Weyl Group Multiple Dirichlet Series: Type A Combinatorial Theory

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Preface

An *L-function*, as the term is generally understood, is a Dirichlet series in one complex variable s with an Euler product that has (at least conjecturally) analytic continuation to all complex s and a functional equation under a single reflection $s \mapsto 1-s$. The coefficients are in particular multiplicative.

By contrast *Weyl group multiple Dirichlet series* are a new class of Dirichlet series with arithmetic content that differ from L-functions in two ways. First, although the coefficients of the series are not multiplicative in the usual sense, they are *twisted multiplicative*, the multiplicativity being modified by some n-th power residue symbols – see (1.3) below. Second, they are Dirichlet series in several complex variables $s_1, \cdots, s_r$. They have (at least conjecturally) meromorphic continuation to all $\mathbb{C}^r$ and groups of functional equations that are finite reflection groups.

The data needed to define such a series in r complex variables are a root system Φ of rank r with Weyl group W, a fixed integer $n > 1$, and a global ground field F containing the n-th roots of unity; in some of the literature (including this work) the ground field F is assumed to contain the $2n$-th roots of unity. Twisted multiplicativity implies that it is sufficient to describe the prime-power coefficients of such a series.

In this work we consider the case that Φ is of Cartan type A_r. In this case a class of multiple Dirichlet series, convergent for $\Re(s_i)$ sufficiently large, was described in [10], where the analytic continuation and functional equations were conjectured. Their definition is given in detail in Chapter 1 below. The prime-power coefficients are sums of products of n-th order Gauss sums, with the individual terms indexed by Gelfand-Tsetlin patterns. It is not clear from this definition that these series have analytic continuation and functional equations. However it was shown in [9] that this global property would be a consequence of a conjectured purely local property of a combinatorial and number-theoretic nature.

Specifically, two distinct versions of the Gelfand-Tsetlin definition were given. It is not apparent that they are equal. Either of these definitions is purely local in that it specifies the p-part of the multiple Dirichlet series, and this then determines the global Dirichlet series by twisted multiplicativity. It was proved in [9] that if these two definitions are equivalent, then the analytic continuation and functional equations of the multiple Dirichlet series follows. The argument from [9] is summarized in Chapter 5 below. It is ultimately based on the analytic continuation in the rank one case, which was treated by Kubota using the theory of Eisenstein series on the metaplectic covers of SL_2. The reduction to the rank one case makes use of Bochner's tube domain theorem from several complex variables. In this work we establish the desired local equality, that is, the equality of the two definitions of the p-part.

The assignment of number-theoretic quantities to a given Gelfand-Tsetlin pattern can be described representation-theoretically; to do so it is helpful to pass to an alternative description presented in [7], where the coefficients were reinterpreted as sums over crystal bases of type A_r. After translating our main result into the language of crystals, this equivalence takes on another meaning. The crystal basis definition of the multiple Dirichlet series depends on one choice – that of a "long word," by which we mean a decomposition of minimal length of the long element w_0 of the Weyl group into a product of simple reflections. Once this choice is made, there is, for every element of the crystal a canonical path to the lowest weight vector. The lengths of the "straight-line" segments of this path (in the sense of Figures 2.1 and 2.2) are the basic data from which its number-theoretic contribution to the Dirichlet series is computed. See Chapter 2 for details.

The desired local equality turns out to be equivalent to the equality of the local factors obtained from two particular choices of long word. Comparing the contributions from these two choices, we prove that there exists a bijection preserving the number-theoretic quantity attached to "most" vertices in the crystal. However, there is no bijection on the entire crystal (or equivalently, on all Gelfand-Tsetlin patterns of fixed top row) preserving the number-theoretic quantity – exceptional vertices that cannot be bijectively matched appear on the boundary of the polytope that parametrizes a weight space in the crystal, although the bijective matching works perfectly on the interior of this polytope. It is only after summing over all contributions from vectors of equal weight that the equality of the two definitions results. Moreover the equality is more than just combinatorial, since it makes use of number-theoretic facts related to Gauss sums.

There are many long words; for type A_r, with $r = 1, 2, 3, 4, \cdots$, there are $1, 2, 16, 768, \cdots$ words, respectively. For each rank, only two of these are actually needed for the proof of the analytic continuation. In some sense these two decompositions are as "far apart" as possible; for example, they are the first and last such decompositions in the lexicographical order. Our proof demonstrates equivalent definitions of the multiple Dirichlet series for several additional decompositions of the long word along the way, and it is probable that one can extend the results proven here to the set of all reduced decompositions of the long element.

Some progress has been made in establishing similar results for other Cartan types. For n sufficiently large (the "stable case") the authors have given a satisfactory theory in [6, 8]. For the remaining small n, the situation is more difficult but also extremely interesting. There have been two distinct approaches to developing a theory for general n: one based on a novel variant of the Weyl character formula due to Chinta and Gunnells [18], and the other, the subject of this book, based on crystal graphs (or Gelfand-Tsetlin patterns). Proving that the two approaches actually define the same Dirichlet series is a central problem in this field. Chinta and Gunnells treat all root systems. At this writing, the crystal graph approach can be used to define Weyl group multiple Dirichlet series for Type C_r (with n odd) as in Beineke, Brubaker and Frechette [1] and in Type B_r (with n even) as in Brubaker, Bump, Chinta and Gunnells [5] whose conjectured analytic properties can be proved in a number of special cases for both types.

We believe that ultimately the crystal graph approach will be extended to all root systems, all n, and all long words. We hope to extend this theory to all Cartan types and ultimately to symmetrizable Kac-Moody root systems. The alternative approach of Chinta and Gunnells [18] also should lend itself to the Kac-Moody case, and though the two approaches are different, it is to be expected that they will eventually be unified in a single theory. For a connection between the two approaches in certain cases, see Chinta, Friedberg and Gunnells [16].

A first case of multiple Dirichlet series having infinite group of functional equations (the affine Weyl group $D_4^{(1)}$ in Kac's classification) may be found in the work of Bucur and Diaconu [11]. (Their result requires working over the rational function field; it builds on work of Chinta and Gunnells [18].) If one could establish the analytic properties of such series in full generality, one would have a potent tool for studying moments of L-functions.

It is expected that Weyl group multiple Dirichlet series for finite Weyl groups can be identified with the Whittaker coefficients on metaplectic groups. For example in [7] we show that the multiple Dirichlet series of Type A_r can be identified with Whittaker coefficients of Eisenstein series. Also, working locally, Chinta and Offen [19] relate the Chinta-Gunnells construction to metaplectic Whittaker functions on $\mathrm{GL}_{r+1}(F)$ when F is nonarchimedean. However there is reason to avoid identifying this program too closely with the question of metaplectic Whittaker functions. For if the theory of Weyl group multiple Dirichlet series is to be extended to infinite Kac-Moody Weyl groups, one will not have the interpretation of the multiple Dirichlet series as Whittaker functions of automorphic forms, so a combinatorial approach will be necessary. The results of this work are a proof of concept that a combinatorial approach based on crystal graphs should be viable.

The first six chapters provide a detailed introduction and preparation for the later chapters. The first chapter gives the two definitions of the Type A_r Weyl group multiple Dirichlet series in which the p-parts are sums over Gelfand-Tsetlin patterns. In Chapter 2, these two definitions are translated into the language of crystal bases. Chapter 3 considers the special case $n = 1$, and Chapter 4 describes variants of these definitions and an interesting related geometric property of crystals. Returning to the main theorems, Chapter 5 outlines the proofs, which will occupy most of the book. This chapter introduces many concepts and ideas, and several equivalent forms of the result, called Statements A through F. Each statement is purely combinatorial, but with each statement the nature of the problem changes. The first reduction changes the focus from Gelfand-Tsetlin patterns to "short" Gelfand-Tsetlin patterns, consisting of just three rows. This reduction, based on the Schützenberger involution, is explained in Chapters 6 and 7. A particular phenomenon called *resonance* is isolated, which seems to be at the heart of the difficulty in all the proofs. In Chapters 8 through 13, a reduction to the totally resonant case is accomplished. Now the equality is of two different sums of products of Gauss sums attached to the lattice points in two different polytopes. On the interior of the polytopes, the terms bijectively match but on the boundary a variety of perplexing phenomena occur. Moreover, the polytopes are irregular in shape. It is shown in Chapter 14 by means of an inclusion-exclusion process that these sums can be replaced by sums over lattice points in simplices; the terms that are summed are not the original products of Gauss sums but certain alternating sums of these. Although these terms appear more complicated than the original ones, they lead to an intricate but explicit rule for matching terms in each sum. This results in a final equivalent version of our main theorem, called Statement G, which is formulated in Chapter 15. This is proved in Chapters 16 through 17. See Chapter 5 for a more detailed outline of the proof.

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Contents

Chapter 1

Type A Weyl group multiple Dirichlet series

We begin by defining the basic shape of the class of Weyl group multiple Dirichlet series. To do so, we choose the following parameters.

- Φ, a reduced root system. Let r denote the rank of Φ.
- n, a positive integer,
- F, an algebraic number field containing the group μ_{2n} of $2n$-th roots of unity,
- S, a finite set of places of F containing all the archimedean places, all places ramified over $\mathbb{Q}$, and large enough so that the ring

  $$\mathfrak{o}_S = \{x \in F \,|\, |x|_v \leqslant 1 \text{ for } v \notin S\}$$

  of S-integers is a principal ideal domain,
- $\boldsymbol{m} = (m_1, \cdots, m_r)$, an r-tuple of non-zero S-integers.

We may embed F and $\mathfrak{o}_S$ into $F_S = \prod_{v \in S} F_v$ along the diagonal. Let $(d, c)_{n,S} \in \mu_n$ denote the S-Hilbert symbol, the product of local Hilbert symbols at each place $v \in S$, defined for $c, d \in F_S^\times$. Let $\Psi : (F_S^\times)^r \longrightarrow \mathbb{C}$ be any function satisfying

$$\Psi(\varepsilon_1 c_1, \cdots, \varepsilon_r c_r) = \prod_{i=1}^{r} (\varepsilon_i, c_i)_{n,S} \left\{ \prod_{i<j} (\varepsilon_i, c_j)_{n,S}^{-1} \right\} \Psi(c_1, \cdots, c_r) \tag{1.1}$$

for any $\varepsilon_1, \cdots, \varepsilon_r \in \mathfrak{o}_S^\times(F_S^{\times,n})$ and $c_i \in F_S^\times$. Here $(F_S^{\times,n})$ denotes the set of n-th powers in $F_S^\times$. It is proved in [6] that the set $\mathcal{M}$ of such functions is a nonzero but finite-dimensional vector space.

To any such function Ψ and data chosen as above, Weyl group multiple Dirichlet series are functions of r complex variables $\boldsymbol{s} = (s_1, \cdots, s_r) \in \mathbb{C}^r$ of the form

$$Z_\Psi^{(n)}(\boldsymbol{s};\boldsymbol{m};\Phi) = Z_\Psi(\boldsymbol{s};\boldsymbol{m}) = \sum_{\substack{\boldsymbol{c}=(c_1,\cdots,c_r)\in(\mathfrak{o}_S/\mathfrak{o}_S^\times)^r \\ c_i \neq 0}} \frac{H(\boldsymbol{c};\boldsymbol{m})\Psi(\boldsymbol{c})}{\mathbb{N}c_1^{-2s_1}\cdots\mathbb{N}c_r^{-2s_r}}, \tag{1.2}$$

where $\mathbb{N}c$ is the cardinality of $\mathfrak{o}_S/c\mathfrak{o}_S$, and it remains to define the coefficients $H(\boldsymbol{c};\boldsymbol{m})$ in the Dirichlet series. In particular, the function Ψ is not independent of the choice of representatives in $\mathfrak{o}_S/\mathfrak{o}_S^\times$, so the function H must possess complementary transformation properties for the sum to be well-defined.

Indeed, the function H satisfies a "twisted multiplicativity" in $\boldsymbol{c}$, expressed in terms of n-th power residue symbols and depending on the root system Φ, which specializes to the usual multiplicativity when $n = 1$. Recall that the n-th power residue symbol $\left(\frac{c}{d}\right)_n$ is defined when c and d are coprime elements of $\mathfrak{o}_S$ and $\gcd(n, d) = 1$. It depends only on c modulo d, and satisfies the reciprocity law

$$\left(\frac{c}{d}\right)_n = (d, c)_{n,S} \left(\frac{d}{c}\right)_n.$$

(The properties of the power residue symbol and associated S-Hilbert symbols in our notation are set out in [6].) Then given $\boldsymbol{c} = (c_1, \cdots, c_r)$ and $\boldsymbol{c}' = (c_1', \cdots, c_r')$ with $\gcd(c_1 \cdots c_r, c_1' \cdots c_r') = 1$, the function H satisfies

$$\frac{H(c_1c_1', \cdots, c_rc_r';\boldsymbol{m})}{H(\boldsymbol{c};\boldsymbol{m})\, H(\boldsymbol{c}';\boldsymbol{m})} = \prod_{i=1}^r \left(\frac{c_i}{c_i'}\right)_n^{||\alpha_i||^2} \left(\frac{c_i'}{c_i}\right)_n^{||\alpha_i||^2} \prod_{i<j} \left(\frac{c_i}{c_j'}\right)_n^{2\langle\alpha_i,\alpha_j\rangle} \left(\frac{c_i'}{c_j}\right)_n^{2\langle\alpha_i,\alpha_j\rangle}, \tag{1.3}$$

where α_i, $i = 1, \cdots, r$ denote the simple roots of Φ and we have chosen a Weyl group invariant inner product $\langle\cdot,\cdot\rangle$ for our root system embedded into a real vector space of dimension r. The inner product should be normalized so that for any $\alpha, \beta \in \Phi$, both $||\alpha||^2 = \langle\alpha,\alpha\rangle$ and $2\langle\alpha,\beta\rangle$ are integers. We will devote the majority of our attention to Φ of type A_r, and will assume the inner product is chosen so that all roots have length 1.

The function H possesses a further twisted multiplicativity with respect to the parameter $\boldsymbol{m}$. Given any $\boldsymbol{c} = (c_1, \cdots, c_r)$, $\boldsymbol{m} = (m_1, \cdots, m_r)$ and $\boldsymbol{m}' = (m_1', \cdots, m_r')$

with $\gcd(m_1' \cdots m_r', c_1 \cdots c_r) = 1$, H satisfies the twisted multiplicativity relation

$$H(\boldsymbol{c}; m_1 m_1', \cdots, m_r m_r') = \left(\frac{m_1'}{c_1}\right)_n^{-||\alpha_1||^2} \cdots \left(\frac{m_r'}{c_r}\right)_n^{-||\alpha_r||^2} H(\boldsymbol{c}; \boldsymbol{m}). \tag{1.4}$$

As a consequence of properties (1.3) and (1.4) the specification of H reduces to the case where the components of $\boldsymbol{c}$ and $\boldsymbol{m}$ are all powers of the same prime. Given a fixed prime p of $\mathfrak{o}_S$ and any $\boldsymbol{m} = (m_1, \cdots, m_r)$, let $l_i = \mathrm{ord}_p(m_i)$. Then we must specify $H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r})$ for any r-tuple of non-negative integers $\boldsymbol{k} = (k_1, \cdots, k_r)$. For brevity, we will refer to these coefficients as the "p-part" of H. To summarize, specifying a multiple Dirichlet series $Z_\Psi^{(n)}(\boldsymbol{s}; \boldsymbol{m}; \Phi)$ with chosen data is equivalent to specifying the p-parts of H.

Remark 1 *Both the transformation property of Ψ in (1.1) and the definition of twisted multiplicativity in (1.3) depend on an enumeration of the simple roots of Φ. However the product $H \cdot \Psi$ is independent of this enumeration of roots and furthermore well-defined modulo units, according to the reciprocity law. The p-parts of H are also independent of this enumeration of roots.*

The definitions given above apply to any root system Φ. For the remainder of this text, we now take Φ to be of type A, and provide a definition of the p-part of H for these cases. In fact, we will propose two definitions of H, to be referred to as H_Γ and H_Δ, either of which may be used to define the multiple Dirichlet series $Z(\boldsymbol{s}; \boldsymbol{m}; A_r)$. Both definitions will be given in terms of Gelfand-Tsetlin patterns.

By a *Gelfand-Tsetlin pattern of rank r* we mean an array of integers

$$\mathfrak{T} = \left\{ \begin{array}{ccccccccc} a_{00} & & a_{01} & & a_{02} & & \cdots & & a_{0r} \\ & a_{11} & & a_{12} & & & & a_{1r} & \\ & & \ddots & & & & \iddots & & \\ & & & & a_{rr} & & & & \end{array} \right\} \tag{1.5}$$

where the rows interleave; that is, $a_{i-1,j-1} \geqslant a_{i,j} \geqslant a_{i-1,j}$. Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a dominant integral element for SL_{r+1}, so that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$. In the next chapter, we will explain why Gelfand-Tsetlin patterns with top row $(\lambda_1, \cdots, \lambda_r, 0)$ are in bijection with basis vectors for the highest weight module for $SL_{r+1}(\mathbb{C})$ with highest weight λ.

Given non-negative integers $(l_1, \cdots, l_r)$, the coefficients $H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r})$ in both definitions H_Γ and H_Δ will be described in terms of Gelfand-Tsetlin patterns with top row (or equivalently, highest weight vector)

$$\lambda + \rho = (l_1 + l_2 + \cdots + l_r + r, \cdots, l_{r-1} + l_r + 2, l_r + 1, 0). \tag{1.6}$$

We denote by $\mathrm{GT}(\lambda+\rho)$ the set of all Gelfand-Tsetlin patterns having this top row. Here $\rho = (r, r-1, \cdots, 0)$ and $\lambda = (\lambda_1, \cdots, \lambda_{r+1})$ where $\lambda_i = \sum_{j \geqslant i} l_j$.

To any Gelfand-Tsetlin pattern $\mathfrak{T}$, we associate the following pair of functions with image in $\mathbb{Z}^r_{\geqslant 0}$:

$$k_\Gamma(\mathfrak{T}) = (k_{\Gamma,1}(\mathfrak{T}), \cdots, k_{\Gamma,r}(\mathfrak{T})), \qquad k_\Delta(\mathfrak{T}) = (k_{\Delta,1}(\mathfrak{T}), \cdots, k_{\Delta,r}(\mathfrak{T})),$$

where

$$k_{\Gamma,i}(\mathfrak{T}) = \sum_{j=i}^{r} (a_{i,j} - a_{0,j}) \quad \text{and} \quad k_{\Delta,i}(\mathfrak{T}) = \sum_{j=r+1-i}^{r} (a_{0,j-r-1+i} - a_{r+1-i,j}). \tag{1.7}$$

In the language of representation theory, the weight of the basis vector corresponding to the Gelfand-Tsetlin pattern $\mathfrak{T}$ can be read from differences of consecutive row sums in the pattern, so both k_Γ and k_Δ are expressions of the weight of the pattern up to an affine linear transformation.

Then given a fixed r-tuple of non-negative integers $(l_1, \cdots, l_r)$, we make the following two definitions for p-parts of the multiple Dirichlet series:

$$H_\Gamma(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) = \sum_{\substack{\mathfrak{T} \in \mathrm{GT}(\lambda+\rho) \\ k_\Gamma(\mathfrak{T}) = (k_1, \cdots, k_r)}} G_\Gamma(\mathfrak{T}) \tag{1.8}$$

and

$$H_\Delta(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) = \sum_{\substack{\mathfrak{T} \in \mathrm{GT}(\lambda+\rho) \\ k_\Delta(\mathfrak{T}) = (k_1, \cdots, k_r)}} G_\Delta(\mathfrak{T}), \tag{1.9}$$

where the functions G_Γ and G_Δ on Gelfand-Tsetlin patterns will now be defined.

We will associate with $\mathfrak{T}$ two arrays $\Gamma(\mathfrak{T})$ and $\Delta(\mathfrak{T})$. The entries in these arrays are

$$\Gamma_{i,j} = \Gamma_{i,j}(\mathfrak{T}) = \sum_{k=j}^{r} (a_{i,k} - a_{i-1,k}), \qquad \Delta_{i,j} = \Delta_{i,j}(\mathfrak{T}) = \sum_{k=i}^{j} (a_{i-1,k-1} - a_{i,k}), \tag{1.10}$$

with $1 \leqslant i \leqslant j \leqslant r$, and we often think of attaching each entry of the array $\Gamma(\mathfrak{T})$ (resp. $\Delta(\mathfrak{T})$) with an entry of the pattern $a_{i,j}$ lying below the fixed top row. Thus we think of $\Gamma(\mathfrak{T})$ as applying a kind of *right-hand rule* to $\mathfrak{T}$, since $\Gamma_{i,j}$ involves entries above and to the right of $a_{i,j}$ as in (1.10); in Δ we use a *left-hand rule* where $\Delta_{i,j}$ involves entries above and to the left of $a_{i,j}$ as in (1.10). When we represent these arrays graphically, we will right justify the Γ array and left justify the Δ array. For example, if

$$\mathfrak{T} = \left\{ \begin{array}{ccccccc} 12 & & 9 & & 4 & & 0 \\ & 10 & & 5 & & 3 & \\ & & 7 & & 4 & & \\ & & & 6 & & & \end{array} \right\}$$

then

$$\Gamma(\mathfrak{T}) = \left[\begin{array}{rrr} 5 & 4 & 3 \\ & 3 & 1 \\ & & 2 \end{array} \right] \qquad \text{and} \qquad \Delta(\mathfrak{T}) = \left[\begin{array}{lll} 2 & 6 & 7 \\ 3 & 4 & \\ 1 & & \end{array} \right].$$

To provide the definitions of G_Γ and G_Δ corresponding to each array, it is convenient to *decorate* the entries of the Γ and Δ arrays by boxing or circling certain of them. Using the *right-hand rule* with the Γ array, if $a_{i,j} = a_{i-1,j-1}$ then we say $\Gamma_{i,j}$ is *boxed*, and indicate this when we write the array by putting a box around it, while if $a_{i,j} = a_{i-1,j}$ we say it is *circled* (and we circle it). Using the *left-hand rule* to obtain the Δ array, we box $\Delta_{i,j}$ if $a_{i,j} = a_{i-1,j}$ and we circle it if $a_{i,j} = a_{i-1,j-1}$. For example, if

$$\mathfrak{T} = \left\{ \begin{array}{ccccccc} 12 & & 10 & & 4 & & 0 \\ & 10 & & 5 & & 3 & \\ & & 7 & & 5 & & \\ & & & 6 & & & \end{array} \right\}$$

then the decorated arrays are

$$\Gamma(\mathfrak{T}) = \left[\begin{array}{rrr} \textcircled{4} & 4 & 3 \\ & 4 & \boxed{2} \\ & & 1 \end{array} \right], \qquad \Delta(\mathfrak{T}) = \left[\begin{array}{lll} \boxed{2} & 7 & 8 \\ 3 & \textcircled{3} & \\ 1 & & \end{array} \right].$$

We sometimes use the terms *right-hand rule* and *left-hand rule* to refer to both the direction of accumulation of the row differences, and to the convention for decorating these accumulated differences.

If $m, c \in \mathfrak{o}_S$ with $c \neq 0$ define the Gauss sum

$$g(m,c) = \sum_{a \bmod c} \left(\frac{a}{c}\right)_n \psi\left(\frac{am}{c}\right), \tag{1.11}$$

where ψ is a character of F_S that is trivial on $\mathfrak{o}_S$ and no larger fractional ideal. With p now fixed, for brevity let

$$g(a) = g(p^{a-1}, p^a), \qquad h(a) = g(p^a, p^a). \tag{1.12}$$

These functions will only occur with $a > 0$. The reader may check that $g(a)$ is non-zero for any value of a, while $h(a)$ is non-zero only if $n|a$, in which case $h(a) = (q-1)q^{a-1} = \phi(p^a)$, where q is the cardinality of $\mathfrak{o}_S/p\mathfrak{o}_S$. Thus if $n|a$ then $h(a) = \phi(p^a)$, the Euler phi function for $p^a\mathfrak{o}_S$. Let

$$G_\Gamma(\mathfrak{T}) = \prod_{1\leqslant i\leqslant j\leqslant r} \begin{cases} g(\Gamma_{ij}) & \text{if } \Gamma_{ij} \text{ is boxed but not circled in } \Gamma(\mathfrak{T}); \\ q^{\Gamma_{ij}} & \text{if } \Gamma_{ij} \text{ is circled but not boxed;} \\ h(\Gamma_{ij}) & \text{if } \Gamma_{ij} \text{ neither circled nor boxed;} \\ 0 & \text{if } \Gamma_{ij} \text{ both circled and boxed.} \end{cases}$$

We say that the pattern $\mathfrak{T}$ is non-strict if $a_{i,j} = a_{i,j+1}$ for any i, j in the pattern. It is clear from the definitions that $\mathfrak{T}$ is nonstrict if and only if $\Gamma(\mathfrak{T})$ has an entry that is both boxed and circled, so $G_\Gamma(\mathfrak{T}) = 0$ for nonstrict patterns. Also let

$$G_\Delta(\mathfrak{T}) = \prod_{1\leqslant i\leqslant j\leqslant r} \begin{cases} g(\Delta_{ij}) & \text{if } \Delta_{ij} \text{ is boxed but not circled in } \Delta(\mathfrak{T}); \\ q^{\Delta_{ij}} & \text{if } \Delta_{ij} \text{ is circled but not boxed;} \\ h(\Delta_{ij}) & \text{if } \Delta_{ij} \text{ neither circled nor boxed;} \\ 0 & \text{if } \Delta_{ij} \text{ both circled and boxed.} \end{cases}$$

Inserting these respective definitions into the formulas (1.8) and (1.9) completes the two definitions of the p-parts of H_Γ and H_Δ, and with it two definitions for a multiple Dirichlet series $Z_\Psi(\boldsymbol{s}; \boldsymbol{m})$. In [10], the definition H_Γ was used to define the series, and so we will state our theorem on functional equations and analytic continuation of $Z_\Psi(\boldsymbol{s}; \boldsymbol{m})$ using this choice.

Before stating the result precisely, we need to define certain normalizing factors for the multiple Dirichlet series. These have a uniform description for all root systems (see Section 3.3 of [6]), but for simplicity we state them only for type A here. Let

$$\mathbf{G}_n(s) = (2\pi)^{-2(n-1)s} n^{2ns} \prod_{j=1}^{n-2} \Gamma\left(2s - 1 + \frac{j}{n}\right). \tag{1.13}$$

We will identify the weight space for $\mathrm{GL}(r+1, \mathbb{C})$ with $\mathbb{Z}^{r+1}$ in the usual way. For any $\alpha \in \Phi^+$, there exist $1 \leqslant i < j \leqslant r+1$ such that $\alpha = \alpha_{i,j}$ is the root $(0, \cdots, 0, 1, 0, \cdots, -1, 0, \cdots)$ with the 1 in the i-th place and the -1 in the j-th place. We will also denote the simple roots $\alpha_i = \alpha_{i,i+1}$ for $1 \leqslant i \leqslant r$. If $\alpha = \alpha_{i,j}$ is a positive root, then define

$$\mathbf{G}_\alpha(\mathbf{s}) = \mathbf{G}_n\left(\frac{1}{2} + (s_i + s_{i+1} + \cdots + s_{j-1})\right). \tag{1.14}$$

Further let

$$\zeta_\alpha(\mathbf{s}) = \zeta\left(1 + 2n\left(s_i + \cdots + s_{j-1} - \frac{j-i}{2}\right)\right)$$

where ζ is the Dedekind zeta function attached to the number field F. Then the normalized multiple Dirichlet series is given by

$$Z^*_\Psi(\mathbf{s};\mathbf{m}) = \left[\prod_{\alpha\in\Phi^+} \mathbf{G}_\alpha(\mathbf{s})\zeta_\alpha(\mathbf{s})\right] Z_\Psi(\mathbf{s},\mathbf{m}). \tag{1.15}$$

Theorem 1 *The Weyl group multiple Dirichlet series $Z^*_\Psi(\boldsymbol{s};\boldsymbol{m})$ with coefficients H_Γ as in (1.8) has meromorphic continuation to $\mathbb{C}^r$ and satisfies functional equations*

$$Z^*_\Psi(\mathbf{s};\mathbf{m}) = |m_i|^{1-2s_i} Z^*_{\sigma_i\Psi}(\sigma_i\mathbf{s};\mathbf{m}) \tag{1.16}$$

for all simple reflections $\sigma_i \in W$, where

$$\sigma_i(s_i) = 1 - s_i, \qquad \sigma_i(s_j) = \begin{cases} s_i + s_j - 1/2 & \textit{if } i, j \textit{ adjacent,} \\ s_j & \textit{otherwise.} \end{cases}$$

Here $\sigma_i : \mathcal{M} \to \mathcal{M}$ is a linear map defined in [8].

The endomorphisms σ_i of the space $\mathcal{M}$ of functions satisfying (1.1) are the simple reflections in an action of the Weyl group W on $\mathcal{M}$. See [6] and [8] for more information.

This proves Conjecture 2 of [10]. An explicit description of the polar hyperplanes of Z^*_Ψ can be found in Section 7 of [8]. As we will demonstrate in Chapter 3, this theorem ultimately follows from proving the equivalence of the two definitions of the p-part H_Γ and H_Δ offered in (1.8) and (1.9). Because of this implication, and because it is of interest to construct such functions attached to a representation but independent of choices of coordinates (a notion we make precise in subsequent chapters using the crystal description), we consider the equivalence of these two descriptions to be our main theorem.

Theorem 2 *We have $H_\Gamma = H_\Delta$.*

A special role in proving the equivalence of the two definitions H_Γ and H_Δ is played by the *Schützenberger involution*, originally introduced by Schützenberger [37] in the context of tableaux. It was transported to the setting of Gelfand-Tsetlin patterns by Berenstein and Kirillov [27], and defined for general crystals (to be discussed in Chapter 2) by Lusztig [32]. We give its definition now.

Given a Gelfand-Tsetlin pattern (1.5), the condition that the rows interleave means that each $a_{i,j}$ is constrained by the inequalities

$$\min(a_{i-1,j-1}, a_{i+1,j}) \leqslant a_{i,j} \leqslant \max(a_{i-1,j}, a_{i+1,j+1}).$$

This means that we can reflect this entry across the midpoint of this range and obtain another Gelfand-Tsetlin pattern. Thus we replace every entry $a_{i,j}$ in the i-th row by

$$a'_{i,j} = \min(a_{i-1,j-1}, a_{i+1,j}) + \max(a_{i-1,j}, a_{i+1,j+1}) - a_{i,j}.$$

This requires interpretation if $j = i$ or $j = r + 1$. Thus

$$a'_{i,i} = a_{i-1,j-1} + \max(a_{i-1,j}, a_{i+1,j+1}) - a_{i,i}$$

and

$$a'_{i,r} = \min(a_{i-1,r-1}, a_{i+1,r}) + a_{i-1,r} - a_{i,r},$$

while if $i = j = r$, set $a'_{r,r} = a_{r-1,r-1} + a_{r-1,r} - a_{r,r}$. This operation on the entire row will be denoted by t_{r+1-i}. Note that it only affects this lone row in the pattern. Further involutions on patterns may be built out of the t_i, and will be called q_i following Berenstein and Kirillov. Let q_0 be the identity map, and define recursively $q_i = t_1 t_2 \cdots t_i q_{i-1}$. The t_i have order two. They do not satisfy the braid relation, so $t_i t_{i+1} t_i \neq t_{i+1} t_i t_{i+1}$. However $t_i t_j = t_j t_i$ if $|i - j| > 1$ and this implies that the q_i also have order two. The operation q_r is called the *Schützenberger involution.*

For example, let $r = 2$, and let us compute q_2 of a typical Gelfand-Tsetlin pattern. Following the algorithm outlined above,

$$q_2\left(\left\{\begin{array}{ccccc} 9 & & 3 & & 0 \\ & 7 & & 1 & \\ & & 3 & & \end{array}\right\}\right) = \left\{\begin{array}{ccccc} 9 & & 3 & & 0 \\ & 7 & & 2 & \\ & & 4 & & \end{array}\right\}.$$

Indeed, $q_2 = t_1 t_2 t_1$ and we compute:

$$\left\{\begin{array}{ccccc} 9 & & 3 & & 0 \\ & 7 & & 1 & \\ & & 3 & & \end{array}\right\} \xrightarrow{t_1} \left\{\begin{array}{ccccc} 9 & & 3 & & 0 \\ & 7 & & 1 & \\ & & 5 & & \end{array}\right\} \xrightarrow{t_2} \left\{\begin{array}{ccccc} 9 & & 3 & & 0 \\ & 7 & & 2 & \\ & & 5 & & \end{array}\right\}$$
$$\xrightarrow{t_1} \left\{\begin{array}{ccccc} 9 & & 3 & & 0 \\ & 7 & & 2 & \\ & & 4 & & \end{array}\right\}.$$

We will discuss the the relationship between the Schützenberger involution and Theorem 2 in Chapters 4, 5 and 6; see in particular (5.1).

