to prove a functional equation for σ_1.

Indeed, the base case of this induction, $r = 1$, is handled by the paper [**2**], where a single functional equation is proved for the Dirichlet series with coefficients $H(c;m) = \mathfrak{g}(m,c)$ as defined in (3). One can easily confirm that the Gauss sum decomposes into prime-power pieces according to the twisted multiplicativity defined in [**5**] and [**6**] and that the resulting $H(p^k;p^l)$ for each prime p with $\mathrm{ord}_p(m) = l > 0$ is given by

$$G_R(\mathfrak{T}) = G_R\left(\left\{\begin{array}{ccc} l+1 & & 0 \\ & k & \end{array}\right\}\right) = \mathfrak{g}(p^l, p^k),$$

according to the recipe outlined in (10) above. Hence, the Weyl group multiple Dirichlet series for A_1 coincides with the so-called "Kubota" Dirichlet series defined in [**2**] and the functional equation follows.

We now sketch a proof of the induction step. Note that

$$H(c_1,\cdots,c_r;m_1,\cdots,m_r) = \mu_{\mathbf{c},\mathbf{m}} \prod_p H(p^{k_1(p)},\cdots,p^{k_r(p)};p^{l_1(p)},\cdots,p^{l_r(p)}), \tag{16}$$

where $k_i(p) = \mathrm{ord}_p(c_i)$ and $l_i(p) = \mathrm{ord}_p(m_i)$, and $\mu_{\mathbf{c},\mathbf{m}}$ denotes a certain n-th root of unity depending on $\mathbf{c} = (c_1, c_2, \ldots, c_r)$ and $\mathbf{m}$. More specifically, it may be expressed in terms of the n-th power reciprocity law by the multiplicativity properties of H that are set out in [**3**], [**5**] and [**4**]. The product is essentially finite since unless p divides one of the c_i or m_i we have $k_i(p) = l_i(p) = 0$ and the corresponding factor equals 1 (as one may readily check from (6) and (8)).

Now using the definition $H = H_R$, the coefficient $H(p^{k_1},\cdots,p^{k_r};p^{l_1},\cdots,p^{l_r})$ is a sum of the $G_R(\mathfrak{T}_p;p)$ over Gelfand-Tsetlin patterns, with top row (12) and with the sum of the elements in the $i+1$-st row being k_i minus the sum of the last $r+1-i$ elements of the top row. Now if for each p we are given a Gelfand-Tsetlin pattern

$$\mathfrak{T}_p = \left\{\begin{array}{ccccccccc} a_{00}(p) & & a_{01}(p) & & a_{02}(p) & & \cdots & & a_{0r}(p) \\ & a_{11}(p) & & a_{12}(p) & & & & a_{1r}(p) & \\ & & \ddots & & & & \iddots & & \\ & & & & a_{rr}(p) & & & & \end{array}\right\}$$

we may consider the pattern

$$\boxed{\mathfrak{T}} = \left\{\begin{array}{ccccccccc} a_{00} & & a_{01} & & a_{02} & & \cdots & & a_{0r} \\ & a_{11} & & a_{12} & & & & a_{1r} & \\ & & \ddots & & & & \iddots & & \\ & & & & a_{rr} & & & & \end{array}\right\},$$

where

$$a_{ij} = \prod_p p^{a_{ij}(p)}.$$

This is a "Gelfand-Tsetlin pattern" in which the elements are not rational integers, but rather elements of $\mathfrak{o}_S$, and the inequalities defining a Gelfand-Tsetlin pattern have been replaced by divisibility conditions, namely $a_{i-1,j}|a_{i,j}|a_{i-1,j-1}$. We call $\boxed{\mathfrak{T}}$ a *global Gelfand-Tsetlin pattern*. Now denote

$$G_R\left(\boxed{\mathfrak{T}}\right) = \mu_{\mathbf{c},\mathbf{m}} \prod_p G_R(\mathfrak{T}_p),$$

where $\mu_{\mathbf{c},\mathbf{m}}$ is as in (16). We see that we may write $H_R(\mathbf{c};\mathbf{m})$ as a sum of $G_R\left(\boxed{\mathfrak{T}}\right)$, where now the sum is over global Gelfand-Tsetlin patterns. As explained in Remark 1.1 we are only considering global patterns in which the elements are products of powers of a fixed set of generators of the primes; if we considered $H\Psi$ instead of H, we could sum over ideals.

Now $\tilde{Z}^*_\Psi$ is a sum over global patterns with fixed top row depending on $\mathbf{m} = (m_1, \cdots, m_r)$. We may break this up as follows. Let us fix the top *two* rows and consider the resulting sum. If the second row of the global pattern is

$$\prod_p p^{d_1(p)+\cdots+d_r(p)+r-1}, \prod_p p^{d_2(p)+\cdots+d_r(p)+r-2}, \cdots, \prod_p p^{d_r(p)}$$

then let

$$m_i' = \prod_p p^{d_i(p)}, \qquad (i = 1, \cdots, r-1).$$

One may check that the sum over the patterns with fixed top two rows is a Weyl group multiple Dirichlet series attached to A_{r-1} of the form

$$\tilde{Z}^*_{\Psi'}(s_2, \cdots, s_r; m_1', \cdots, m_{r-1}'; A_{r-1})$$

times a product of Gamma functions and Dedekind zeta functions, and a product of powers of the norms of the m_i that is invariant under the transformations $\sigma_2, \cdots, \sigma_r$. Thus one obtains functional equations for this subset of Weyl group generators for $\tilde{Z}^*_\Psi$.

Now we may explain at last why Conjecture 2.1 implies the FE Conjecture. If one defines Z_Ψ in terms of H_L instead of H_R, the argument proceeds as above, but we obtain instead the functional equations for $\sigma_1, \cdots, \sigma_{r-1}$. Combining these with the functional equations for the simple reflections that we had before, one obtains functional equations with respect to the full Weyl group. □

3. A second reduction to a combinatorial conjecture

Let q be either a complex number or an indeterminate, and let R be a $\mathbb{Z}[q]$-algebra with generators $h(a)$ and $g(a)$, one for each positive integer a, subject to the following relations.

Relation (i). Suppose that $a, b > 0$. Then

$$h(b)h(a+b) = q^b h(b)h(a), \qquad h(b)g(a+b) = q^b h(b)g(a).$$

Relation (ii). Suppose that $a, b \geqslant 0$ and $a + b > 0$. Then

$$g(a+b)h(a)h(b) = h(a+b)g(a)g(b) + h(a+b)g(a+b). \tag{17}$$

Relation (iii). If $a > 0$ we have

$$h(a)^2 = g(a)h(a) + q^a h(a). \tag{18}$$

We also define $h(0) = 1$, but $g(0)$ will never appear.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be as in the previous section. We reiterate that $\mathfrak{g}(a)$ is only defined if $a > 0$. Let q be the cardinality of $\mathfrak{o}/p\mathfrak{o}$.

