Toeplitz Minors

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Abstract. *We give a new proof of the strong Szegö limit theorem estimating the determinants of Toeplitz matrices using symmetric function theory. We also obtain asymptotics for Toeplitz minors.*

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If $f(t) = \sum_{-\infty}^{\infty} d_n t^n$ is a function on the unit circle $\mathbb{T}$ in $\mathbb{C}$ then $D_{n-1}(f)$ will denote the Toeplitz determinant det $T_{n-1}(f)$, where $T_{n-1}(f)$ is the $n \times n$ Toeplitz matrix

$$T_{n-1}(f) = \begin{pmatrix} d_0 & d_1 & \cdots & d_{n-1} \\ d_{-1} & d_0 & \cdots & d_{n-2} \\ \vdots & & & \vdots \\ d_{-(n-1)} & d_{-(n-2)} & \cdots & d_0 \end{pmatrix}.$$

Szegö [Sz1] studied the eigenvalues of large Toeplitz matrices by computing the asymptotics of their determinants. The *strong Szegö limit theorem* asserts that if $\sigma : \mathbb{T} \to \mathbb{C}$ is of the form $\sigma(t) = \exp\left(\sum_{-\infty}^{\infty} c_n t^n\right)$ then (under certain hypotheses on σ)

$$D_{n-1}(\sigma) \sim \exp\left(nc_0 + \sum_{k=1}^{\infty} kc_k c_{-k} \right).$$

This result has many applications. Szegö proved it originally to answer a question of Onsager in statistical physics: the magnetization in the Ising model for an $n \times n$ toroidal grid can be represented as a Toeplitz determinant and Szegö's asymptotics allow the first rigorous proof of a phase transition. Böttcher and Silbermann [BS] give a readable account of this and other classical applications together with historical background, references and other versions.

In recent years combinatorialists have found many new applications for the asymptotics of Toeplitz determinants. Gessel [Ge] shows that many generating functions of combinatorial interest can be expressed as Toeplitz determinants. The celebrated asymptotics of Baik, Deift and Johanssen [BDJ] for the longest increasing subsequence of a random permutation proceeds from this path. Tracy and Widom [TW3] extend these applications to alphabets with repeated values. Their paper has a very readable development of Gessel's theorem. Fulman [F] uses Szegö's theorem to give a card shuffling interpretation of Schur functions.

In this paper we give a simple proof of the strong Szegö limit theorem using the orthogonality relations for the power sum symmetric functions. Our proof was motivated by work of Diaconis and Shahshahani [DS] and Johansson [J2] on eigenvalues of random matrices, but none of this is needed for the present work.

Our proof leads to a generalization to Toeplitz *minors*, whose asymptotics surprisingly involve the representation theory of the symmetric group S_m. To state a result of this type, observe that because Toeplitz matrices are banded, their minors may be obtained by either *striking* rows and columns, or by *shifting* rows and columns. Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$ be a partition of m, that is, a decreasing sequence of nonnegative integers, eventually zero, whose sum is m. Then λ parametrizes a character χ^λ of S_m is a standard way (see Section 1). If $D^\lambda_{n-1}(f) = \det(d_{\lambda_i - i + j})_{1 \leq i,j \leq n}$ then we find that if λ is fixed and $n \to \infty$

$$D^\lambda_{n-1}(\sigma)/D_{n-1}(\sigma) \sim \frac{1}{m!} \sum_{\pi \in S_m} \chi^\lambda(\pi) \prod_{k=1}^{\infty} (kc_k)^{\gamma_k(\pi)},$$

where $\gamma_k(\pi)$ is the number of cycles of length k in π.

The right side in this identity is constant on the conjugacy classes of S_m. Thus D^λ_{n-1} is asymptotic to $\exp(nc_0 + \sum kc_k c_{-k})$ times a correction term which is the sum over these conjugacy classes of monomials involving $c_1, \cdots, c_m$. If λ is the empty partition, the correction term is 1 and this is the Strong Szegö limit theorem. If $\lambda = (1)$, the correction term is c_1 and the left side is the minor of $T_{n-1}(f)$ obtained by striking out the first column and the second row. In general, D^λ_{n-1} is the minor of an $(n+\lambda_1) \times (n+\lambda_1)$ Toeplitz matrix obtained by striking the first λ_1 columns, keeping the first row but striking the next $\lambda_1 - \lambda_2$ rows, keeping the next row, then striking the next $\lambda_2 - \lambda_3$ rows, and so forth. For example if $\lambda = (4, 2, 2)$ we strike the first four columns and rows 2,3,6 and 7. When the smoke clears, the partition λ appears running down the main diagonal.

The result just stated gives the asymptotics of Toeplitz minors obtained by striking or shifting rows only. More generally, we obtain asymptotics for minors obtained by striking or shifting both rows and columns. These have the form $D^{\lambda,\mu}_{n-1}(f) = \det(d_{\lambda_i - \mu_j - i + j})_{1 \leq i,j \leq n}$ where λ and μ are a pair of partitions (of possibly different integers). We will obtain asymptotics for these in Theorem 6.

Our asymptotics include every possible Toeplitz minor. However λ is considered fixed in our proof. It would be desirable to be able to vary λ as $n \to \infty$ but we do not address this uniformity issue.

Toeplitz minors obtained by deleting a single row and column of a Toeplitz matrix occur in the inverse matrix, and as such fall into a standard body of theory. See for example Widom [W]. For more general minors, Tracy and Widom [TW2] independently found asymptotics for the same minors $D^{\lambda,\mu}_{n-1}$ as in our Theorem 6. Their results express the asymptotic as a determinant involving the Fourier coefficients of the Wiener-Hopf factorization of σ. Since their expression is very different from ours, comparing their results with ours gives a nontrivial algebraic identity.

In Section 1, we will review the results from symmetric function theory which we need. In Section 2, we prove and generalize a classical formula of Heine and Szegö, expressing the Toeplitz minors as integrals over the unitary group. In Section 3, we prove our main asymptotic results. In Section 4, we consider the special case of a triangular Toeplitz matrix, where we relate our theorems to the representation theory of the symmetric group and Pólya theory, and obtain a formula for skew Schur functions.