Chapter 2

Crystals and Gelfand-Tsetlin Patterns

We will translate the definitions of the Γ and Δ arrays in (1.10), and hence of the multiple Dirichlet series, into the language of crystal bases. The entries in these arrays and the accompanying boxing and circling rules will be reinterpreted in terms of the Kashiwara operators. Thus, what appeared as a pair of unmotivated functions on Gelfand-Tsetlin patterns in the previous chapter now takes on intrinsic representation theoretic meaning. Despite the conceptual importance of this reformulation, the reader can skip this chapter and the subsequent chapters devoted to crystals with no loss of continuity. For further background information on crystals, we recommend Hong and Kang [23] and Kashiwara [25] as basic references.

We will identify the weight lattice Λ of $\mathfrak{gl}_{r+1}(\mathbb{C})$ with $\mathbb{Z}^{r+1}$. We call the weight $\lambda = (\lambda_1, \cdots, \lambda_{r+1}) \in \mathbb{Z}^{r+1}$ *dominant* if $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$. If furthermore $\lambda_{r+1} \geqslant 0$ we call the dominant weight *effective*. (An effective dominant weight is just a partition of length $\leqslant r+1$.) If λ is a dominant weight then there is a *crystal graph* $\mathcal{B}_\lambda$ with highest weight λ. It is equipped with a *weight function* $\mathrm{wt} : \mathcal{B}_\lambda \longrightarrow \mathbb{Z}^{r+1}$ such that if μ is any weight and if $m(\mu, \lambda)$ is the multiplicity of μ in the irreducible representation of $\mathrm{GL}_{r+1}(\mathbb{C})$ with highest weight λ then $m(\mu, \lambda)$ is also the number of $v \in \mathcal{B}_\lambda$ with $\mathrm{wt}(v) = \mu$. It has operators $e_i, f_i : \mathcal{B}_\lambda \longrightarrow \mathcal{B}_\lambda \cup \{0\}$ $(1 \leqslant i \leqslant r)$ such that if $e_i(v) \neq 0$ then $v = f_i(e_i(v))$ and $\mathrm{wt}(e_i(v)) = \mathrm{wt}(v) + \alpha_i$, and if $f_i(v) \neq 0$ then $e_i(f_i(v)) = v$ and $\mathrm{wt}(f_i(v)) = \mathrm{wt}(v) - \alpha_i$. Here $\alpha_1 = (1, -1, 0, \cdots, 0)$, $\alpha_2 = (0, 1, -1, 0, \cdots, 0)$ etc. are the simple roots in the usual order. These root operators give $\mathcal{B}_\lambda$ the structure of a directed graph with edges labeled from the set $1, 2, \cdots, r$. The vertices v and w are connected by an edge $v \xrightarrow{i} w$ or $v \xrightarrow{f_i} w$ if $w = f_i(v)$.

Remark 2 *The elements of $\mathcal{B}_\lambda$ are basis vectors for a representation of the quantized enveloping algebra $U_q(\mathfrak{gl}_{r+1}(\mathbb{C}))$. Strictly speaking we should reserve f_i for the root operators in this quantized enveloping algebra and so distinguish between f_i and $\tilde{f}_i$ as in [26]. However we will not actually use the quantum group but only the crystal graph, so we will simplify the notation by writing f_i instead of $\tilde{f}_i$, and similarly for the e_i. We will use the terms crystal, crystal base and crystal graph interchangeably.*

The crystal graph $\mathcal{B}_\lambda$ has an involution $\mathrm{Sch} : \mathcal{B}_\lambda \longrightarrow \mathcal{B}_\lambda$ that such that

$$\mathrm{Sch} \circ e_i = f_{r+1-i} \circ \mathrm{Sch}, \qquad \mathrm{Sch} \circ f_i = e_{r+1-i} \circ \mathrm{Sch}. \tag{2.1}$$

In addition to the involution Sch there is a bijection $\psi_\lambda : \mathcal{B}_\lambda \longrightarrow \mathcal{B}_{-w_0\lambda}$ such that

$$\psi_\lambda \circ f_i = e_i \circ \psi_\lambda, \qquad \psi_\lambda \circ e_i = f_i \circ \psi_\lambda. \tag{2.2}$$

Here w_0 is the long Weyl group element. If $\lambda = (\lambda_1, \cdots, \lambda_{r+1})$ is a dominant weight then $-w_0\lambda = (-\lambda_{r+1}, \cdots, -\lambda_1)$ is also a dominant weight so there is a crystal $\mathcal{B}_{-w_0\lambda}$ with that highest weight. The map ψ_λ commutes with Sch and the composition $\phi_\lambda = \mathrm{Sch} \circ \psi_\lambda = \psi_\lambda \circ \mathrm{Sch}$ has the effect

$$\phi_\lambda \circ f_i = f_{r+1-i} \circ \phi_\lambda, \qquad \phi_\lambda \circ e_{r+1-i} = e_i \circ \phi_\lambda. \tag{2.3}$$

The involution Sch was first described by Schützenberger [37] in the context of tableaux. It was transported to the setting of Gelfand-Tsetlin patterns by Berenstein and Kirillov [27], and defined for general crystals by Lusztig [32]. Another useful reference for the involutions is Lenart [30].

If we remove all edges of type r from the crystal graph $\mathcal{B}_\lambda$, then we obtain a crystal graph of rank $r-1$. It inherits a weight function from $\mathcal{B}_\lambda$, which we compose with the projection $\mathbb{Z}^{r+1} \longrightarrow \mathbb{Z}^r$ onto the first r coordinates.

The restricted crystal may be disconnected, in which case it is a disjoint union of crystals of type A_{r-1}, and the crystals that appear in this restriction are described by the following *branching rule*:

$$\mathcal{B}_\lambda = \bigcup_{\substack{\mu = (\mu_1, \cdots, \mu_r) \\ \mu \text{ dominant} \\ \lambda, \mu \text{ interleave}}} \mathcal{B}_\mu \tag{2.4}$$

where the "interleave" condition means that μ runs through dominant weights such that

$$\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_{r+1}.$$

The branching rule is multiplicity-free, meaning that no crystal $\mathcal{B}_\mu$ occurs more than once. Since representations of the quantized enveloping algebra $U_q\big(\mathfrak{gl}_{r+1}(\mathbb{C})\big)$ correspond exactly to representations of $\mathrm{GL}_{r+1}(\mathbb{C})$, follows from the well-known branching rule from $\mathrm{GL}_{r+1}(\mathbb{C})$ to $\mathrm{GL}_r(\mathbb{C})$. See for example Bump [12], Chapter 44.

There is a bijection between the crystal graph with highest weight λ and Gelfand-Tsetlin patterns with top row λ. There are several ways of seeing this. The first way is that, given $v \in \mathcal{B}_\lambda$, we first branch down from A_r to A_{r-1} by the branching rule, which means selecting the unique crystal $\mathcal{B}_\mu$ from (2.4) with $v \in \mathcal{B}_\mu$, that is, the connected component of the restricted crystal which contains v. Then λ and μ are the first two rows of the Gelfand-Tsetlin pattern. Continuing to branch down to $A_{r-2}, A_{r-3}, \cdots$ we may read off the remaining rows of the pattern. Let $\mathfrak{T}_v$ be the resulting Gelfand-Tsetlin pattern.

The crystal $\mathcal{B}_\lambda$ contains $\mathcal{B}_\mu$ if and only if λ and μ interleave, which is equivalent to $-w_0\lambda$ and $-w_0\mu$ interleaving, and hence if and only if $\mathcal{B}_{-w_0\lambda}$ contains $\mathcal{B}_{-w_0\mu}$. The operation ψ_λ in (2.2) which reverses the root operators must be compatible with this branching rule, and so each row of $\mathfrak{T}_{\psi_\lambda v}$ is obtained from the corresponding row of $\mathfrak{T}_v$ by reversing the entries and changing their sign. Thus, denoting by "rev" the operation of reversing an array from left to right and by $-\mathfrak{T}$ the pattern with all entries negated, we have

$$\mathfrak{T}_{\psi_\lambda v} = -\mathfrak{T}_v^{\mathrm{rev}}. \tag{2.5}$$

An alternative way of getting this bijection comes from the interpretation of crystals as crystals of tableaux. We will assume that λ is effective, that is, that its entries are nonnegative.

We recall that Gelfand-Tsetlin patterns with top row λ are in bijection with semi-standard Young tableau with shape λ and labels in $\{1, 2, 3, \cdots, r+1\}$. In this bijection, one starts with a tableau, and successively reduces to a series of smaller tableaux by eliminating the entries. Thus if $r + 1 = 4$, starting with the tableau

$$\mathcal{T} = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

and eliminating $4, 3, 2, 1$ successively one has the following sequence of tableaux:

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} .$$

Now reading off the shapes of these tableaux gives $r+1$ shapes which are the rows of a Gelfand-Tsetlin pattern $\mathfrak{T}$. In this example:

$$\mathcal{T} \mapsto \mathfrak{T} = \left\{ \begin{array}{ccccccc} 4 & & 2 & & 1 & & 0 \\ & 4 & & 1 & & 1 & \\ & & 4 & & 1 & & \\ & & & 2 & & & \end{array} \right\}.$$

In discussing the bijection between Gelfand-Tsetlin patterns and tableaux, we have assumed that λ is effective, but what if it is not? If λ is a dominant weight, so is

$$\lambda + (n^{r+1}) = (\lambda_1 + n, \cdots, \lambda_{r+1} + n)$$

for any n. (As usual, $(n^k) = (n, \cdots, n)$ is the partition with k parts each equal to n.) We will denote the corresponding crystal $\mathcal{B}_{\lambda+(n^{r+1})} = \det^n \otimes \mathcal{B}_\lambda$ since this operation corresponds to tensoring with the determinant character for representations of $\mathfrak{gl}_{r+1}(\mathbb{C})$. There is a bijection from $\mathcal{B}_\lambda$ to $\det^n \otimes \mathcal{B}_\lambda$ which is compatible with the root operators and which shifts the weight by (n^{r+1}). If λ is not effective, still $\lambda + (n^{r+1})$ is effective for sufficiently large n. On the other hand, if λ *is* effective (so there is a bijection with tableaux of shape λ) then it is instructive to consider the effect of this operation on tableaux corresponding to the bijection $\mathcal{B}_\lambda \longrightarrow \det^n \otimes \mathcal{B}_\lambda$. It simply adds n columns of the form

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline r+1 \\ \hline \end{array}$$

at the beginning of the tableau. So if λ is not effective, we may still think of $\mathcal{B}_\lambda$ as being in bijection with a crystal of tableaux with the weight operator shifted by (n^{r+1}), which amounts to "borrowing" n columns of this form.

Returning to the effective case, the tableau $\mathcal{T}$ parametrizes a vector in a tensor power of the standard module of the quantum group $U_q(\mathfrak{sl}_{r+1}(\mathbb{C}))$ as follows. Following the notations in Kashiwara and Nakashima [26] the *standard crystal* (corresponding to the standard representation) has basis $\boxed{i}$ $(i = 1, 2, \cdots, r+1)$. The highest weight vector is $\boxed{1}$ and the root operators have the effect $\boxed{i} \xrightarrow{f_i} \boxed{i+1}$.

The tensor product operation on crystals is described in Kashiwara and Nakashima [26], or in Hong and Kang [23]. If $\mathcal{B}$ and $\mathcal{B}'$ are crystals, then $\mathcal{B} \otimes \mathcal{B}'$ consists of all pairs $x \otimes y$ with $x \in \mathcal{B}$ and $y \in \mathcal{B}'$. The root operators have the following effect:

$$f_i(x \otimes y) = \begin{cases} f_i(x) \otimes y & \text{if } \phi_i(x) > \varepsilon_i(y), \\ x \otimes f_i(y) & \text{if } \phi_i(x) \leqslant \varepsilon_i(y), \end{cases}$$
$$e_i(x \otimes y) = \begin{cases} e_i(x) \otimes y & \text{if } \phi_i(x) \geqslant \varepsilon_i(y), \\ x \otimes e_i(y) & \text{if } \phi_i(x) < \varepsilon_i(y). \end{cases}$$

Here $\phi_i(x)$ is the largest integer ϕ such that $f_i^{\phi}(x) \neq 0$ and similarly $\varepsilon_i(x)$ is the largest integer ε such that $e_i^{\varepsilon}(x) \neq 0$.

Now tableaux are turned into elements of a tensor power of the standard crystal by reading the columns from top to bottom, and taking the columns in order from right to left. Thus the tableau

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \qquad \text{becomes} \qquad \boxed{2} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{4} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{3}\,.$$

The set of vectors coming from tableaux with shape λ form a subcrystal of the tensor power of the standard crystal. This crystal of tableaux has highest weight λ and is isomorphic to $\mathcal{B}_\lambda$. Thus we get bijections

$$\mathcal{B}_\lambda \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{Tableau in } 1, \cdots, r \\ \text{with shape } \lambda \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{Gelfand-Tsetlin patterns} \\ \text{with top row } \lambda \end{array} \right\}. \tag{2.6}$$

This is the same as the bijection between $\mathcal{B}_\lambda$ and Gelfand-Tsetlin patterns that was described previously in terms of branching rules. Indeed, the branching rule for tableaux is as follows. Beginning with a tableau $\mathcal{T}$ in $1, \cdots, r$ of shape λ, erase all r's. This produces a tableau $\mathcal{T}'$ of shape μ where λ and μ interleave, and the Gelfand-Tsetlin pattern of $\mathcal{T}'$ is the Gelfand-Tsetlin pattern corresponding to $\mathcal{T}$ minus its top row.

We will soon explain yet another way of relating the Gelfand-Tsetlin pattern to $v \in \mathcal{B}_\lambda$. This is based on ideas in Lusztig [33], Berenstein and Zelevinsky [3, 2] and Littelmann [31]. Let w be an element of the Weyl group W, and let us give a reduced decomposition of w into simple reflections. That is, if $l(w)$ is the length of w, let $1 \leqslant \Omega_i \leqslant r$ be given ($1 \leqslant i \leqslant l(w)$) such that

$$w = \sigma_{\Omega_1} \sigma_{\Omega_2} \cdots \sigma_{\Omega_N}.$$

We call the sequence $\Omega_1, \cdots, \Omega_N$ a *word*, and if $N = l(w)$ we call the word *reduced.* A reduced word for w_0 is a *long word.* Now if $v \in \mathcal{B}_\lambda$ let us apply f_{Ω_1} to v as many times as we can. That is, let b_1 be the largest integer such that $f_{\Omega_1}^{b_1} v \neq 0$. Then let b_2 be the largest integer such that $f_{\Omega_2}^{b_2} f_{\Omega_1}^{b_1} v \neq 0$. Let $v' = f_{\Omega_N}^{b_N} \cdots f_{\Omega_1}^{b_1} v$. We summarize this situation symbolically as follows:

$$v \begin{bmatrix} b_1 & \cdots & b_N \\ \Omega_1 & \cdots & \Omega_N \end{bmatrix} v' . \tag{2.7}$$

We refer to this as a *path*.

The crystal $\mathcal{B}_\lambda$ has a unique highest weight vector v_{high} such that $\text{wt}(v_{\text{high}}) = \lambda$, and a unique lowest weight vector v_{low} such that $\text{wt}(v_{\text{low}}) = w_0(\lambda)$. Thus $w_0(\lambda) = (\lambda_{r+1}, \cdots, \lambda_1)$.

Lemma 1 *If $w = w_0$ then (2.7) implies that $v' = v_{\text{low}}$. In this case the integers $(b_1, \cdots, b_N)$ determine the vector v.*

Proof See Littelmann [31] or Berenstein and Zelevinsky [2] for the fact that $v' = v_{\text{low}}$. (We are using f_i instead of the e_i that Littelmann uses, but the methods of proof are essentially unchanged.) Alternatively, the reader may prove this directly by pushing the arguments in Proposition 1 below a bit further. The fact that the b_i determine v follows from $v_{\text{low}} = f_{\Omega_N}^{b_N} \cdots f_{\Omega_1}^{b_1} v$ since then $v = e_{\Omega 1}^{b_1} \cdots e_{\Omega_N}^{b_N} v_{\text{low}}$. □

The Gelfand-Tsetlin pattern can be recovered intrinsically from the location of a vector in the crystal as follows. Assume (2.7) with $w = w_0$. Let

$$\text{BZL}_\Omega(v) = \text{BZL}_\Omega^{(f)}(v) = (b_1, b_2, \cdots, b_N).$$

There are many reduced words representing w_0, but two will be of particular concern for us. If either

$$\Omega = \Omega_\Gamma = (1, 2, 1, 3, 2, 1, \cdots, r, r-1, \cdots, 3, 2, 1) \tag{2.8}$$

or

$$\Omega = \Omega_\Delta = (r, r-1, r, r-2, r-1, r, \cdots, 1, 2, 3, \cdots, r), \tag{2.9}$$

then Littelmann showed that

$$\begin{array}{rcl} b_1 & \geqslant & 0 \\ b_2 \geqslant b_3 & \geqslant & 0 \\ b_4 \geqslant b_5 \geqslant b_6 & \geqslant & 0 \\ \vdots & & \end{array} \tag{2.10}$$

and that these inequalities characterize the possible patterns BZL_Ω. See in particular Theorem 4.2 of Littelmann [31], and Theorem 5.1 for this exact statement for Ω_Γ.

Proposition 1 *Let $v \longmapsto \mathfrak{T} = \mathfrak{T}_v$ be the bijection defined by (2.6) from $\mathcal{B}_\lambda$ to the set of Gelfand-Tsetlin patterns with top row λ.*
(i) If Ω is the word Ω_Γ defined by (2.8) in Lemma 1, so

$$v \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & \cdots & b_{N-r+1} & b_{N-r+2} & \cdots & b_{N-2} & b_{N-1} & b_N \\ 1 & 2 & 1 & 3 & 2 & 1 & \cdots & r & r-1 & \cdots & 3 & 2 & 1 \end{bmatrix} v_{\text{low}},$$

where $N = \frac{1}{2}r(r+1)$. Then, with $\Gamma(\mathfrak{T})$ as defined in (1.10),

$$\Gamma(\mathfrak{T}_v) = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots \\ & b_4 & b_5 & b_6 \\ & & b_2 & b_3 \\ & & & b_1 \end{bmatrix}.$$

(ii) If Ω is the word Ω_Δ defined in (2.9) in Lemma 1, so

$$v \begin{bmatrix} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & \cdots & z_{N-r+1} & z_{N-r+2} & \cdots & z_{N-2} & z_{N-1} & z_N \\ r & r-1 & r & r-2 & r-1 & r & \cdots & 1 & 2 & \cdots & r-2 & r-1 & r \end{bmatrix} v_{\text{low}},$$

then, with $\Delta(\mathfrak{T})$ as in (1.10),

$$\Delta(q_r\mathfrak{T}_v) = \begin{bmatrix} \vdots & \vdots & \vdots & \iddots \\ z_6 & z_5 & z_4 & \\ z_3 & z_2 & & \\ z_1 & & & \end{bmatrix}.$$

(iii) We have $\mathfrak{T}_{\text{Sch}(v)} = q_r\mathfrak{T}_v$.

Proof Most of this is in Littelmann [31], Berenstein and Zelevinsky [2] and Berenstein and Kirillov [27]. However it is also possible to see this directly from Kashiwara's description of the root operators by translating to tableaux, and so we will explain this.

Let $\Omega = \Omega_\Gamma$. We consider a Gelfand-Tsetlin pattern

$$\mathfrak{T} = \mathfrak{T}_v = \left\{ \begin{array}{ccccc} \ddots & & \vdots & & \iddots \\ & a_{r-1,r-1} & & a_{r-1,r} & \\ & & a_{rr} & & \end{array} \right\}$$

with corresponding tableau $\mathcal{T}$. Then a_{rr} is the number of 1's in $\mathcal{T}$, all of which must occur in the first row since $\mathcal{T}$ is column strict. In the tensor these correspond to $\boxed{1}$'s. Applying f_1 will turn some of these to $\boxed{2}$'s. In fact it follows from the definitions that the number b_1 of times that f_1 can be applied is the number of 1's in the first row of $\mathcal{T}$ that are not above a 2 in the second row. Now the number of 2's in the second row is $a_{r-1,r}$. Thus $b_1 = a_{rr} - a_{r-1,r}$.

For example if

$$\mathfrak{T} = \left\{ \begin{array}{ccccccc} 10 & & 5 & & 3 & & 0 \\ & 9 & & 4 & & 2 & \\ & & 7 & & 3 & & \\ & & & 5 & & & \end{array} \right\} \leftrightarrow \mathcal{T} = \begin{array}{cccccccccc} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\ 2 & 2 & 2 & 3 & 4 \\ 3 & 3 & 4 \end{array}$$

then we can apply f_1 twice (so $b_1 = a_{33} - a_{23} = 5 - 3 = 2$) and we obtain

$$\begin{array}{cccccccccc} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 4 \\ 2 & 2 & 2 & 3 & 4 \\ 3 & 3 & 4 \end{array}$$

Now b_2 is the number of times we can apply f_2. This will promote $\boxed{2} \longrightarrow \boxed{3}$ but only if the 2 in the tableau is not directly above a 3. One 2 will be promoted from the second row ($1 = a_{23} - a_{13} = 3 - 2$) and three will be promoted from the first row ($3 = a_{22} - a_{12} = 7 - 4$). Thus $b_2 = a_{22} + a_{23} - a_{12} - a_{13}$. This produces the tableau

$$\begin{array}{cccccccccc} 1 & 1 & 1 & 2 & 3 & 3 & 3 & 3 & 3 & 4 \\ 2 & 2 & 3 & 3 & 4 \\ 3 & 3 & 4 \end{array}$$

After this, we can apply f_1 once ($1 = a_{23} - a_{13} = 3 - 2$) promoting one 1 and giving

$$\begin{array}{cccccccccc} 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 4 \\ 2 & 2 & 3 & 3 & 4 \\ 3 & 3 & 4 \end{array}$$

Thus $b_3 = a_{23} - a_{13}$. After this, we apply f_3 seven times promoting two 3's in the third row ($2 = 2 - 0 = a_{31} - a_{30}$), one 3 in the second row ($1 = 4 - 3 = a_{12} - a_{02}$) and and four 3's in the first row ($4 = 9 - 5 = a_{11} - a_{01}$) to obtain

$$\begin{array}{cccccccccc} 1 & 1 & 2 & 2 & 3 & 4 & 4 & 4 & 4 & 4 \\ 2 & 2 & 3 & 4 & 4 \\ 4 & 4 & 4 \end{array}$$

Thus $b_4 = a_{11} + a_{12} + a_{13} - a_{01} - a_{02} - a_{03}$. One continues in this way.

From this discussion (i) is clear. We refer to Berenstein and Kirillov [27] for the computation of the involution Sch. Thus we will refer to [27] for (iii) and using (iii) we will prove (ii). By (iii) and (2.5) the map $\phi_\lambda : \mathcal{B}_\lambda \longrightarrow \mathcal{B}_{-w_0\lambda}$ satisfying (2.3) has the effect

$$\mathfrak{T}_{\phi_\lambda v} = q_r \mathfrak{T}_{\psi_v} = (-q_r \mathfrak{T}_v)^{\text{rev}}. \tag{2.11}$$

Since ϕ_λ changes f_i to f_{r+1-i} it replaces $b_1, \cdots, b_N$ (computed for $\phi_\lambda v \in \mathcal{B}_{-w_0\lambda}$) by $z_1, \cdots, z_N$. It is easy to see from the definition (1.10) that $\Gamma(-\mathfrak{T}^{\text{rev}}) = \Delta(\mathfrak{T})^{\text{rev}}$, and (ii) follows. $\square$

Now let us reinterpret the factors $G_\Gamma(\mathfrak{T}_v)$ and $G_\Delta(q_r\mathfrak{T}_v)$ defined in Chapter 1. It follows from Proposition 1 that the numbers Γ_{ij} and Δ_{ij} that appear in the respective arrays are exactly the quantities that appear in $\text{BZL}_\Omega(v)$ when $\Omega = \Omega_\Gamma$ or Ω_Δ, and we have only to describe the circling and boxing decorations.

The circling is clear: we circle b_i if either $i \in \{1, 3, 6, 10, \cdots\}$ (so b_i is the first element of its row) and $b_i = 0$, or if $i \notin \{1, 3, 6, 10, \cdots\}$ and $b_i = b_{i+1}$. This is a direct translation of the circling definition in Chapter 1.

Let us illustrate this with an example. In Figure 2.1 we compute $\Gamma(\mathfrak{T}_v)$ for a vertex of the A_2 crystal with highest weight (5,3,0).

$$v \begin{bmatrix} 0 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix} v_{\text{low}}$$

so that $b_1 = 0$, $b_2 = b_3 = 2$. The inequalities (2.10) assert that $b_1 \geqslant 0$ and $b_2 \geqslant b_3 \geqslant 0$. Since two of these are sharp, we circle b_1 and b_2 and

$$\Gamma(\mathfrak{T}_v) = \begin{bmatrix} \textcircled{2} & 2 \\ & \textcircled{0} \end{bmatrix}, \qquad \mathfrak{T}_v = \left\{ \begin{matrix} 5 & & 3 & & 0 \\ & 3 & & 2 & \\ & & 2 & & \end{matrix} \right\}.$$

As for the boxing, the condition has an interesting reformulation in terms of the crystal, which we describe next. Given the path

$$v, f_{\Omega_1}v, f_{\Omega_1}^2 v, \cdots, f_{\Omega_1}^{b_1}v, f_{\Omega_2}f_{\Omega_1}^{b_1}v, \cdots, f_{\Omega_2}^{b_2}f_{\Omega_1}^{b_1}v, f_{\Omega_3}f_{\Omega_2}^{b_2}f_{\Omega_1}^{b_1}v, \cdots, f_{\Omega_N}^{b_N} \cdots f_{\Omega_1}^{b_N}v = v_{\text{low}}$$

through the crystal from v to v_{low}, the b_j are the lengths of consecutive moves along edges f_{Ω_j} in the path. (These are depicted by straight-line segments in the figure.) If u is any vertex and $1 \leqslant i \leqslant r$, then the *$i$-string through u* is the set of vertices that can be obtained from u by repeatedly applying either e_i or f_i. The boxing condition

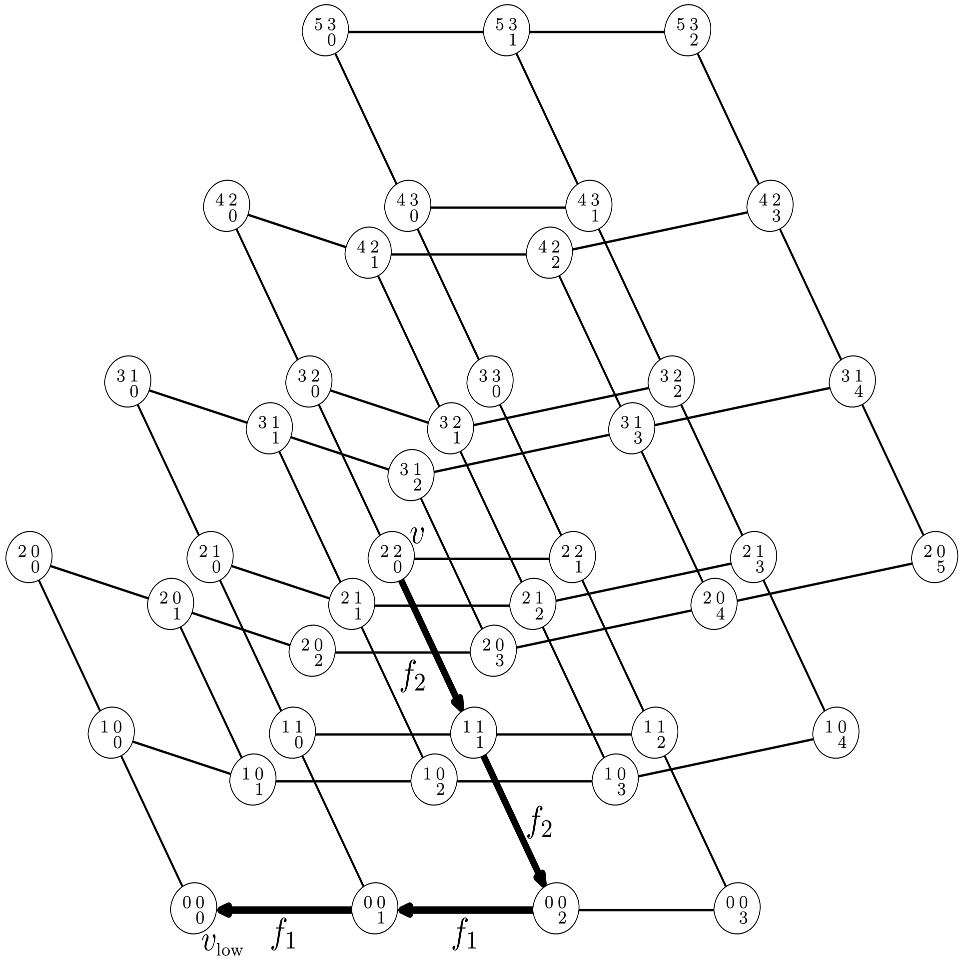


Figure 2.1: **The circling rule.** The crystal graph has highest weight $\lambda = (5, 3, 0)$. The element v_{low} has lowest weight $w_0\lambda = (0, 3, 5)$, and v has weight $(2, 0, 3)$. The labels of the vertices are the Γ arrays. The word $\Omega_\Gamma = 121$ is used to compute $\Gamma(\mathfrak{T}_v)$. The root operator f_1 moves left along crystal edges, and the root operator f_2 moves down and to the right. The crystal has been drawn so that elements of a given weight are placed in diagonally aligned clusters.

then amounts to the assumption that the canonical path contains the entire Ω_t string through $f_{\Omega_{t-1}}^{b_{t-1}} \cdots f_{\Omega_1}^{b_1} v$. That is, the condition for b_t to be boxed is that

$$e_{\Omega_t} f_{\Omega_{t-1}}^{b_{t-1}} \cdots f_{\Omega_1}^{b_1} v = 0.$$

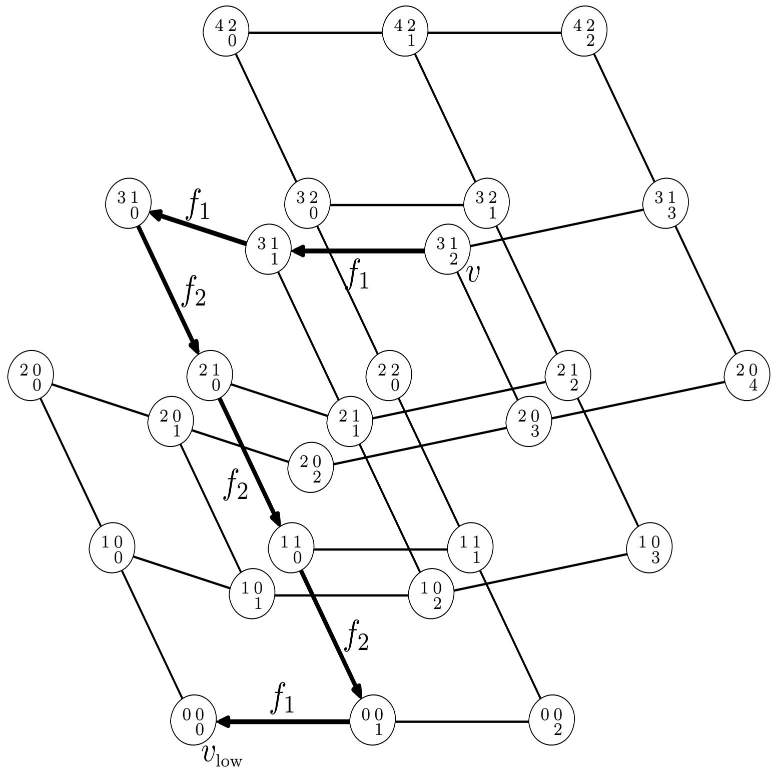


Figure 2.2: **The boxing rule.** The crystal graph $\mathcal{B}_\lambda$ with $\lambda = (4, 2, 0)$. The element v_{low} has lowest weight $w_0\lambda = (0, 2, 4)$, and v has weight $(3, 2, 1)$. The word $\Omega_\Gamma = 121$ is used to compute $\Gamma(\mathfrak{T}_v)$. The root operator f_1 moves left along edges, and the root operator f_2 moves down and to the right. The crystal has been drawn so that elements of a given weight are placed in diagonally aligned clusters.