LEMMA 3.1. *If $n|a$ and $a > 0$ then*

$$\mathfrak{h}(a) = \phi(p^a) = q^{a-1}(q-1), \qquad \mathfrak{g}(a) = -q^{a-1},$$

while if $n \nmid a$ then $\mathfrak{h}(a) = 0$ and $|\mathfrak{g}(a)| = q^{a-\frac{1}{2}}$. If $n \nmid a, b$ but $n|a+b$ then

$$\mathfrak{g}(a)\mathfrak{g}(b) = q^{a+b-1}.$$

If $a \equiv b$ modulo n with $a, b > 0$, then $\mathfrak{h}(a) = q^{a-b}\mathfrak{h}(b)$ and $\mathfrak{g}(a) = q^{a-b}\mathfrak{g}(b)$.

These facts about Gauss sums are standard.

The Lemma implies that there is an algebra homomorphism from the algebra R to the $\mathbb{Z}[q]$-algebra generated by $\mathfrak{g}(a)$ and $\mathfrak{h}(a)$ for $a > 0$, for if one replaces $g(a)$ by $\mathfrak{g}(a)$ and $h(a)$ by $\mathfrak{h}(a)$ then Relations (i), (ii) and (iii) are still satisfied. The Relations can be proved on a case-by-case basis depending on the divisibility properties of a, b and $a+b$. Relation (i) follows since if $\mathfrak{h}(b) = 0$, both sides of both equations vanish; otherwise $n|b$, and so Relation (i) follows from the last statement of Lemma 3.1. Both sides of (18) vanish unless $n|a$ since $\mathfrak{h}(a)$ appears in every term; and if $n|a$, then by Lemma 3.1 we have $\mathfrak{h}(a) = \mathfrak{g}(a) + q^a$, so (iii) follows. As for (ii), equation (17) is harder than (18) but also follows from Lemma 3.1.

REMARK 3.2. When one begins to investigate in this area, one encounters a bewildering array of identities. One could base their combinatorial study on either Lemma 3.1 or on Relations (i), (ii) and (iii). There are two advantages of the latter approach. First, it avoids a descent into case-by-case considerations depending on the divisibility of various parameters by n. A second very important advantage is that it allows one to work in R; as we will illustrate later the relations that define R are precisely those needed to prove the combinatorial Conjecture 3.4 below in the rank one case, and it is our experience that no further identities will be needed in higher rank. There are also advantages to working directly with $\mathfrak{g}$ and $\mathfrak{h}$, and we may do so in subsequent papers, but to clarify the issues we formalize the relations in this one.

By a *short Gelfand-Tsetlin pattern* (or *short pattern*) we mean an array

$$\mathfrak{t} = \left\{ \begin{array}{ccccccccc} l_1 & & l_2 & & l_3 & & \cdots & & l_{r+1} \\ & a_1 & & a_2 & & & & a_r & \\ & & b_1 & & \cdots & & b_{r-1} & & \end{array} \right\}, \tag{19}$$

where the rows are nonincreasing sequences of integers that interleave, that is,

$$l_i \geqslant a_i \geqslant l_{i+1}, \qquad a_i \geqslant b_i \geqslant a_{i+1}. \tag{20}$$

We will refer to $l_1, \cdots, l_{r+1}$ as the *top row* of $\mathfrak{t}$, $a_1, \cdots, a_r$ as the *middle row* and $b_1, \cdots, b_{r-1}$ as the *bottom row*.

Our aim is to define two R-valued functions G_Γ and G_Δ on the set of short Gelfand-Tsetlin patterns. Let us assume momentarily that $\mathfrak{t}$ is strict. We will associate with $\mathfrak{t}$ two arrays

$$\Gamma = \Gamma(\mathfrak{t}) = \left\{ \begin{array}{ccccccc} \Gamma_{1,1} & & \Gamma_{1,2} & & & & \Gamma_{1,r} \\ & \Gamma_{2,1} & & \cdots & & \Gamma_{2,r-1} & \end{array} \right\}$$

and

$$\Delta = \Delta(\mathfrak{t}) = \left\{ \begin{array}{ccccccc} \Delta_{1,1} & & \Delta_{1,2} & & & & \Delta_{1,r} \\ & \Delta_{2,1} & & \cdots & & \Delta_{2,r-1} & \end{array} \right\},$$

where $\Gamma_{i,j} = \Gamma_{i,j}(\mathfrak{t})$ and $\Delta_{i,j} = \Delta_{i,j}(\mathfrak{t})$ are to be defined. If $\mathfrak{t}$ is as in (19) then

$$\Gamma_{1,j} = \Gamma_{1,j}(\mathfrak{t}) = \sum_{k=j}^{r}(a_k - l_{k+1}), \qquad \Gamma_{2,j} = \Gamma_{2,j}(\mathfrak{t}) = \sum_{k=1}^{j}(a_k - b_k), \tag{21}$$

and

$$\Delta_{1,j} = \Delta_{1,j}(\mathfrak{t}) = \sum_{k=1}^{j}(l_k - a_k), \qquad \Delta_{2,j} = \Delta_{2,j}(\mathfrak{t}) = \sum_{k=j}^{r-1}(b_k - a_{k+1}). \tag{22}$$

Let

$$\tilde{\Gamma}_{1,j} = \tilde{\Gamma}_{1,j}(\mathfrak{t}) = \begin{cases} q^{\Gamma_{1,j}} & \text{if } a_j = l_{j+1}; \\ g(\Gamma_{i,j}) & \text{if } a_j = l_j; \\ h(\Gamma_{1,j}) & \text{otherwise,} \end{cases} \qquad \tilde{\Gamma}_{2,j} = \tilde{\Gamma}_{2,j}(\mathfrak{t}) = \begin{cases} q^{\Gamma_{2,j}} & \text{if } b_j = a_j; \\ g(\Gamma_{2,j}) & \text{if } b_j = a_{j+1}; \\ h(\Gamma_{2,j}) & \text{otherwise,} \end{cases}$$

and also let

$$\tilde{\Delta}_{1,j} = \begin{cases} q^{\Delta_{1,j}} & \text{if } a_j = l_j; \\ g(\Delta_{1,j}) & \text{if } a_j = l_{j+1}; \\ h(\Delta_{1,j}) & \text{otherwise,} \end{cases} \qquad \tilde{\Delta}_{2,j} = \begin{cases} q^{\Delta_{2,j}} & \text{if } b_j = a_{j+1} \\ g(\Delta_{2,j}) & \text{if } b_j = a_j; \\ h(\Delta_{2,j}) & \text{otherwise.} \end{cases}$$

Now define

$$G_\Gamma(\mathfrak{t}) = \prod_{i,j} \tilde{\Gamma}_{i,j}(\mathfrak{t}), \qquad G_\Delta(\mathfrak{t}) = \prod_{i,j} \tilde{\Delta}_{i,j}(\mathfrak{t}). \tag{23}$$

The above definition assumed that $\mathfrak{t}$ is strict. If $\mathfrak{t}$ is not strict we define $G_\Gamma(\mathfrak{t}) = G_\Delta(\mathfrak{t}) = 0$.