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1. Review of symmetric functions. The facts we need from symmetric function theory may be found, for example in Macdonald [M] or Chapter 7 in volume 2 of Stanley [S1]. We will therefore summarize these facts without proof.

Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a partition. Thus the λ_i are nonnegative integers and $\lambda_1 \geq \ldots \geq \lambda_r$. We do not distinguish between two partitions if they are the same except for trailing entries equal to zero. The *length* of λ is the largest j such that $\lambda_j > 0$; we will denote $\lambda_j = 0$ if j exceeds the length of λ, so λ_j is defined for all positive integers. We will call $|\lambda| = \sum \lambda_i$ the *weight* of the partition, and if $|\lambda| = m$ we call λ a *partition of m.*

The *conjugate partition* $\mu = \lambda'$ is characterized by the property that μ_i is the number of j such that $\lambda_j \geq i$.

Let us fix an integer n. We will be concerned with symmetric polynomials in n variables. Such a function f gives rise to a function $\mathbf{f}$ on $U(n)$ whose value at g having eigenvalues $t_1, \cdots, t_n$ is

$$\mathbf{f}(g) = f(t_1, \cdots, t_n). \tag{1.1}$$

There is then an inner product on symmetric polynomials defined by

$$\langle f_1, f_2 \rangle = \int_{U(n)} \mathbf{f}_1(g)\, \overline{\mathbf{f}_2(g)}\, dg \tag{1.2}$$

when $\mathbf{f}_1$ and $\mathbf{f}_2$ are the functions on $U(n)$ associated with the symmetric polynomials f_1 and f_2 by (1.1).

Particular symmetric polynomials of importance are the *elementary* symmetric polynomials

$$e_r(t_1, \cdots, t_n) = \sum_{k_1 < \cdots < k_r} t_{k_1} \cdots t_{k_r},$$

the *complete* symmetric polynomials

$$h_r(t_1, \cdots, t_n) = \sum_{k_1 \leq \cdots \leq k_r} t_{k_1} \cdots t_{k_r},$$

and the *power sum* polynomials

$$p_r(t_1, \cdots, t_n) = t_1^r + \ldots + t_n^r.$$

If λ is a partition of m, we will denote

$$e_\lambda = \prod e_{\lambda_i}, \qquad h_\lambda = \prod h_{\lambda_i}, \qquad p_\lambda = \prod p_{\lambda_i}.$$

These are homogeneous polynomials of degree m.

Let λ be a partition and μ its conjugate partition. The *Jacobi-Trudi identity* asserts that if $|\lambda| \leq n$ and $|\mu| \leq p$, then

$$\det(h_{\lambda_i - i + j})_{1 \leq i,j \leq n} = \det(e_{\mu_i - i + j})_{1 \leq i,j \leq p}. \tag{1.3}$$

Here it is understood that if $\lambda_i - i + j < 0$ for any (i, j), then $h_{\lambda_i - i + j}$ is interpreted as zero, and similarly for $e_{\mu_i - i + j}$. The symmetric polynomial (1.3) is the *Schur polynomial* s_λ. It is nonzero as long as n is at least equal to the length of λ. The nonzero Schur polynomials s_λ with $|\lambda| = m$ are an orthonormal basis of the space of symmetric polynomials of degree m. The function $\mathbf{s}_\lambda$ associated with s_λ by (1.1), if nonzero, is an irreducible character of $U(n)$. The Jacobi-Trudi identity is proved in Macdonald [M], I.3 (p. 41), or Stanley [S1], Section 7.16 (p. 342 of volume 2).

We will denote by k^r the partition $(k, k, \cdots, k)$ of length r, and assuming λ is a permutation of length $\leq r$, we will denote

$$\lambda + k^r = (\lambda_1 + k, \cdots, \lambda_r + k). \tag{1.4}$$

We have

$$s_{\lambda + k^n} = e_n^k \, s_\lambda. \tag{1.5}$$

It is sufficient to prove this when $k = 1$, since the general case then follows by repeated applications of the special. Assuming thus that $k = 1$, by Pieri's formula ((5.16) in Macdonald [M] I.5 or p. 340 of Stanley [S1] vol. 2) the product on the right is a sum of s_ν where ν runs over partitions of weight $|\lambda| + n$ such that $\lambda_j \leq \nu_j$ and $\nu_j - \lambda_j \leq 1$ for all j. Only one such permutation has length $\leq n$, namely μ. Since we are considering symmetric functions in exactly n variables, the remaining s_ν vanish, whence (1.5).

In terms of characters, (1.5) means that on $U(n)$

$$\mathbf{s}_{\lambda + k^n}(g) = \det(g)^k \mathbf{s}_\lambda(g) \tag{1.6}$$

The *dual Cauchy identity* asserts that

$$\sum_\lambda s_\lambda(\alpha) \, s_{\lambda'}(\beta) = \prod_{i,j} (1 + \alpha_i \beta_j). \tag{1.7}$$

See Macdonald [M], I.4 (4.3') on p. 65 or Stanley [S1], Theorem 7.14.3 on p. 332 of volume 2. We note for all but but finitely many λ either λ or its conjugate λ' will have length greater than p. Thus the sum on the left side is actually finite.

Frobenius-Schur duality is a relationship between the irreducible representations of $U(n)$ and the irreducible representations of the symmetric group S_m. Both $U(n)$ and

S_m act on the m-fold tensor product $\otimes^m \mathbb{C}^n$, the group $U(n)$ acting linearly and the symmetric group by permuting the factors. These actions commute with each other, so if ρ is a representation of S_m then $\left(\otimes^m \mathbb{C}^n\right) \otimes_{\mathbb{C}[S_m]} \rho$ is a module for $U(n)$. It is irreducible if nonzero. The irreducible representations of S_m may be parametrized by partitions in such a way that if ρ^λ is the irreducible representation parametrized by a partition λ of m, then the character $\left(\otimes^m \mathbb{C}^n\right) \otimes_{\mathbb{C}[S_m]} \rho^\lambda$ is the character $\mathbf{s}_\lambda$ of $U(n)$ introduced previously. We will denote the character of ρ^λ by χ^λ.