Here is an example. Let $\lambda = (4, 2, 0)$, and let $\Omega = \Omega_\Gamma = (1, 2, 1)$. Then (see Figure 2.2) we have $b_1 = 2$, $b_2 = 3$ and $b_3 = 1$. Since the path includes the entire 2-string through $f_1^2 v$ (or equivalently, since $e_2 f_1^2 v = 0$) we box b_2 and

$$\Gamma(\mathfrak{T}_v) = \left\{ \begin{array}{cc} \boxed{3} & 1 \\ & 2 \end{array} \right\}, \qquad \mathfrak{T}_v = \left\{ \begin{array}{ccccc} 4 & & 2 & & 0 \\ & 4 & & 1 & \\ & & 3 & & \end{array} \right\}.$$

It is not hard to see that the decorations of $\Gamma(\mathfrak{T}_v)$ described this way agree with those already defined in Chapter 1.

Now that we have explained the boxing and circling rules geometrically for the

BZL pattern, it is natural to make the definition

$$G_\Gamma(v) = G_\Gamma^{(f)}(v) = \prod_{i=1}^{\frac{1}{2}r(r+1)} \begin{cases} q^{b_i} & \text{if } b_i \text{ is circled,} \\ g(b_i) & \text{if } b_i \text{ is boxed,} \\ h(b_i) & \text{if } b_i \text{ is neither circled nor boxed,} \\ 0 & \text{if } b_i \text{ is both circled and boxed,} \end{cases} \tag{2.12}$$

where the b_i are as in Proposition 1 (i), and the boxing and circling rules are to be discussed. The definition of $G_\Delta(v) = G_\Delta^{(f)}(v)$ is identical, except that we use the z_i in Proposition 1 (ii).

$$G_\Gamma(v) = G_\Gamma(\mathfrak{T}_v), \qquad G_\Delta(v) = G_\Delta(\mathfrak{T}_v). \tag{2.13}$$

In order to finish describing the p-part of the multiple Dirichlet series in completely we need to describe k_Γ and k_Δ. If $\mathfrak{T}$ is a Gelfand-Tsetlin pattern, let $d_i = d_i(\mathfrak{T})$ be the sum of the i-th row of $\mathfrak{T}$, so that if $\mathfrak{T}$ is as in (1.5), then $d_i = \sum_{j=i}^r a_{ij}$.

Proposition 2 *If $v \in \mathcal{B}_\lambda$ and d_i are the row sums of $\mathfrak{T}_v$, then*

$$\mathrm{wt}(v) = (d_r, d_{r-1} - d_r, \cdots, d_0 - d_1). \tag{2.14}$$

Moreover if $b_1, \cdots, b_{\frac{1}{2}r(r+1)}$ are as in Proposition 1 (i),

$$\langle \mathrm{wt}(v) - w_0(\lambda), \rho\rangle = \sum_i b_i. \tag{2.15}$$

Proof Let $b_1, \cdots, b_{\frac{1}{2}r(r+1)}$ be as in Proposition 1 (i). Thus they are the entries in $\Gamma(\mathfrak{T})$. Since v_{low} is obtained from v by applying f_1 $b_1 + b_3 + b_6 + \ldots$ times, f_2 $b_2 + b_5 + b_9$ times, etc., we have

$$\mathrm{wt}(v) - w_0(\lambda) = \mathrm{wt}(v) - \mathrm{wt}(v_{\text{low}}) = k_r\alpha_1 + k_{r-1}\alpha_2 + \ldots + k_1\alpha_r \tag{2.16}$$

where $k_r = b_1 + b_3 + b_6 + \ldots$, $k_{r-1} = b_2 + b_5 + \ldots$. Now (2.15) follows since $\langle \alpha_i, \rho\rangle = 1$.

By Proposition 1 (i) the b_i are the entries in $\Gamma(\mathfrak{T}_v)$. In particular $b_1 = a_{r,r} - a_{r-1,r}$, $b_3 = a_{r-1,r} - a_{r-2,r}$, etc. and so the sum defining k_r collapses, and since $a_{0,i} = \lambda_{i+1}$ we have

$$k_r = a_{r,r} - a_{0,r} = d_r - \lambda_{r+1}.$$

Similarly

$$\begin{aligned} k_{r-1} &= d_{r-1} - \lambda_r - \lambda_{r+1} \\ k_{r-2} &= d_{r-2} - \lambda_{r-1} - \lambda_r - \lambda_{r+1} \\ &\vdots \end{aligned}$$

In view of (1.7) the definitions of the previous chapter, $k_i = k_{\Gamma,i}$.

Remembering that $\mathrm{wt}(v_{\mathrm{low}}) = w_0(\lambda) = (\lambda_{r+1}, \lambda_r, \cdots, \lambda_1)$, this shows that

$$\begin{aligned} \mathrm{wt}(v) - (\lambda_r, \lambda_{r-1}, \cdots, \lambda_1) &= \\ (d_r - \lambda_{r+1})\alpha_1 + (d_{r-1} - \lambda_r - \lambda_{r+1})\alpha_2 + \ldots. + (d_1 - \lambda_2 - \ldots - \lambda_r)\alpha_r &= \\ (d_r - \lambda_r, d_{r-1} - d_r - \lambda_{r-1}, \cdots, -d_1 + \lambda_2 + \ldots + \lambda_r). \end{aligned}$$

Since $d_0 = \lambda_1 + \ldots + \lambda_r$, we get (2.14). □

Proposition 3 *Let $v \in \mathcal{B}_\lambda$, and let $k_i = k_{\Gamma,i}(\mathfrak{T}_v)$. The k_i are the unique integers such that*

$$\sum_{i=1}^{r} k_i \alpha_i = \lambda - w_0\left(\mathrm{wt}(v)\right). \tag{2.17}$$

Proof Equation (2.17) applies from applying w_0 to (2.16), since $w_0\alpha_i = \alpha_{r+1-i}$. The uniqueness of the k_i comes from the linear independence of the k_i. □

Theorem 3 *We have*

$$H_\Gamma(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) = \sum_{\substack{v \in \mathcal{B}_{\lambda+\rho} \\ \mathrm{wt}(v) = w_0\left(\lambda - \sum_i k_i\alpha_i\right)}} G_\Gamma(v),$$

$$H_\Delta(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) = \sum_{\substack{v \in \mathcal{B}_{\lambda+\rho} \\ \mathrm{wt}(v) = w_0\left(\lambda - \sum_i k_i\alpha_i\right)}} G_\Delta(v),$$

where $\lambda = (\lambda_1, \cdots, \lambda_{r+1})$ with $\lambda_i = \sum_{j \geqslant i} l_i$.

In view of Theorem 2, these two expressions are equal.

Proof This is clear in view of (2.13) and (2.17). □

Chapter 3

Tokuyama's Theorem

It was shown in [7] that the p-parts of the Weyl group multiple Dirichlet series $Z_\Psi(\boldsymbol{s};\boldsymbol{m})$ can be identified with Whittaker functions of Eisenstein series on the n-fold metaplectic cover of GL_{r+1}. When $n = 1$, these can be computed explicitly by the formula of Shintani [38] and Casselman and Shalika [14]: they are the values of the characters of irreducible representations of the L-group $\mathrm{GL}_{r+1}(\mathbb{C})$.

Let $\boldsymbol{z} = (z_1, \cdots, z_{r+1})$ be complex numbers that in the Weyl character formula are the eigenvalues of an element of $\mathrm{GL}_{r+1}(\mathbb{C})$. In the application to the Casselman-Shalika formula we will take the z_i to be the Langlands parameters. (In terms of the s_i the z_i are determined by the conditions that $\prod z_i = 1$ and $z_i/z_{i+1} = q^{1-2s_{r+1-i}}$.) If $\mu = (\mu_1, \cdots, \mu_{r+1})$ is a weight and $\boldsymbol{z} = (z_1, \cdots, z_{r+1})$ we will denote $\boldsymbol{z}^\mu = \prod z_i^{\mu_i}$. By the Weyl character formula, an irreducible character of $\mathrm{GL}_{r+1}(\mathbb{C})$ is of the form

$$\frac{\sum_{w\in W}(-1)^{l(w)}\boldsymbol{z}^{w(\lambda+\rho)}}{\boldsymbol{z}^{-\rho}\prod_{\alpha\in\Phi^+}(1-\boldsymbol{z}^\alpha)}.$$

The p-part of the normalizing factor of the Eisenstein series is

$$\frac{1}{\prod_{\alpha\in\Phi^+}(1-q^{-1}\boldsymbol{z}^\alpha)}, \tag{3.1}$$

which resembles the denominator in the Weyl character formula – except for a "shift," i.e. the q^{-1}). By (1.15) this is also the p-part of the normalizing factor of $Z_\Psi(\mathbf{s};\mathbf{m})$.

Tokuyama [41] gave a deformation of the Weyl character formula; in this deformation, the denominator has a parameter t, which can be specialized to $-q^{-1}$ so that Tokuyama's denominator matches (3.1). The numerator is a sum not over the Weyl group, but over Gelfand-Tsetlin patterns. When $n = 1$, the numerator in Tokuyama's formula is exactly the p-part of $Z_\Psi(\mathbf{s};\mathbf{m})$.

We will state and prove Tokuyama's formula, then translate it into the language of crystals.

If $\mathfrak{T}$ is a Gelfand-Tsetlin pattern in the notation (1.5) let $s(\mathfrak{T})$ be the number of entries a_{ij} with $i > 0$ such that $a_{i-1,j-1} < a_{ij} < a_{i-1,j}$. Let $l(\mathfrak{T})$ be the number of entries a_{ij} with $i > 0$ such that $a_{ij} = a_{i-1,j-1}$. Thus $l(\mathfrak{T})$ is the number of boxed elements in $\Gamma(\mathfrak{T})$, and $s(\mathfrak{T})$ is the number of entries that are neither boxed nor circled. For $0 \leqslant i \leqslant r$ let $d_i(\mathfrak{T}) = \sum_{j=i}^{r} a_{ij}$ be the i-th row sum of $\mathfrak{T}$.

We will denote by $s_\lambda(z_1, \cdots, z_{r+1})$ the standard Schur polynomial (Macdonald [34]). Thus s_λ is a symmetric polynomial and if z_i are the eigenvalues of $g \in \mathrm{GL}_{r+1}(\mathbb{C})$ then $s_\lambda(z_1, \cdots, z_{r+1}) = \chi_\lambda(g)$, where χ_λ is the character of the irreducible representation with highest weight λ. In this Chapter there will be an induction on r, so we will sometimes write $\rho = \rho_r = (r, r-1, \cdots, 0)$. Let $\mathrm{GT}(\lambda) = \mathrm{GT}_r(\lambda)$ be the set of Gelfand-Tsetlin patterns with $r+1$ rows having top row λ.

Theorem 4 (Tokuyama [41]) *We have*

$$\sum_{\substack{\mathfrak{T} \in \mathrm{GT}(\lambda+\rho) \\ \mathfrak{T} \text{ strict}}} (t+1)^{s(\mathfrak{T})} t^{l(\mathfrak{T})} z_1^{d_r} z_2^{d_{r-1}-d_r} \cdots z_{r+1}^{d_0-d_1} = \left\{ \prod_{i<j} (z_j + z_i t) \right\} s_\lambda(z_1, \cdots, z_{r+1}). \tag{3.2}$$

For comparison with the original paper, we note that we have reversed the order of the z_i, which does not affect the Schur polynomial since it is symmetric. The following proof is essentially Tokuyama's original one.

Proof There is a homomorphism $\Lambda^{(r+1)} \longrightarrow \Lambda^{(r)}$ in which one sets $z_{r+1} \longmapsto 1$. The homomorphism is not injective, but its restriction to the homogeneous part of Λ of fixed degree r is injective. We note that as polynomials in the z_i both sides of (3.2) are homogeneous of degree $d_0 = |\lambda| + \frac{1}{2} r(r+1)$. (If λ is a partition then $|\lambda| = \sum_i \lambda_i|$. See Macdonald [34] for background on partitions and symmetric functions.) Two homogeneous polynomials of the same degree are equal if they are equal when $z_{r+1} = 1$, so it is sufficient to show that (3.2) is true when $z_{r+1} = 1$, and for this we may assume inductively that the formula is true for $r-1$.

The branching rule from $\mathrm{GL}_{r+1}(\mathbb{C})$ to $\mathrm{GL}_r(\mathbb{C})$, already mentioned in connection with (2.4), may be expressed as

$$s_\lambda(z_1, \cdots, z_r, 1) = \sum_{\mu \text{ interleaves } \lambda} s_\mu(z_1, \cdots, z_r). \tag{3.3}$$

See for example Bump [12], Chapter 44.

Also on setting $z_{r+1} = 1$,

$$\left\{ \prod_{1 \leqslant i < j \leqslant r+1} (z_j + z_i t) \right\} \text{ becomes } \left\{ \prod_{1 \leqslant i < j \leqslant r} (z_j + z_i t) \right\} \left[\sum_{k=0}^{r} t^k e_k(z_1, \cdots, z_r) \right],$$

where e_k is the k-th elementary symmetric polynomial, that is, the sum of all square-free monomials of degree k.

Now we recall Pieri's formula in the form

$$e_k s_\mu = \sum_{\substack{\nu \perp |\mu| + k \\ \nu \backslash \mu \text{ is a vertical strip}}} s_\nu.$$

See for example Bump [12], Chapter 42. The notation $\nu \perp |\mu| + k$ means that ν is a partition of $|\mu| + k$. The condition that $\nu \backslash \mu$ is a vertical strip means that the Young diagram of ν contains the Young diagram of μ, and that the skew-diagram $\nu \backslash \mu$ has no two entries in the same row.

The condition means that each $\nu_i = \mu_i$ or $\mu_i + 1$, and that $\nu_i = \mu_i + 1$ exactly k times. Thus when $z_{r+1} = 1$ the right-hand side of (3.2) becomes

$$\sum_k \left\{ \prod_{1 \leqslant i < j \leqslant r} (z_j + z_i t) \right\} t^k \sum_{\mu \text{ interleaves } \lambda} \sum_{\substack{\nu \perp |\mu| + k \\ \nu \backslash \mu \text{ is a vertical strip}}} s_\nu(z_1, \cdots, z_r).$$

Now by induction

$$\left\{ \prod_{1 \leqslant i < j \leqslant r} (z_j + z_i t) \right\} s_\nu(z_1, \cdots, z_r) = \sum_{\substack{\mathfrak{T}' \in \mathrm{GT}_{r-1}(\nu + \rho_{r-1}) \\ \mathfrak{T}' \text{ strict}}} (t+1)^{s(\mathfrak{T}')} t^{l(\mathfrak{T}')} z_1^{d'_{r-1}} z_2^{d'_{r-2} - d'_{r-1}} \cdots z_r^{d'_0 - d'_1},$$

where d'_i are the row sums of $\mathfrak{T}'$. We substitute this and interchange the order of summation and make the summation over μ the innermost sum. The condition that $\nu \backslash \mu$ is a vertical strip means that $\mu_i \leqslant \nu_i \leq \mu_i + 1$. Combining this with the fact that μ interleaves λ we have

$$\lambda_i + 1 \geq \mu_i + 1 \geq \nu_i \geq \mu_i \geq \lambda_{i+1} \tag{3.4}$$

and therefore $\nu + \rho_{r-1}$ interleaves $\lambda + \rho_r$. Since $k = |\nu| - |\mu|$, the right hand side of (3.2) is

$$\sum_{\substack{\nu \\ \nu + \rho_{r-1} \text{ interleaves } \lambda + \rho_r}} \sum_{\substack{\mathfrak{T}' \in \mathrm{GT}_{r-1}(\nu + \rho_{r-1}) \\ \mathfrak{T}' \text{ strict}}} (t+1)^{s(\mathfrak{T}')} t^{l(\mathfrak{T}')}$$

$$\left[\sum_{\substack{\mu \text{ interleaving } \lambda \\ \nu \supset \mu \\ \nu \backslash \mu \text{ is a vertical strip}}} t^{|\nu| - |\mu|} \right] z_1^{d'_{r-1}} z_2^{d'_{r-2} - d'_{r-1}} \cdots z_r^{d'_0 - d'_1}.$$

Now we assemble $\lambda + \rho_r$ and the Gelfand-Tsetlin pattern $\mathfrak{T}'$ into a big Gelfand-Tsetlin pattern $\mathfrak{T}$. The row sums of $\mathfrak{T}$ and $\mathfrak{T}'$ are the same except the top row, so $d_i = d'_{i-1}$ for $1 \leqslant i \leqslant r$. We may just as well sum over $\mathfrak{T}$, in which case $\mathfrak{T}'$ is the pattern obtained by discarding the top row of $\mathfrak{T}$. We get

$$\sum_{\substack{\mathfrak{T} \in \mathrm{GT}_r(\lambda + \rho) \\ \mathfrak{T} \text{ strict}}} (t+1)^{s(\mathfrak{T}')} t^{l(\mathfrak{T}')} \left[\sum_{\substack{\mu \text{ interleaving } \lambda \\ \nu \supset \mu \\ \nu \backslash \mu \text{ is a vertical strip}}} t^{|\nu| - |\mu|} \right] z_1^{d_r} z_2^{d_{r-1} - d_r} \cdots z_{r-1}^{d_1 - d_2}.$$

We evaluate the term in brackets. It is

$$\prod_{i=1}^{r} \left[\sum_{\substack{\lambda_i \geqslant \mu_i \geqslant \lambda_{i+1} \\ \mu_i = \nu_i \text{ or } \nu_i - 1}} t^{\nu_i - \mu_i} \right].$$

We now show that this equals $(t+1)^{s(\mathfrak{T}) - s(\mathfrak{T}')} t^{l(\mathfrak{T}) - l(\mathfrak{T}')}$. By (3.4), if $\nu_i = \lambda_i + 1$ then $\nu_i = \mu_i + 1$ and so $t^{\nu_i - \mu_i} = 1$. This is the case that the i-th entry of the second row in $\mathfrak{T}$ is boxed. The number of such terms is $l(\mathfrak{T}) - l(\mathfrak{T}')$ and so we have a contribution of $t^{l(\mathfrak{T}) - l(\mathfrak{T})}$; this is the case where the i-th entry of the second row in $\mathfrak{T}$ is circled. If $\nu_i = \lambda_{i+1}$, then (3.4) implies that $\nu_i = \mu_i$ and there is no contribution from these factors. In the remaining cases we have $\lambda_i + 1 > \nu_i > \lambda_{i+1}$ both t and 1 can occur. The number of such terms is $s(\mathfrak{T}) - s(\mathfrak{T}')$, and so we have a contribution of $(t+1)^{s(\mathfrak{T})-s(\mathfrak{T}')}$.

$$\sum_{\substack{\lambda_i \geqslant \mu_i \geqslant \lambda_{i+1} \\ \mu_i = \nu_i \text{ or } \nu_i - 1}} t^{\nu_i - \mu_i} = \begin{cases} t & \text{if the boxed case,} \\ t+1 & \text{in the unboxed, uncircled case,} \\ 1 & \text{in the circled case.} \end{cases}$$

Hence the term in brackets equals $t^{l_{\text{top}}(\mathfrak{T})}(t+1)^{s_{\text{top}}(\mathfrak{T})}$, where $l_{\text{top}}(\mathfrak{T})$ is the number of boxed entries in the top row of $\Gamma(\mathfrak{T})$ and $s_{\text{top}}(\mathfrak{T})$ is the number of entries in the top row of $\Gamma(\mathfrak{T})$ that are neither boxed nor circled. Clearly $l(\mathfrak{T}) = l(\mathfrak{T}') + l_{\text{top}}(\mathfrak{T})$ and $s(\mathfrak{T}) = s(\mathfrak{T}') + s_{\text{top}}(\mathfrak{T})$ so we obtain the left-hand side of (3.2), with $z_{r+1} = 1$. This completes the induction. □

We will give a version of Tokuyama's formula for crystals. We will take $n = 1$ in (1.12). In this case the Gauss sums may be evaluated explicitly and

$$g(a) = -q^{a-1}, \qquad h(a) = (q-1)q^{a-1}.$$

Theorem 5 *If λ is a dominant weight, and if $z_1, \cdots, z_{r+1}$ are the eigenvalues of $g \in \mathrm{GL}_{r+1}(\mathbb{C})$, then*

$$\prod_{\alpha \in \Phi^+} (1 - q^{-1}\boldsymbol{z}^{\alpha})\chi_{\lambda}(g) = \sum_{v \in \mathcal{B}_{\rho+\lambda}} G_{\Gamma}(v) q^{-\langle \mathrm{wt}(v) - w_0(\lambda+\rho), \rho \rangle} \boldsymbol{z}^{\mathrm{wt}(v) - w_0\rho}.$$

Proof We will prove this in the form

$$\sum_{v \in \mathcal{B}_{\rho+\lambda}} G_{\Gamma}(v) q^{-\langle \mathrm{wt}(v) - w_0(\lambda+\rho), \rho \rangle} \boldsymbol{z}^{\mathrm{wt}(v)} = \prod_{i<j} (z_i - q^{-1} z_j) \chi_{\lambda}(g). \tag{3.5}$$

This is equivalent since $\boldsymbol{z}^{w_0\rho} \prod_{\alpha \in \Phi^+} (1 - q^{-1}\boldsymbol{z}^{\alpha}) = z_2 z_3^2 \cdots z_{r+1}^r \prod_{i<j} (1 - q^{-1} z_i / z_j)$. By Theorem 4 with $t = -q^{-1}$, the right-hand side of (3.5) equals

$$\sum_{\substack{\mathfrak{T} \in \mathrm{GT}(\lambda+\rho) \\ \mathfrak{T} \text{ strict}}} (1 - q^{-1})^{s(\mathfrak{T})} (-q^{-1})^{l(\mathfrak{T})} z_1^{d_r} z_2^{d_{r-1} - d_r} \cdots z_r^{d_0 - d_1}.$$

Turning to the left-hand side of (3.5), let $v \in \mathcal{B}_{\rho+\lambda}$. By (2.14) we have $\boldsymbol{z}^{\mathrm{wt}(v)} = z_1^{d_r} z_2^{d_{r-1}-d_r} \cdots z_r^{d_0-d_1}$ while (with b_i as in Proposition 1 (i))

$$q^{-\langle \mathrm{wt}(v) - w_0(\lambda+\rho), \rho \rangle} = \prod_i q^{-b_i}. \tag{3.6}$$

We may discard those v for which $\mathfrak{T}_v$ is nonstrict, since for these $G_\Gamma(\mathfrak{T}_v)=0$. Indeed if $\mathfrak{T}_v$ is nonstrict then some $a_{i,j}=a_{i,j+1}$. This means that $a_{i,j}=a_{i+1,j+1}=a_{i,j+1}$ and so the entry $\Gamma_{i+1,j+1}$ is both boxed and circled in $\Gamma(\mathfrak{T}_v)$, which implies that $G_\Gamma(v)=G_\Gamma(\mathfrak{T}_v)=0$. On the other hand if $\mathfrak{T}_v$ is strict then by (3.6) and the definition of $G(v)$

$$G_\Gamma(v)q^{-\langle \mathrm{wt}(v)-w_0(\lambda+\rho),\rho\rangle}=\prod_i q^{-b_i}\prod_i\begin{cases}-q^{b_i-1} & \text{if } b_i \text{ is boxed,}\\ q^{b_i} & \text{if circled,}\\ (q-1)q^{b_i-1} & \text{neither}\end{cases}$$

which evidently equals $(-q)^{l(\mathfrak{T}_v)}(1-q^{-1})^{s(\mathfrak{T}_v)}$ and the statement is proved. □

The Casselman-Shalika formula for GL_{r+1} may be written as follows. Let ψ_0 be a character of the nonarchimedean local field K that is trivial on the ring $\mathfrak{o}$ of integers but no larger fractional ideal. Let ϖ be a prime element in $\mathfrak{o}$, and let q be the cardinality of the residue field. Let $\lambda=(\lambda_1,\cdots,\lambda_{r+1})$ be a dominant weight and write $\lambda_i=\sum_{j\geqslant i}l_j$. Define a character of the group N_- of lower triangular unipotent elements by

$$\psi_\lambda\begin{pmatrix}1&&&\\x_{21}&1&&\\\vdots&\iddots&\ddots&\\x_{r+1,1}&\cdots&x_{r+1,r}&1\end{pmatrix}=\psi_0(\varpi^{l_1}x_{r+1,r}+\ldots+\varpi^{l_r}x_{21}).$$

Let f° be the function on $\mathrm{GL}_{r+1}(F)$ defined by

$$f^\circ\left(\begin{pmatrix}\varpi^{\mu_1}&*&\cdots&*\\&\varpi^{\mu_2}&&*\\&&\ddots&\vdots\\&&&\varpi^{\mu_{r+1}}\end{pmatrix}k\right)=\prod q^{-\frac{1}{2}(r+2-2i)}\boldsymbol{z}^\mu,\qquad k\in\mathrm{GL}_{r+1}(\mathfrak{o})$$

where $\boldsymbol{z}=(z_1,\cdots,z_{r+1})$ are the Langlands parameters. Then the Casselman-Shalika formula can be written

$$\int_{N_-}f^\circ(n)\psi_\lambda(n)\,dn=\boldsymbol{z}^{-w_0\lambda}\left[\prod_{\alpha\in\Phi^+}(1-q^{-1}\boldsymbol{z}^\alpha)\right]s_\lambda(z_1,\cdots,z_{r+1}).\tag{3.7}$$

The integral is absolutely convergent if $|\boldsymbol{z}^\alpha|<1$.

On the other hand, Tokuyama's formula describes the right-hand side of this equation. So what we have proved is the formula

$$\int_{N_-} f^\circ(n)\psi_\lambda(n)\,dn = \sum_{v\in\mathcal{B}_{\rho+\lambda}} G_\Gamma(v) q^{-\langle \mathrm{wt}(v)-w_0(\lambda+\rho),\rho\rangle} \boldsymbol{z}^{\mathrm{wt}(v)-w_0(\rho+\lambda)}. \tag{3.8}$$

We observe:

- The identical formula appears to be true if f is taken on the n-th order metaplectic group, in which case the $G_\Gamma(v)$ are products of Gauss sums as in Chapter 2.
- There is a homomorphism $\mathcal{B}_\infty \longrightarrow \mathcal{B}_{\lambda+\rho}$ of crystals, where $\mathcal{B}_\infty$ is the crystal base on the quantized enveloping algebra of N_-, as considered by Lusztig [32] and by Kashiwara [24]. So the right-hand side may be regarded as a sum over $\mathcal{B}_\infty$.
- This is suggestive, since the left-hand side is an integral over the unipotent group N_-, and the right-hand side is a sum over the crystal basis on the quantized universal enveloping algebra of $\mathrm{Lie}(N_-)$.

Chapter 4

Duality

In Chapter 2 we used the notation

$$v \begin{bmatrix} b_1 & \cdots & b_N \\ i_1 & \cdots & i_N \end{bmatrix} v' \qquad \text{or} \qquad v \begin{bmatrix} b_1 & \cdots & b_N \\ i_1 & \cdots & i_N \end{bmatrix}^{(f)} v'$$

to mean that $v' = f_{i_N}^{b_N} \cdots f_{i_1}^{b_1} v$ where each integer b_k is as large as possible in the sense that $f_{i_k}^{b_k+1} f_{i_{k-1}}^{b_{k-1}} \cdots f_{i_1}^{b_1} v = 0$. In this chapter, we will exclusively use the second notation – the superscript (f) will be needed to avoid confusion since we now analogously define

$$v \begin{bmatrix} b_1 & \cdots & b_N \\ i_1 & \cdots & i_N \end{bmatrix}^{(e)} v' \tag{4.1}$$

to mean that $v' = e_{i_N}^{b_N} \cdots e_{i_1}^{b_1} v$ where $e_{i_k}^{b_k+1} e_{i_{k-1}}^{b_{k-1}} \cdots e_{i_1}^{b_1} v = 0$ for all $1 \leqslant k \leqslant N$. Thus

$$v \begin{bmatrix} b_1 & \cdots & b_N \\ i_1 & \cdots & i_N \end{bmatrix}^{(e)} v' \qquad \text{if and only if} \qquad v' \begin{bmatrix} b_N & \cdots & b_1 \\ i_N & \cdots & i_1 \end{bmatrix}^{(f)} v.$$

Let us assume that (4.1) holds, where $\Omega = (i_1, \cdots, i_N)$ and $N = \frac{1}{2}r(r+1)$. Assuming that $\Omega = \Omega_\Gamma$ or Ω_Δ, defined in (2.8) and (2.9), we may decorate the values b_k with circling and boxing, just as in Chapter 2. Thus b_k is circled if and only if either $b_k = 0$ (when k is a triangular number) or $b_k = b_{k+1}$ (when it is not). And b_k is boxed if and only if $f_{i_k} e_{i_{k-1}}^{b_{k-1}} \cdots e_{i_1}^{b_1} v = 0$, which is equivalent to saying that the path (4.1) includes the entire i_k string through $e_{i_{k-1}}^{b_{k-1}} \cdots e_{i_1}^{b_1} v$. In this case, dual to Lemma 1, we have $v' = v_{\text{high}}$.