Remark 3.3. Observe the following important difference between how we define the products G_R and G_L for ordinary ($r+1$-rowed) Gelfand-Tsetlin patterns and the products G_Γ and G_Δ for short patterns. Referring to (5), if $\mathfrak{T}$ is an ordinary Gelfand-Tsetlin pattern, we use a "right-leaning rule" to define $R_{i,j}$ and $G_R(R_{i,j})$ in every row; similarly we use a "left-leaning rule" to define the L array. (The reader will understand the meaning of "left-leaning" and "right-leaning" in this context after computing an example.) In contrast, for the short pattern, the Γ array is obtained by using a right-leaning rule in the middle row and a left-leaning rule in the bottom row, while the Δ array is obtained using a left-leaning rule in the middle row and a right-leaning rule in the bottom row.

We define the *weight* k of $\mathfrak{t}$ to be the sum of the a_i. Finally, if $\mathfrak{t}$ is a short pattern we define another short pattern $\mathfrak{t}'$ as the image of $\mathfrak{t}$ under t_1 defined in (13) and (14). More explicitly,

$$\mathfrak{t}' = \left\{ \begin{array}{ccccccccc} l_1 & & l_2 & & l_3 & & \cdots & & l_{r+1} \\ & a_1' & & a_2' & & & & a_r' & \\ & & b_1 & & \cdots & & b_{r-1} & & \end{array} \right\}, \tag{24}$$

where

$$a_i' = \min(l_i, b_{i-1}) + \max(l_{i+1}, b_i) - a_i, \qquad (i = 2, \cdots, r-1), \tag{25}$$

$$a_1' = l_1 + \max(l_2, b_1) - a_1, \qquad a_r' = \min(l_r, b_{r-1}) + l_{r+1} - a_r. \tag{26}$$

By a short pattern *type* $\mathfrak{S}$ of rank r we mean a triple $(\mathbf{l}, \mathbf{b}, k)$ specifying the following data: a top row consisting of an integer sequence $\mathbf{l} = (l_1, \cdots, l_{r+1})$, a bottom row consisting of a sequence $\mathbf{b} = (b_1, \cdots, b_{r-1})$, and a positive integer k. It is assumed that $l_1 > l_2 > \ldots > l_{r+1}$, that $b_1 > b_2 > \ldots > b_{r-1}$, that $l_i > b_i > l_{i+2}$, and that $\sum l_i > k > \sum b_i$.

We say that a short pattern $\mathfrak{t}$ of rank r *belongs to the type* $\mathfrak{S}$ if it has the prescribed top and bottom rows, and its weight is k (so $\sum_i a_i = k$). By abuse of notation, we will use the notation $\mathfrak{t} \in \mathfrak{S}$ to mean that $\mathfrak{t}$ belongs to the type $\mathfrak{S}$.

Conjecture 3.4. *We have*

$$\sum_{\mathfrak{t}\in\mathfrak{S}} G_\Gamma(\mathfrak{t}) = \sum_{\mathfrak{t}\in\mathfrak{S}} G_\Delta(\mathfrak{t}'). \tag{27}$$

Theorem 3.5. *Conjecture 3.4 implies Conjecture 2.1.*

Hence by Theorem 2.2 we see that Conjecture 3.4 implies the FE Conjecture.

Proof. We begin by observing that the operations $t_1, \cdots, t_r$ on Gelfand-Tsetlin patterns that were previously defined in (13) and (14) of our discussion of the Schützenberger involution boil down to the operation $\mathfrak{t} \longmapsto \mathfrak{t}'$ on short Gelfand-Tsetlin patterns. Indeed, if $1 \leqslant k \leqslant r$, we extract from the Gelfand-Tsetlin pattern $\mathfrak{T}$ given by (2) the short pattern made with the $(r-k)$-, $(r+1-k)$- and $(r+2-k)$-th rows of $\mathfrak{T}$. We apply the operation $\mathfrak{t} \longmapsto \mathfrak{t}'$ to this short pattern and reinsert it into $\mathfrak{T}$; the resulting pattern we call $t_k\mathfrak{T}$. Note that only the $(r+1-k)$-th row is changed by this procedure. If $k = 1$, the $(r+1)$-th row of $\mathfrak{T}$ is empty, so t_1 should be interpreted as just replacing a_{rr} by $a_{r-1,r-1} + a_{r-1,r} - a_{r,r}$. We have $t_1 \cdots t_{i-1} = q_{i-1}q_{i-2}^{-1} = q_{i-1}q_{i-2}$ and therefore

$$q_i = q_{i-1}q_{i-2}t_iq_{i-1}. \tag{28}$$

Next let us consider the effect of the Schützenberger involution on the weights. Let $A_0 = \sum_j a_{i,j}$ be the sum of the i-th row of $\mathfrak{T}$. It may be checked that the row sums of $q_r\mathfrak{T}$ are (in order)

$$A_0, A_0 - A_r, A_0 - A_{r-1}, \cdots, A_0 - A_1.$$

From this it may easily be deduced that

$$(q_r\mathfrak{T}) = k_L(\mathfrak{T}). \tag{29}$$

From this we see that Conjecture 2.1 will follow if we prove

$$\sum_{k_R(\mathfrak{T})=\mathbf{k}} G_R(\mathfrak{T}) = \sum_{k_R(\mathfrak{T})=\mathbf{k}} G_L(q_r\mathfrak{T}). \tag{30}$$

We note that the sum is over all patterns with fixed *top row* and *row sums*. The proof of will involve an induction on r, and so we introduce the temporary notations ${}^{(r)}G_R(\mathfrak{T}) = G_R(\mathfrak{T})$ and ${}^{(r)}G_L(\mathfrak{T}) = G_L(\mathfrak{T})$.

Note that the sum in (29) is over all patterns with fixed *top row* and *row sums*. The proof of will involve an induction on r, and so we introduce the temporary notations ${}^{(r)}G_R(\mathfrak{T}) = G_R(\mathfrak{T})$ and ${}^{(r)}G_L(\mathfrak{T}) = G_L(\mathfrak{T})$.