Let R_m denote the vector space of functions on S_m which are constant on conjugacy classes, with the usual inner product

$$\langle f, g\rangle = \frac{1}{m!} \sum_{x\in S_m} f(x)\,\overline{g(x)}, \tag{1.8}$$

and let $\Lambda_m^{(n)}$ be vector space of symmetric polynomials of degree m in n variables. Then $\chi^\lambda \to s_\lambda$ extends to a map $\mathrm{ch} : R_m \to \Lambda_m^{(n)}$. This correspondence, known as the *characteristic map* is an *isometry* for the inner products (1.2) and (1.8) if $n \geq m$. More generally, for $f \in R_m$ we always have

$$\langle f, f\rangle \geq \langle \mathrm{ch}(f), \mathrm{ch}(f)\rangle\,,$$

with equality when $n \geq m$.

Every partition μ of m determines a conjugacy class c_μ of S_m, consisting of disjoint cycles of length μ_j. We call the partition μ the *cycle type* of this conjugacy class. Let z_μ be the order of the centralizer of an element of the conjugacy class μ. Thus if μ contains α_1 1's, α_2 2's, and so forth, so that $\sum j\,\alpha_j = m$, then

$$z_\mu = \prod j^{\alpha_j}\,\alpha_j!\;.$$

Let f_μ denote the characteristic function of the conjugacy class c_μ. We will denote the value of the character χ^λ on c_μ by χ^λ_μ.

As was known to Frobenius, we have

$$\mathrm{ch}(z_\mu f_\mu) = p_\mu. \tag{1.9}$$

Equivalently,

$$\chi^\lambda_\mu = \left\langle \chi^\lambda, z_\mu f_\mu \right\rangle = \langle s_\lambda, p_\mu\rangle\,, \tag{1.10}$$

where the second equality is true assuming $n \geq m$. Here the first inner product is the one (1.2) for symmetric functions, so the first equality in (1.10) is equivalent to (1.9) given the definition (1.8) of the inner product. We are using the fact that the characteristic map is an isometry when $n \geq m$. The identity (1.10) is proved in Macdonald [M], I.7 (formula (7.7) on p.114), or Stanley [S1] (Corollary 7.17.4 on p. 347 of volume 2, together with the discussion in Section 7.18).

Thus

$$\langle p_\lambda, p_\mu\rangle = \langle z_\lambda f_\lambda, z_\mu f_\mu\rangle = \begin{cases} z_\lambda & \text{if } \lambda = \mu; \\ 0 & \text{otherwise,} \end{cases} \tag{1.11}$$

again assuming $n \geq m$. If $n < m$ we still have

$$\langle p_\lambda, p_\mu \rangle \leq \langle z_\lambda f_\lambda, z_\mu f_\mu \rangle = \begin{cases} z_\lambda & \text{if } \lambda = \mu; \\ 0 & \text{otherwise,} \end{cases} \tag{1.12}$$

We have

$$s_\lambda = \sum_\mu z_\mu^{-1} \chi_\mu^\lambda p_\mu, \tag{1.13}$$

and

$$p_\mu = \sum_\lambda \chi_\mu^\lambda s_\lambda. \tag{1.14}$$

In view of the orthogonality properties of the s_λ and of the p_μ, these are equivalent to (1.10). See Macdonald [M], I.7 following 7.6, or Stanley [S1] volume 2, Corollary 7.17.5.

Finally, we will need some of the theory of *skew Schur functions*. If ν and μ are partitions of k and l respectively, Then the *Littlewood-Richardson* coefficients $c_{\nu\mu}^\lambda$ are defined for partitions λ of $k+l$ such that

$$s_\nu \, s_\mu = \sum_\lambda c_{\nu\mu}^\lambda \, s_\lambda. \tag{1.15}$$

(The sum is over partitions of $k+l$.) If λ is a partition of $k+l$ and μ is a partition of k, we denote

$$s_{\lambda/\mu} = \sum_\nu c_{\nu\mu}^\lambda \, s_\nu,$$

where the sum is over partitions ν of $|\lambda| - |\mu|$. This is zero unless $\lambda \supset \mu$. If $\lambda \supset \mu$ then $s_{\lambda/\mu}$ is called a *skew Schur function*. We also denote

$$\mathbf{s}_{\lambda/\mu} = \sum_\nu c_{\nu\mu}^\lambda \, \mathbf{s}_\nu.$$

Let n be greater than or equal to the lengths of both λ and μ. We have the following generalization of the Jacobi-Trudi identity:

$$s_{\lambda/\mu} = \det(h_{\lambda_i - \mu_j - i + j})_{1 \leq i,j \leq n} = \det(e_{\lambda'_i - \mu'_j - i + j})_{1 \leq i,j \leq n}. \tag{1.16}$$

See Macdonald [M], (5.4) and (5.5) in I.5 on pp. 70–71, or Stanley [S1] volume 2, Theorem 7.16.1 on p. 342 and Corollary 7.16.2 on p. 344.

The multiplicative structure in the ring of symmetric polynomials has the following interpretation. Let R_m denote the vector space of class functions on S_m, as in Section 1. Then $R = \bigoplus_m R_m$ has the structure of a graded ring defined as follows. It is sufficient to describe the product in this ring of two characters of χ^μ and S_p and χ^ν of S_{m-p}. $(\pi, \rho) \to \chi^\mu(\pi)\chi^n(\rho)$ is then a character of $S_p \times S_{m-p} \subset S_m$. If $\chi^\mu * \chi^\nu$ denotes the character of S_m induced from this character of $S_p \times S_{m-p} \subset S_m$, then then the $*$ multiplication makes R a graded ring. The characteristic map is then a homomorphism from R to the ring $\Lambda^{(n)}$ of symmetric polynomials in n variables. Using the fact that the characteristic map is an isometry when $R_m \to \Lambda^{(m)}$ for sufficiently large $n \geq m$ and the orthonormality of the Schur functions, (1.15) now implies that

$$c^{\lambda}_{\mu\nu} = \langle \chi^{\mu} * \chi^{\nu}, \chi^{\lambda} \rangle . \tag{1.17}$$

2. The Heine-Szegö identity. A basic identity expresses Toeplitz determinants as integrals over the unitary group. A closely related formula is is in Heine [H]. The first appearance of this exact formula that we are aware of is in Szegö [Sz2], pages 27 and 288. We will therefore refer to this result (Theorem 1 below) as the *Heine-Szegö formula.* We will give a proof of this identity showing its relationship to the Jacobi-Trudi identity; this in turn suggests generalizations for Toeplitz minors.