The definition of the multiple Dirichlet series can be made equally well with respect to the e_i. Indeed, we adapt (2.12) and define

$$G_{\Delta}^{(e)}(v) = \prod_{i=1}^{\frac{1}{2}r(r+1)} \begin{cases} q^{b_i} & \text{if } b_i \text{ is circled,} \\ g(b_i) & \text{if } b_i \text{ is boxed,} \\ h(b_i) & \text{if } b_i \text{ is neither circled nor boxed,} \\ 0 & \text{if } b_i \text{ is both circled and boxed,} \end{cases}$$

assuming that $\Omega = \Omega_\Delta$; if instead $\Omega = \Omega_\Gamma$, then $G_\Gamma^{(e)}$ is defined by the same formula.

Corollary to Theorem 3. *We have*

$$\begin{aligned} H_\Gamma(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) &= \sum_{\substack{v \in \mathcal{B}_{\lambda+\rho} \\ \mathrm{wt}(v) = \lambda - \sum_i k_i \alpha_i}} G_\Delta^{(e)}(v). \\ H_\Delta(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) &= \sum_{\substack{v \in \mathcal{B}_{\lambda+\rho} \\ \mathrm{wt}(v) = \lambda - \sum_i k_i \alpha_i}} G_\Gamma^{(e)}(v). \end{aligned}$$

Proof We replace v by $\mathrm{Sch}(v)$ in Theorem 3. It follows from (2.1) that

$$G_\Delta^{(e)}(\mathrm{Sch}(v)) = G_\Gamma^{(f)}(v),$$

and $\mathrm{wt}(\mathrm{Sch}(v)) = w_0(\mathrm{wt}(v))$, and the statement follows. □

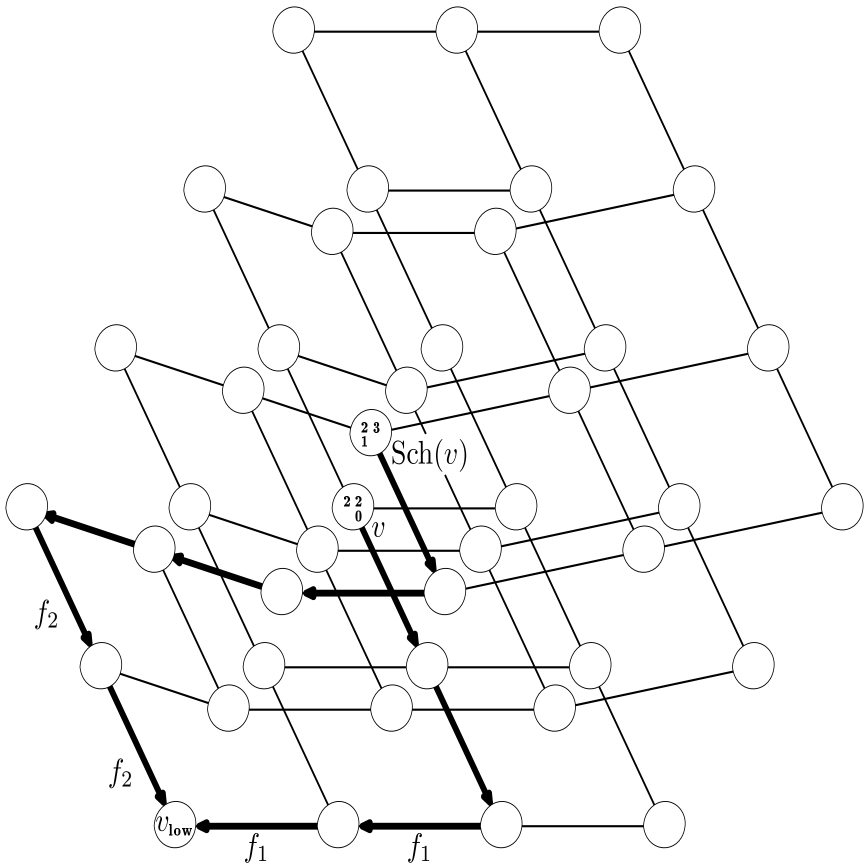
The circling and boxing rules seem quite different from each other, but actually they are closely related, and the involution sheds light on this fact also. Let us apply Proposition 1 (i) to $\mathfrak{T}_{\mathfrak{v}}$ and Proposition 1 (ii) to $\mathfrak{T}_v = q_r \mathfrak{T}_{\mathrm{Sch}(v)}$. For the latter, we see that

$$\Delta(\mathfrak{T}_v) = \begin{bmatrix} \vdots & \vdots & \vdots & \iddots \\ l_6 & l_5 & l_4 & \\ l_3 & l_2 & & \\ l_1 & & & \end{bmatrix},$$

where by (2.1)

$$\mathrm{Sch}(v) \begin{bmatrix} l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & \cdots & l_{N-2} & l_{N-1} & l_N \\ r & r-1 & r & r-2 & r-1 & r & \cdots & r-2 & r-1 & r \end{bmatrix}^{(f)} v_{\mathrm{low}}.$$

Now by definition an entry is circled in $\Gamma(\mathfrak{T}_v)$ if and only if the corresponding entry is boxed in $\Delta(\mathfrak{T}_v)$. This means that if we use the word Ω_Γ for v and Ω_Δ for $\mathrm{Sch}(v)$,



$$\mathfrak{T}_v = \left\{ \begin{array}{ccccc} 5 & & 3 & & 0 \\ & 3 & = & 2 & \\ & & 2 & = & \end{array} \right\}$$

$$\Gamma(\mathfrak{T}_v) = \begin{bmatrix} \textcircled{2} & 2 \\ & \textcircled{0} \end{bmatrix}$$

$$\Delta(\mathfrak{T}_v) = \begin{bmatrix} \boxed{2} & 3 \\ \boxed{1} & \end{bmatrix}$$

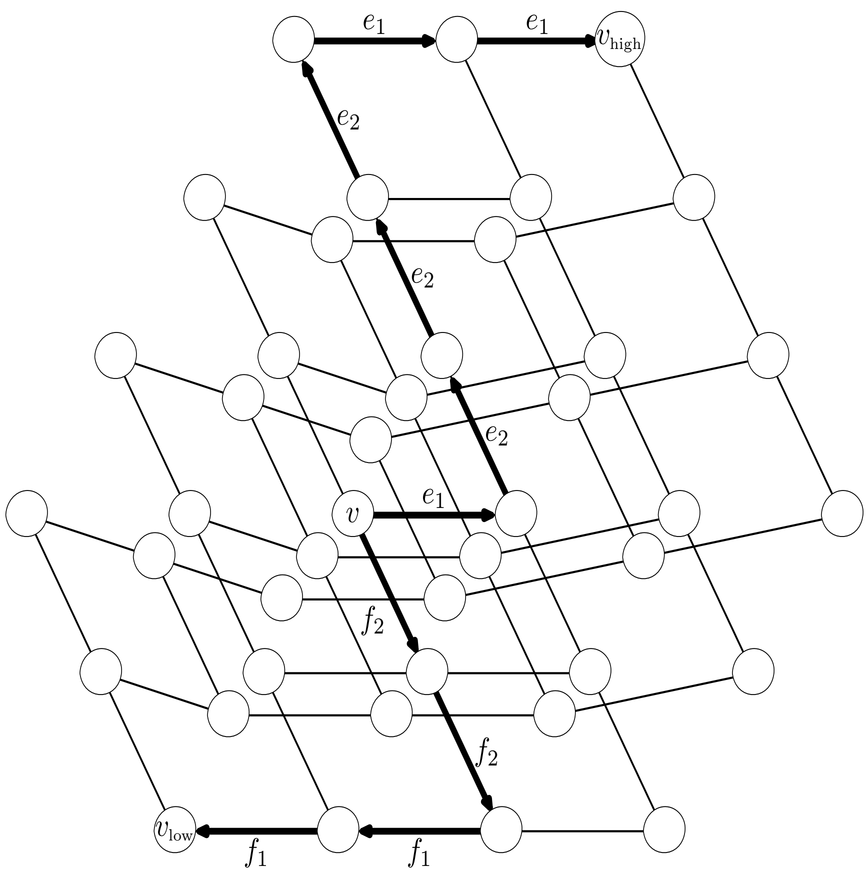
Figure 4.1: **Box-Circle Duality.** Here v is the marked element of $\mathcal{B}_\lambda$, $\lambda = (5,3,0)$. With $\mathfrak{T} = \mathfrak{T}_v$, $\Gamma(\mathfrak{T})$ and $\Delta(\mathfrak{T})$ are obtained from v and $\mathrm{Sch}(v)$ as BZL patterns for the words $\Omega_\Gamma = (1,2,1)$ and $\Omega_\Delta = (2,1,2)$. Boxes in one array correspond to circles in the other.

then we obtain two BZL patterns in which circled entries in one correspond to boxed entries in the other!

Figure 4.1 illustrates this with an example. The two equalities marked in $\mathfrak{T}_v$ give rise to two circles in $\Gamma(\mathfrak{T}_v)$ and two boxes in $\Delta(\mathfrak{T}_v)$. We can see these in the marked paths from v and $\mathrm{Sch}(v)$ to v_{low}.

But there is another way to look at this. It follows from (2.1) that

$$v \begin{bmatrix} l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & \cdots & l_{N-2} & l_{N-1} & l_N \\ 1 & 2 & 1 & 3 & 2 & 1 & \cdots & 3 & 2 & 1 \end{bmatrix}^{(e)} v_{\mathrm{high}}.$$



$$\mathfrak{T}_v = \left\{ \begin{array}{ccccc} 5 & & 3 & & 0 \\ & 3 & & 2 & \\ & & 2 & & \end{array} \right\}$$

$$\Gamma(\mathfrak{T}_v) = \begin{bmatrix} \textcircled{2} & 2 \\ & \textcircled{0} \end{bmatrix}$$

$$\Delta(\mathfrak{T}_v) = \begin{bmatrix} \boxed{2} & 3 \\ \boxed{1} & \end{bmatrix}$$

Figure 4.2: **More Box-Circle Duality.** Here v is the marked element of $\mathcal{B}_\lambda$, $\lambda = (5, 3, 0)$. With $\mathfrak{T} = \mathfrak{T}_v$, $\Gamma(\mathfrak{T})$ and $\Delta(\mathfrak{T})$ are obtained from v as BZL patterns for the word $\Omega_\Gamma = (1, 2, 1)$, but using f root operators for Γ and e root operators for Δ. The cirling of entries in one path corresponds to boxing of entries in the other path, a striking combinometrical property of the crystal.

This means that we may generate $\Gamma(\mathfrak{T}_v)$ and $\Delta(\mathfrak{T}_v)$ from the same element v and the same word Ω_Γ, but applying f_i successively to generate the entries b_i of $\Gamma(\mathfrak{T}_v)$ and applying the e_i (in the same order) to generate the entries l_i of $\Delta(\mathfrak{T}_v)$. The boxing and circling rules are defined analogously for the $\Delta(\mathfrak{T}_v)$ as for $\Gamma(\mathfrak{T}_v)$. Now *box-circle duality* means that there a bijection between the b_i and the l_i in which b_i is circled if and only if the corresponding l_i is boxed. Note that this bijection changes the order:

$$\begin{array}{ccccccc} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ l_1 & l_3 & l_2 & l_6 & l_5 & l_4 & \cdots \end{array}$$

This is a rather striking property of the crystal graph.

Chapter 5

Outline of the Proof

The proof of Theorem 1 involves many remarkable phenomena, and we wish to explain its structure in this chapter. To this end, will give the first of a succession of statements, each of which implies the theorem. Passing from each statement to the next is a nontrivial reduction that changes the nature of the problem to be solved. We will outline the ideas of these reductions here and tackle them in detail in subsequent chapters.

Statement A. *We have* $H_\Gamma = H_\Delta$.

This reduction was already mentioned in the first chapter, where Statement A appeared as Theorem 2. Owing to results from the previous chapter, H_Γ and H_Δ may be regarded as sums parametrized by either Gelfand-Tsetlin patterns or crystal bases.

The proof that this implies Theorem 1 is Theorem 1 of [9]. We review the idea of the proof. To prove the functional equations that $Z_\Psi(\boldsymbol{s}; \boldsymbol{m})$ is to satisfy, using the method described in [13], [4], [6] and [8] based on Bochner's convexity prinicple, one must prove meromorphic continuation to a larger region and a functional equation for each generator $\sigma_1, \cdots, \sigma_r$ of the Weyl group – the simple reflections. These act on the coordinates by

$$\sigma_i(s_j) = \begin{cases} 1 - s_j & \text{if } j = i, \\ s_i + s_j - \frac{1}{2} & \text{if } j = i \pm 1, \\ s_j & \text{if } |j - i| > 1. \end{cases}$$

We proceed inductively. Taking $H = H_\Gamma$ as the definition of the series, all but one of these functional equations may be obtained by collecting the terms to produce a series whose terms are multiple Dirichlet series of lower rank. To see this reduction, note that we have described the p-part of H as a sum over Gelfand-Tsetlin patterns, extended this to a definition to $H(c_1, \cdots, c_r; m_1, \cdots, m_r)$ by (twisted) multiplicativity. Equivalently, one may define $H(c_1, \cdots, c_r; m_1, \cdots, m_r)$ by specifying a Gelfand-Tsetlin pattern $\mathfrak{T}_p$ for each prime; for all but finitely many p the pattern must be the minimal one

$$\left\{\begin{array}{ccccccc} r & & r-1 & & \cdots & & 0 \\ & r-1 & & \cdots & & 0 & \\ & & \ddots & & \iddots & & \\ & & & 0 & & & \end{array}\right\}.$$

Summing over such data with (for each p) fixed top row (determined by the $\mathrm{ord}_p(m_i)$) and fixed row sums (determined by $\mathrm{ord}_p(c_i)$) gives $H(c_1, \cdots, c_r; m_1, \cdots, m_r)$. More precisely we may group the terms as follows. For each prime p of $\mathfrak{o}_S$, fix a partition λ_p of length r into unequal parts such that for almost all p we have $\lambda_p = (r, r-1, \cdots, 0)$; then collect the terms in which for each p the top row of $\mathfrak{T}_p$ is λ_p. These produce an exponential factor times a term $Z(\boldsymbol{s}; \boldsymbol{m}'; A_{r-1})$ where $\boldsymbol{m}'$ depends on λ_p (for each p). This expansion gives, by induction, the functional equations for the subgroup of W generated by $\sigma_2, \cdots, \sigma_r$. Similarly starting with $H = H_\Delta$ gives functional equations for the subgroup generated by $\sigma_1, \cdots, \sigma_{r-1}$. Notice that these two sets of reflections generate all of W. If Statement A holds, then combining these analytic continuations and functional equations and invoking Bochner's convexity principle gives the required analytic continuation and functional equations. We refer to [9] for further details.

Since H_Γ and H_Δ satisfy the same twisted multiplicativity, it suffices to work at powers of a single prime p. We see that there are two ways in which these coefficients differ. First, given a lattice point $\boldsymbol{k} = (k_1, \ldots, k_r)$ in the polytope defined by the Gelfand-Tsetlin patterns of given top row, there are two ways of attaching a set of Gelfand-Tsetlin patterns to $\boldsymbol{k}$, namely the set of $\mathfrak{T}$ with $k_\Gamma(\mathfrak{T}) = \boldsymbol{k}$, or with $k_\Delta(\mathfrak{T}) = \boldsymbol{k}$. Second, given a pattern $\mathfrak{T}$, there are two ways of attaching numbers to it: $G_\Gamma(\mathfrak{T})$, resp. $G_\Delta(\mathfrak{T})$.

An attack on Statement A can be formulated using the *Schützenberger involution* on Gelfand-Tsetlin patterns. This is the involution q_r, which was already defined in Chapter 1. It interchanges the functions k_Δ and k_Γ, and one may thus formulate Statement A as saying that

$$\sum_{k_\Gamma(\mathfrak{T})=(k_1,\cdots,k_r)} G_\Gamma(\mathfrak{T}) = \sum_{k_\Gamma(\mathfrak{T})=(k_1,\cdots,k_r)} G_\Delta(q_r\mathfrak{T}). \tag{5.1}$$

In many cases, for example if $\mathfrak{T}$ is on the interior of the polytope of Gelfand-Tsetlin patterns with fixed $k_\Gamma(\mathfrak{T})$, it can be proved that $G_\Gamma(\mathfrak{T}) = G_\Delta(q_r\mathfrak{T})$. If this were always true there would be no need to sum in (5.1). In general, however, this is false. What *is* ultimately true is that the patterns may be partitioned into fairly small "packets" such that if one sums over a packet, $\sum G_\Gamma(\mathfrak{T}) = \sum G_\Delta(q_r\mathfrak{T})$. The packets, we observe, can be identified empirically in any given case, but are difficult to characterize in general, and not even uniquely determined in some cases. See [9].

To proceed further, we introduce the notion of a *short Gelfand-Tsetlin pattern* or (for short) *short pattern*. By this we mean an array with just three rows

$$\mathfrak{t} = \left\{ \begin{array}{ccccccccc} l_0 & & l_1 & & l_2 & & \cdots & & l_{d+1} \\ & a_0 & & a_1 & & & & a_d & \\ & & b_0 & & \cdots & & b_{d-1} & & \end{array} \right\}. \tag{5.2}$$

where the rows are nonincreasing sequences of integers that interleave, that is,

$$l_i \geqslant a_i \geqslant l_{i+1}, \qquad a_i \geqslant b_i \geqslant a_{i+1}. \tag{5.3}$$

We will refer to $l_0, \cdots, l_{d+1}$ as the *top* or *zero-th row* of $\mathfrak{t}$, $a_0, \cdots, a_d$ as the *first* or *middle row* and $b_0, \cdots, b_{d-1}$ as the *second* or *bottom row*. We may assume that the top and bottom rows are strict, but we need to allow the first row to be nonstrict. We define the *weight* k of $\mathfrak{t}$ to be the sum of the a_i.

If $\mathfrak{t}$ is a short pattern we define another short pattern

$$\mathfrak{t}' = \left\{ \begin{array}{ccccccccc} l_0 & & l_1 & & l_2 & & \cdots & & l_{d+1} \\ & a'_0 & & a'_1 & & & & a'_d & \\ & & b_0 & & \cdots & & b_{d-1} & & \end{array} \right\}, \tag{5.4}$$

where

$$a'_j = \min(l_j, b_{j-1}) + \max(l_{j+1}, b_j) - a_j, \qquad 0 < j < d, \tag{5.5}$$

$$a'_0 = l_0 + \max(l_1, b_0) - a_0, \qquad a'_d = \min(l_d, b_{d-1}) + l_{d+1} - a_d. \tag{5.6}$$

We call $\mathfrak{t}'$ the (Schützenberger) *involute* of $\mathfrak{t}$. To see why this definition is reasonable, note that if the top and bottom rows of $\mathfrak{t}$ are specified, then a_i are constrained by the inequalities

$$\min(l_j, b_{j-1}) \geqslant a_j \geqslant \max(l_{j+1}, b_j), \qquad 0 < j < d, \tag{5.7}$$

$$l_0 \geqslant a_0 \geqslant \max(l_1, b_0), \qquad \min(l_d, b_{d-1}) \geqslant a_d \geqslant l_{d+1}. \tag{5.8}$$

These inequalities express the assumption that the three rows of the short pattern interleave. The array $\mathfrak{t}'$ is obtained by reflecting a_j in its permitted range.

The Schützenberger involution of full Gelfand-Tsetlin patterns is built up from operations involving three rows at a time, based on the operation $\mathfrak{t} \longmapsto \mathfrak{t}'$ of short

Gelfand-Tsetlin patterns. This is done $\frac{1}{2}r(r+1)$ times to obtain the Schützenberger involution. Using this decomposition and induction, we prove that to establish Statement A one needs only the equivalence of two sums of Gauss sums corresponding to Gelfand-Tsetlin patterns that differ by a single involution. This allows us to restrict our attention, within Gelfand-Tsetlin patterns, to short patterns. To be more precise and to explain what must be proved, we make the following definitions.

By a short pattern *prototype* $\mathfrak{S}$ of length d we mean a triple $(\boldsymbol{l}, \boldsymbol{b}, k)$ specifying the following data: a top row consisting of an integer sequence $\boldsymbol{l} = (l_0, \cdots, l_{d+1})$, a bottom row consisting of a sequence $\boldsymbol{b} = (b_0, \cdots, b_d)$, and a positive integer k. It is assumed that $l_0 > l_1 > \ldots > l_{d+1}$, that $b_0 > b_1 > \ldots > b_{d-1}$ and that $l_i > b_i > l_{i+2}$.

We say that a short pattern $\mathfrak{t}$ of length d *belongs to the prototype* $\mathfrak{S}$ if it has the prescribed top and bottom rows, and its weight is k (so $\sum_i a_i = k$). By abuse of notation, we will use the notation $\mathfrak{t} \in \mathfrak{S}$ to mean that $\mathfrak{t}$ belongs to the prototype $\mathfrak{S}$. Prototypes were called *types* in [9], but we will reserve that term for a more restricted equivalence class of short patterns.

Given a short Gelfand-Tsetlin pattern, we may define two two-rowed arrays $\Gamma_\mathfrak{t}$ and $\Delta_\mathfrak{t}$, to be called *preaccordions*, which display information used in the evaluations we must make. These are defined analogously to the patterns associated with a full Gelfand-Tsetlin pattern, which were denoted $\Gamma(\mathfrak{t})$ and $\Delta(\mathfrak{t})$. There is an important distinction in that in $\Gamma_\mathfrak{t}$ we use the right-hand rule on the first row, and the left-hand rule on the second row, and for $\Delta_\mathfrak{t}$ we reverse these. In the full-pattern $\Gamma(\mathfrak{T})$ we used the right-hand rule for every row, and in $\Delta(\mathfrak{T})$ we used the left-hand rule for every row. Specifically

$$\Gamma_\mathfrak{t} = \left\{ \begin{matrix} \mu_0 & & \mu_1 & & \cdots & & \mu_d \\ & \nu_0 & & \cdots & & \nu_{d-1} & \end{matrix} \right\}, \tag{5.9}$$

and

$$\Delta_\mathfrak{t} = \left\{ \begin{matrix} \kappa_0 & & \kappa_1 & & \cdots & & \kappa_d \\ & \lambda_0 & & \cdots & & \lambda_{d-1} & \end{matrix} \right\}, \tag{5.10}$$

where

$$\mu_j = \sum_{k=j}^{d} (a_k - l_{k+1}), \qquad \nu_j = \sum_{k=0}^{j} (a_k - b_k),$$

and

$$\kappa_j = \sum_{k=0}^{j} (l_k - a_k), \qquad \lambda_j = \sum_{k=j}^{d-1} (b_k - a_{k+1}).$$

We also use the right-hand rule to describe the circling and boxing of the elements of the first row of $\Gamma_\mathfrak{t}$, and the left-hand rule to describe the circling and boxing of elements of the bottom row, reversing these for $\Delta_{\mathfrak{t}}$. This means we circle μ_j if $a_j = l_{j+1}$ and box μ_i if $a_j = l_j$; we circle ν_j if $b_j = a_j$ and box ν_j if $b_j = a_{j+1}$. The boxing and circling rules are reversed for $\Delta_{\mathfrak{t}}$: we box κ_j if $\alpha_j = l_{j+1}$ and circle α_i if $\alpha_j = l_j$; we box λ_j if $b_j = a_j$ and box λ_j if $b_j = a_{j+1}$.

We give an example to illustrate these definitions. Suppose that

$$\mathfrak{t} = \left\{ \begin{array}{ccccccccccc} 23 & & 15 & & 12 & & 5 & & 2 & & 0 \\ & 20 & & 12 & & 5 & & 4 & & 2 & \\ & & 14 & & 9 & & 5 & & 3 & & \end{array} \right\}.$$

Then

$$\Gamma_{\mathfrak{t}} = \left\{ \begin{array}{ccccccccc} 9 & & \textcircled{4} & & \textcircled{4} & & 4 & & \boxed{2} \\ & 6 & & 9 & & \textcircled{9} & & 10 & \end{array} \right\}.$$

We have indicated the circling and boxing of entries. Now applying the involution,

$$\mathfrak{t}' = \left\{ \begin{array}{ccccccccccc} 23 & & 15 & & 12 & & 5 & & 2 & & 0 \\ & 18 & & 14 & & 9 & & 4 & & 0 & \\ & & 14 & & 9 & & 5 & & 3 & & \end{array} \right\},$$

and

$$\Delta_{\mathfrak{t}'} = \left\{ \begin{array}{ccccccccc} 5 & & 6 & & 9 & & 10 & & \boxed{12} \\ & \textcircled{4} & & \textcircled{4} & & 4 & & 3 & \end{array} \right\}.$$

We observe the following points.

- The first row of $\Gamma_{\mathfrak{t}}$ is decreasing and the bottom row is increasing; these are reversed for $\Delta_{\mathfrak{t}'}$, just as the boxing and circling conventions are reversed.

- The involution does not preserve strictness. If $\mathfrak{t}$ is strict, no element can be both boxed and circled, but if $\mathfrak{t}$ is not strict, an entry in the bottom row is both boxed and circled, and the same is true for $\Delta_{\mathfrak{t}'}$: if $\mathfrak{t}'$ is not strict, then an entry in the bottom row of $\Delta_{\mathfrak{t}'}$ is both boxed and circled.

If $\mathfrak{T}$ is a Gelfand-Tsetlin pattern, then in $\Gamma(\mathfrak{T})$ we use the right-hand rule in every row, and in $\Delta(\mathfrak{T})$ we use the left-hand row in every row. But if $\mathfrak{t}$ is a *short* Gelfand-Tsetlin pattern then in $\Gamma_{\mathfrak{t}}$ and $\Delta_{\mathfrak{t}}$ one row uses the right-hand rule, the other the left-hand rule.

Let us define

$$G_\Gamma(\mathfrak{t}) = \prod_{x\in\Gamma_\mathfrak{t}} \begin{cases} g(x) & \text{if } x \text{ is boxed in } \Gamma_\mathfrak{t}\text{, but not circled;} \\ q^x & \text{if } x \text{ is circled, but not boxed;} \\ h(x) & \text{if } x \text{ is neither boxed nor circled;} \\ 0 & \text{if } x \text{ is both boxed and circled.} \end{cases}$$

Thus if $\mathfrak{t}$ is not strict, then $G_\Gamma(\mathfrak{t}) = 0$. Similarly, let

$$G_\Delta(\mathfrak{t}') = \prod_{x\in\Delta_{\mathfrak{t}'}} \begin{cases} g(x) & \text{if } x \text{ is boxed in } \Delta_{\mathfrak{t}'}\text{, but not circled;} \\ q^x & \text{if } x \text{ is circled, but not boxed;} \\ h(x) & \text{if } x \text{ is neither boxed nor circled;} \\ 0 & \text{if } x \text{ is both boxed and circled.} \end{cases}$$

Thus in our examples,

$$G_\Gamma(\mathfrak{t}) = h(9)\cdot q^4\cdot q^4\cdot h(4)\cdot g(2)\cdot h(6)\cdot h(9)\cdot q^9\cdot h(10)$$

and

$$G_\Delta(\mathfrak{t}') = h(5)\cdot h(6)\cdot h(9)\cdot h(12)\cdot g(12)\cdot q^4\cdot q^4\cdot h(4)\cdot h(3).$$

Statement B. *Let $\mathfrak{S}$ be a short pattern prototype. Then*

$$\sum_{\mathfrak{t}\in\mathfrak{S}} G_\Gamma(\mathfrak{t}) = \sum_{\mathfrak{t}\in\mathfrak{S}} G_\Delta(\mathfrak{t}'). \tag{5.11}$$

A reinterpretation of Statement B in terms of crystal bases will be given in Chapter 7.

This was conjectured in [9]. By Theorems 7 and 8 below we actually have, for many $\mathfrak{t}$ (in some sense most)

$$G_\Gamma(\mathfrak{t}) = G_\Delta(\mathfrak{t}'). \tag{5.12}$$

However this is not always true, so the summation in (5.11) is needed.

The reduction to Statement B was proved in [9], which was written before Statement B was proved. We will repeat this argument (based on the Schützenberger involution) in Chapter 6. In a nutshell, Statement A can be deduced from Statement B because the Schützenberger involution q_r is built up from the involution $\mathfrak{t}\longmapsto\mathfrak{t}'$ of short Gelfand-Tsetlin patterns by repeated applications, and this will be explained in detail in Chapter 6.

Just as in (5.12), it is in some sense *usually* true that the individual terms agree in this identity, that is, $G_\Gamma(\mathfrak{T}) = G_\Delta(\mathfrak{T}')$. In many case, for example if $\mathfrak{T}$ is on the interior of the polytope of Gelfand-Tsetlin patterns with fixed $k_\Gamma(\mathfrak{T})$, it can be proved that $G_\Gamma(\mathfrak{T}) = G_\Delta(q_r\mathfrak{T})$. If this were always true there would be no need to sum in (5.1). In general, however, this is false. What *is* true is that the patterns may be partitioned into fairly small "packets" such that if one sums over a packet we have $\sum G_\Gamma(\mathfrak{T}) = \sum G_\Delta(q_r\mathfrak{T})$. The packets, we observe, can be identified empirically in any given case, but are difficult to characterize in general, and not even uniquely determined in some cases. These phenomena occur in microcosm for the short Gelfand-Tsetlin patterns, so studying the phenomena that occur in connection with (5.11) gives us insight into (5.1).

Instead of pursuing the identification of packets as suggested in [9], we proceed by using (5.1) to reduce Statement A to Statement B, and the mysterious packets will eventually be sorted out by further combinatorial transformations of the problem that we will come to presently (Statements C, D, E, F and G).

Most our effort will be devoted to the proof of Statement B, to which we now turn. We call the short pattern *totally resonant* if the bottom row repeats the top row, that is, if it has the form

$$\mathfrak{t} = \left\{ \begin{array}{ccccccccc} l_0 & & l_1 & & l_2 & & \cdots & & l_{d+1} \\ & a_0 & & a_1 & & & & a_d & \\ & & l_1 & & \cdots & & l_d & & \end{array} \right\}. \tag{5.13}$$

Notice that this is a property of the prototype, whose data consists of the top and bottom rows, and the middle row sum.

Statement C. *Statement B is true for totally resonant short pattern prototypes.*

The fact that Statement C implies Statement B will be proved in Theorem 9, which requires the introduction of some new concepts. Given a short pattern (5.2), we will associate with its prototype a certain graph called the *cartoon* by connecting a_i to either l_i if $i = 0$, or else to either l_i or b_{i-1}, whichever is numerically closer to a_i. If $b_{i-1} = l_i$, then we connect a_i to both. We will also connect a_i to either l_{i+1} or b_i, whichever is numerically closer to a_i, or to both if they are equal, provided $i < d$; and a_d is connected to l_{d+1}. The connected components of the cartoon are called *episodes*.

We will subdivide the prototype into smaller equivalence classes called *types*. Two patterns $\mathfrak{t}_1$ and $\mathfrak{t}_2$ have the same type if, first of all, they have the same prototype (hence the same cartoon), and if for each episode $\mathcal{E}$, the sum of the a_i that lie in $\mathcal{E}$ are the same for $\mathfrak{t}_1$ and $\mathfrak{t}_2$. Statement B follows from the stronger statement that (5.11) holds when we sum over a type. We will reduce this statement to a series of separate problems, one for each episode. But each of these problems will be reduced to a single common problem (Statement D below) that is equivalent to Statement B for totally resonant prototypes.