If we discard the top row of $\mathfrak{T}$, we obtain a Gelfand-Tsetlin pattern $\mathfrak{T}_{r-1}$ of rank $r-1$; similarly let $\mathfrak{T}_{r-2}$ be the pattern obtained by discarding the top two rows. We will denote

$$G_R^i(\mathfrak{T}) = \prod_{j=i}^{r} G_R(R_{i,j})(\mathfrak{T}), \qquad G_L^i(\mathfrak{T}) = \prod_{j=i}^{r} G_L(L_{i,j})(\mathfrak{T}).$$

Let us show that if $\mathbf{k} = (k_1, \cdots, k_r)$ is a fixed r-tuple then

$$\sum_{k_R(\mathfrak{T})=\mathbf{k}} G_R(q_{r-1}\mathfrak{T}) = \sum_{k_R(\mathfrak{T})=\mathbf{k}} G_R^r(\mathfrak{T}) G_L^{r-1}(\mathfrak{T}) \prod_{i=1}^{r-2} G_L^i(\mathfrak{T}) \tag{31}$$

and

$$\sum_{k_R(\mathfrak{T})=\mathbf{k}} G_L(q_{r-1}q_{r-2}\mathfrak{T}) = \sum_{k_R(\mathfrak{T})=\mathbf{k}} G_L^r(\mathfrak{T}) G_R^{r-1}(\mathfrak{T}) \prod_{i=1}^{r-2} G_L^i(\mathfrak{T}). \tag{32}$$

To prove (31), the left-hand side equals

$$\sum_{k_R(\mathfrak{T})=\mathbf{k}} G_R^r(q_{r-1}\mathfrak{T}) \prod_{i=1}^{r-1} G_R^i(q_{r-1}\mathfrak{T}).$$

We have $G_R^r(q_{r-1}\mathfrak{T}) = G_R^r(\mathfrak{T})$, because q_{r-1} does not affect the top two rows of $\mathfrak{T}$. Also, let $\mathfrak{T}_{r-1}$ be the pattern of rank $r-1$ obtained by discarding the top row of $\mathfrak{T}$. Then we obtain

$$\begin{aligned} \sum_{k_R(\mathfrak{T})=\mathbf{k}} G_R(q_{r-1}\mathfrak{T}) &= \\ \sum_{k_R(\mathfrak{T})=\mathbf{k}} G_R^r(\mathfrak{T}) \cdot {}^{(r-1)}G_R(q_{r-1}\mathfrak{T}_{r-1}) &= \sum_{k_R(\mathfrak{T})=\mathbf{k}} G_R^r(\mathfrak{T}) \cdot {}^{(r-1)}G_L(\mathfrak{T}_{r-1}), \end{aligned}$$

where we have used our induction hypothesis, and this is the right-hand side of (31). This step must be understood as follows: fix the top row of $\mathfrak{T}_{r-1}$ (that is, the top two rows) and then sum over all remaining rows, with the row sums fixed (depending on $\mathbf{k}$). In this summation $G_R^r(\mathfrak{T})$ is constant and may be pulled out of the inner summation (over all rows but the top). The inner sum may be treated using (30) with r replaced by $r-1$. This proves (31).

Next we note that $q_{r-1}q_{r-2}$ does not affect the top two rows of $\mathfrak{T}$, so $G_L^i(q_{r-1}q_{r-2}\mathfrak{T}) = G_L^i(\mathfrak{T})$ when $i = r$. Thus the left-hand side of (32) equals

$$\sum_{k_R(\mathfrak{T})=\mathbf{k}} G_L^r(q_{r-1}q_{r-2}\mathfrak{T}) \cdot {}^{(r-1)}G_L(q_{r-1}q_{r-2}\mathfrak{T}_{r-1}) = \sum_{k_R(\mathfrak{T})=\mathbf{k}} G_L^r(\mathfrak{T}) \cdot {}^{(r-1)}G_R(q_{r-2}\mathfrak{T}_{r-1}),$$

where we have used the induction hypothesis. Let $\mathfrak{T}_{r-2}$ be the pattern obtained by omitting the top two rows of $\mathfrak{T}$; we see that the left-hand side of (32) equals

$$\sum_{k_R(\mathfrak{T})=\mathbf{k}} G_L^r(\mathfrak{T}) G_R^{r-1}(q_{r-2}\mathfrak{T}) \cdot {}^{(r-2)}G_R(q_{r-2}\mathfrak{T}_{r-2}).$$

We note that q_{r-2} does not change the top three rows of $\mathfrak{T}$, and that by induction we have ${}^{(r-2)}G_R(q_{r-2}\mathfrak{T}_{r-2}) = {}^{(r-2)}G_L(\mathfrak{T}_{r-2})$. Equation (32) follows.

Now Conjecture 3.4 implies that

$$\sum_{k_R(\mathfrak{T})=\mathbf{k}} G_R^r(\mathfrak{T}) G_L^{r-1}(\mathfrak{T}) \prod_{i=1}^{r-2} G_L^i(\mathfrak{T}) = \sum_{k_R(\mathfrak{T})=\mathbf{k}} G_L^r(t_r\mathfrak{T}) G_R^{r-1}(t_r\mathfrak{T}) \prod_{i=1}^{r-2} G_L^i(\mathfrak{T}). \tag{33}$$

Indeed, let $\mathfrak{t}$ be the short pattern obtained by discarding all but the top three rows of $\mathfrak{T}$. Then

$$G_R^r(\mathfrak{T})G_L^{r-1}(\mathfrak{T}) = G_\Gamma(\mathfrak{t}), \qquad G_L^r(t_r\mathfrak{T})G_R^{r-1}(t_r\mathfrak{T}) = G_\Delta(\mathfrak{t}').$$

So we may fix all rows but the second of $\mathfrak{T}$, and sum over all patterns with that row allowed to vary, but with fixed row sum. In the inner summation $\prod_{i=1}^{r-2} G_L^i(\mathfrak{T})$ is constant, since it does not depend on the second row of $\mathfrak{T}$, so it may be pulled out of the inner summation. Applying Conjecture 3.4 to the inner sum proves (33).

Now (31), (32) and (33) show that

$$\sum_{k_R(\mathfrak{T})=\mathbf{k}} G_R(q_{r-1}\mathfrak{T}) = \sum_{k_R(\mathfrak{T})=\mathbf{k}} G_L(q_{r-1}q_{r-2}t_r\mathfrak{T}).$$

Replacing $\mathfrak{T}$ by $q_{r-1}\mathfrak{T}$ (and changing $\mathbf{k}$) gives

$$\sum_{k_R(\mathfrak{T})=\mathbf{k}} G_R(\mathfrak{T}) = \sum_{k_R(\mathfrak{T})=\mathbf{k}} G_L(q_{r-1}q_{r-2}t_rq_{r-1}\mathfrak{T}).$$

The theorem follows by (28). □

4. Gauss sum combinatorics

With the reduction to Conjecture 3.4 we have entered into a very rich combinatorial landscape. We will only mention a few features.