Let $\Phi_{n,f}$ be the function on the unitary group $U(n)$ whose value on a matrix g with eigenvalues $t_1, \cdots, t_n$ is $f(t_1)\cdots f(t_n)$. Let $\int_{U(n)} dg$ be the Haar integral, normalized so that the volume of $U(n)$ is 1.

Theorem 1. *If $f \in L^1(\mathbb{T})$ has Fourier coefficients d_n $(n \in \mathbb{Z})$, then*

$$D_{n-1}(f) = \int_{U(n)} \Phi_{n,f}(g)\, dg. \tag{2.1}$$

Proof. Since f may be approximated in $L^1(\mathbb{T})$ by polynomials, it is sufficient to prove this in the special case where

$$f(t) = t^{-N} \prod_{j=1}^{M} (1 + \alpha_j t), \tag{2.2}$$

where $\alpha_1, \cdots, \alpha_M$ are complex numbers. Here N may be positive or negative, but if it is negative, a slight change to the following argument will show that both sides of (2.1) are zero, so we will assume that $N \geq 0$. We have

$$d_k = \begin{cases} e_{k+N}(\alpha_1, \cdots, \alpha_M) & \text{if } k \geq -N, \\ 0 & \text{otherwise,} \end{cases}$$

in terms of the elementary symmetric polynomials. According to the Jacobi-Trudi identity, the Toeplitz determinant $D_{n-1}(f)$ is then equal to the Schur polynomial $s_{(n^N)}(\alpha)$, where (n^N) denotes the partition $(n, \cdots, n)$ of length N.

The integrand on the right-hand side of (2.1) is equal to

$$\det(g)^{-N} \prod_{k=1}^{n} \prod_{j=1}^{M} (1 + \alpha_j\, t_k).$$

Let λ be a partition of length $\leq n$. There is a character $\mathbf{s}_\lambda$ of $U(n)$ such that $\mathbf{s}_\lambda(g) = s_\lambda(t_1 \cdots, t_n)$, where t_i are the eigenvalues of g. Using the dual Cauchy identity (1.7) we may rewrite the right side of (2.1)

$$\sum_\lambda s_{\lambda'}(\alpha) \int_{U(n)} \mathbf{s}_\lambda(g) \det(g)^{-N} \, dg.$$

By (1.6) we have $\det(g)^N = \mathbf{s}_{(N^n)}(g)$. Integrating over the group picks off the single contribution where $\lambda = (N^n)$, $\lambda' = (n^N)$, whence (2.1) equals $s_{(n^N)}(\alpha)$, as required. ∎

We will now generalize (2.1). Let λ be a partition of length $\leq n$, and let

$$D_{n-1}^\lambda(f) = \begin{vmatrix} d_{\lambda_1} & d_{\lambda_1+1} & \cdots & d_{\lambda_1+n-1} \\ d_{\lambda_2-1} & d_{\lambda_2} & \cdots & d_{\lambda_2+n-2} \\ \vdots & & & \vdots \\ d_{\lambda_n-(n-1)} & d_{\lambda_n-(n-2)} & \cdots & d_{\lambda_n} \end{vmatrix}. \tag{2.3}$$

This is essentially a Toeplitz determinant with some of the rows shifted. Note that if λ has length $< n$, then the trailing λ_j are interpreted as zero.

Theorem 2. *With the hypotheses of Theorem 1,*

$$D_{n-1}^\lambda(f) = \int_{U(n)} \Phi_{n,f}(g) \, \overline{\mathbf{s}_\lambda(g)} \, dg. \tag{2.4}$$

Proof. Indeed, using the same test function (2.2), invoking the dual Cauchy identity (1.7) and (1.6), the right side of (2.4) equals

$$\sum_\nu s_{\nu'}(\alpha) \int_{U(n)} \mathbf{s}_\nu(g) \, \overline{\mathbf{s}_{\lambda+N^n}(g)} \, dg,$$

where $\lambda+N^n$ has the meaning defined in (1.4). The only nonvanishing term has $\nu = \lambda+N^n$, and by the Jacobi-Trudi identity, this contribution equals the left side of (2.4). ∎

Noting that $D_{n-1}^\lambda(f)$ is a minor in a larger Toeplitz matrix, one seeks a generalization which gives an arbitrary Toeplitz minor. One thought would be to replace $\mathbf{s}_\lambda$ in (2.4) by a skew Schur function. We caution the reader that this sometimes produces a Toeplitz minor, but not always. Luckily, we will find a satisfactory alternative construction in Theorem 3 below.

Let λ and μ be partitions of length $\leq n$. Define

$$D_{n-1}^{\lambda,\mu}(f) = \det(d_{\lambda_i-\mu_j-i+j})_{1\leq i,j\leq n}. \tag{2.5}$$

Note that despite the resemblance to (1.16), we are *not* assuming $\lambda \supset \mu$.

Lemma. *Let d_j ($j \in \mathbb{Z}$) be complex numbers, and let λ, μ be partitions of m and p, respectively, both of length $\leq n$. If N is any sufficiently large integer, we have*

$$\sum_\nu c_{\nu\mu}^{\lambda+N^n} \det(d_{\nu_i-N+j-i})_{1\leq i,j\leq n} = \det(d_{\lambda_i-\mu_j+j-i})_{1\leq i,j\leq n}, \tag{2.6}$$

where the summation is over partitions of $m+N^n-p$ of length $\leq n$, and the $c_{\nu\mu}^{\lambda+N^n}$ are the Littlewood-Richardson coefficients.