The reduction to the totally resonant case involves some quite fascinating combinatorial phenomena. A key point is a combinatorial Lemma, which was called the "Snake Lemma" in [9], but which is not the familiar Snake Lemma from homological algebra. This Lemma says the elements in $\Gamma_{\mathfrak{t}}$ can be matched up with the elements in $\Delta_{\mathfrak{t}'}$ in a bijection that has quite surprising properties. (The "snakes" appear in graphing this bijection.)

In addition to such combinatorial phenomena, number theory enters as well, in particular in the "Knowability Lemma" (Proposition 14), which we now briefly discuss. The term "knowability" refers to the fact that Gauss sums such as $g(a)$ when $n \nmid a$ have known absolute values, but their arguments as complex numbers are still mysterious. They are "unknowable." We will refer to an expression that is a product of terms of the form q^a, $h(a)$ and $g(a)$ as "knowable" if it can given a closed expression as a polynomial in q. Thus $g(a)$, taken in isolation, is unknowable unless $n|a$, in which case $g(a) = -q^{a-1}$. But even if $n \nmid a$ the product $g(a)g(b)$ is knowable if $n|a+b$ since then $g(a)g(b) = q^{a+b-1}$.

There is a strong tendency for the Gauss sums that appear in the terms $G_\Gamma(\mathfrak{t})$ (for short patterns $\mathfrak{t}$) or in $G_\Gamma(\mathfrak{T})$ (for full Gelfand-Tsetlin patterns $\mathfrak{T}$), to appear in knowable combinations. The Knowability Lemma give an explanation for this. Moreover it gives key information that is needed for the sequel. Stable patterns are an important exception. We recall that a pattern is *stable* if every entry (except those in the top row) equals one of the two directly above it. The stable patterns are in a sense the most important ones, since they are the *only* patterns that contribute in the stable case (when n is large). If the Gelfand-Tsetlin patterns with fixed top row are embedded into a Euclidean space, the stable patterns are the extremal ones. The Gauss sums that appear in the stable terms are unknowable. Thus when $r = 1$, $Z_\Psi(\boldsymbol{s}; \boldsymbol{m})$ is Kubota's Dirichlet series [28], and its use by Heath-Brown and Patterson [22] to study the distribution of cubic Gauss sums exploited precisely the appearance of such unknowable terms.

In order to prove Statement C, we must work with the evaluations of $G_\Gamma(\mathfrak{t})$ and $G_\Delta(\mathfrak{t}')$. To do so, it is convenient to describe these evaluations by introducing some new notation and terminology. A Γ-*accordion* (of length d and weight s) is an array of nonnegative integers

$$\mathfrak{a} = \left\{ \begin{array}{ccccccccc} s & & \mu_1 & & \mu_2 & & \cdots & & \mu_d \\ & \nu_1 & & \nu_2 & & \cdots & & \nu_d & \end{array} \right\}, \tag{5.14}$$

in which the first row is decreasing, the second increasing, and $\mu_i + \nu_i = s$. Thus if

$\mathfrak{t}$ is a short Gelfand-Tsetlin pattern, then the preaccordion $\Gamma_{\mathfrak{t}}$ is an accordion if the condition $\mu_i + \nu_i = s$ is satisfied. We will sometimes write

$$\mu_0 = \nu_{d+1} = s, \qquad \nu_0 = \mu_{d+1} = 0. \tag{5.15}$$

Also, by a Δ-*accordion* (of length $d+1$ and weight s) we mean an array

$$\mathfrak{a}' = \left\{ \begin{matrix} \nu_1 & & \nu_2 & & \cdots & & \nu_d & & s \\ & \mu_1 & & \mu_2 & & \cdots & & \mu_d & \end{matrix} \right\}, \tag{5.16}$$

where the first row is increasing, the bottom decreasing, and $\mu_i + \nu_i = s$. We will make use of the map $\mathfrak{a} \longmapsto \mathfrak{a}'$ that takes Γ-accordions to Δ-accordions.

The significance of these definitions is that if $\mathfrak{t}$ is a totally resonant short Gelfand-Tsetlin pattern, then its Γ-preaccordion is a Γ-accordion. Moreover the Δ-pre-accordion of $\mathfrak{t}'$ is the Δ-accordion $\mathfrak{a}'$.

We have already described the decoration of preaccordions with boxes and circles. In the special case of an accordion, the decorations have some pleasant additional properties because they come from totally resonant short patterns.

- No entry of the first row is both boxed and circled. An entry of the bottom row may be both boxed and circled, in which case we say the accordion is *nonstrict*.

- In a Γ-accordion, a bottom row entry is circled if and only if the entry above it and to the left is circled, and a bottom row entry is boxed if and only if the entry above it and to the right is boxed. Thus in (5.14), ν_i is circled if and only if the μ_{i-1} is circled, and ν_i is boxed if and only if μ_i is boxed.

- In a Δ-accordion, a bottom row entry is circled if and only if the entry above it and to the right is circled, and a bottom row entry is boxed if and only if the entry above it and to the left is boxed.

- In either (5.14) or (5.16) μ_i is circled if and only if $\mu_i = \mu_{i+1}$, and ν_i is circled if and only if $\nu_i = \nu_{i-1}$. (But note that μ_i is in the first row in (5.14) but in the second row in (5.16).) Invoking (5.15), special cases of this rule are that s is circled if and only if $s = \mu_1$, μ_d is circled if and only if $\mu_d = 0$, and ν_1 is circled if and only if $\nu_1 = 0$.

- There is no corresponding rule for the boxing. Thus the circling is determined by $\mathfrak{a}$ but the boxing is not.

We also note that the Knowability Lemma mentioned above allows us to assume that $n|s$.

Our goal is to systematically describe all decorated accordions that arise and their corresponding evaluations, so that we may sum over them and establish Statement C. If $\mathfrak{a}$ is a decorated accordion (of either kind), then in view of the second and third rules, the decoration of the second row is determined by the decoration of the top row. We encode this by a *signature*, which is by definition a string $\sigma = \sigma_0 \cdots \sigma_d$, where each σ_i is one of the symbols $\bigcirc, \square$ or $*$. We associate a signature with a decorated accordion by taking $\sigma_i = \bigcirc$ if μ_i is circled in the first row (with $\mu_0 = s$, of course), $\square$ if μ_i is boxed, and $*$ if it is neither circled or boxed. We say the accordion $\mathfrak{a}$ and the signature σ are *compatible* if the following *circling compatibility condition* is satisfied (for conformity with the rules already stated for the decorations). For Γ-accordions labeled as in (5.14) the condition is

$$\sigma_i = \bigcirc \text{ if and only if } \mu_i = \mu_{i+1}. \tag{5.17}$$

In view of (5.15), if $i = 0$, this means $s = \mu_1$, and if $i = d$ it means $\mu_d = 0$. For Δ-accordions labeled as in (5.16) the condition is

$$\sigma_i = \bigcirc \text{ if and only if } \nu_i = \nu_{i-1},$$

which means that $\sigma_0 = \bigcirc$ if and only if $\nu_1 = 0$, and $\sigma_d = \bigcirc$ if and only if $s = \nu_1$.

Since the signature determines the decoration, we will denote by $\mathfrak{a}_\sigma$ the decorated accordion, where σ is a signature compatible with the accordion $\mathfrak{a}$. We will apply the same signature σ to the involute $\mathfrak{a}'$. Thus if $\sigma = * \square * * \bigcirc *$ and

$$\mathfrak{a} = \left\{ \begin{array}{ccccccccccc} 9 & & 7 & & 6 & & 4 & & 2 & & 2 \\ & 2 & & 3 & & 5 & & 7 & & 7 & \end{array} \right\}$$

then

$$\mathfrak{a}_\sigma = \left\{ \begin{array}{ccccccccccc} 9 & & \boxed{7} & & 6 & & 5 & & \textcircled{2} & & 2 \\ & \boxed{2} & & 3 & & 5 & & 7 & & \textcircled{7} & \end{array} \right\},$$

which is a decorated Γ-accordion, while

$$\mathfrak{a}'_\sigma = \left\{ \begin{array}{ccccccccccc} 2 & & \boxed{3} & & 5 & & 7 & & \textcircled{7} & & 9 \\ & 7 & & \boxed{6} & & 4 & & \textcircled{2} & & 2 & \end{array} \right\},$$

which is a decorated Δ-accordion. Observe that the signature encodes the decoration of the first row, and since we use the same signature in both $\mathfrak{a}$ and $\mathfrak{a}'$, it follows that the location of the boxes and circles in the first row is the same for $\mathfrak{a}$ as for the Δ-accordion $\mathfrak{a}'$. However the decoration of the bottom rows are different. This is because the boxing and circling rules are different for Γ-accordions and Δ-accordions.

Now if $\mathfrak{a}_\sigma$ is a decorated Γ-accordion, let

$$\mathcal{G}_\Gamma(\mathfrak{a},\sigma)=\mathcal{G}_\Gamma(\mathfrak{a}_\sigma)=\prod_{x\in\mathfrak{a}}\begin{cases} g(x) & \text{if } x \text{ is boxed in } \mathfrak{a}_\sigma \text{ (but not circled)},\\ q^x & \text{if } x \text{ is circled (but not boxed)},\\ h(x) & \text{if } x \text{ is neither boxed nor circled},\\ 0 & \text{if } x \text{ is both boxed and circled}.\end{cases}$$

The notations $\mathcal{G}_\Gamma(\mathfrak{a}_\sigma)$ and $\mathcal{G}_\Gamma(\mathfrak{a},\sigma)$ are synonyms; we will prefer the former when working with the free abelian group on the decorated accordions, the latter when $\mathfrak{a}$ is fixed and σ is allowed to vary.

If $\mathfrak{a}'_\sigma$ is a decorated Δ-accordion, we define $\mathcal{G}_\Delta(\mathfrak{a}'_\sigma)$ by the same formula. We retain the subscripts Γ and Δ since $\mathcal{G}_\Gamma$ and $\mathcal{G}_\Delta$ have different domains. Thus in the last example

$$\mathcal{G}_\Gamma(\mathfrak{a}_\sigma)=h(9)g(7)g(2)h(6)h(3)h(5)h(5)q^2h(7)h(2)q^7,$$
$$\mathcal{G}_\Delta(\mathfrak{a}'_\sigma)=h(2)h(7)g(3)g(6)h(5)h(4)h(7)q^2q^7h(2)h(9).$$

Now let positive integers $s, c_0, c_1, \cdots, c_d$ be given. By the Γ-*resotope* (of length d), to be denoted $\mathcal{A}_s^\Gamma(c_0, c_1, \cdots, c_d)$, we mean the sum, in the free abelian group $\mathfrak{Z}_\Gamma$ on the set of decorated accordions, of such $\mathfrak{a}_\sigma$ such that the parameters in $\mathfrak{a}$ satisfy

$$0\leqslant s-\mu_1\leqslant c_0,\quad 0\leqslant \mu_1-\mu_2\leqslant c_1,\quad \cdots\quad, 0\leqslant \mu_d\leqslant c_d, \tag{5.18}$$

and

$$\sigma_i=\begin{cases} \circ & \text{if } \mu_i-\mu_{i+1}=0;\\ \square & \text{if } \mu_i-\mu_{i+1}=c_i;\\ * & \text{if } 0<\mu_i-\mu_{i+1}<c_i.\end{cases}$$

If $\mathcal{A}=\mathcal{A}_s^\Gamma(c_0, c_1, \cdots, c_d)$, by abuse of notation we may write $\mathfrak{a}_\sigma\in\mathcal{A}$ to mean that $\mathfrak{a}_\sigma$ appears with nonzero coefficient in $\mathcal{A}$ as described above. Let $\mathcal{A}'$ be the image of $\mathcal{A}$ under the involution $\mathfrak{a}_\sigma\longmapsto\mathfrak{a}'_\sigma$; we call $\mathcal{A}'$ a Δ-*resotope*.

The set of Γ-accordions of the form (5.14), embedded into Euclidean space by mapping $\mathfrak{a}\longmapsto(\mu_1,\cdots,\mu_d)$, may be regarded as the set of lattice points in a polytope. The resotopes we have just described correspond to these points with compatible decorations and signatures attached. We will sometimes discuss the geometry of the underlying polytope without making explicit note of the additional data attached to each point.

Given a totally resonant type, we will prove in the Corollary to Proposition 16 below that the accordion $\Gamma_\mathfrak{t}$ runs through a resotope $\mathcal{A}$, and $\Delta_{\mathfrak{t}'}$ runs through $\mathcal{A}'$. This allows us to pass from Statement C to the following statement.

Statement D. *Let $\mathcal{A}$ be a Γ-resotope. Then*

$$\sum_{\mathfrak{a}_\sigma \in \mathcal{A}} \mathcal{G}_\Gamma(\mathfrak{a}_\sigma) = \sum_{\mathfrak{a}'_\sigma \in \mathcal{A}'} \mathcal{G}_\Delta(\mathfrak{a}'_\sigma). \tag{5.19}$$

We turn to the proof of Statement D. We will show that if $\mathfrak{a}$ lies in the interior of the resotope, then its signature is just $* \cdots *$, and we have $\mathcal{G}_\Gamma(\mathfrak{a}_\sigma) = h(s) \prod_{i=1}^d h(\mu_i)h(\nu_i) = \mathcal{G}_\Delta(\mathfrak{a}'_{\sigma'})$. This may fail, however, when $\mathfrak{a}$ is on the boundary, so this is the remaining obstacle to proving Statement D. The approach suggested in [9] is to try to partition the boundary into small "packets" such that the sums over each packet are equal. In practice one can carry this out in any given case, but giving a coherent theory of packets along these lines seems unpromising. First, the resotopes themselves are geometrically complex. Second, even when the resotope is geometrically simple, the identification of the packets can be perplexing, and devoid of any apparent pattern. An example is done at the end of [9].

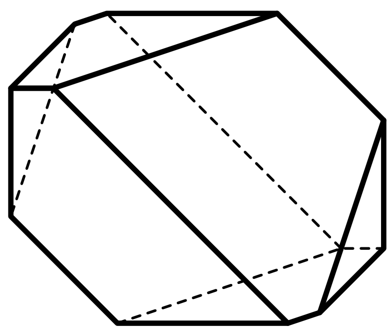


Figure 5.1: A resotope, when $d = 3$.

Geometrically, a resotope is a figure obtained from a simplex by chopping off some of the corners; the pieces removed are themselves simplices. But the resulting polytopes are quite varied. Figure 5.1 shows a resohedron (2-dimensional resotope) with five pentagonal faces and three triangular ones. To avoid these geometric difficulties we develop an approach, based on the Principle of Inclusion-Exclusion, that allows us to replace the complicated geometry of a general polytope with the simple geometry of a simplex.

The process of passing from the simplex of all accordions to an arbitrary resotope is complex. Indeed, as one chops corners off the simplex of all accordions to obtain a general resotope, interior accordions become boundary accordions, so their signatures change. Sometimes the removed simplices overlap, so one must restore any part that has been removed more than once. As we shall show, there is nonetheless a good way of handling it. Before we formulate this precisely, let us consider an example. The set of all Γ-resohedra, with $d = 2$ and fixed value s is represented in Figure 5.2 by the triangle $\triangle\, \mathfrak{abc}$ with vertices

$$\mathfrak{a} = \left\{ \begin{array}{ccccc} s & & s & & 0 \\ & 0 & & s & \end{array} \right\}, \qquad \mathfrak{b} = \left\{ \begin{array}{ccccc} s & & 0 & & 0 \\ & s & & s & \end{array} \right\}, \qquad \mathfrak{c} = \left\{ \begin{array}{ccccc} s & & s & & s \\ & 0 & & 0 & \end{array} \right\}.$$

We are concerned with the shaded resotope $\mathcal{A} = \mathcal{A}_s^\Gamma(c_0, c_1, \infty)$, which is obtained by truncating the simplex $\triangle\, \mathfrak{abc}$ by removing $\triangle\, \mathfrak{aeg}$ and $\triangle\, \mathfrak{dbh}$. We use ∞ to mean any value of c_2 that is so large that the inequality $\mu_2 \leqslant c_2$ is automatically true (and strict) for all Γ-accordions; indeed any $c_2 > s$ can be replaced by ∞ without changing $\mathcal{A}$.

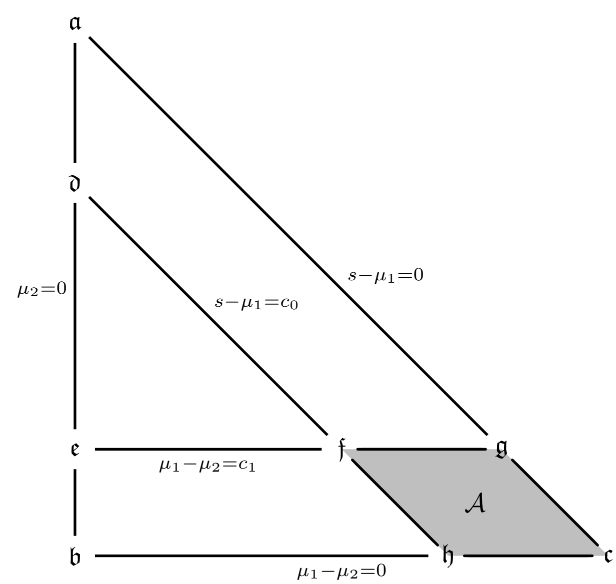


Figure 5.2: Inclusion-Exclusion.

Then, since the $\triangle\, \mathfrak{def}$ has been removed twice, it must be restored, and we may write

$$\mathcal{A} = \triangle\, \mathfrak{abc} - \triangle\, \mathfrak{aeg} - \triangle\, \mathfrak{dbh} + \triangle\, \mathfrak{def}.$$

Now in this equation, $\triangle\, \mathfrak{def}$ (for example) should be regarded as an element of $\mathfrak{Z}_\Gamma$, and in addition to specifying its *support* – its underlying set – we must also specify what signatures occur with each accordion that appears in it, and with what sign. For example, in $\triangle\, \mathfrak{def}$ the accordion $\mathfrak{f}$ will occur with four different signatures: the actual contribution of $\mathfrak{f}$ to $\triangle\,\mathfrak{def}$ is

$$\mathfrak{f}_{\Box\Box *} - \mathfrak{f}_{\Box **} - \mathfrak{f}_{*\Box *} + \mathfrak{f}_{***}\,.$$

Now let us give a formal description of this setup. A signature η is called *nodal* if each η_i is either $\circ$ or $\Box$. We fix a nodal signature η. Let $\mathrm{CP}_\eta(c_0, \cdots, c_d)$ be the "cut and paste" simplex, which is the set of Γ-accordions

$$\mathfrak{a} = \left\{ \begin{matrix} s & & \mu_1 & & \cdots & & \mu_d \\ & \nu_1 & & \cdots & & \nu_d & \end{matrix} \right\} \tag{5.20}$$

that satisfy the inequalities

$$\mu_i - \mu_{i+1} \geqslant c_i', \qquad c_i' = \left\{ \begin{array}{ll} c_i & \text{if } \sigma_i = \Box, \\ 0 & \text{if } \sigma_i = \circ\,, \end{array} \right. \tag{5.21}$$

To see that this is truly a simplex, embed it into Euclidean space via the map

$$\mathfrak{a} \longmapsto (a_0, a_1, \cdots, a_d), \qquad a_i = \mu_i - \mu_{i+1} - c_i'.$$

The image of this map is the set of integer points in the simplex defined by the inequalities

$$a_i \geqslant 0, \qquad a_0 + \ldots + a_d \leqslant N, \qquad N = s - \sum c_i'.$$

Thus in the above example, with $\eta = \Box\Box\circ$ we have

$$\triangle\,\mathfrak{abc} = \mathrm{CP}_{**\circ}(c_0, c_1, \infty), \qquad \triangle\,\mathfrak{aeg} = \mathrm{CP}_{*\Box\circ}(c_0, c_1, \infty),$$

$$\triangle\,\mathfrak{dbh} = \mathrm{CP}_{\Box *\circ}(c_0, c_1, \infty), \qquad \triangle\,\mathfrak{def} = \mathrm{CP}_{\Box\Box\circ}(c_0, c_1, \infty).$$

If σ and τ are signatures, we say that τ is a *subsignature* of σ if $\tau_i = \sigma_i$ whenever $\tau_i \neq *$, and we write $\tau \subset \sigma$ in this case. In other words, $\tau \subset \sigma$ if τ is obtained from σ by changing some $\Box$'s or $\circ$'s to $*$'s. If τ is a signature, we will denote $\mathrm{sgn}(\tau) = (-1)^\varepsilon$ where ε is the number of boxes in τ.

Returning to the general case, let $\mathfrak{a}$ be a Γ-accordion, and let σ be a compatible signature for $\mathfrak{a}$. Define

$$\Lambda_\Gamma(\mathfrak{a}, \sigma) = \sum_{\mathfrak{a}\text{-compatible } \tau \subset \sigma} \mathrm{sgn}(\tau)\mathcal{G}_\Gamma(\mathfrak{a}, \tau), \qquad \Lambda_\Delta(\mathfrak{a}', \sigma) = \sum_{\mathfrak{a}\text{-compatible } \tau \subset \sigma} \mathrm{sgn}(\tau)\mathcal{G}_\Delta(\mathfrak{a}', \tau).$$

In the definition of subsignature we allow either $\Box$'s or $\circ$'s to be changed to $*$'s (needed for later purposes). But in this summation, because τ and σ are both required to be compatible with the same accordion $\mathfrak{a}$, only $\Box$'s are changed to $*$ between σ and τ in the τ that appear in this definition.

Thus in the above example

$$\Lambda_\Gamma(\mathfrak{f}, \Box\Box*) = \mathcal{G}_\Gamma(\mathfrak{f}_{\Box\Box*}) - \mathcal{G}_\Gamma(\mathfrak{f}_{\Box**}) - \mathcal{G}_\Gamma(\mathfrak{f}_{*\Box*}) + \mathcal{G}_\Gamma(\mathfrak{f}_{***})\ .$$

Now let η be a nodal signature, and we may take $c_i = \infty$ if $\eta_i = \circ$. Let $\mathfrak{a} \in \mathrm{CP}_\eta(c_0, \cdots, c_d)$. Let $\sigma = \sigma(\mathfrak{a})$ be the subsignature of η obtained by changing η_i to $*$ when the inequality (5.21) is strict.

Statement E. *Assume that $n|s$. We have*

$$\sum_{\mathfrak{a}\in\mathrm{CP}_\eta(c_0,\cdots,c_d)} \Lambda_\Gamma(\mathfrak{a}, \sigma) = \sum_{\mathfrak{a}\in\mathrm{CP}_\eta(c_0,\cdots,c_d)} \Lambda_\Delta(\mathfrak{a}', \sigma). \tag{5.22}$$

We reiterate that in this sum σ depends on $\mathfrak{a}$, and we have described the nature of the dependence above.

We have already noted that $n|s$ can be imposed in Statement D and now we impose it explicitly. We will show in Chapter 14 that Statement E implies Statement D by application of the Inclusion-Exclusion principle. We hope for the purpose of this outline of the proof, the above example will make that plausible. In that example, the four triangles $\triangle$ $\mathfrak{abc}$, $\triangle$ $\mathfrak{aeg}$, $\triangle$ $\mathfrak{dbc}$ and $\triangle$ $\mathfrak{def}$ are examples of cut and paste simplices.

We have already mentioned that in the context of Statements B, C or D, it is empirically possible to partition the sum into a disjoint union of smaller units called *packets* such that the identity is true when summation is restricted to a packet.Yet it is also true that in those contexts, a general rule describing the packets is notoriously slippery to nail down. However, in the context of Statement E we are able to describe the packets explicitly. The d-dimensional simplex $\mathrm{CP}_\eta(c_0, \cdots, c_d)$ is partitioned into *facet*, which are subsimplices of lower dimension. Specifically, there are $\binom{d+1}{f+1}$ facets that are simplices of dimension f; we will call these *f-facets*. We will define the packets so that if $\mathfrak{a}$ lies on the interior of an f-facet, then the packet containing $\mathfrak{a}$ has $\binom{d+1}{f+1}$ elements, one chosen from the interior of each r-facet.

Let us make this precise. First we observe the following description of the $d+1$ vertices $\mathfrak{a}_i$ of $\mathrm{CP}_\eta(c_0, \cdots, c_d)$. If $0 \leqslant i \leqslant d$ let $\mathfrak{a}_i$ be the accordion whose coordinates μ_i are determined by the equations

$$\mu_j - \mu_{j+1} = c'_j, \qquad \text{for all } 0 \leqslant j \leqslant d \text{ with } j \neq i.$$

A *closed (resp. open) f-facet* of $\mathrm{CP}_\eta(c_0, \cdots, c_d)$ will be the set of integer points in the closed (resp. open) convex hull of a subset with cardinality $f+1$ of the set

$\{\mathfrak{a}_0, \cdots, \mathfrak{a}_d\}$ of vertices. Clearly every element of $\mathrm{CP}_\eta(c_0, \cdots, c_d)$ lies in a unique open facet.

We associate the facets with subsignatures σ of η; if σ is obtained by replacing η_i ($i \in S$) by $*$, where S is some subset of $\{0, 1, 2, \cdots, d\}$, then we will denote the set of integer points in the closed (resp. open) convex hull of $\mathfrak{a}_i$ ($i \in S$) by $\overline{\mathcal{S}_\sigma}$ (resp. $\mathcal{S}_\sigma$). The facet $\overline{\mathcal{S}_\sigma}$ is itself a simplex, of dimension f. Thus if σ is a signature with exactly $f+1$ $*$'s, we call σ an *f-signature* or an *f-subsignature of* η.

Now if σ and τ are f-subsignatures of η, then we will define a bijection $\phi_{\sigma,\tau} : \overline{\mathcal{S}_\sigma} \longrightarrow \overline{\mathcal{S}_\tau}$. It is the unique affine linear map that takes the vertices of $\overline{\mathcal{S}_\sigma}$ to the vertices of $\overline{\mathcal{S}_\tau}$ in order. This means that if

$$S = \{s_0, \cdots, s_f\}, \qquad 0 \leqslant s_0 < s_1 < \cdots < s_f \leqslant d$$

is the set of i such that $\sigma_i = *$, and similarly if

$$T = \{t_0, \cdots, t_f\}, \qquad 0 \leqslant t_0 < t_1 < \cdots < t_f \leqslant d$$

is the set of i such that $\tau_i = *$, then $\phi_{\sigma,\tau}$ takes $\mathfrak{a}_{s_i}$ to $\mathfrak{a}_{t_i}$, and this map on vertices is extended by affine linearity to a map on all of $\overline{\mathcal{S}_\sigma}$.

It is obvious from the definition that $\phi_{\sigma,\sigma}$ is the identity map on $\mathcal{S}_\sigma$ and that if σ, τ, θ are f-subsignatures of η then $\phi_{\tau,\theta} \circ \phi_{\sigma,\tau} = \phi_{\sigma,\theta}$. This means that we may define an equivalence relation on $\mathrm{CP}_\eta(c_0, \cdots, c_d)$ as follows. Let $\mathfrak{a}, \mathfrak{b} \in \mathrm{CP}_\eta(c_0, \cdots, c_d)$. Let $\mathcal{S}_\sigma$ and $\mathcal{S}_\tau$ be the (unique) open facets such that $\mathfrak{a} \in \mathcal{S}_\sigma$ and $\mathfrak{b} \in S_\tau$. Then $\mathfrak{a}$ is equivalent to $\mathfrak{b}$ if and only if $\phi_{\sigma,\tau}(\mathfrak{a}) = \mathfrak{b}$. The equivalence classes are called *packets*. It is clear from the definitions that the number $f+1$ of $*$'s in σ is constant for σ that appear in a packet Π, and we will call Π a *f-packet*. Clearly every f-packet contains exactly one element from each f-simplex.

Statement F. *Assume that $n|s$. Let Π be a packet. Then*

$$\sum_{\mathfrak{a} \in \Pi} \Lambda_\Gamma(\mathfrak{a}, \sigma) = \sum_{\mathfrak{a} \in \Pi} \Lambda_\Delta(\mathfrak{a}', \sigma). \tag{5.23}$$

As in Statement E, σ depends on $\mathfrak{a}$ in this sum, and from the definition of packets, no σ appears more than once; in fact, if Π is an f-packet, then every f-subsignature σ of η appears exactly once on each side of the equation.

It is obvious that Statement F implies Statement E. There is one further sufficient condition that we call **Statement G**, but a proper formulation requires more notation than we want to give at this point. We will therefore postpone Statement G to the end of Chapter 15, describing it here in informal terms.

Due to the knowability property of the products of Gauss sums that make up $\mathcal{G}_\Gamma(\mathfrak{a}, \sigma)$, these can be evaluated explicitly when $n|s$ (Proposition 21) and this leads to an evaluation of $\Lambda_\Gamma(\mathfrak{a}, \sigma)$ and a similar evaluation of $\Lambda_\Delta(\mathfrak{a}', \sigma)$ (Theorems 11 and 12). However, these evaluations depend on the divisibility properties of the μ_i that appear in the top row of $\mathfrak{a}$ given by (5.20); more precisely, if Σ is a subset of $\{1, 2, \cdots, d\}$, let $\delta_n(\Sigma; \mathfrak{a})$ be 1 if $n|\mu_i$ for all $i \in \Sigma$ and 0 otherwise. Then there is a sum over certain such subsets Σ of $\mathfrak{a}$ – only those i such that $\sigma_i \neq \bigcirc$ can appear – and the terms that appear are with a coefficient $\delta_n(\Sigma; \mathfrak{a})$. We recall that each σ appears only once on each side of (5.23), and hence $\mathfrak{a}$ is really a function of σ. Thus we are reduced to proving Statement G, amounting to an identity (15.9) in which there is first a sum over all f-subsignatures σ of η, and then a sum over subsets Σ, of $\{1, 2, \cdots, d\}$.

The identity (15.9) seems at first perplexing since $\delta_n(\Sigma; \mathfrak{a})$ depends on $\mathfrak{a}$. It won't work to simply interchange the order of summation since then $\delta_n(\Sigma; \mathfrak{a})$ will not be constant on the inner sum over $\mathfrak{a}$ (or equivalently σ). However we are able to identify an equivalence relation that we call *concurrence* on pairs (σ, Σ) such that $\delta_n(\Sigma; \mathfrak{a})$ is constant on concurrence classes (Proposition 22). We will then need a result that implies that some groups of terms from the same side of (15.9) involve concurrent data (Proposition 23). These concurrent data are called Γ-*packs* for the left-hand side or Δ-*packs* for the right-hand side. Then we will need a rather more subtle result (Proposition 24) giving a bijection between the Γ-packs and the Δ-packs that also matches concurrent data. With these combinatorial preparations, we will be able to prove (15.9) and therefore Statement G.

Chapter 6

Statement B implies Statement A

In this chapter we will recall the use of the Schützenberger involution on Gelfand-Tsetlin patterns in [9] to prove that Statement B implies Statement A. We will return to the involution in the next chapter and we reinterpret this proof in terms of crystal bases.