A short pattern $\mathfrak{t}$ is called *superstrict* if each defining inequality (20) is strict. We call $\mathfrak{t}$ *nonresonant* if $l_{i+1} \neq b_i$ for $1 \leqslant i \leqslant r-1$. Finally, we call $\mathfrak{t}$ *stable* if each element of the middle and bottom rows is equal to one of the two elements above it. (Note that a short pattern may satisfy more than one of these conditions at once.)

THEOREM 4.1. *If $\mathfrak{t}$ is superstrict, nonresonant or stable, then*

$$G_\Gamma(\mathfrak{t}) = G_\Delta(\mathfrak{t}'). \tag{34}$$

Thus in Conjecture 3.4 there is no need to sum over patterns of these three classes. The proof of Theorem 4.1 will be given in a subsequent paper. To give some feeling for the combinatorial nature of this situation, we discuss briefly the superstrict case.

Lemma. *There exists orderings of the $\Gamma_{ij}(\mathfrak{t})$ and $\Delta_{ij}(\mathfrak{t}')$ such that*

$$\{\Gamma_{ij}(\mathfrak{t})\} = \{\gamma_1, \gamma_2, \cdots, \gamma_{2r-1}\}, \qquad \{\Delta_{ij}(\mathfrak{t}')\} = \{\delta'_1, \delta'_2, \cdots, \delta'_{2r-1}\}$$

with the following property. Extend the labelings by letting $\gamma_0 = \gamma_{2r} = 0$. Then

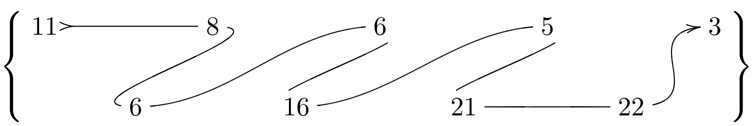
$$\delta'_k = \begin{cases} \gamma_k & \text{if } k \text{ is even,} \\ \gamma_k + \gamma_{k-1} - \gamma_{k+1} & \text{if } k \text{ is odd.} \end{cases} \tag{35}$$

The proof of this somewhat tricky combinatorial Lemma will be given elsewhere, but let us do an example. Suppose that

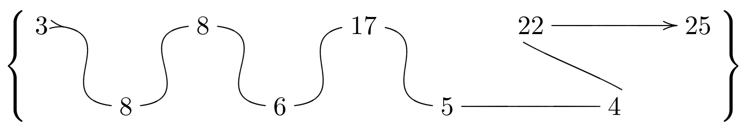
$$\mathfrak{t} = \left\{ \begin{array}{ccccccccccc} 45 & & 37 & & 28 & & 14 & & 5 & & 0 \\ & 40 & & 30 & & 15 & & 7 & & 3 & \\ & & 34 & & 20 & & 10 & & 6 & & \end{array} \right\},$$

$$\mathfrak{t}' = \left\{ \begin{array}{ccccccccccc} 45 & & 37 & & 28 & & 14 & & 5 & & 0 \\ & 42 & & 32 & & 19 & & 9 & & 2 & \\ & & 34 & & 20 & & 10 & & 6 & & \end{array} \right\}.$$

The arrays $\Gamma(\mathfrak{t})$ and $\Delta(\mathfrak{t}')$ are as follows:

$\Gamma(\mathfrak{t}) =$ 

,

$\Delta(\mathfrak{t}') =$ 

.

We have indicated the ordering of the entries with a pair of "snakes." If we compute the ordered sets Γ_{ij} and Δ_{ij} defined above, we have:

k	0	1	2	3	4	5	6	7	8	9	10
γ_k	0	11	8	6	6	16	5	21	22	3	0
δ_k		3	8	8	6	17	5	4	22	25	

we see that the Lemma is satisfied. We note that the snakes depend on the original data, and we will give a recipe for finding them in a later paper, where the proof of this "Snake Lemma" is given. Assuming the lemma and proceeding with the proof, we now assume that $\mathfrak{t}$ is superstrict. We have

$$G_\Gamma(\mathfrak{t}) = \prod_i h(\gamma_i), \qquad G_\Delta(\mathfrak{t}') = \prod_i h(\delta_i).$$

Assuming the Lemma, we rewrite the latter product

$$\prod_{i \text{ even}} h(\gamma_i) \prod_{i \text{ odd}} h(\gamma_i + \gamma_{i-1} - \gamma_{i+1}) = \prod_{i \text{ even}} h(\gamma_i) \prod_{i \text{ odd}} h(\gamma_i) q^{\gamma_{i-1} - \gamma_{i+1}},$$

where we have used Relation (i). The powers of q cancel, so (34) is satisfied. This shows that the Lemma implies Theorem 4.1 in the superstrict case. The nonresonant and stable cases require other ideas.

Although the patterns handled in Theorem 4.1 are in some sense most, still (34) is not true for all patterns. For example, it may be that $\mathfrak{t}$ is non-strict while $\mathfrak{t}'$ is strict. Then $G_\Gamma(\mathfrak{t}) = 0$ but $G_\Delta(\mathfrak{t}') \neq 0$. Thus (27) is not true without the summation.

Resonance is a phenomenon that occurs when, in the notation (19), we have $l_{i+1} = b_i$ for one or several values of i. The terminology is suggested by quantum chemistry, where some compounds such as benzene can occur with two or more structures contributing to the wave function. In the context of small Gelfand-Tsetlin patterns, patterns associated with a resonance can generally be grouped into fairly small sets called *packets* such that if Π is a packet then all members of Π belong to the same type, and

$$\sum_{\mathfrak{t} \in \Pi} G_\Gamma(\mathfrak{t}) = \sum_{\mathfrak{t} \in \Pi} G_\Delta(\mathfrak{t}'). \tag{36}$$

The first example that occurs is with A_2. We consider (short) patterns in the resonant type

$$\left\{ \begin{array}{ccccc} l_1 & & l_2 & & l_3 \\ & a_1 & & a_2 & \\ & & l_2 & & \end{array} \right\},$$

with a fixed row sum $a_1 + a_2 = k$. Among these patterns there will be two extremal ones, one in which a_1 is as large as possible (so either $a_1 = l_1$ or $a_2 = l_3$ or both), and one in which a_1 is as small as possible (so a_2 equals a_1 or a_2 or both).