Proof. Since the length of μ is $\leq n$, we may choose N so large that $\mu \subset \lambda + N^n$. We consider the skew Schur function $s_{\lambda+N^n/\mu}$ in many variables (possibly $> n$). This equals $\sum_\nu c_{\nu\mu}^{\lambda+N^n} s_\nu$, where only ν of length $\leq n$ occur, because any ν with $c_{\nu\mu}^{\lambda+N^n} \neq 0$ must be contained in $\lambda+N^n$. Using the Jacobi-Trudi identities (1.3) and (1.16), we have

$$\sum_\nu c_{\nu\mu}^{\lambda+N^n} \det(h_{\nu_i+j-i})_{1\leq i,j\leq n} = \sum_\nu c_{\nu\mu}^{\lambda+N^n} s_\nu = s_{\lambda+N^n/\mu} = \det(h_{\lambda_i+N-\mu_j+j-i})_{1\leq i,j\leq n}.$$

If N is sufficiently large, then $\nu_i+j-i \geq 0$ and $\lambda_i+N-\mu_j+j-i \geq 0$ for every i,j and every ν in this expression such that $c_{\nu\mu}^{\lambda+N^n} \neq 0$. Assuming this, and working with Schur functions in sufficiently many variables, the parameters h_j occuring here are algebraically independent, so this is an algebraic identity. We may then replace h_j by d_{j-N} to obtain (2.6). ■

Theorem 3. *With the hypotheses of Theorem 1,*

$$D_{n-1}^{\lambda,\mu}(f) = \int_{U(n)} \Phi_{n,f}(g)\,\overline{\mathbf{s}_\lambda(g)}\,\mathbf{s}_\mu(g)\,dg. \tag{2.7}$$

Proof. Once again, we use the test function (2.2). The integral on the right side of (2.7) equals

$$\sum_\nu s_{\nu'}(\alpha) \int_{U(n)} s_\nu(g)\,s_\mu(g)\,\overline{s_{\lambda+N^n}(g)}\,dg = \sum_\nu c_{\nu\mu}^{\lambda+N^n} s_{\nu'}(\alpha).$$

Using (1.3) this equals

$$\sum_\nu c_{\nu\mu}^{\lambda+N^n} \det(e_{\nu_i-i+j}) = \sum_\nu c_{\nu\mu}^{\lambda+N^n} \det(d_{\nu_i-N-i+j}).$$

The result now follows from the Lemma. ■

Baxter [Ba], Lemma 7.4 proved that

$$D_{n-1}(1/UV) = \prod_{i,j}(1-\alpha_i\beta_j)^{-1}$$

where $U(t) = \prod(1 - \alpha_j t)$, $V(t) = \prod(1 - \beta_j t^{-1})$, $|\alpha_i|, |\beta_j| < 1$. Although this identity does not appear in the above proofs, it is related. A special case of this identity was used by Szegö in the proof of the strong Szegö limit theorem. See Szegö [Sz1] and Grenander and Szegö [GZ], p. 78. Johansson [J2] also applied this identity of Szegö and Baxter. The identity was rediscovered by Gessel [Ge] who used it to define generating functions for longest increasing subsequences. This identity has become a standard tool in random matrix theory. A nice exposition with applications and extensions appears in Tracy and Widom [TW1]. Borodin and Okounkov [BO] have used this identity to show that Toeplitz determinants can be expressed as Fredholm determinants.

3. The strong Szegö limit theorem. We will prove:

Theorem 4 (the strong Szegö limit theorem). *Let c_k $(k \in \mathbb{Z})$ satisfy*

$$\sum |c_k| < \infty \tag{3.1}$$

and

$$\sum |k|\, |c_k|^2 < \infty. \tag{3.2}$$

Let $\sigma(t) = \exp(\sum c_k t^k)$ for $t \in \mathbb{T}$. Then

$$D_{n-1}(\sigma) \sim \exp\left(nc_0 + \sum_{k=1}^{\infty} kc_k c_{-k}\right). \tag{3.3}$$

Proof of Theorem 4. First we only assume (3.1). Using (2.1) we have

$$D_{n-1}(\sigma) = \int_{U(n)} \Phi(g)\, dg,$$

where, if $t_1, \cdots, t_n$ are the eigenvalues of g, we define $\Phi(g) = \prod_{j=1}^{n} e^{\sigma(t_i)}$. Assuming (3.1), since each trace $\mathrm{tr}(g^k)$ is bounded by n, we have

$$\int_{U(n)} \exp\Big(\sum |c_k|\, |\mathrm{tr}(g^k)|\Big)\ dg < \infty,$$

and this absolute convergence justifies the following manipulations. We can write

$$D_{n-1}(\sigma) = \int_{U(n)} \exp\Big(\sum c_k\, \mathrm{tr}(g^k)\Big)\ dg.$$

Substituting the power series for the exponential function and grouping together the terms with $k = 0$, $k > 0$ and $k < 0$ we get

$$e^{nc_0} \int_{U(n)} \prod_{k=1}^{\infty} \sum_{\alpha_k=0}^{\infty} \frac{\big(c_k\, \mathrm{tr}(g^k)\big)^{\alpha_k}}{\alpha_k!} \sum_{\beta_k=0}^{\infty} \frac{\big(c_{-k}\, \overline{\mathrm{tr}(g^k)}\big)^{\beta_k}}{\beta_k!}\, dg.$$

Now expanding this and invoking (1.11) and (1.12), the only terms which survive have $\alpha_k = \beta_k$. Given a sequence $\alpha_1, \alpha_2, \cdots$ of nonnegative integers, only finitely many of which are nonzero, let λ_α denote the partition having α_k values of λ_j equal to k. Then

$$D_{n-1}(\sigma) = e^{nc_0} \sum \frac{\langle p_{\lambda_\alpha}, p_{\lambda_\alpha} \rangle \; (c_k \, c_{-k})^{\alpha_k}}{(\alpha_k!)^2}. \tag{3.4}$$

We compare this with

$$e^{nc_0} \sum \frac{\langle z_{\lambda_\alpha} f_{\lambda_\alpha}, z_{\lambda_\alpha} f_{\lambda_\alpha} \rangle \; (c_k \, c_{-k})^{\alpha_k}}{(\alpha_k!)^2}. \tag{3.5}$$

We remind the reader that in (3.4), the inner product is the one defined by (1.2), while in (3.5), the inner product is defined by (1.8). By (1.11), (3.5) equals

$$e^{nc_0} \sum \frac{(k \, c_k \, c_{-k})^{\alpha_k}}{\alpha_k!} = \exp\left(nc_0 + \sum_{k=1}^{\infty} kc_k \, c_{-k} \right).$$