We observe that the Schützenberger involution q_r can be formulated in terms of operations on short Gelfand-Tsetlin patterns. If

$$\mathfrak{T} = \left\{ \begin{matrix} a_{00} & & a_{01} & & a_{02} & \cdots & & a_{0r} \\ & a_{11} & & a_{12} & & & a_{1r} & \\ & & \ddots & & & \iddots & & \\ & & & & a_{rr} & & & \end{matrix} \right\}$$

is a Gelfand-Tsetlin pattern and $1 \leqslant k \leqslant r$, then extracting the $r-k$, $r+1-k$ and $r+2-k$ rows gives a short Gelfand-Tsetlin pattern $\mathfrak{t}$. Replacing this with the pattern $\mathfrak{t}'$ gives a new Gelfand-Tsetlin pattern which is the one denoted $t_r\mathfrak{T}$ in Chapter 1. Thus

$$t_1 \left\{ \begin{matrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & a & & b & \\ & & c & & \end{matrix} \right\} = \left\{ \begin{matrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & a & & b & \\ & & a+b-c & & \end{matrix} \right\}$$

and

$$t_2 \left\{ \begin{matrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & a & & b & \\ & & c & & \end{matrix} \right\} = \left\{ \begin{matrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & a' & & b' & \\ & & c & & \end{matrix} \right\}$$

where $a' = \lambda_1 + \max(\lambda_2, c) - a$ and $b' = \lambda_3 + \min(\lambda_2, c) - b$.

We defined q_0 to be the identity map, and defined recursively $q_i = t_1 t_2 \cdots t_i q_{i-1}$. The t_i have order two. They do not satisfy the braid relation, so $t_i t_{i+1} t_i \neq t_{i+1} t_i t_{i+1}$.

However $t_i t_j = t_j t_i$ if $|i-j|>1$ and this implies that the q_i also have order two. One may check easily that

We note that

$$q_i = q_{i-1} q_{i-2} t_i q_{i-1}. \tag{6.1}$$

Let $A_i = \sum_j a_{i,j}$ be the sum of the i-th row of $\mathfrak{T}$. It may be checked that the row sums of $q_r\mathfrak{T}$ are (in order)

$$A_0, A_0 - A_r, A_0 - A_{r-1}, \cdots, A_0 - A_1.$$

From this it follows that

$$k_\Gamma(q_r \mathfrak{T}) = k_\Delta(\mathfrak{T}).$$

From this we see that Statement A will follow if we prove

$$\sum_{k_r(\mathfrak{T})=\boldsymbol{k}} G_\Gamma(\mathfrak{T}) = \sum_{k_r(\mathfrak{T})=\boldsymbol{k}} G_\Delta(q_r\mathfrak{T}). \tag{6.2}$$

We note that the sum is over all patterns with fixed *top row* and *row sums*.

Let us denote

$$G_R^i(\mathfrak{T}) = \prod_{j=i}^{r} \begin{cases} g(\Gamma_{ij}) & \text{if } \Gamma_{ij} \text{ is boxed but not circled in } \Gamma(\mathfrak{T}); \\ q^{\Gamma_{ij}} & \text{if } \Gamma_{ij} \text{ is circled but not boxed;} \\ h(\Gamma_{ij}) & \text{if } \Gamma_{ij} \text{ neither circled nor boxed;} \\ 0 & \text{if } \Gamma_{ij} \text{ both circled and boxed} \end{cases}$$

and

$$G_L^i(\mathfrak{T}) = \prod_{j=i}^{r} \begin{cases} g(\Delta_{ij}) & \text{if } \Delta_{ij} \text{ is boxed but not circled in } \Gamma(\mathfrak{T}); \\ q^{\Delta_{ij}} & \text{if } \Delta_{ij} \text{ is circled but not boxed;} \\ h(\Delta_{ij}) & \text{if } \Delta_{ij} \text{ neither circled nor boxed;} \\ 0 & \text{if } \Delta_{ij} \text{ both circled and boxed,} \end{cases}$$

where Γ_{ij} and Δ_{ij} are given by (1.10). Thus

$$G_\Gamma(\mathfrak{T}) = \prod_{i=1}^{r} G_R^i(\mathfrak{T}), \qquad G_\Delta(\mathfrak{T}) = \prod_{i=1}^{r} G_L^i(\mathfrak{T}).$$

To facilitate our inductive proof we denote ${}^{(r)}G_\Gamma(\mathfrak{T}) = G_\Gamma(\mathfrak{T})$ and ${}^{(r)}G_\Delta(\mathfrak{T}) = G_\Delta(\mathfrak{T})$. Also if $i \leqslant r$ let $\mathfrak{T}_i$ denote the pattern formed with the bottom $i+1$ rows of $\mathfrak{T}$.

$$\begin{aligned} \textstyle\sum_{k_\Gamma=\boldsymbol{k}} G_\Gamma(\mathfrak{T}) &= \textstyle\sum_{k_\Gamma=\boldsymbol{k}} G_R^r(\mathfrak{T}) \cdot {}^{(r-1)}G_\Gamma(\mathfrak{T}) \\ &= \textstyle\sum_{k_\Gamma=\boldsymbol{k}} G_R^r(\mathfrak{T}) \cdot {}^{(r-1)}G_\Delta(q_{r-1}\mathfrak{T}) \\ &= \textstyle\sum_{k_\Gamma=\boldsymbol{k}} G_R^r(\mathfrak{T}) \cdot G_L^{r-1}(q_{r-1}\mathfrak{T}) \cdot {}^{(r-2)}G_\Delta(q_{r-1}\mathfrak{T}) \\ &= \textstyle\sum_{k_\Gamma=\boldsymbol{k}} G_R^r(\mathfrak{T}) \cdot G_L^{r-1}(q_{r-1}\mathfrak{T}) \cdot {}^{(r-2)}G_\Gamma(q_{r-2}q_{r-1}\mathfrak{T}) \\ &= \textstyle\sum_{k_\Gamma=\boldsymbol{k}} G_R^r(q_{r-2}q_{r-1}\mathfrak{T}) \cdot G_L^{r-1}(q_{r-2}q_{r-1}\mathfrak{T}) \cdot {}^{(r-2)}G_\Gamma(q_{r-2}q_{r-1}\mathfrak{T}). \end{aligned}$$

Here the first step is by definition; the second step is by applying the induction hypothesis that Statement A is true for $r-1$ to $\mathfrak{T}_{r-1}$; the third step is by definition; the fourth step is by induction, using Statement A for $r-2$ applied to $\mathfrak{T}_{r-2}$; and the last step is because $q_{r-2}q_{r-1}$ does not change the top two rows of $\mathfrak{T}$, hence does not affect the value of G_R^r, and similarly q_{r-2} does not change the value of G_R^{r-1}.

On the other hand we have

$$\begin{aligned}
\textstyle\sum_{k_\Gamma=\boldsymbol{k}} G_\Delta(q_r\mathfrak{T}) &= \textstyle\sum_{k_\Gamma=\boldsymbol{k}} \; G_L^r(q_r\mathfrak{T}) \cdot {}^{(r-1)}G_\Delta(q_r\mathfrak{T}) \\
&= \textstyle\sum_{k_\Gamma=\boldsymbol{k}} \; G_L^r(q_r\mathfrak{T}) \cdot {}^{(r-1)}G_\Gamma(q_{r-1}q_r\mathfrak{T}) \\
&= \textstyle\sum_{k_\Gamma=\boldsymbol{k}} \; G_L^r(q_r\mathfrak{T}) \cdot G_R^{r-1}(q_{r-1}q_r\mathfrak{T}) \cdot {}^{(r-2)}G_\Gamma(q_{r-1}q_r\mathfrak{T}) \\
&= \textstyle\sum_{k_\Gamma=\boldsymbol{k}} \; G_L^r(q_{r-1}q_r\mathfrak{T}) \cdot G_R^{r-1}(q_{r-1}q_r\mathfrak{T}) \cdot {}^{(r-2)}G_\Gamma(t_rq_{r-1}q_r\mathfrak{T}).
\end{aligned}$$

Here the first step is by definition, the second by induction, the third by definition, and the fourth because q_{r-1} does not affect the top two rows of $q_{r-1}q_r\mathfrak{T}$, and t_r does not affect the rows of $(q_{r-1}q_r\mathfrak{T})_{r-2}$. Now we use the assumption that Statement B is true. Statement B implies that

$$\sum G_L^r(q_{r-1}q_r\mathfrak{T}) \cdot G_R^{r-1}(q_{r-1}q_r\mathfrak{T}) = \sum G_R^r(t_rq_{r-1}q_r\mathfrak{T}) \cdot G_L^{r-1}(t_rq_{r-1}q_r\mathfrak{T})$$

where in this summation we may collect together all $q_{r-1}q_r\mathfrak{T}$ with the same first, third, fourth, ... rows and let only the second row vary to form a summation over short Gelfand-Tsetlin pattern. Substituting this back into the last identity gives

$$\sum_{k_\Gamma=\boldsymbol{k}} G_\Delta(q_r\mathfrak{T}) = \sum_{k_\Gamma=\boldsymbol{k}} G_R^r(t_rq_{r-1}q_r\mathfrak{T}) \cdot G_L^{r-1}(t_rq_{r-1}q_r\mathfrak{T}) \cdot {}^{(r-2)}G_\Gamma(t_rq_{r-1}q_r\mathfrak{T}).$$

Now we make use of (6.1) in the form $t_rq_{r-1}q_r = q_{r-2}q_{r-1}$ to complete the proof of Statement A, assuming Statement B.

Chapter 7

Accordions and Crystal Graphs

We will translate Statements A and B into Statements A′ and B′ in the language of crystal bases, and explain how Statement B′ implies Statement A′. This sheds light on the last chapter but is not used later, so the reader may skip this chapter with no loss of continuity.

Paralleling the definition on Gelfand-Tsetlin patterns, we now define

$$G_\Omega(v) = \prod_{b_i \in \mathrm{BZL}_\Omega(v)} \begin{cases} g(b_i) & \text{if } b_i \text{ is boxed but not circled in } \mathrm{BZL}_\Omega(v), \\ q^{b_i} & \text{if } b_i \text{ is circled but not boxed,} \\ h(b_i) & \text{if } b_i \text{ is neither circled nor boxed,} \\ 0 & \text{if } b_i \text{ is both boxed and circled.} \end{cases}$$

Then Statement A (Section 5) can be paraphrased as follows.

Statement A′. *We have*

$$\sum_{\mathrm{wt}(v)=\mu} G_{\Omega_\Gamma}(v) = \sum_{\mathrm{wt}(v)=\mu} G_{\Omega_\Delta}(v).$$

We believe that if the correct definition of the boxing and circling decorations can be given, we could say that $\sum_{\mathrm{wt}(v)=\mu} G_\Omega(v)$ is independent of the choice of Ω. However the description of the boxing and circling might be different for Ω other than Ω_Δ and Ω_Γ, and we will limit our discussion to those two words. This need for caution may be related to assumptions required by Littelmann [31] in order to specify sets of BZL patterns associated to a particular "good" long word. Littelmann found that for particular choices of "good" decompositions, including $\Omega = \Omega_\Gamma, \Omega_\Delta$, one can easily compute explicit inequalities which describe a polytope whose integer lattice points parametrize the set of all BZL patterns in a highest weight crystal. The decoration rules are closely connected to the location of $\mathrm{BZL}_\Omega(v)$ in this polytope.

The crystal graph formulation in Statement A′ is somewhat simpler than its Gelfand-Tsetlin counterpart. In particular, in the formulation of Statement A, we had two different Gelfand-Tsetlin patterns $\mathfrak{T}$ and $\mathfrak{T}'$ that were related by the Schützenberger involution, but the equality in Statement A was further complicated because the involution changes the weight of the pattern. In the crystal graph formulation, different decompositions of the long element simply result in different paths from the same vertex v to the lowest weight vector.

We will explain how Statement A′ can be proved inductively. First we must explain the interpretation of the short Gelfand-Tsetlin patterns $\mathfrak{t}$ and their associated preaccordions $\Gamma_{\mathfrak{t}}$ and $\Delta_{\mathfrak{t}}$ in the crystal language.

Removing all edges labeled either 1 or r from the crystal graph results in a disjoint union of crystals of type A_{r-2}. The root operators for one of these subcrystals have indices shifted – they are $f_2, \cdots, f_{r-1}$ and $e_2, \cdots, e_{r-1}$ – but this is unimportant. Each such subcrystal has a unique lowest weight vector, characterized by $f_i(v) = 0$ for all $1 < i < r$. If $v \in \mathcal{B}_\lambda$ we will say that v is a *short end* if $f_i(v) = 0$ for all $1 < i < r$. Thus there is a bijection between these subcrystals and the short ends.

Now consider the words

$$\omega_\Gamma = (1, 2, 3, \cdots, r-1, r, r-1, \cdots, 3, 2, 1)$$

and

$$\omega_\Delta = (r, r-1, r-2, \cdots, 3, 2, 1, 2, 3, \cdots, r-1, r).$$

Identifying the Weyl group with the symmetric group S_{r+1} and the simple reflections $\sigma_i \in W$ with transpositions $(i, i+1)$, these give reduced decompositions of the long element expressed as the transposition $(1, r+1)$. That is, if $\omega = \omega_\Gamma$ or ω_Δ and

$$\omega = (b_1, \cdots, b_{2r-1})$$

then $\sigma_{b_1} \cdots \sigma_{b_{r+1}} = (1, r+1)$.

The following result interprets the preaccordions $\Gamma_{\mathfrak{t}}$ and $\Delta_{\mathfrak{t}'}$ of a short Gelfand-Tsetlin pattern, which have occupied so much space in this document, as paths in the crystal.

Theorem 6 *Let v be a short end, and let $\omega = \omega_\Gamma$ or ω_Δ. Then we have*

$$v \begin{bmatrix} b_1 & \cdots & b_{2r-1} \\ \omega_1 & \cdots & \omega_{2r-1} \end{bmatrix} v' \tag{7.1}$$

with $v' = v_{\text{low}}$. Moreover, the b_i satisfy the inequalities

$$b_1 \geqslant b_2 \geqslant \ldots \geqslant b_{r-1} \geqslant 0, \qquad b_r \geqslant b_{r+1} \geqslant \ldots \geqslant b_{2r-1} \geqslant 0.$$

Let $\mathfrak{t} = \mathfrak{t}(v)$ be the short Gelfand-Tsetlin pattern obtained by discarding all but the top three rows of $q_{r-1}\mathfrak{T}_v$. Then if $\omega = \omega_\Gamma$ we have in the notation (5.9)

$$\Gamma_{\mathfrak{t}} = \left\{ \begin{array}{ccccccccccc} b_r & & b_{r+1} & & b_{r+2} & & \cdots & & b_{2r-2} & & b_{2r-1} \\ & b_{r-1} & & b_{r-2} & & \cdots & & b_2 & & b_1 & \end{array} \right\} \tag{7.2}$$

where $d = r - 1$. On the other hand if $\omega = \omega_\Delta$ then in the notation (5.10)

$$\Delta_{\mathfrak{t}'} = \left\{ \begin{array}{ccccccccccc} b_{2r-1} & & b_{2r-2} & & b_{2r-3} & & \cdots & & b_{r+1} & & b_r \\ & b_1 & & b_2 & & \cdots & & b_{r-2} & & b_{r-1} & \end{array} \right\}. \tag{7.3}$$

If v_1 and v_2 are two short ends such that $\mathfrak{t}(v_1)$ and $\mathfrak{t}(v_2)$ are in the same short pattern prototype, then $\mathrm{wt}(v_1) = \mathrm{wt}(v_2)$.

Proof Let $\mathcal{B}_\mu$ be the A_{r-1} crystal containing v which is obtained from $\mathcal{B}_\lambda$ by deleting the r-labeled edges. We make use of the word

$$\Omega_{\Delta,r-1} = (r-1, r-2, r-1, r-3, r-2, r-1, \cdots, 1, 2, 3, \cdots, r-1)$$

which represents the long element of A_{r-1} and obtain a path

$$v \left[\begin{array}{cccccccccc} 0 & 0 & 0 & \cdots & 0 & b_1 & b_2 & b_3 & \cdots & b_{r-1} \\ r-1 & r-2 & r-1 & \cdots & r-1 & 1 & 2 & 3 & \cdots & r-1 \end{array} \right] v'$$

where the initial string of 0's is explained by the fact that $f_i v = 0$ when $2 \leqslant i \leqslant r-1$. Thus we could equally well write

$$v \left[\begin{array}{ccccc} b_1 & b_2 & b_3 & \cdots & b_r \\ 1 & 2 & 3 & \cdots & r \end{array} \right] v'.$$

By Proposition 1, v' is the lowest weight vector of $\mathcal{B}_\mu$, so $f_1 v' = \cdots = f_{r-1} v' = 0$. Next we make use of the word

$$\Omega_\Gamma = (1, 2, 1, 3, 2, 1, \cdots, r, r-1, \cdots, 3, 2, 1)$$

and apply it to v'. Again, the first f_i that actually "moves" v' is f_r, and so we obtain a path

$$v' \left[\begin{array}{cccccccccc} 0 & 0 & 0 & \cdots & 0 & b_r & b_{r+1} & b_{r+2} & \cdots & b_{2r-1} \\ 1 & 2 & 1 & \cdots & 1 & r & r-1 & r-2 & \cdots & 1 \end{array} \right] v_{\mathrm{low}}$$

which we could write

$$v' \left[\begin{array}{ccccc} b_r & b_{r+1} & b_{r+2} & \cdots & b_{2r-1} \\ r & r-1 & r-2 & \cdots & 1 \end{array} \right] v_{\mathrm{low}}.$$

Splicing the two paths we get (7.1).

Next we prove (7.2). We note that the top row of $\Gamma_{\mathfrak{t}}$ depends only on the top two rows of $\mathfrak{t}$, which are the same as the top two rows of $\mathfrak{T} = \mathfrak{T}_v$ since q_{r-1} does not affect these top two rows and $\mathfrak{t}$ consists of the top three rows of $q_{r-1}\mathfrak{t}$. The top row of $\Gamma_{\mathfrak{t}}$ is obtained from the top two rows of $\mathfrak{t}$ by the right-hand rule (see Chapter 1), and so it agrees with the top row of $\Gamma_{\mathfrak{T}}$.

Now we regard v' as an element of the crystal $\mathcal{B}_\mu$ and apply the word $\Omega_{\Delta,r-1}$. We see that $b_1, \cdots, b_r$ are the top row of $\Delta(q_{r-1}\mathfrak{T}_{r-1})$ where $\mathfrak{T}_{r-1}$ is the Gelfand-Tsetlin pattern obtained by discarding the top row of $\mathfrak{T}$. Now the top two rows of $q_{r-1}\mathfrak{T}_{r-1}$ are the middle and bottom rows of $\mathfrak{t}$, which in $\Gamma_{\mathfrak{t}}$ is read by the left-hand rule, which is the same as $\Delta(q_{r-1}\mathfrak{T}_{r-1})$. It follows that $b_1, \cdots, b_r$ form the top row of $\Gamma_{\mathfrak{t}}$, as required. This proves (7.2).

It remains for us to prove (7.3). As in Proposition 1 we can make use of ϕ_v which interchanges the words ω_Γ and ω_Δ. Using (2.11) and arguing as at the end of Proposition 1 we see that the right-hand side of (7.3) equals $\Gamma_{\mathfrak{u}}^{\text{rev}}$, where $\mathfrak{u}$ is the short Gelfand-Tsetlin pattern obtained by taking the top three rows of $-q_{r-1}q_r\mathfrak{T}_v^{\text{rev}}$. Now we make use of (6.1) in the form $q_{r-1}q_r = q_{r-2}t_r q_{r-1}$ to see that $\mathfrak{u}$ is the short Gelfand-Tseltin pattern obtained by taking the top three rows of $-q_{r-2}t_r q_{r-1}\mathfrak{T}_v^{\text{rev}}$, and since q_{r-2} does not affect these top three rows, we see that $\mathfrak{u}$ is $-(\mathfrak{t}')^{\text{rev}}$. Now $\Gamma_{\mathfrak{u}}^{\text{rev}} = \Delta_{\mathfrak{t}'}$ which concludes the proof. □

Having identified the $\Gamma_{\mathfrak{t}}$ and $\Delta_{\mathfrak{t}'}$ that appear in Statement B, let us paraphrase Statement B as follows. If v is a short end, we may define decorations on $\Gamma_{\mathfrak{t}(v)}$ and $\Delta_{\mathfrak{t}(v)'}$. These may be described alternatively as in Chapter 5 or geometrically as in this chapter: b_i is circled if $i = r-1$ or $2r-1$ and $b_i = 0$ or if $i \neq r-1$ or $2r-1$ and $b_i = b_{i+1}$. Also b_i is boxed if the segment of length b_i that occurs in the canonical path contains the entire i-segment.

Statement B′. *We have*

$$\sum_{\substack{\text{short end } v\\ \text{wt}(v)=\mu}} G_{\omega_\Gamma}(v) = \sum_{\substack{\text{short end } v\\ \text{wt}(v)=\mu}} G_{\omega_\Delta}(v),$$

where the sum is over short ends of a given weight.

This statement is equivalent to Statement B and is thus proved in the preceding chapters. We now explain how Statement B′ implies Statement A′.

This is proved by induction on r. It will perhaps be clearer if we explain this point with a fixed r, say $r = 4$; the general case follows by identical methods. We have two paths from v to v_{low}, each of which we may decorate with boxing and circling. These paths will be denoted

$$v \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} \\ 1 & 2 & 1 & 3 & 2 & 1 & 4 & 3 & 2 & 1 \end{bmatrix} v_{\text{low}} \tag{7.4}$$

and

$$v \begin{bmatrix} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 & z_9 & z_{10} \\ 4 & 3 & 4 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \end{bmatrix} v_{\text{low}}. \tag{7.5}$$

We have

$$G_\Gamma(\mathfrak{T}_v) = G_{\Omega_\Gamma}(v) = \prod_i \begin{cases} g(b_i) & \text{if } b_i \text{ is boxed but not circled,} \\ q^{b_i} & \text{if } b_i \text{ is circled but not boxed,} \\ h(b_i) & \text{if } b_i \text{ is neither circled nor boxed,} \\ 0 & \text{if } b_i \text{ is both boxed and circled,} \end{cases}$$

and similarly for $G_\Delta(\mathfrak{T}'_v) = G_{\Omega_\Delta}(v)$. We split the first path into two:

$$v \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ 1 & 2 & 1 & 3 & 2 & 1 \end{bmatrix} v', \qquad v' \begin{bmatrix} b_7 & b_8 & b_9 & b_{10} \\ 4 & 3 & 2 & 1 \end{bmatrix} v_{\text{low}}.$$

Since $1, 2, 1, 3, 2, 1$ is a reduced decomposition of the long element in the Weyl group of type $A_3 = A_{r-1}$ generated by the $1, 2, 3$ root operators, v' is the lowest weight vector in the connected component containing v of the subcrystal obtained by discarding the edges labeled r. This means that we may replace

$$v \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ 1 & 2 & 1 & 3 & 2 & 1 \end{bmatrix} v' \qquad \text{by} \qquad v \begin{bmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ 3 & 2 & 3 & 1 & 2 & 3 \end{bmatrix} v'$$

and we obtain a new path:

$$v \begin{bmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & b_7 & b_8 & b_9 & b_{10} \\ 3 & 2 & 3 & 1 & 2 & 3 & 4 & 3 & 2 & 1 \end{bmatrix} v_{\text{low}}.$$

We split this again:

$$v \begin{bmatrix} c_1 & c_2 & c_3 \\ 3 & 2 & 3 \end{bmatrix} v'', \qquad v'' \begin{bmatrix} c_4 & c_5 & c_6 & b_7 & b_8 & b_9 & b_{10} \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 \end{bmatrix} v_{\text{low}}.$$

Now $3, 2, 3$ is a reduced word for the Weyl group of type $A_2 = A_{r-2}$ whose crystals are obtained by discarding edges labeled 1 and 4, and so v'' is a short end. It follows that we may replace the path from v'' to v_{low} by

$$v'' \begin{bmatrix} d_4 & d_5 & d_6 & z_7 & z_8 & z_9 & z_{10} \\ 4 & 3 & 2 & 1 & 2 & 3 & 4 \end{bmatrix} v_{\text{low}}.$$

(We have labeled some of these z since we will momentarily see that these z_i are the same as $z_7 - z_{10}$ in (7.5).) We may also replace the path from v to v'' by

$$v\begin{bmatrix} d_1 & d_2 & d_3 \\ 2 & 3 & 2 \end{bmatrix} v''$$

because both 323 and 232 are reduced words for A_2 and v'' is the lowest weight vector in an A_2 crystal. Combining these paths through v'' we obtain a path

$$v\begin{bmatrix} d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & z_7 & z_8 & z_9 & z_{10} \\ 2 & 3 & 2 & 4 & 3 & 2 & 1 & 2 & 3 & 4 \end{bmatrix} v_{\text{low}}.$$

Now we split the path again:

$$v\begin{bmatrix} d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ 2 & 3 & 2 & 4 & 3 & 2 \end{bmatrix} v''', \qquad v\begin{bmatrix} z_7 & z_8 & z_9 & z_{10} \\ 1 & 2 & 3 & 4 \end{bmatrix} v_{\text{low}}.$$

We observe that $2, 3, 2, 4, 3, 2$ is a reduced decomposition of the long element in the Weyl group of type $A_3 = A_{r-1}$ whose crystals are obtained by discarding edges labeled 1, and so v''' is a lowest weight vector of one of these, so we have also a path

$$v\begin{bmatrix} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ 4 & 3 & 4 & 2 & 3 & 4 \end{bmatrix} v''',$$

which we splice in and now we have obtained the path (7.5) by the following sequence alterations of (7.4).

$$v\begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} \\ 1 & 2 & 1 & 3 & 2 & 1 & 4 & 3 & 2 & 1 \end{bmatrix} v_{\text{low}}$$
$$v\begin{bmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & b_7 & b_8 & b_9 & b_{10} \\ 3 & 2 & 3 & 1 & 2 & 3 & 4 & 3 & 2 & 1 \end{bmatrix} v_{\text{low}}$$
$$v\begin{bmatrix} c_1 & c_2 & c_3 & d_4 & d_5 & d_6 & z_7 & z_8 & z_9 & z_{10} \\ 2 & 3 & 2 & 4 & 3 & 2 & 1 & 2 & 3 & 4 \end{bmatrix} v_{\text{low}}$$
$$v\begin{bmatrix} d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & z_7 & z_8 & z_9 & z_{10} \\ 2 & 3 & 2 & 4 & 3 & 2 & 1 & 2 & 3 & 4 \end{bmatrix} v_{\text{low}}$$
$$v\begin{bmatrix} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 & z_9 & z_{10} \\ 4 & 3 & 4 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \end{bmatrix} v_{\text{low}}$$

To each of these paths we may assign in a now familiar way a set of decorations and hence a value

$$G(\text{path}) = \prod_{x \in \text{path}} \begin{cases} g(x) & \text{if } x \text{ is boxed but not circled,} \\ q^x & \text{if } x \text{ is circled but not boxed,} \\ h(x) & \text{if } x \text{ is neither circled nor boxed,} \\ 0 & \text{if } x \text{ is both boxed and circled.} \end{cases}$$

Now if we sum $G(\text{path})$ over all v of given weight, each of these terms contributes equally to the next. For the second step, this is by Statement B′; for the others, this is by inductive hypothesis. Putting everything together, we obtain Statement A′.

Finally, we characterize accordions among preaccordions. Let $\alpha_1, \cdots, \alpha_r$ be the simple roots.

Proposition 4 *Let v be a short end in $\mathcal{B}_\lambda$. Then the associated preaccordions $\Gamma_{\mathfrak{T}_v}$ and $\Delta_{\mathfrak{T}'_v}$ are accordions if and only if $\mathrm{wt}(v) - w_0(\lambda)$ is a multiple of the longest root $\alpha_1 + \alpha_2 + \ldots + \alpha_r$.*

Thus the phenomenon of resonance can be understood as relating to the "diagonal" short ends, whose weights have equal components for all roots.

Proof By Theorem 6 we have

$$v \left[\begin{array}{cccccccc} b_1 & b_2 & \cdots & b_{r-1} & b_r & b_{r+1} & \cdots & b_{2r-1} \\ 1 & 2 & \cdots & r-1 & r & r-1 & \cdots & 1 \end{array} \right] v_{\text{low}}$$

This means that the path from v to v_{low} involves b_1+b_{2r-1} applications of f_1, b_2+b_{2r-2} applications of f_2, and so forth, and b_r applications of f_r. Since $\mathrm{wt}(f_i(x)) = \mathrm{wt}(x) - \alpha_i$, and since $\mathrm{wt}(v_{\text{low}}) = w_0(\lambda)$, this means that

$$\mathrm{wt}(v) - w_0(\lambda)(b_1 + b_{2r-1})\alpha_1 + (b_2 + b_{2r-2})\alpha_2 + \ldots + b_r\alpha_r.$$

Since the roots are linearly independent, this means $\mathrm{wt}(v) - w_0(\lambda)$ is a multiple of $\alpha_1 + \ldots + \alpha_r$ if and only if $b_1 + b_{2r-1} = \ldots = b_{r-1} + b_{r+1} = b_r$. This is precisely the condition for (7.2) to be an accordion. □

Chapter 8

Cartoons

Proposition 5 *(i) If* $n \nmid a$ *then* $h(a) = 0$*, while if* $n|a$ *we have*

$$h(a+b) = q^a h(b), \qquad g(a+b) = q^a h(b).$$

(ii) If $n|a$ *then*

$$h(a) = \phi(p^a) = q^{a-1}(q-1), \qquad g(a) = -q^{a-1},$$

while if $n \nmid a$ *then* $h(a) = 0$ *and* $|g(a)| = q^{a-\frac{1}{2}}$*. If* $n \nmid a, b$ *but* $n|a+b$ *then*

$$g(a)g(b) = q^{a+b-1}.$$

Proof This is easily checked using standard properties of Gauss sums. □

For the reduction to totally resonant prototypes – that is, the fact that Statement C implies Statement B – only (i) is used. The properties in (ii) become important later.

To define the cartoon, we will take a slightly more formal approach to the short Gelfand-Tsetlin patterns. Let

$$\Theta = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} |\, 0 \leqslant i \leqslant 2, 0 \leqslant j \leqslant d+1-i\}.$$

We call this set the *substrate*, and divide Θ into three *rows*, which are

$$\begin{aligned} \Theta_0 &= \{(0,j) \in \Theta | 0 \leqslant j \leqslant d+1\}, \\ \Theta_1 &= \{(1,j) \in \Theta | 0 \leqslant j \leqslant d\}, \\ \Theta_2 &= \{(2,j) \in \Theta | 0 \leqslant j \leqslant d-1\}, \end{aligned}$$

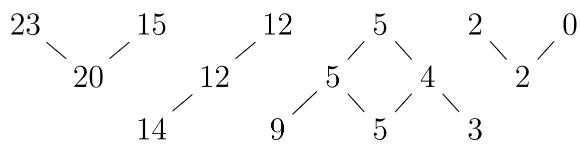
Let $\Theta_B = \Theta_1 \cup \Theta_2$. Each row has an order in which $(i, j) \leqslant (i, j')$ if and only if $j \leqslant j'$.

Now we can redefine a *short Gelfand-Tsetlin pattern* to be an integer valued function $\mathfrak{t}$ on the substrate, subject to the conditions that we have already stated. Thus the pattern (5.2) corresponds to the function on Θ such that $l_i = \mathfrak{t}(0, i)$, $a_i = \mathfrak{t}(1, i)$ and $b_i = \mathfrak{t}(2, i)$. The Γ and Δ preaccordions then become functions on Θ_B (the bottom and middle rows) in the same way. Specifying the circled and boxed elements just means specifying subsets of Θ_B.