Proposition 4.2. *These two extremal patterns form a packet.*

Proof. (Sketch) To prove this, there are several cases, and we take a typical one. Suppose that the extremal patterns are

$$\mathfrak{t}_1 = \left\{ \begin{array}{ccccc} l_1 & & l_2 & & l_3 \\ & l_1 & & x & \\ & & l_2 & & \end{array} \right\}, \qquad \mathfrak{t}_2 = \left\{ \begin{array}{ccccc} l_1 & & l_2 & & l_3 \\ & l_2 & & y & \\ & & l_2 & & \end{array} \right\},$$

where $x + l_1 = y + l_2$ and $l_2 > x > l_3$, $l_2 > y > l_3$. Then

$$\mathfrak{t}_1' = \left\{ \begin{array}{ccccc} l_1 & & l_2 & & l_3 \\ & l_2 & & l_2 + l_3 - x & \\ & & l_2 & & \end{array} \right\}, \qquad \mathfrak{t}_2' = \left\{ \begin{array}{ccccc} l_1 & & l_2 & & l_3 \\ & l_1 & & l_2 + l_3 - y & \\ & & l_2 & & \end{array} \right\}.$$

We find that

$$G_\Gamma(\mathfrak{t}_1) = \left\{ \begin{array}{ccc} \boxed{c} & & a \\ & b & \end{array} \right\}, \qquad G_\Gamma(\mathfrak{t}_2) = \left\{ \begin{array}{ccc} \boxed{c} & & c \\ & \textcircled{0} & \end{array} \right\},$$

$$G_\Gamma(\mathfrak{t}_1') = \left\{ \begin{array}{ccc} \boxed{b} & & c \\ & \boxed{a} & \end{array} \right\}, \qquad G_\Delta(\mathfrak{t}_2') = \left\{ \begin{array}{ccc} \textcircled{0} & & c \\ & c & \end{array} \right\},$$

where $a = x - l_3$, $b = l_1 - l_2$, $c = a + b = y - l_3 = l_1 + x - l_2 - l_3$, and our convention is that we circle an entry w if it contributes q^w; box it if it contributes $g(w)$, and leave it unboxed and uncircled if it contributes $h(w)$ in the definitions (23). Thus if $\Pi = \{\mathfrak{t}_1, \mathfrak{t}_2\}$, then (36) boils down to the identity

$$g(c)h(a)h(b) + g(c)h(c) = g(b)h(c)g(a) + h(c)^2, \qquad c = a + b,$$

which the reader will easily deduce from our relations. Even for A_2 there are several more cases, and the reader may treat them as an exercise. □

For higher resonances, in which $l_{i+1} = b_i$ for several consecutive i, many fascinating phenomena occur. We will give just one example, cautioning the reader that in some ways it is not entirely typical. We now consider short patterns in the resonant type

$$\mathfrak{t} = \left\{ \begin{array}{ccccccc} l_1 & & l_2 & & l_3 & & l_4 \\ & x & & y & & z & \\ & & l_2 & & l_3 & & \end{array} \right\}$$

where $l_1 \geqslant l_2 \geqslant l_3 \geqslant l_4$ and $x + y + z = k$.

We will assume first that

$$2l_2 > l_1 + l_3, \qquad \max(l_1 + l_2 + l_4, l_1 + 2l_3) < k < 2l_2 + l_3.$$

For example, one could take the top row to be $(63, 47, 25, 0)$ and the row sum to be 117. Let $u = k - l_2 - l_3 - l_4 = \alpha_1 + \alpha_2 = \beta_1 + \beta_2 = \gamma_1 + \gamma_2 = \delta_1 + \delta_2$, where

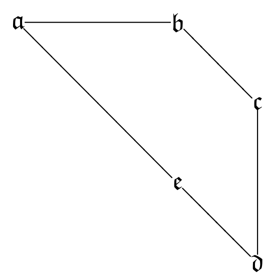
$$\alpha_1 = l_1 - l_2, \qquad \beta_1 = l_1 - l_3, \qquad \gamma_1 = l_2 - l_3, \qquad \delta_1 = l_3 - l_4.$$

We will also denote

$$g_\alpha = g(\alpha_1)g(\alpha_2), \qquad h_\alpha = h(\alpha_1)h(\alpha_2),$$

and similarly define $g_\beta, g_\gamma, g_\delta$ and h_β, h_γ, h_δ.

It is convenient to visualize the patterns in a type by mapping them into $\mathbb{Z}^r$ by means of the second row. Since the row sum is fixed, one element is redundant, and we may vizualize the type as the set of lattice points in some polytope. In this case with $r = 3$, we embed the patterns into $\mathbb{Z}^2$ by mapping this $\mathfrak{t}$ to (y, z). Assuming (4) the patterns form a trapezoid, which we can diagram thus:



The labeled spots are the following five short patterns:

$\mathfrak{t}$	$G_\Gamma(\mathfrak{t})$	$G_\Delta(\mathfrak{t}')$
$\mathfrak{a} = \left\{\begin{array}{ccccccc} l_1 & & l_2 & & l_3 & & l_4 \\ & l_1 & & l_2 & & k-l_1-l_2 & \\ & & l_2 & & l_3 & & \end{array}\right\}$	$g(u)g_\alpha h_\beta$	$h(u)g_\alpha g_\beta$
$\mathfrak{b} = \left\{\begin{array}{ccccccc} l_1 & & l_2 & & l_3 & & l_4 \\ & l_2 & & l_2 & & k-2l_2 & \\ & & l_2 & & l_3 & & \end{array}\right\}$	0	$h(u)^2 g_\gamma$
$\mathfrak{c} = \left\{\begin{array}{ccccccc} l_1 & & l_2 & & l_3 & & l_4 \\ & l_2 & & k-l_2-l_3 & & l_3 & \\ & & l_2 & & l_3 & & \end{array}\right\}$	$q^u h(u) g_\delta$	$g(u)h(u)h_\delta$
$\mathfrak{d} = \left\{\begin{array}{ccccccc} l_1 & & l_2 & & l_3 & & l_4 \\ & l_1 & & k-l_1-l_3 & & l_3 & \\ & & l_2 & & l_3 & & \end{array}\right\}$	$g(u)h_\alpha g_\delta$	$g(u)g_\alpha h_\delta$
$\mathfrak{e} = \left\{\begin{array}{ccccccc} l_1 & & l_2 & & l_3 & & l_4 \\ & l_1 & & 2l_2-l_1 & & k-2l_2 & \\ & & l_2 & & l_3 & & \end{array}\right\}$	$g(u)h_\alpha h_\gamma$	$h(u)g_\alpha h_\gamma$

Any pattern in the interior of the trapezoid is superstrict, hence consists of a singleton packet. On the other hand, the patterns on the boundary must be grouped into packets of size 2, 3 and 5.