Note that $\sum kc_k c_{-k}$ converges absolutely by (3.2) and the Cauchy-Schwarz inequality, so (3.5) is absolutely convergent. By (1.12) it dominates (3.4) termwise, and as n is increased, each term in (3.4) eventually becomes its corresponding term in (3.5). Evidently (3.4) converges to (3.5), which completes the proof. ■

The space of functions $f = \sum d_k t^k$ on the circle whose Fourier coefficients satisfy

$$\sum |d_k| < \infty \tag{3.6}$$

and

$$\sum |k| \, |d_k|^2 < \infty, \tag{3.7}$$

if given the norm $\sum |d_k| + \sqrt{\sum |d_k|^2}$, is a Banach algebra $\mathcal{D}$ under pointwise multiplication whose maximal ideal space may be identified with $\mathbb{T}$. This fact was found independently by Hirschman [Hi] and Kreĭn [K]. Proofs of this and other basic relevant facts may be found in Böttcher and Silberman [BS], p. 123. Condition (3.6) implies easily that f is continuous. If σ is an element of $\mathcal{D}$ which is nonvanishing and has winding number zero around the origin, then its logarithm is also an element of $\mathcal{D}$. Consequently the strong Szegö limit theorem as we have stated it is equivalent to the formulations in Hirschman [Hi] and in Böttcher and Silberman [BS] (Theorem 5.2 on p. 124).

Johansson [J1], beginning on p. 267, has given an argument which shows in a similar situation that the conclusion (3.3) follows assuming (3.2) but not (3.1). Thus it is possible that this hypothesis may be lifted from our results, though we have not tried to do so. His paper also proves the strong Szegö limit theorem using the Heine-Szegö identity, though it is very different from the above proof.

We now *generalize* the strong Szegö limit theorem. Let λ be a fixed partition of m, and let γ_k be the number of λ_j equal to k. We will find the asymptotics of $D_{n-1}^{\lambda}(\sigma)$ in the notation (2.3). Note that this is essentially a Toeplitz determinant with a certain fixed set of rows shifted by a predetermined amount (independent of n).

Theorem 5. *Let $\sigma(t) = \exp(\sum c_k t^k)$ be a function on $\mathbb{T}$ satisfying (3.1) and (3.2). Let $m \le n$, let λ be a partition of m, and let χ^{λ} be the character of S_m parametrized by λ. If π is an element of the symmetric group S_m, let $\gamma_k = \gamma_k(\pi)$ equal the number of k-cycles in the decomposition of π into disjoint cycles, and define*

$$\Delta(\sigma, \pi) = \prod_{k=1}^{\infty} (k c_k)^{\gamma_k}.$$

(The product is actually finite.) With notation as in (2.3), with λ fixed and $n \to \infty$, we have

$$D_{n-1}^{\lambda}(\sigma) \sim \frac{1}{m!} \sum_{\pi \in S_m} \chi^{\lambda}(\pi)\, \Delta(\sigma, \pi) \exp\left(n c_0 + \sum_{k=1}^{\infty} k c_k c_{-k} \right). \tag{3.8}$$

We remark that Δ only involves c_k with k *positive.* This is because in the definition of $D_{n-1}^{\lambda}(\sigma)$ the rows which have been shifted have all been shifted to the *left.*

Proof. Substituting (2.4) for (2.1) in the preceeding proof, and making use of (1.13), and proceeding as before we obtain

$$\sum_{\mu} z_{\mu}^{-1} \chi_{\mu}^{\lambda} e^{n c_0} \int_{U(n)} \prod_{k=1}^{\infty} \sum_{\alpha_k=0}^{\infty} \frac{\left(c_k \operatorname{tr}(g^k)\right)^{\alpha_k}}{\alpha_k!} \sum_{\beta_k=0}^{\infty} \frac{\left(c_{-k} \overline{\operatorname{tr}(g^k)}\right)^{\beta_k}}{\beta_k!} \overline{\left(\operatorname{tr}(g^k)\right)^{\gamma_k}}\, dg,$$

where γ_k is the number of μ_j equal to k. Remembering that S_m contains $m!/z_{\mu}$ elements with cycle type (γ_k), we may write this

$$\frac{1}{m!} \sum_{\pi \in S_m} \chi^{\lambda}(\pi)\, e^{n c_0} \prod_{k=1}^{\infty} \sum_{\alpha_k=0}^{\infty} \sum_{\beta_k=0}^{\infty} \frac{c_k^{\alpha_k}}{\alpha_k!} \frac{c_{-k}^{\beta_k}}{\beta_k!} \int_{U(n)} \operatorname{tr}(g^k)^{\alpha_k}\, \overline{\operatorname{tr}(g^k)}^{\beta_k+\gamma_k}\, dg.$$

The only terms which survive have $\alpha_k = \beta_k + \gamma_k$. Continuing as in the proof of Theorem 4, we obtain (3.8). ∎

Finally, we have an asymptotic result for the most general Toeplitz minors. Let λ and μ be partitions of m and p, respectively. Let $\pi \in S_m$ and $\rho \in S_p$. Let γ_k be the number of k-cycles in π, and let δ_k be the number of k-cycles in ρ. Recall that the *Laguerre polynomials* are defined by

$$L_n^{(\alpha)}(t) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-t)^k}{k!}.$$

See Szegö [Sz2] Chapter 5 or Rainville [Rv], Chapter 12. Let

$$\Delta(\sigma,\pi,\rho) = \prod_{k=1}^{\infty} \begin{cases} k^{\gamma_k}\, c_k^{\gamma_k-\delta_k}\, \delta_k!\, L_{\delta_k}^{(\gamma_k-\delta_k)}(-kc_kc_{-k}) & \text{if } \gamma_k \ge \delta_k, \\ k^{\delta_k}\, c_{-k}^{\delta_k-\gamma_k}\, \gamma_k!\, L_{\gamma_k}^{(\delta_k-\gamma_k)}(-kc_kc_{-k}) & \text{if } \delta_k \ge \gamma_k. \end{cases} \tag{3.9}$$

Theorem 6. *Let $\sigma(t) = \exp(\sum c_k t^k)$ be a function on $\mathbb{T}$ satisfying (3.1) and (3.2). Let λ and μ be partitions of m and p, respectively. With λ and μ fixed, as $n \to \infty$, we have*

$$D_{n-1}^{\lambda,\mu}(\sigma) \sim \frac{1}{m!} \sum_{\pi\in S_m} \frac{1}{p!} \sum_{\rho\in S_p} \chi^{\lambda}(\pi)\, \chi^{\mu}(\rho)\, \Delta(\sigma,\pi,\rho) \exp\left(nc_0 + \sum_{k=1}^{\infty} kc_kc_{-k} \right). \tag{3.10}$$