Now the vertices of the cartoon will be the elements of the substrate Θ, and we have only to define the edges. With $\mathfrak{t}$ as in (5.2) we connect $(1, i)$ to $(0, i)$ if either $i = 0$ or $l_i \leqslant b_{i-1}$, and we connect $(1, i)$ to $(2, i-1)$ if $i > 1$ and $b_{i-1} \leqslant l_i$. Furthermore we connect $(1, i)$ to $(0, i+1)$ if either $i = d$ or if $i < d$ and $l_{i+1} \geqslant b_i$, and we connect $(1, i)$ to $(2, i)$ if $i < d$ and $b_i \geqslant l_{i+1}$. For example, consider the short pattern of rank 5:

$$\mathfrak{t} = \left\{ \begin{array}{ccccccccccc} 23 & & 15 & & 12 & & 5 & & 2 & & 0 \\ & 20 & & 12 & & 5 & & 4 & & 2 & \\ & & 14 & & 9 & & 5 & & 3 & & \end{array} \right\} \tag{8.1}$$

It is convenient to draw the cartoon as a graph on top of the preaccordion representing $\mathfrak{t}$, as follows:



. (8.2)

- The cartoon depends only on the top and bottom rows of $\mathfrak{t}$, so it is really a function of the prototype $\mathfrak{S}$ to which $\mathfrak{t}$ belongs.
- The cartoon encodes the relationship between $\mathfrak{t}$ and $\mathfrak{t}'$. Indeed, suppose that the cartoon has a subgraph $x —— y —— z$ where y is in the middle row, x and z are each in either the top or bottom row, with x is to the left of y and z is to the right. Then in $\mathfrak{t}'$, y is replaced by $x + z - y$.

For example, if $\mathfrak{t}$ is given by (8.1), then the cartoon (8.2) tells us how to compute

$$\mathfrak{t}' = \left\{ \begin{array}{ccccccccccc} 23 & & 15 & & 12 & & 5 & & 2 & & 0 \\ & 18 & & 14 & & 9 & & 4 & & 0 & \\ & & 14 & & 9 & & 5 & & 3 & & \end{array} \right\};$$

the middle row entries are $18 = 23 + 15 - 20$, $14 = 12 + 14 - 12$, $9 = 5 + 9 - 5$, $4 = 5 + 3 - 4$ and $0 = 2 + 0 - 2$.

The connected components of the cartoon are called *episodes*. These may be arranged in an order $\mathcal{E}_1, \cdots, \mathcal{E}_N$ so that if $i < j$ and $\alpha \in \mathcal{E}_i$, $\beta \in \mathcal{E}_j$, and if α and β are in the same row of the substrate Θ, then $\alpha < \beta$. With this partial order, if $\alpha \in \mathcal{E}_i$, $\beta \in \mathcal{E}_j$ then $\mathfrak{t}(\alpha) > \mathfrak{t}(\beta)$ regardless of whether or not α and β are in the same row.

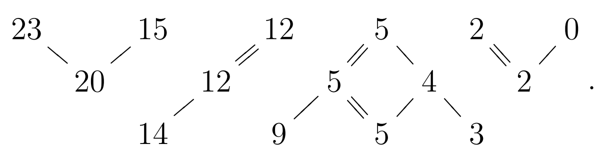
We call short pattern (5.2) *resonant* at i if $l_{i+1} = b_i$. A *resonance* of order k is a sequence $R = \{i, i+1, \cdots, i+k-1\}$ such that $\mathfrak{t}$ is resonant at each $j \in R$; the sequence must be maximal with this property, so that $\mathfrak{t}$ is not resonant at $i-1$ or $i+k$. In the example (8.1), a resonance at 2 can be recognized from the cartoon by the diamond shape between $l_3 = b_2 = 5$.

We will next describe another kind of diagram related to the cartoon in which we mark certain edges with double bonds, and box and circle certain vertices. We will refer to the diagram in which the bonded edges and circled vertices are marked as the *bond-marked* cartoon. See (8.3) and (8.4) below for examples.

- Unlike the cartoon, the bond-marked cartoon really depends on $\mathfrak{t}$, not just on its prototype.

- The bond-marked cartoon is useful since the circling and boxing of the Γ and Δ preaccordions can be read off from it.

The edge joining $\alpha, \beta \in \Theta$ will be called *distinguished* if $\mathfrak{t}(\alpha) = \mathfrak{t}(\beta)$. In representing the bond-marked cartoon graphically we will mark the distinguished edges by double bonds, which may be read as equal signs. Thus in the example (8.1), the cartoon of $\mathfrak{t}$ becomes



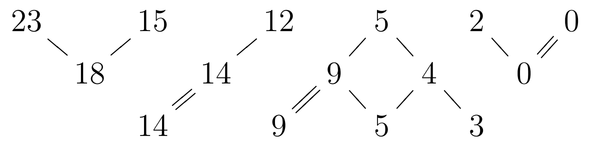
(We have drawn this labeling on top of $\mathfrak{t}$ itself, but ultimately we will draw it on top of the Γ or Δ preaccordions.)

We observe that while the original bond-unmarked cartoon only depends on the pattern prototype $\mathfrak{S}$ to which $\mathfrak{t}$ belongs, this diagram does depend on $\mathfrak{t}$. In particular, $\mathfrak{t}$ and $\mathfrak{t}'$ no longer have the same cartoon, since the double bonds move under the involution $\mathfrak{t} \longmapsto \mathfrak{t}'$. However the rule is quite simple:

Lemma 2 *Suppose the bond-marked cartoon of $\mathfrak{t}$ has a subgraph of the form x — z == z, where the first z is in the middle row, so that x and the second z are in the top or bottom row. Then in the bond-marked cartoon the double bond moves to the other edge, so the bond-marked cartoon of $\mathfrak{t}'$ contains a subgraph x == x —— z.*

Proof Immediate from the definitions. □

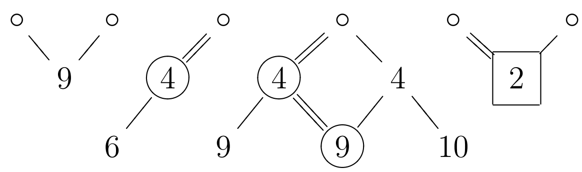
In this example, the bond-marked cartoon of $\mathfrak{t}'$ is



As we have already mentioned, the cartoon is very useful when superimposed on the Γ and Δ' preaccordions, where $\Gamma = \Gamma_{\mathfrak{t}}$ and $\Delta' = \Delta_{\mathfrak{t}'}$. Since these arrays have only two rows, we add a third row at the top. We will also box and circle certain entries, by a convention that we will explain after giving an example. Thus in this example

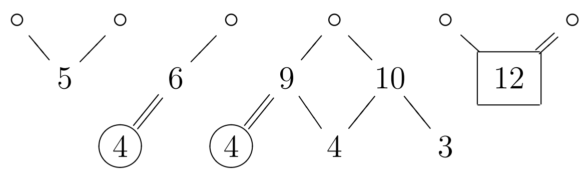
$$\Gamma = \left\{ \begin{array}{ccccccccc} 9 & & 4 & & 4 & & 4 & & 2 \\ & 6 & & 9 & & 9 & & 10 & \end{array} \right\}, \qquad \Delta' = \left\{ \begin{array}{ccccccccc} 5 & & 6 & & 9 & & 10 & & 12 \\ & 4 & & 4 & & 4 & & 3 & \end{array} \right\}.$$

We superimpose the cartoon on these, representing Γ thus:



(8.3)

and Δ' as



(8.4)

We've inserted a row of $\circ$'s in the top (0-th) row since the Γ and Δ' preaccordions have first and second rows but no 0-th row; we supply these for the purpose of drawing the bond-marked cartoon. When the bond-marked cartoon is thus placed on top of the Γ and Δ' preaccordions, the circling and boxing conventions can be conveniently understood.

- In the first row of Γ or the second row of Δ' we circle an entry if a double bond is above it and to the right. We box an entry if a double bond is above it and to the left. Thus:

- In the second row of Γ or the first row of Δ' we circle an entry if a double bond is above it and to the left. We box an entry if a double bond is above it and to the right. Thus:

Now we have the basic language that will allow us to prove the reduction to the totally resonant case.

Chapter 9

Snakes

The key lemma of this chapter was stated without proof in [9]. There it was called the "Snake Lemma." Here we will recall it, prove it, and use it to prove the statement made in Chapter 5, that (5.12) is "often" true.

By an *indexing* of the Γ preaccordion we mean a bijection

$$\phi : \{1, 2, \cdots, 2d+1\} \longrightarrow \Theta_B.$$

With such an indexing in hand, we will denote $\Gamma_{\mathfrak{t}}(\alpha)$ by $\gamma_k(\mathfrak{t})$ or just γ_k if $\alpha = \phi(k)$ corresponds to k. Thus

$$\{\gamma_1, \gamma_2, \cdots, \gamma_{2d+1}\} = \{\Gamma(\alpha) \,|\, \alpha \in \Theta_B\}.$$

We will also consider an indexing ψ of the Δ' preaccordion, and we will denote $\Delta'(\alpha)$ by δ'_k if $\alpha = \psi(k)$. It will be convenient to extend the indexings by letting $\gamma_0 = \gamma_{2d+2} = 0$ and $\delta'_0 = \delta'_{2d+2} = 0$.

Proposition 6 *There exist indexings of the* Γ *and* Δ' *preaccordions such that*

$$\delta'_k = \begin{cases} \gamma_k & \text{if } k \text{ is even,} \\ \gamma_k + \gamma_{k-1} - \gamma_{k+1} & \text{if } k \text{ is odd.} \end{cases} \tag{9.1}$$

If $i \in \{1, 2, \cdots, 2d+2\}$*, and if* $\phi(i) \in \mathcal{E}_k$*, then* $\psi(i) \in \mathcal{E}_k$ *also. Moreover if* $\phi(j), \psi(j) \in \mathcal{E}_l$ *and* $k < l$ *then* $i < j$.

Before we prove this, let us confirm it in the specific example at hand. With Γ

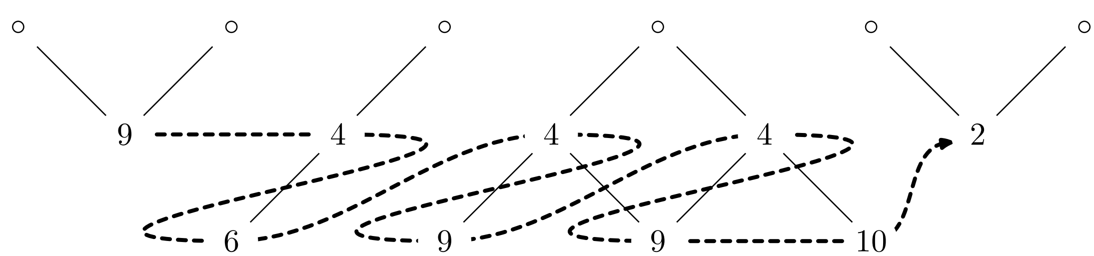
and Δ' as in (8.3) and (8.4), we may take the correspondence as follows:

k	0	1	2	3	4	5	6	7	8	9	10
(i,j) in the Γ ordering		$(1,0)$	$(1,1)$	$(2,0)$	$(1,2)$	$(2,1)$	$(1,3)$	$(2,2)$	$(2,3)$	$(1,4)$	
episode		$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_3$	$\mathcal{E}_3$	$\mathcal{E}_3$	$\mathcal{E}_3$	$\mathcal{E}_4$	
γ_k	0	9	4	6	4	9	4	9	10	2	0
(i,j) in the Δ' ordering		$(1,0)$	$(2,0)$	$(1,1)$	$(2,1)$	$(1,2)$	$(2,2)$	$(2,3)$	$(1,3)$	$(1,4)$	
δ'_k	0	5	4	6	4	9	4	3	10	12	0
episode		$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_3$	$\mathcal{E}_3$	$\mathcal{E}_3$	$\mathcal{E}_3$	$\mathcal{E}_4$	

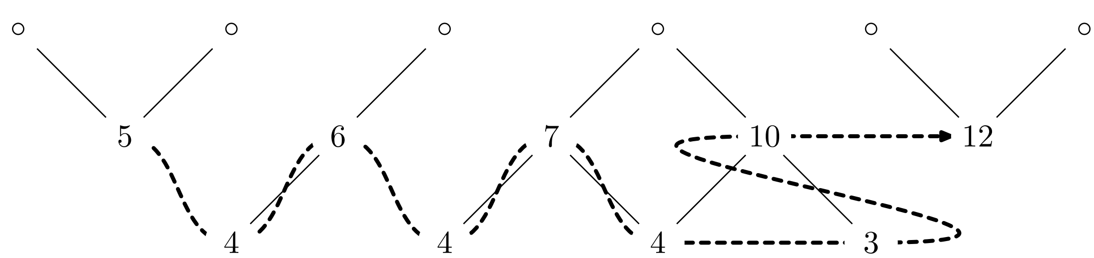
(9.2)

The meaning of the last assertion of Proposition 6 is that each indexing visits the episodes of the sequence in order from left to right, and after it is finished with an episode, it moves on to the next with no skipping around. Thus both indexings must be in the same episode.

The reason that Proposition 6 was called the "Snake Lemma" in [9] is that if one connects the nodes of Θ_B in the indicated orderings, a pair of "snakes" becomes visible. Thus in the example (9.2) the paths will look as follows. The Γ indexing is represented:



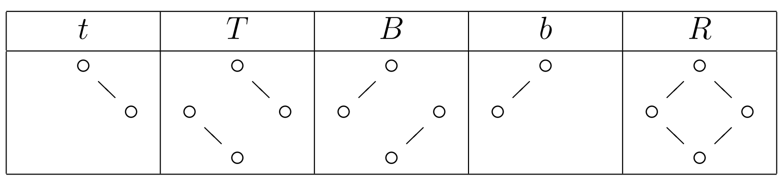
and the Δ' indexing:



- The proof will provide a particular description of the pair of snakes; in applying Proposition 6 will will sometimes need this particular description. We will describe the pair of indexings, or "snakes," as *canonical* if they are produced by the method described in the proof, which is expressed in Table 9.1 below. Thus we implicitly use the proof as well as the statement of the Lemma.

- If there are resonances, there will be more than one possible pair of snakes. (Indeed, the reader will find another way of drawing the snakes in the preceding example.) These will be obtained through a process of *specialization* that will be described in the proof. Any one of these pairs of snakes will be described as canonical.

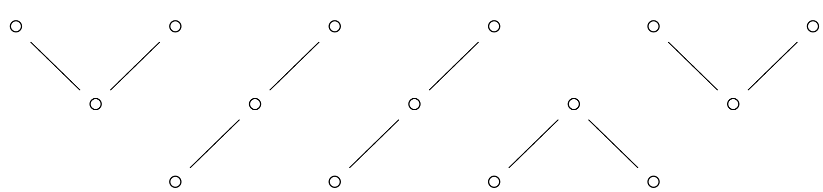
Proof For this proof, double bonds are irrelevant, and we will work with the bond-unmarked cartoon. Thus both Γ and Δ' are again represented by the same cartoon, which in the example (8.1) was the cartoon (8.2). Resonances are a minor complication, which we eliminate as follows. We divide the cartoon into *panels*, each being of one five types:



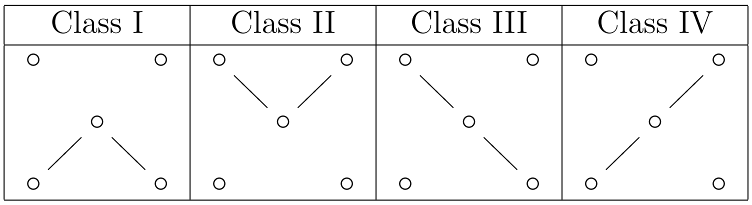
The first panel is always of type t and the last one of type b. We call a cartoon *simple* if it contains no panels of type R.

The panel type R occurs at each resonance. Including it in our discussion would unnecessarily increase the number of cases to be considered, so we resolve each resonance by arbitrarily replacing each R by either a T or B. This will produce a simple cartoon. We refer to this process as *specialization*.

For example the cartoon (8.2) corresponds to the word $tBBRTb$, meaning that these panels appear in sequence from left to right. We replace the resonant panel R arbitrarily by either T or B; for example if we choose B we obtain the simple cartoon:

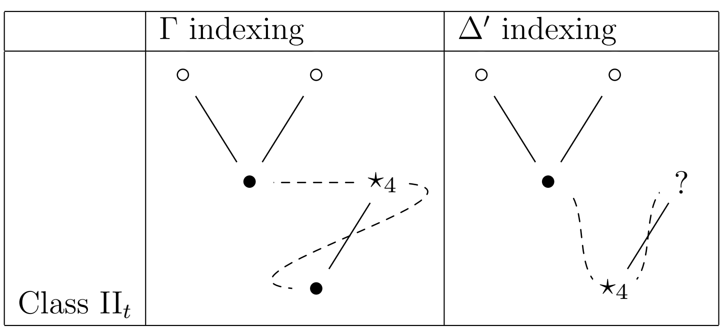


Now, from the simple cartoon we may describe the algorithm for finding the pair of snakes, that is, the Γ and Δ' indexings. Each connected component (episode) in the simple cartoon has three vertices, the middle one being in the second row. We may classify these episodes into four classes as follows:



Now we can describe the snakes. For each episode, we select a path from Table 9.1. The nodes labeled $\star$ will turn out to be indexed by even integers, and the nodes labeled $\bullet$ will be indexed by odd integers. We've subscripted the $\star$'s to indicate which entries are corresponding in the Γ and Δ'. A ? means that the information at hand does not determine whether the entry will be even or odd in the indexing, so we do not attempt to assign it a $\star$ or $\bullet$.

There are modifications at the left and right edges of the pattern: for example, if the first two panels are tB then the first connected component is of Class II, and the left parts of the Γ and Δ' indexings indicated in the table are missing. Thus we have a modification of the Class II pattern that we call II_t.



Similarly there are Classes III_t, II_b and IV_b that can occur at the left or right edge of the pattern. In every case these are obtained by simply deleting part of the corresponding pattern, and we will not enumerate these for this reason.

It is necessary to see that these paths are assigned consistently. For example, suppose that the cartoon contains consecutive panels TBT. Inside the resulting configuration are both a Class II connected component and a Class I component. Referring to Table 9.1, both these configurations mandate the following dashed line

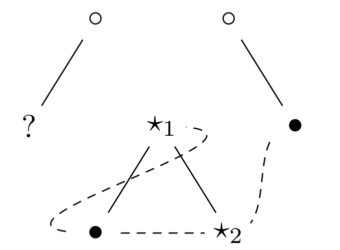
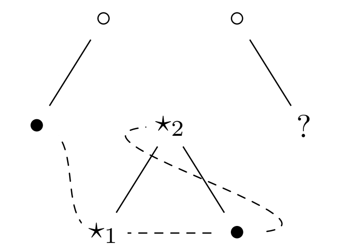
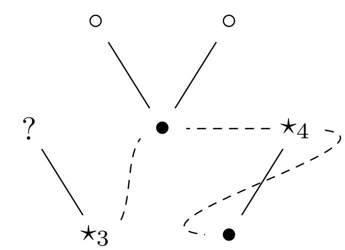
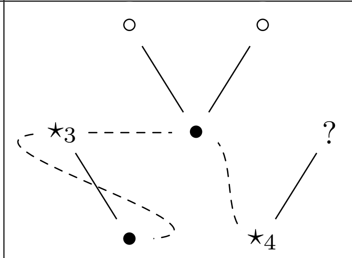
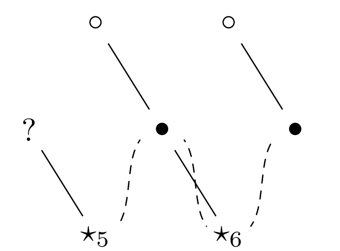
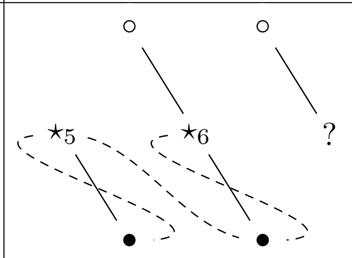
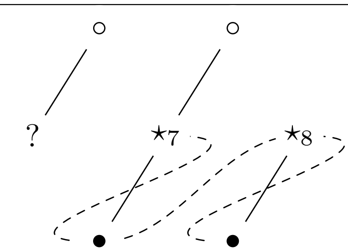
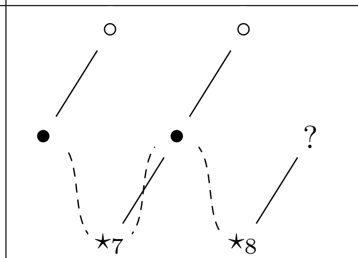
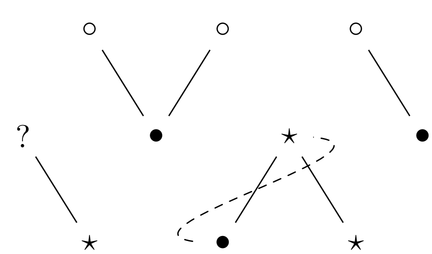
	Γ labeling	Δ′ labeling
Class I		
Class II		
Class III		
Class IV		

Table 9.1: Snake taxonomy.

in the Γ diagram.



This sort of consistency must be checked in eight cases, which we leave to the reader. Then it is clear that splicing together the segments prescribed this way gives a consistent pair of snakes, and the indexings can be read off from left to right.

It remains, however, for us to prove (9.1). This is accomplished by four Lemmas. There are many things to verify; we will do one and leave the rest to the reader.

Lemma 3 *If the j-th connected component is of Class I, then*

$$\begin{aligned} \Delta'(1,j) &= \Gamma(2,j) \\ \Delta'(2,j-1) &= \Gamma(1,j) \\ \Delta'(2,j) &= \Gamma(2,j-1)+\Gamma(1,j)-\Gamma(2,j). \end{aligned}$$

This asserts a part of (9.1), namely the equality $\delta'_k = \gamma_k$ for the vertices labeled $\star_1$ and $\star_2$ in Table 9.1, and the equality $\delta'_k = \gamma_k + \gamma_{k-1} - \gamma_{k+1}$ for the unstarred vertex in the connected component.

We prove that $\Delta'(1,j) = \Gamma(2,j)$. With a'_i defined by (5.5) and (5.6) we have

$$\Delta'(1,j) = \left(\sum_{i\leqslant j} l_i\right) - \left(\sum_{i\leqslant j} a'_i\right).$$

Our assumption that the j-th connected component is of Class I means that $l_j \geqslant b_{j-1}$ and that $l_{j+1} \geqslant b_j$, so $a'_j = b_{j-1} + b_j - a_j$. Moreover, since $l_j \geqslant b_{j-1}$ we have

$$\begin{aligned} \sum_{i<j} [\min(l_i, b_{i-1}) + \max(l_{i+1}, b_i)] &= \\ \max(l_j, b_{j-1}) + \sum_{i<j} [\min(l_i, b_{i-1}) + \max(l_i, b_{i-1})] &= \\ l_j + \sum_{i<j} (l_i + b_{i-1}) &= \sum_{i\leqslant j} l_i + \sum_{i\leqslant j-2} b_i. \end{aligned}$$

By (5.5) we therefore have

$$\sum_{i\leqslant j} a'_i = \sum_{i\leqslant j} l_i + b_i - a_i, \qquad \Delta'(1,j) = \sum_{i\leqslant j} a_i - b_i = \Gamma(2,j).$$

We leave the remaining two statements to the reader.

Lemma 4 *If the j-th connected component is of Class II, then*

$$\Delta'(1,j) = \Gamma(1,j) + \Gamma(2,j-1) - \Gamma(1,j+1).$$

Lemma 5 *If the j-th connected component is of Class III, then*

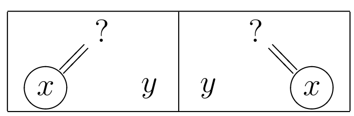
$$\begin{aligned}\Delta'(1,j) &= \Gamma(2,j),\\ \Delta'(2,j) &= \Gamma(1,j)+\Gamma(2,j-1)-\Gamma(2,j).\end{aligned}$$

Lemma 6 *If the j-th connected component is of Class IV, then*

$$\begin{aligned}\Delta'(2,j-1) &= \Gamma(1,j),\\ \Delta'(1,j) &= \Gamma(2,j-1)+\Gamma(1,j)-\Gamma(1,j+1).\end{aligned}$$

We leave the proofs of the last Lemmas to the reader. The assertions in (9.1) are contained in the Lemmas, for each δ'_k whose corresponding vertex is in the given connected component. It must lie in the first (middle) or second row of the cartoon, which is why there are three identities for Class I, one for Class II and two for Classs III and IV. Thus (9.1) is proved for every δ'_k. The final assertion, that the episodes of the cartoon are visited from left to right in order by both indexings, can be seen by inspection from Table 9.1. □

Lemma 7 (Circling Lemma) *Assume that $\mathfrak{t}$ is strict.*
(i) Suppose that either of the following two configurations occurs in either $\Gamma_{\mathfrak{t}}$ or $\Delta_{\mathfrak{t}}$. Then $x = y$.



(ii) If x occurs circled in either the Γ or Δ preaccordions of a strict pattern $\mathfrak{t}$, then either the same value x also occurs uncircled (and unboxed) at another location, or $x = 0$.

Proof The first statement follows from the definition. To prove the second statement, we note that $y = x$ is unboxed since the pattern is strict. If it is uncircled, (ii) is proved. If it is circled, we continue to the right (if y is to the right of x) or to the left (if y is to the left of x) until we come to an uncircled one. This can only fail if we come to the edge of the pattern. If this happens, then $x = 0$. □

Chapter 10

Noncritical Resonances

We recall that a short pattern (5.2) is *resonant* at i if $l_{i+1} = b_i$. This property depends only on the associated prototype, so resonance is actually a property of prototypes. We also call a first (middle) row entry a_i *critical* if it is equal to one of its four *neighbors*, which are l_i, l_{i+1}, b_i and b_{i-1}. We say that the resonance at i is *critical* if either a_i or a_{i+1} is critical.

Theorem 7 *Suppose that* $\mathfrak{t}$ *is a strict pattern with no critical resonances; then* $\mathfrak{t}'$ *is also strict with no critical resonances. Choose canonical indexings* γ_i *and* δ'_i *as in Proposition 6. Then either* $G_\Gamma(\mathfrak{t}) = G_\Delta(\mathfrak{t}) = 0$ *or* $n|\gamma_i$*. In any case, we have*

$$G_\Gamma(\mathfrak{t}) = G_\Delta(\mathfrak{t}').$$

As an example, the pattern is called *superstrict* if the inequalities (5.7) and (5.8) are strict, that is, if

$$\min(l_j, b_{j-1}) > a_j > \max(l_{j+1}, b_j), \qquad 0 < j < d, \tag{10.1}$$

$$l_1 > a_1 > \max(l_2, b_1), \qquad \min(l_d, b_{d-1}) > a_d > l_{d+1}. \tag{10.2}$$

Thus if the patterns within a type are regarded as lattice points in a polytope, the superstrict patterns are the interior points. Again, the pattern (or prototype) is called *nonresonant* if there are no resonances. The theorem is clearly applicable if $\mathfrak{t}$ is either superstrict or nonresonant.

Proof To see that $\mathfrak{t}'$ is strict, let a_i, b_i, l_i and a'_i be as in (5.2) and (5.4). If $\mathfrak{t}'$ is not strict, we must have $a'_i = a'_{i-1}$ for some i, and it is easy to see that this implies that $l_i = b_{i-1}$, and that $\mathfrak{t}$ has a critical resonance at i. It is also easy to see that if $\mathfrak{t}'$ has a critical resonance at i so does $\mathfrak{t}$.

Choose a pair of canonical indexings of $\Gamma = \Gamma_{\mathfrak{t}}$ and $\Delta' = \Delta_{\mathfrak{t}'}$. Our first task is to show that either $G_\Gamma(\mathfrak{t}) = G_\Delta(\mathfrak{t}') = 0$ or $n|\gamma_i$ for all even i. It is easy to see that γ_i and δ'_i are not boxed, since if it were, examination of every case in Table 9.1 shows that it would be at the terminus of a double bond in the bond-marked cartoon that is not one of the marked bonds in the figures. This could conceivably happen since in the proof of Proposition 6 we began by replacing the cartoon by a simple cartoon, a process that can involve discarding some parallel pairs of the bonds; however it would force γ_i (or δ'_i) to be a neighbor of a critical resonance, and we are assuming that $\mathfrak{t}$ has no critical resonances.

Suppose that γ_i is not circled (i even). Then $G_\Gamma(\mathfrak{t})$ is a multiple of $h(\gamma_i)$, which vanishes unless $n|\gamma_i$. If γ_i is circled, we must argue differently. By Lemma 7, either the same value γ_i occurs uncircled and unboxed somewhere in the Γ preaccordion, in which case $G_\Gamma(\mathfrak{t})$ is again a multiple of $h(\gamma_i)$, or $\gamma_i = 0$. Since $n|\gamma_i$ if $\gamma_i = 0$ the conclusion that $G_\Gamma(\mathfrak{t}) = 0$ or $n|\gamma_i$ is proved. Since $\gamma_i = \delta'_i$ when n is even, we may also conclude that $G_\Delta(\mathfrak{t}') = 0$ unless the γ_i (i even) are all divisible by n.

We assume for the remainder of the proof that $n|\gamma_i$ when i is even. Let us denote

$$\tilde{\gamma}_i = \begin{cases} q^{\gamma_i} & \text{if } \gamma_i \text{ is circled in the } \Gamma \text{ indexing,} \\ g(\gamma_i) & \text{if } \gamma_i \text{ is boxed in the } \Gamma \text{ indexing,} \\ h(\gamma_i) & \text{otherwise,} \end{cases}$$

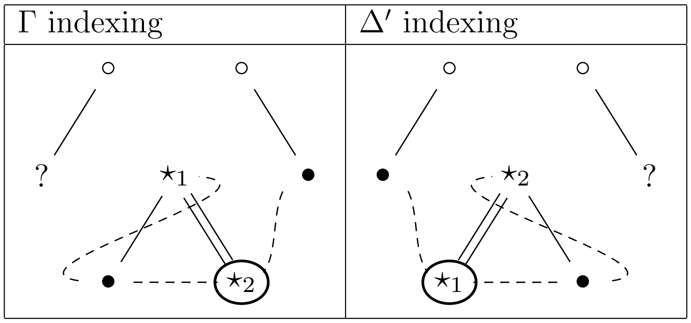
with $\tilde{\delta}'_i$ defined similarly. Thus

$$G_\Gamma(\mathfrak{t}) = \prod \tilde{\gamma}_i, \qquad G_\Delta(\mathfrak{t}') = \prod \tilde{\delta}'_i.$$

We next show that

$$\prod_{i \text{ even}} \tilde{\gamma}_i = \prod_{i \text{ even}} \tilde{\delta}'_i. \tag{10.3}$$

Since $\gamma_i = \delta'_i$ when i is even, and since as we have noted these entries are never boxed, the only way this could fail is if one of $\widetilde{\gamma}_i$ and $\tilde{\delta}'_i$ is circled and the other not. We look at the connected component in the bond-unmarked cartoon containing $\widetilde{\gamma}_i$. In Table 9.1, this entry is starred and must correspond to one of $\star_1$, $\star_2$, $\star_6$ or $\star_7$. (Since the snake is obtained by splicing pieces together different pieces of Table 9.1 it may also appear $\star_3$, $\star_4$, $\star_5$ or $\star_8$.) If it is $\star_6$, then it is circled in the Γ indexing if and only if the bond above it is doubled, and by Lemma 2 the bond above $\star_6$ in the Δ' indexing is also doubled, so $\star_6$ is circled in both indexings; and similarly with $\star_7$. Turning to the Class I components, it is impossible for $\star_1$ to be circled in the Γ indexing, since this would imply a critical resonance; and $\star_2$ is never circled in the Δ' indexing for the same reasoning. Nevertheless it is possible for $\star_2$ to be circled in the Γ indexing but not the Δ' indexing. In this case, Lemma 2 shows that $\star_2$ is starred in the Δ' indexing but not the Γ indexing. This happens when the labeling of a Class I component looks like:



Thus if $\star_1$ is the i-th vertex in both orderings we have

$$\tilde{\gamma}_i = h(\gamma_i), \qquad \tilde{\gamma}_{i+2} = q^{\gamma_{i+2}}, \qquad \tilde{\delta}'_i = q^{\delta'_i} = q^{\gamma_i}, \qquad \tilde{\delta}'_{i+2} = h(\delta'_{i+2}) = h(\gamma'_{i+2}),$$

and it is still true that $\tilde{\gamma}_i\tilde{\gamma}_{i+2} = \tilde{\delta}'_i\tilde{\delta}'_{i+2}$. This proves (10.3).