We claim that $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}\}$ is a packet. Indeed, from the above table this means

$$\begin{aligned} g(u)g_\alpha h_\beta + q^u h(u)g_\delta + g(u)h_\alpha g_\delta + g(u)h_\alpha h_\gamma &= \\ h(u)g_\alpha g_\beta + h(u)^2 g_\gamma + g(u)h(u)h_\delta + g(u)g_\alpha h_\delta + h(u)g_\alpha h_\gamma. \end{aligned}$$

By Relation (iii) it is enough to prove the vanishing of

$$\begin{array}{c} g(u)g_\alpha h_\beta + h(u)^2 g_\delta - h(u)g(u)g_\delta + g(u)h_\alpha g_\delta + g(u)h_\alpha h_\gamma \\ -h(u)g_\alpha g_\beta - h(u)^2 g_\gamma - g(u)h(u)h_\delta - g(u)g_\alpha h_\delta - h(u)g_\alpha h_\gamma . \end{array}$$

Indeed, this may be rewritten

$$\begin{array}{r} +(h_\gamma + g_\delta)[g(u)h_\alpha - h(u)g_\alpha - h(u)g(u)] \\ +h(u)[g(u)h_\gamma - h(u)g_\gamma - h(u)g(u)] \\ -(h(u) + g_\alpha)[g(u)h_\delta - h(u)g_\delta - h(u)g(u)] \\ +g_\alpha[g(u)h_\beta - h(u)g_\beta - h(u)g(u)], \end{array}$$

which vanishes by Relation (ii).

The patterns on the interior of the segments from $\mathfrak{c}$ to $\mathfrak{b}$ and from $\mathfrak{e}$ to $\mathfrak{d}$ are equal in number, and they can be grouped into packets of order two by combining two patterns that lie on the same vertical line. To see this, let $k - 2l_2 < a \leqslant l_3$. Let $u = k - l_2 - l_3 - l_4$ as before, and write $u = \alpha_1 + \alpha_2 = \varepsilon_1 + \varepsilon_2$, where $\alpha_1 = l_1 - l_2$ and $\varepsilon_1 = a - l_4$. As before we will denote $g_\alpha = g(\alpha_1)g(\alpha_2)$, $h_\alpha = h(\alpha_1)h(\alpha_2)$ and similarly for g_ε and h_ε. Consider the following two patterns.

$\mathfrak{t}$	$G_\Gamma(\mathfrak{t})$	$G_\Delta(\mathfrak{t})$
$\mathfrak{t}_{[\mathfrak{d},\mathfrak{e}]}(a) = \left\{\begin{array}{ccccccc} l_1 & & l_2 & & l_3 & & l_4 \\ & l_1 & & k-l_1-a & & a & \\ & & l_2 & & l_3 & & \end{array}\right\}$	$g(u)h_\alpha h_\varepsilon$	$h(u)h_\varepsilon g_\alpha$
$\mathfrak{t}_{[\mathfrak{c},\mathfrak{b}]}(a) = \left\{\begin{array}{ccccccc} l_1 & & l_2 & & l_3 & & l_4 \\ & l_2 & & k-l_2-a & & a & \\ & & l_2 & & l_3 & & \end{array}\right\}$	$q^u g(u)h_\varepsilon$	$h(u)^2 h_\varepsilon$

The notation indicates which segment each pattern lies in. It is straightforward to check that the relations imply

$$g(u)h_\alpha + q^u g(u) \quad = \quad h(u)g_\alpha + h(u)^2.$$

Thus $\{\mathfrak{t}_{[\mathfrak{d},\mathfrak{e}]}(a), \mathfrak{t}_{[\mathfrak{c},\mathfrak{b}]}(a)\}$ is a packet.

On the other hand, the three segments from $\mathfrak{a}$ to $\mathfrak{b}$, from $\mathfrak{a}$ to $\mathfrak{e}$ and from $\mathfrak{c}$ to $\mathfrak{d}$ each have the same number of patterns, and these can be grouped together in packets of order 3. To see this, denote again $u = k - l_2 - l_3 - l_4 = \alpha_1 + \alpha_2 = \varepsilon_1 + \varepsilon_2 = \theta_1 + \theta_2 = \delta_1 + \delta_2$ where $\alpha_1 = l_1 - l_2$, $\varepsilon_1 = a - l_4$, $\theta_1 = a + l_2 - l_3 - l_4$ and $\delta_1 = l_3 - l_4$. Consider the following three patterns, one from each of these segments.

$\mathfrak{t}$	$G_\Gamma(\mathfrak{t})$	$G_\Delta(\mathfrak{t}')$
$\mathfrak{t}_{[\mathfrak{a},\mathfrak{e}]}(a) = \left\{\begin{array}{ccccccc} l_1 & & l_2 & & l_3 & & l_4 \\ & l_1 & & k-l_1-a & & a & \\ & & l_2 & & l_3 & & \end{array}\right\}$	$g(u)h_\alpha h_\varepsilon$	$h(u)g_\alpha h_\varepsilon$
$\mathfrak{t}_{[\mathfrak{a},\mathfrak{b}]}(a) = \left\{\begin{array}{ccccccc} l_1 & & l_2 & & l_3 & & l_4 \\ & k-l_2-a & & l_2 & & a & \\ & & l_2 & & l_3 & & \end{array}\right\}$	$h(u)g_\theta h_\varepsilon$	$h(u)h_\theta g_\varepsilon$
$\mathfrak{t}_{[\mathfrak{d},\mathfrak{c}]}(a) = \left\{\begin{array}{ccccccc} l_1 & & l_2 & & l_3 & & l_4 \\ & k-l_2-a & & a+l_2-l_3 & & l_3 & \\ & & l_2 & & l_3 & & \end{array}\right\}$	$h(u)h_\theta g_\delta$	$g(u)h_\theta h_\delta$

To see that these three form a packet, one must prove

$$\begin{aligned} g(u)h_\alpha h_\varepsilon + h(u)g_\theta h_\varepsilon + h(u)h_\theta g_\delta &= \\ h(u)g_\alpha h_\varepsilon + h(u)h_\theta g_\varepsilon + g(u)h_\theta h_\delta. & \end{aligned}$$

This may be deduced from the relations as follows. The right-hand side minus the left-hand side equals

$$\begin{aligned} h_\varepsilon[g(u)h_\alpha - h(u)g_\alpha] + h(u)[g_\theta h_\varepsilon - h_\theta g_\varepsilon] + h_\theta[h(u)g_\delta - g(u)h_\delta] &= \\ h_\varepsilon h(u)g(u) + h(u)[g_\theta h_\varepsilon - h_\theta g_\varepsilon] - h(u)g(u)h_\theta &= \\ h_\varepsilon[h(u)g(u) + h(u)g_\theta] - h_\theta[h(u)g_\varepsilon + g(u)h_\theta] &= \\ g(u)h_\theta h_\varepsilon - g(u)h_\theta h_\varepsilon &= 0. \end{aligned}$$

The above considerations verify Conjecture 3.4 for this particular resonant type.

References

[1] Bernšteĭn, I. N. and Zelevinskiĭ, A. V. Induced representations of the group $GL(n)$ over a p-adic field. *Funkcional. Anal. i Priložen.*, 10(3):74–75, 1976.

[2] B. Brubaker and D. Bump. On Kubota's Dirichlet series. *J. Reine Angew. Math.*, 598:159–184, 2006.