Proof. Proceeding as in the proof of Theorem 5 gives easily

$$e^{nc_0} \frac{1}{m!} \sum_{\pi\in S_m} \chi^{\lambda}(\pi) \frac{1}{p!} \sum_{\rho\in S_p} \chi^{\mu}(\rho) \prod_k \sum_{\alpha_k,\beta_k} \frac{c_k^{\alpha_k}}{\alpha_k!} \frac{c_{-k}^{\beta_k}}{\beta_k!} \int_{U(n)} \mathrm{tr}(g^k)^{\alpha_k+\delta_k}\, \mathrm{tr}(g^{-k})^{\beta_k+\gamma_k}\, dg,$$

where γ_k is the number of k-cycles in π, and δ_k is the number of k-cycles in ρ. As $n \to \infty$ this becomes asymptotically

$$e^{nc_0} \frac{1}{m!} \sum_{\pi\in S_m} \chi^{\lambda}(\pi) \frac{1}{p!} \sum_{\rho\in S_p} \chi^{\mu}(\rho) \prod_k \sum_{\alpha_k+\delta_k=\beta_k+\gamma_k} \frac{c_k^{\alpha_k}\, c_{-k}^{\beta_k} (\beta_k+\gamma_k)!\, k^{\beta_k+\gamma_k}}{\alpha_k!\, \beta_k!}.$$

We have:

$$\sum_{\alpha} \frac{t^{\alpha}\,(\alpha+\delta)!}{\alpha!(\alpha+\delta-\gamma)!} = \gamma!\, L_{\gamma}^{(\delta-\gamma)}(-t)\, e^t, \qquad (\delta \ge \gamma \ge 0)$$

This is equivalent for the Rodrigues formula for the Laguerre polynomials, (5.1.5) on p. 101 of Szegö [Sz2], or p. 203 of Rainville [Rv]. Using this we obtain (3.10). ■

4. The triangular case. When the Toeplitz matrix is upper triangular, Our viewpoint bears a strong relationship to Exercise 7.91 on page 381 of Stanley [S1] volume 2, which is based on the approach to Schur functions taken by Littlewood [Li].

In the special case of an upper triangular Toeplitz matrix, Theorems 2 and 3 are implicit in Littlewood's definition of the Schur functions, if one bears in mind the "unitary" interpretation (1.2) of the Hall inner product on symmetric functions. In the triangular case Stanley has already pointed out the relevance of the Jacobi-Trudi identity to Toeplitz minors. See [S2], and the remarks on p. 544 of [S1], where Schur functions are related to the result of Aissen, Edrei, Schoenberg and Whitney [AESW] characterizing triangular Toeplitz matrices all of whose nontrivial minors are positive.

In the case of a triangular Toeplitz matrix, the formulas of Section 3 are connected with Pólya theory, which is concerned with cycle enumeration in permutation groups. See

Stanley [S1], part 2 Section 7.24, and [S2]. Since this point is worth understanding, we review the basics.

In this section we will study the case where $c_k = 0$ if $c \le 0$. Then if $\sigma(t) = \exp(\sum c_k t^k)$, the Fourier coefficients d_k of σ satisfy $d_0 = 1$ and $d_k = 0$ when $k < 0$. A first important observation is that in this case case, the Toeplitz minors $D^{\lambda}_{n-1}(\sigma)$ and $D^{\lambda,\mu}_{n-1}(\sigma)$ become constant when n is at least the lengths of μ and ν. Thus although Theorems 5 and 6 only assert asymptotic results, they are exact in this case.

We will see that this case is intimately connected with the representation theory of the symmetric group. Let $x_1, x_2, \cdots$ be indeterminates. Recall that the cycle index polynomial is given by

$$f_m(x_1, \cdots, x_m) = \frac{1}{m!} \sum_{\pi \in S_m} \prod_{k=1}^{m} x_k^{\gamma_k(\pi)}, \tag{4.1}$$

where as in Section 3, $\gamma_k(\pi)$ is the number of cycles of length k in the permutation π. We also define $f_0 = 1$, and $f_m = 0$ if $m < 0$.

Pólya proved an identity for the generating function of these polynomials:

$$\sum_{m=0}^{\infty} f_m t^m = \exp\left(\sum_{k=1}^{\infty} \frac{x_k t^k}{k} \right). \tag{4.2}$$

This can be used to derive limit theorems for the joint distribution of the γ_k. See Diaconis and Shahshahani [DS] and Shepp and Lloyd [SL]. The proof of (4.2) is relevant, so we recall it. Let h_m, p_m etc. be symmetric polynomials in sufficiently many variables $\alpha_1, \cdots, \alpha_n$. We specialize the variables $x_k \to p_k$, and interpret f_m as a function on S_m via the characteristic map (Section 2). Since there are $m!/z_\lambda$ elements of S_m with cycle type λ, using (1.9), this function equals $\sum_\lambda f_\lambda$, which is the constant function equal to 1, that is, the trivial character of S_m. This corresponds to the symmetric polynomial h_m under the characteristic map, so under the specialization $x_k \to p_k$ we have $f_m \to h_m$. Now exponentiating the identity

$$\log\left(\sum_k h_k t^k \right) = \log \prod (1 - \alpha_j t)^{-1} = \sum_k \frac{p_k\, t^k}{k}$$

gives (4.2).