[3] B. Brubaker, D. Bump, G. Chinta, S. Friedberg, and J. Hoffstein. Weyl group multiple Dirichlet series I. In *Multiple Dirichlet Series, Automorphic Forms, and Analytic Number Theory*, Proc. Sympos. Pure Math., 75. Amer. Math. Soc., Providence, R.I., 2006.

[4] B. Brubaker, D. Bump, and S. Friedberg. Weyl group multiple Dirichlet series: The stable twisted case. In *Eisenstein series and applications (Gan, Kudla and Tchinkel, eds.)*. Birkhäuser Progress in Mathematics 258 (in press).

[5] B. Brubaker, D. Bump, and S. Friedberg. Weyl group multiple Dirichlet series. II. The stable case. *Invent. Math.*, 165(2):325–355, 2006.

[6] B. Brubaker, D. Bump, S. Friedberg, and J. Hoffstein. Weyl group multiple Dirichlet series III: Eisenstein series and twisted unstable A_r. *Annals of Mathematics*, to appear.

[7] W. Casselman and J. Shalika. The unramified principal series of p-adic groups. II. The Whittaker function. *Compositio Math.*, 41(2):207–231, 1980.

[8] G. Chinta. Mean values of biquadratic zeta functions. *Invent. Math.*, 160(1):145–163, 2005.

[9] G. Chinta, S. Friedberg, and P. Gunnells. On the p-parts of quadratic Weyl group multiple Dirichlet series. *J. Reine Angew. Math.*, to appear.

[10] G. Chinta and P. Gunnells. Weyl group multiple Dirichlet series constructed from quadratic characters. *Invent. Math.*, 167(2), 2007.

[11] S. Gelbart. *Automorphic forms on adèle groups*. Princeton University Press, Princeton, N.J., 1975. Annals of Mathematics Studies, No. 83.

[12] S. Gelbart. *Weil's representation and the spectrum of the metaplectic group*. Springer-Verlag, Berlin, 1976. Lecture Notes in Mathematics, Vol. 530.

[13] S. Gelbart, R. Howe, and I. Piatetski-Shapiro. Uniqueness and existence of Whittaker models for the metaplectic group. *Israel J. Math.*, 34(1-2):21–37 (1980), 1979.

[14] S. Gelbart and H. Jacquet. A relation between automorphic forms on GL(2) and GL(3). *Proc. Nat. Acad. Sci. U.S.A.*, 73(10):3348–3350, 1976.

[15] S. Gelbart and H. Jacquet. A relation between automorphic representations of GL(2) and GL(3). *Ann. Sci. École Norm. Sup. (4)*, 11(4):471–542, 1978.

[16] S. Gelbart and I. Piatetski-Shapiro. Distinguished representations and modular forms of half-integral weight. *Invent. Math.*, 59(2):145–188, 1980.

[17] S. Gelbart and I. Piatetski-Shapiro. Automorphic L-functions of half-integral weight. *Proc. Nat. Acad. Sci. U.S.A.*, 75(4):1620–1623, 1978.

[18] S. Gelbart and I. Piatetski-Shapiro. On Shimura's correspondence for modular forms of half-integral weight. In *Automorphic forms, representation theory and arithmetic (Bombay, 1979)*, volume 10 of *Tata Inst. Fund. Res. Studies in Math.*, pages 1–39. Tata Inst. Fundamental Res., Bombay, 1981.

[19] S. Gelbart and I. Piatetski-Shapiro. Some remarks on metaplectic cusp forms and the correspondences of Shimura and Waldspurger. *Israel J. Math.*, 44(2):97–126, 1983.

[20] S. Gelbart and D. Soudry. On Whittaker models and the vanishing of Fourier coefficients of cusp forms. *Proc. Indian Acad. Sci. Math. Sci.*, 97(1-3):67–74 (1988), 1987.
[21] I. M. Gelfand and D. A. Kajdan. Representations of the group $\mathrm{GL}(n, K)$ where K is a local field. In *Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971)*, pages 95–118. Halsted, New York, 1975.
[22] R. Godement and H. Jacquet. *Zeta functions of simple algebras*. Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 260.
[23] H. Jacquet and R. Langlands. *Automorphic forms on GL*(2). Springer-Verlag, Berlin, 1970. Lecture Notes in Mathematics, Vol. 114.
[24] D. A. Kazhdan and S. J. Patterson. Metaplectic forms. *Inst. Hautes Études Sci. Publ. Math.*, (59):35–142, 1984.
[25] A. Kirillov and A. Berenstein. Groups generated by involutions, Gelfand-Tsetlin patterns and combinatorics of Young tableaux. *Algebra i Analiz*, 1:92–152, 1995.
[26] T. Kubota. Topological covering of SL(2) over a local field. *J. Math. Soc. Japan*, 19:114–121, 1967.
[27] P. Littelmann Cones, crystals, and patterns. *Transform. Groups*, 3:145–179, 1998.
[28] H. Matsumoto. Sur les sous-groupes arithmétiques des groupes semi-simples déployés. *Ann. Sci. École Norm. Sup. (4)*, 2:1–62, 1969.
[29] I. I. Pjateckij-Šapiro. Euler subgroups. In *Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971)*, pages 597–620. Halsted, New York, 1975.
[30] M.-P. Schützenberger. La correspondance de Robinson. In *Combinatoire et représentation du groupe symétrique (Actes Table Ronde CNRS, Univ. Louis-Pasteur Strasbourg, Strasbourg, 1976)*, pages 59–113. Lecture Notes in Math., Vol. 579. Springer, Berlin, 1977.
[31] J. Shalika. The multiplicity one theorem for GL_n. *Ann. of Math. (2)*, 100:171–193, 1974.
[32] G. Shimura. On modular forms of half integral weight. *Ann. of Math. (2)*, 97:440–481, 1973.
[33] G. Shimura. On the holomorphy of certain Dirichlet series. *Proc. London Math. Soc. (3)*, 31(1):79–98, 1975.
[34] T. Shintani. On an explicit formula for class-1 "Whittaker functions" on GL_n over P-adic fields. *Proc. Japan Acad.*, 52(4):180–182, 1976.
[35] T. Tokuyama. A generating function of strict Gelfand patterns and some formulas on characters of general linear groups. *J. Math. Soc. Japan*, 40(4):671–685, 1988.
[36] J.-L. Waldspurger. Sur les coefficients de Fourier des formes modulaires de poids demi-entier. *J. Math. Pures Appl. (9)*, 60(4):375–484, 1981.
[37] A. Weil. Sur certains groupes d'opérateurs unitaires. *Acta Math.*, 111:143–211, 1964.

Department of Mathematics, MIT, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, **email:** `brubaker@math.mit.edu`

Department of Mathematics, Stanford University, Stanford, CA 94305-2125, USA **email:** `bump@math.stanford.edu`

Department of Mathematics, Boston College, Chestnut Hill, MA 02467-3806, USA. **email:** `friedber@bc.edu`