The relation with Theorem 5 may be seen by setting $c_{-k} = 0$ when $k \ge 0$ and $c_k = x_k/k$ when $1 \le k \le m$ and $c_k = 0$ for $j > m$. Thus

$$\sigma(t) = \exp\left(\sum_{k=1}^{m} \frac{x_k t^k}{k} \right).$$

From (4.2), the first m coefficients of $\sigma(t)$ equal $f_1, \cdots, f_m$. Fix a partition λ of m. Assuming that n is greater than or equal to the length of λ, the left side of (3.8) in

Theorem 5 is an $n \times n$ determinant whose value depends on λ but not on n, while the right side is asymptotically given by (3.8). Since both sides are stable for all large n we have

$$\frac{1}{m!} \sum_{\pi \in S_m} \chi^\lambda(\pi) \prod_{k=1}^{m} x_k^{\gamma_k(\pi)} = \begin{vmatrix} f_{\lambda_1} & f_{\lambda_1+1} & \cdots & f_{\lambda_1+n-1} \\ f_{\lambda_2-1} & f_{\lambda_2} & \cdots & f_{\lambda_2+n-2} \\ \vdots & & & \vdots \\ f_{\lambda_n+1} & f_{\lambda_n-n+2} & \cdots & f_{\lambda_n} \end{vmatrix}, \tag{4.3}$$

Under the specialization $x_k \to p_k$, $f_k \to h_k$, the left side becomes s_λ by (1.13). Since $f_k \to h_k$, this is the Jacobi-Trudi identity.

We would like a similar interpretation of Theorem 6. Assuming that $c_k = 0$ when $k < 0$, we may simplify (3.9). In this case, $\Delta(\sigma, \pi, \rho)$ vanishes unless $\delta_k = \gamma_k(\rho) \geq \gamma_k = \gamma_k(\pi)$ for all k, that is, the cycle type of π must be contained in the cycle type of ρ. Assuming this, the argument of the Laguerre polynomial is zero, and using $L_n^{(\alpha)}(0) = \binom{\alpha+n}{n}$, we have

$$\Delta(\sigma, \pi, \rho) = \prod_{k=1}^{\infty} k^{\delta_k} \, c_k^{\delta_k - \gamma_k} \, \frac{\delta_k!}{(\delta_k - \gamma_k)!}.$$

Using the specialization described above (so $c_k \to p_k/k$) we obtain:

Theorem 7. *Let λ and μ be partitions of m and p If $\pi \in S_m$ and $\rho \in S_p$, let $\gamma_k = \gamma_k(\pi)$ and $\delta_k = \gamma_k(\rho)$ be the number of k-cycles in π and ρ respectively, and define*

$$C(\pi, \rho) = \begin{cases} \prod_{k=1}^{\infty} k^{\delta_k} \, p_k^{\gamma_k - \delta_k} \, \frac{\gamma_k!}{(\gamma_k - \delta_k)!} & \textit{if } \gamma_k \geq \delta_k \textit{ for all } k; \\ 0 & \textit{otherwise.} \end{cases}$$

Then

$$\frac{1}{m!p!} \sum_{\substack{\pi \in S_m \\ \rho \in S_p}} C(\pi, \rho) \, \chi^\lambda(\pi) \, \chi^\mu(\rho) = \begin{cases} s_{\lambda/\mu} & \textit{if } \lambda \supset \mu, \\ 0 & \textit{otherwise.} \end{cases} \tag{4.4}$$

To put this result into the context of the representation theory of the symmetric group, we will now give a second proof of Theorem 7.

Proof. Let π be a permutation of m, and let θ (a partition of m) be its cycle type, so the number $\gamma_k = \gamma_k(\pi)$ of cycles in π of length k equals the number of parts of θ of length k. Given $\rho \in S_p$, let ϕ (depending on ρ) be its cycle type, and let δ_k be the number of cycles of length k in ϕ. We will assume that $\gamma_k \geq \delta_k$ for all k, and we will express this assumption with the notation $\rho|\pi$. Let ψ (depending on π and ρ) be the partition of $m-p$ having $\gamma_k - \delta_k$ components of length k. We express the relationship between ϕ, ψ and θ as $\theta = \phi \cup \psi$, since θ is obtained by taking the set-theoretic union of the components of ϕ and ψ, then arranging them in descending order to obtain a partition. If τ is an element of S_{m-p} with cycle type ψ and if π has cycle type θ, then π is conjugate to $\rho\tau$ in S_m.

Let ν be a partition of $m-p$. We will take the inner product on both sides of (4.4) with s_ν. The inner product with $s_{\lambda/\mu}$ is the Littlewood-Richardson coefficient $c^\lambda_{\mu\nu}$. The inner product with the left-hand side is

$$\frac{1}{m!p!} \sum_{\substack{\pi\in S_m \\ \rho\in S_p \\ \rho|\pi}} \chi^\lambda(\pi)\,\chi^\mu(\rho)\,\chi^\nu_\psi \prod_{k=1}^{\infty} \left(k^{\delta_k} \frac{\gamma_k!}{(\gamma_k-\delta_k)!} \right), \tag{4.5}$$

where γ_k, δ_k, and the dependence of ψ on π and ρ, are as explained above; only pairs π and ρ with $\gamma_k \geq \delta_k$ for all k are summed. In view of (1.17), what we must show is that

$$\pi \mapsto \frac{1}{p!} \sum_{\substack{\rho\in S_p \\ \rho|\pi}} \chi^\mu(\rho)\,\chi^\nu_\psi \prod_{k=1}^{\infty} \left(k^{\delta_k} \frac{\gamma_k!}{(\gamma_k-\delta_k)!} \right) \tag{4.6}$$

is the character of $\chi^\mu * \chi^\nu$ induced from the character $\chi^\mu \otimes \chi^\nu$ of $S_p \times S_{m-p}$. It follows from the definitions that the right side of (4.6) is

$$\sum_{\substack{\phi,\psi \\ \phi\cup\psi=\theta}} \frac{z_\theta}{z_\phi\, z_\psi} \chi^\mu_\phi\, \phi^\nu_\psi.$$

Taking representatives ρ and τ with cycle types ϕ and ψ respectively, the order of the centralizer of $\rho\tau$ in S_m is z_θ, while the order or the centralizer of $\rho\tau$ in $S_p \times S_{m-p}$ is $z_\phi\, z_\psi$. It follows that (4.6) is the value of the induced character at π. ∎

Richard Stanley has pointed out to us that this result may also be obtained from $\langle s_{\lambda\backslash\mu}, p_\alpha \rangle = \langle s_\lambda, p_\alpha s_\mu \rangle$. Indeed, the left side gives the coefficient of the p_α when $s_{\lambda\backslash\mu}$ is expanded in power sums. Expanding both Schur functions on the right side using (1.13) and (1.11) then produces (4.4).

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