Now we prove

$$\prod_{i \text{ odd}} \tilde{\gamma}_i = \prod_{i \text{ odd}} \tilde{\delta}'_i. \tag{10.4}$$

When i is odd, it follows from Lemma 2 that γ_i is circled or boxed in the Γ indexing if and only if δ'_i is. Using (9.1), and remembering that since $i-1$ and $i+1$ are even we are now assuming γ_{i-1} and γ_{i+1} are multiples of n, we obtain

$$\tilde{\delta}'_i = q^{\gamma_{i-1}-\gamma_{i+1}}\tilde{\gamma}_i.$$

Thus taking the product over odd i, the powers of q will cancel in pairs, giving (10.4). Combining this with (10.3), the theorem is proved. □

There is another important case where (5.12) is true. This is case where the pattern $\mathfrak{t}$ is *stable*. We say that $\mathfrak{t}$ is *stable* if each a_i equals either l_i or l_{i+1}, and each b_i equals either a_i or a_{i+1}. Thus every element of the Γ preaccordion is either circled or boxed. If this is true then it follows from Lemma 2 that $\mathfrak{t}'$ is also stable. Theorem 7 does not apply to stable patterns since they usually have critical resonances.

Theorem 8 *Suppose that* $\mathfrak{t}$ *is stable. Then* $G_\Gamma(\mathfrak{t}) = G_\Delta(\mathfrak{t}')$.

Proof It is easy to see that every element of the Γ and Δ' preaccordions is either circled or boxed, and that the circled entries are precisely the ones that equal zero.

As we will explain, the boxed elements are precisely the same for the Γ and Δ' preaccordions.

Let S be the set of elements of the top row of $\mathfrak{t}$. Between the top row and the row below it, one element is omitted; call it a. Between this row and the next, another element is omitted; call this b. In $\mathfrak{t}'$ the same two elements are dropped, but in the reverse order. The boxed entries that appear in Γ are

first row:	$\{x - a \mid x \in S, x > a\}$
second row:	$\{b - x \mid x \in S, x < b, x \neq a\}$

The boxed entries that appear in Δ' are:

first row:	$\{b - x \mid x \in S, x > b\}$
second row:	$\{x - a \mid x \in S, x > a, x \neq b\}$

The entry $a - b$ appears in both cases only if $a > b$. The statement is now clear. $\square$

Chapter 11

Types

We now divide the prototypes into much smaller units that we call *types*. We fix a top and bottom row, and therefore a cartoon. For each episode $\mathcal{E}$ of the cartoon, we fix an integer $k_{\mathcal{E}}$. Then the set $\mathfrak{S}$ of all short Gelfand-Tsetlin patterns (5.2) with the given top and bottom rows such that for each $\mathcal{E}$

$$\sum_{\alpha\in\Theta_1\cap\mathcal{E}} \mathfrak{t}(\alpha) = k_{\mathcal{E}} \tag{11.1}$$

is called a *type*. Thus two patterns are in the same type if and only if they have the same top and bottom rows (and hence the same cartoon), and if the sum of the first (middle) row elements in each episode is the same for both patterns.

Let us choose Γ and Δ' indexings as in Proposition 6 (Proposition 6). With notations as in that Lemma, and $\mathcal{E}$ a fixed episode of the corresponding cartoon, there exist k and l such that $\phi(i)\in\mathcal{E}$ and $\psi(i)\in\mathcal{E}$ precisely when $k\leqslant i\leqslant l$. let

$$L_{\mathcal{E}} = \begin{cases} k & \text{if } k \text{ is even,} \\ k-1 & \text{if } k \text{ is odd,} \end{cases} \qquad R_{\mathcal{E}} = \begin{cases} l & \text{if } l \text{ is even,} \\ l+1 & \text{if } l \text{ is odd.} \end{cases}$$

Then Proposition 6 implies that

$$\sum_{i=k}^{l} \delta_i(\mathfrak{t}') = \left(\sum_{i=k}^{l} \gamma_i(\mathfrak{t})\right) + \gamma_{L_{\mathcal{E}}}(\mathfrak{t}) - \gamma_{R_{\mathcal{E}}}(\mathfrak{t}), \tag{11.2}$$

for all elements of the type. We recall that our convention was that $\gamma_0 = \gamma_{2d+2} = 0$. We take $L_{\mathcal{E}} = 0$ for the first (leftmost) cartoon and $R_{\mathcal{E}} = 2d+2$ for the last episode.

We may classify the possible episodes into four classes generalizing the classification in Table 9.1, and indicate in each case the locations of $\gamma_{L_{\mathcal{E}}}$ and $\gamma_{R_{\mathcal{E}}}$ in the

Γ preaccordions, which may be checked by comparison with Table 9.1. Indeed, it must be remembered that in that proof, every panel of type R is replaced by one of type T or type B. Whichever choice is made, Table 9.1 gives the same location for $L_{\mathcal{E}}$ and $R_{\mathcal{E}}$. The classification of the episode into one of four types is given in Table 11.1.

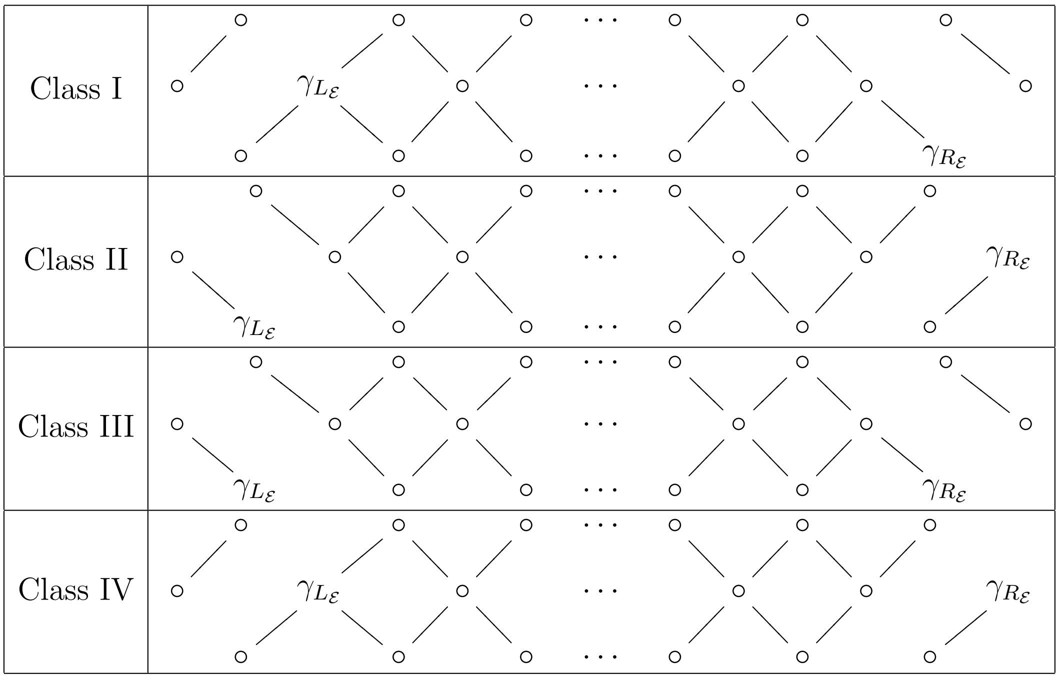


Table 11.1: The four classes of episodes in $\Gamma_{\mathfrak{t}}$.

The location of $\delta'_{L_{\mathcal{E}}}$ and $\delta'_{R_{\mathcal{E}}}$ in the Δ preaccordion of $\mathfrak{t}'$ may also be read off from Table 9.1. The classification of the episode into one of four classes is given in Table 11.2.

Proposition 7 *If $\mathcal{F}$ is the episode that consecutively follows $\mathcal{E}$, then $R_{\mathcal{E}} = L_{\mathcal{F}}$. The values $\gamma_{L_{\mathcal{E}}}(\mathfrak{t})$ and $\gamma_{R_{\mathcal{E}}}(\mathfrak{t})$ are constant on each type. Moreover $G_\Gamma(\mathfrak{t}) = G_\Delta(\mathfrak{t}') = 0$ for all patterns $\mathfrak{t}$ in the type unless the $\gamma_{L_{\mathcal{E}}}$ are divisible by n. In the Γ and Δ′ preaccordions, $\gamma_{L_{\mathcal{E}}}$ and $\delta'_{L_{\mathcal{E}}}$ may be circled or not, but never boxed.*

Proof From Tables 11.1 and 11.2, it is clear that $R_{\mathcal{E}} = L_{\mathcal{F}}$ for consecutive episodes.

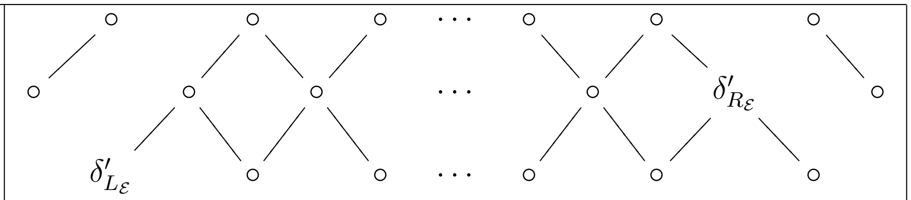
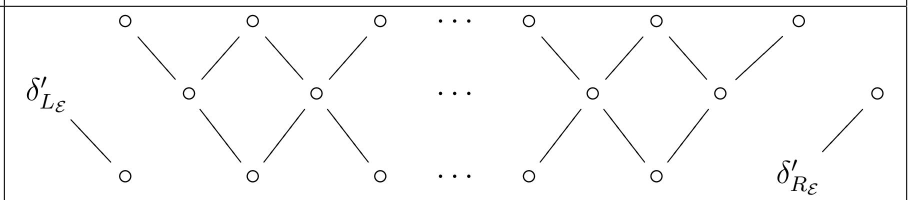
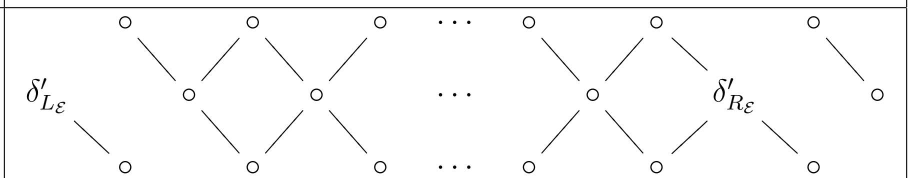
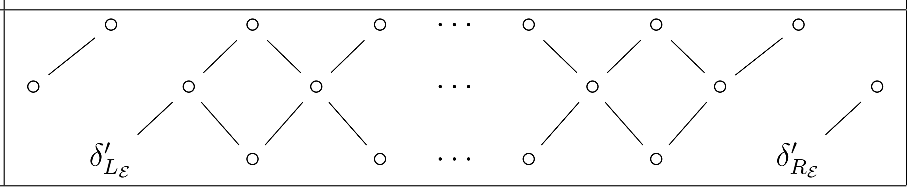
Class I	
Class II	
Class III	
Class IV	

Table 11.2: The four classes of episodes in $\Delta_{\mathfrak{t}'}$.

If $\mathcal{E}$ is of Class I or Class IV, then we see that

$$\gamma_{L_\mathcal{E}} = \sum_{\mathcal{F} \geqslant \mathcal{E}} \left(\sum_{\alpha \in \Theta_1 \cap \mathcal{F}} \mathfrak{t}(\alpha) - \sum_{\alpha \in \Theta_0 \cap \mathcal{F}} \mathfrak{t}(\alpha) \right),$$

where the notation means that we sum over all episodes to the right of $\mathcal{E}$ (including $\mathcal{E}$ itself). If $\mathcal{E}$ is of Class II or III, we have

$$\gamma_{L_\mathcal{E}} = \sum_{\mathcal{F} \leqslant \mathcal{E}} \left(\sum_{\alpha \in \Theta_1 \cap \mathcal{F}} \mathfrak{t}(\alpha) - \sum_{\alpha \in \Theta_2 \cap \mathcal{F}} \mathfrak{t}(\alpha) \right).$$

In either case, these formulas imply that $\gamma_{L_\mathcal{E}}$ is constant on the patterns of the type.

Given their described locations, the fact that $\gamma_{L_\mathcal{E}} = \delta'_{L_\mathcal{E}}$ is never boxed in either the Γ or Δ' preaccordions may be seen from the definitions.

We now show that $G_\Gamma(\mathfrak{t}) = G_\Delta(\mathfrak{t}') = 0$ unless $n|\gamma_{L_\mathcal{E}}$. Indeed, it follows from an examination of the locations of $\gamma_{L_\mathcal{E}}$ in the Γ preaccordions and in the Δ' preaccordions (where it appears as $\delta'_{L_\mathcal{E}}$) that it is unboxed in both Γ and Δ'. If it is uncircled in the Γ preaccordion, then $G_\Gamma(\mathfrak{t})$ is divisible by $h(\gamma_{L_\mathcal{E}})$, hence vanishes unless $n|\gamma_{L_\mathcal{E}}$. If it is circled, then we apply Lemma 7 to conclude that the same value appears somewhere else uncircled and unboxed, unless $\gamma_{L_\mathcal{E}} = 0$ (which is divisible by n), which again forces $G_\Gamma(\mathfrak{t}) = 0$ if $n \nmid \gamma_{L_\mathcal{E}}$; and similarly for $G_\Delta(\mathfrak{t}') = 0$. □

- Due to this result, we may impose the assumption that $n|\gamma_{L\mathcal{E}}$ for every episode. This assumption is in force for the rest of the book.

Now let $\mathcal{E}_1, \cdots, \mathcal{E}_N$ be the episodes of the cartoon arranged from left to right, and let $k_i = k(\mathcal{E}_i)$. By a *local pattern* on $\mathcal{E}_i$ subordinate to $\mathfrak{S}$ we mean an integer-valued function on $\mathcal{E}_i$ that can occur as the restriction of an element of $\mathfrak{S}$ to $\mathcal{E}_i$. Its top and bottom rows are thus the restrictions of the given top rows, and it follows from the definition of the episode that if $(0,t)$ and $(2,t-1)$ are both in $\mathcal{E}_i$ then $\mathfrak{t}(0,t) = \mathfrak{t}(2,t-1)$; that is, if both an element of the top row and the element of the bottom row that is directly below it are in the same episode, then $\mathfrak{t}$ has the same value on both, and patterns in the type are resonant at t. The local pattern is subject to the same inequalities as a short pattern, and by (11.2) the sum of its first (middle) row elements must be k_i. Let $\mathfrak{S}_i$ be the set of local patterns subordinate to $\mathfrak{S}$. We call $\mathfrak{S}_i$ a *local type*.

Lemma 8 *A pattern is in $\mathfrak{S}$ if and only if its restriction to $\mathcal{E}_i$ is in $\mathfrak{S}_i$ for each i and so we have a bijection*

$$\mathfrak{S} \cong \mathfrak{S}_1 \times \cdots \times \mathfrak{S}_N.$$

Proof This is obvious from the definitions, since the inequalities (11.1) for the various episodes are independent of each other. □

Now if $\mathfrak{t}$ is a short pattern let us define for each episode $\mathcal{E}$

$$\begin{aligned} G_\Gamma^\mathcal{E}(\mathfrak{t}) &= \prod_{\alpha\in\mathcal{E}\cap\Theta_B} \begin{cases} g(\alpha) & \text{if } \alpha \text{ is boxed in } \Gamma_\mathfrak{t}, \\ q^\alpha & \text{if } \alpha \text{ is circled in } \Gamma_\mathfrak{t}, \\ h(\alpha) & \text{otherwise}, \end{cases} \\ G_\Delta^\mathcal{E}(\mathfrak{t}) &= \prod_{\alpha\in\mathcal{E}\cap\Theta_B} \begin{cases} g(\alpha) & \text{if } \alpha \text{ is boxed in } \Delta_\mathfrak{t}, \\ q^\alpha & \text{if } \alpha \text{ is circled in } \Delta_\mathfrak{t}, \\ h(\alpha) & \text{otherwise}, \end{cases} \end{aligned} \tag{11.3}$$

provided $\mathfrak{t}$ is *locally strict* at $\mathcal{E}$, by which we mean that if $\alpha, \beta \in \mathcal{E}\cap\Theta_1$ and α is to the left of β then $\mathfrak{t}(\alpha) > \mathfrak{t}(\beta)$. If $\mathfrak{t}$ is not locally strict, then we define $G_\Gamma^\mathcal{E}(\mathfrak{t}) = G_\Delta^\mathcal{E}(\mathfrak{t}) = 0$.

Proposition 8 *Suppose that $n|\gamma_{L\mathcal{E}}$ for every episode. Assume also that for each $\mathfrak{S}_i$ we have*

$$\sum_{\mathfrak{t}_i \in \mathfrak{S}_i} G_\Delta^{\mathcal{E}_i}(\mathfrak{t}_i') = q^{\gamma_{L\mathcal{E}_i} - \gamma_{R\mathcal{E}_i}} \sum_{\mathfrak{t} \in \mathfrak{S}_i} G_\Gamma^{\mathcal{E}_i}(\mathfrak{t}_i). \tag{11.4}$$

Then

$$\sum_{\mathfrak{t} \in \mathfrak{S}} G_\Delta(\mathfrak{t}') = \sum_{\mathfrak{t} \in \mathfrak{S}} G_\Gamma(\mathfrak{t}). \tag{11.5}$$

This proposition is the bridge between types and local types. Two observations are implicit in the statement of equation (11.4).

- Since by its definition $G_\Gamma^{\mathcal{E}_i}(\mathfrak{t})$ depends only on the restriction $\mathfrak{t}_i$ of $\mathfrak{t}$ to $\mathfrak{S}_i$, we may write $G_\Gamma^{\mathcal{E}_i}(\mathfrak{t}_i)$ instead of $G_\Gamma^{\mathcal{E}_i}(\mathfrak{t})$, and this is well-defined.
- The statement uses the fact that $\gamma_{L_\mathcal{E}}(\mathfrak{t})$ and $\gamma_{R_\mathcal{E}}(\mathfrak{t})$ are constant on the type, since otherwise $q^{\gamma_{L\mathcal{E}_i} - \gamma_{R\mathcal{E}_i}}$ would be inside the summation.

Proof If $\mathfrak{t}_i \in \mathfrak{S}_i$ is the restriction of $\mathfrak{t} \in \mathfrak{S}$, we have

$$\sum_{\mathfrak{t} \in \mathfrak{S}} G_\Gamma(\mathfrak{t}) = \prod_i \sum_{\mathfrak{t}_i \in \mathfrak{S}_i} G_\Gamma^{\mathcal{E}_i}(\mathfrak{t}_i) = \prod_i q^{\gamma_{L\mathcal{E}_i} - \gamma_{R\mathcal{E}_i}} \sum_{\mathfrak{t}_i \in \mathfrak{S}_i} G_\Delta^{\mathcal{E}_i}(\mathfrak{t}_i').$$

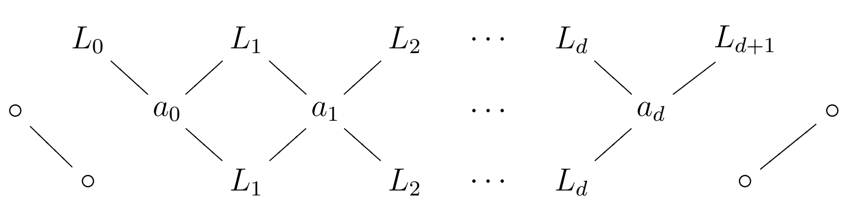
By Proposition 7, the factors $q^{\gamma_{L\mathcal{E}_i} - \gamma_{R\mathcal{E}_i}}$ cancel each other $R_{\mathcal{E}_i} = L_{\mathcal{E}_{i+1}}$, and since our convention is that $\gamma_0 = \gamma_{2d+2} = 0$. Thus we obtain

$$\prod_i \sum_{\mathfrak{t}_i \in \mathfrak{S}_i} G_\Delta^{\mathcal{E}_i}(\mathfrak{t}_i') = \sum_{\mathfrak{t} \in \mathfrak{S}} G_\Delta(\mathfrak{t}').$$

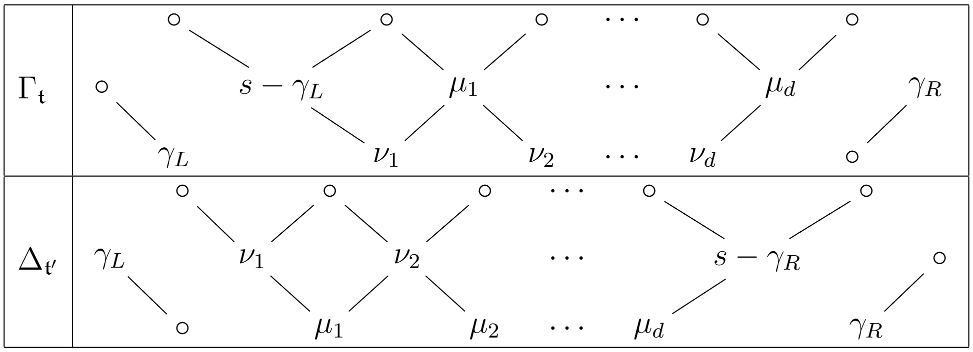
□

In the rest of the chapter we will fix an episode $\mathcal{E} = \mathcal{E}_i$, and let $L = L_\mathcal{E}$ and $R = R_\mathcal{E}$ to simplify the notation for the four remaining Propositions, which describe more precisely the relations between the Γ and Δ' preaccordions within the episode $\mathcal{E}$.

Proposition 9 *Let $\mathfrak{t}$ be a short pattern whose cartoon contains the following Class II resonant episode $\mathcal{E}$ of order d:*



Then there exist integers $s, \mu_1, \nu_1, \mu_2, \nu_2, \cdots, \mu_d, \nu_d$ such that $\mu_i + \nu_i = s$ ($i = 1, \cdots, d$), and the Γ and Δ' preaccordions are given in the following table.



The values s, γ_L and γ_R are constant on the type containing the pattern.

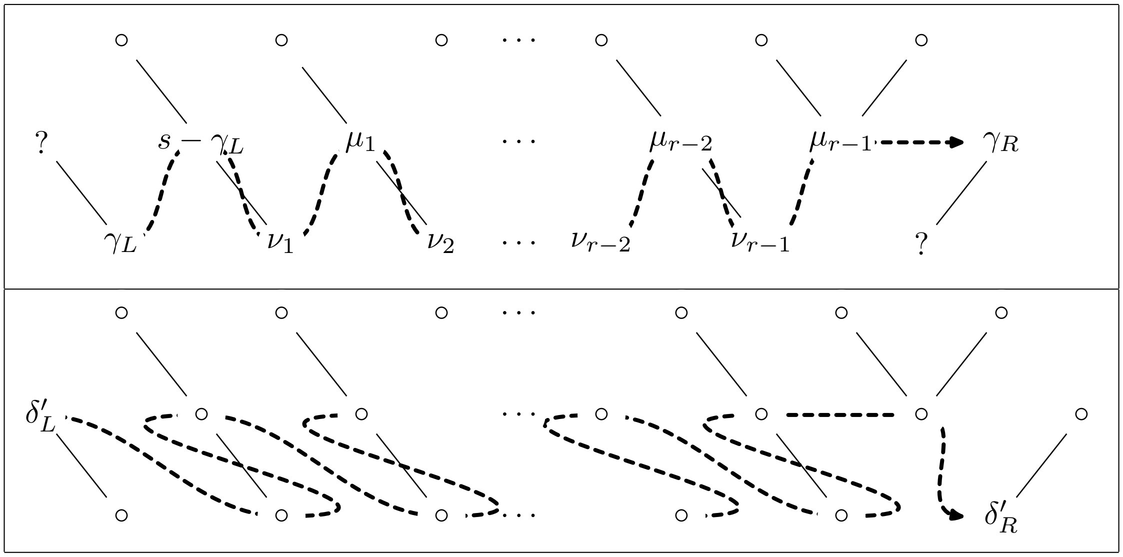
Note: If the episode $\mathcal{E}$ occurs at the left edge of the cartoon, then our convention is that $\gamma_L = \gamma_0 = 0$, and if $\mathcal{E}$ occurs at the right edge of the cartoon, then $\gamma_R = \gamma_{2d+2} = 0$. We would modify the picture by omitting γ_L or γ_R in these cases, but the proof below is unchanged.

Proof Let $\gamma_L = \gamma_{L_\mathcal{E}}$ and $\gamma_R = \gamma_{R_\mathcal{E}}$ in the notation of the previous chapter, and let s, μ_i, ν_i be defined by their locations in the Γ preaccordion. Let $s = \hat{s} + \gamma_R + \gamma_L$, $\mu_i = \hat{\mu}_i + \gamma_R$ and $\nu_i = \hat{\nu}_i + \gamma_L$. It is immediate from the definitions that

$$\hat{s} = \sum_{j=0}^{d} (a_j - L_{j+1}), \qquad \widehat{\mu_i} = \sum_{j=i}^{d} (a_j - L_{j+1}), \qquad \widehat{\nu_i} = \sum_{j=0}^{i-1} (a_j - L_{j+1}).$$

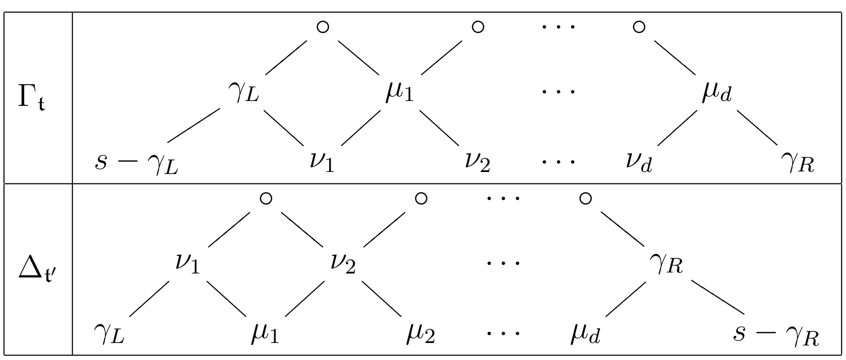
From this it we see that $\mu_i + \nu_i = s$ and $\hat{\mu}_i + \hat{\nu}_i = \hat{s}$.

In order to check the correctness of the Δ' diagram, we observe that the resonance contains d panels of type R, each of which may be specialized to a panel of type T or B. We specialize these to panels of type T. We obtain the following canonical snakes, representing the Γ and Δ' preaccordions.



Now looking at the even numbered locations in these indexings, starting with $\gamma_L = \delta'_L$, Proposition 6 asserts the values $\nu_1, \cdots, \nu_d$ are as advertised in the Δ' labeling. Looking at the first odd numbered location, which is the first spot in the bottom row of the episode, Proposition 1 asserts its value to be $(s - \gamma_L) + \gamma_L - \nu_1 = \mu_1$; the second odd numbered location gets the value $\mu_1 + \nu_1 - \nu_2 = \mu_2$, and so forth. $\square$

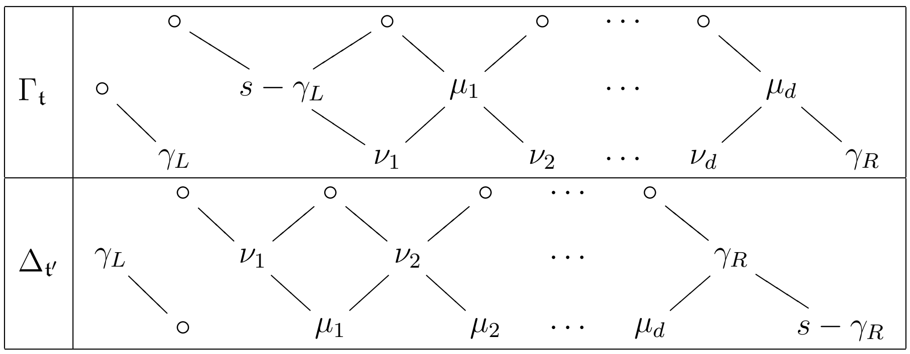
Proposition 10 *Let $\mathfrak{t}$ be a short pattern whose cartoon contains a Class I resonant episode $\mathcal{E}$ of order d. Then there exist integers $s, \mu_1, \nu_1, \mu_2, \nu_2, \cdots, \mu_d, \nu_d$ such that $\mu_i + \nu_i = s$ $(i = 1, \cdots, d)$, and the portions of in Γ and Δ' preaccordions in $\mathcal{E}$ are given in the following table.*



The values s, γ_L and γ_R are constant on the type containing the pattern.

Proof We define s, μ_i and ν_i to be the quantities that make the Γ preaccordion correct. The correctness of the second diagram may be proved using snakes as in Proposition 9. The proof that $\mu_i + \nu_i = s$ is also similar to Proposition 9. □

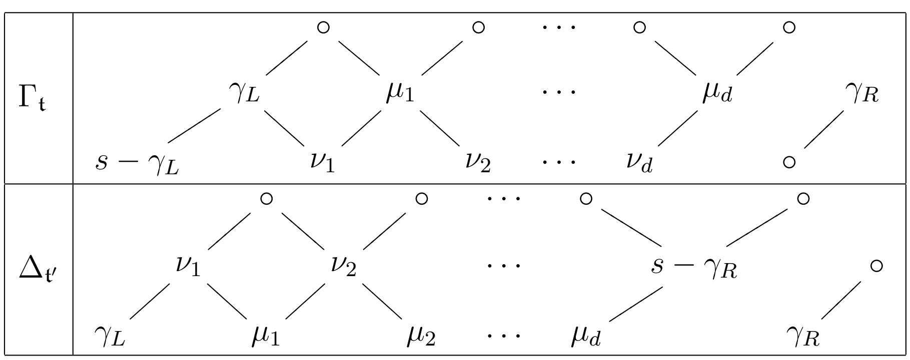
Proposition 11 *Let* $\mathfrak{t}$ *be a short pattern whose cartoon contains a Class III resonant episode* $\mathcal{E}$ *of order* d*. Then there exist integers* $s, \mu_1, \nu_1, \mu_2, \nu_2, \cdots, \mu_d, \nu_d$ *such that* $\mu_i + \nu_i = s$ $(i = 1, \cdots, d)$*, and the portions of the* Γ *and* Δ' *preaccordions in* $\mathcal{E}$ *are given in the following table.*



The values s, γ_L*,* γ_R *and* ξ *are constant on the type containing the pattern.*

Proof We define s, μ_i and ν_i to be the quantities that make the Γ preaccordion correct. The correctness of the second diagram may be proved using snakes as in Proposition 9. The proof that $\mu_i + \nu_i = s$ is also similar to Proposition 9. □

Proposition 12 *Let* $\mathfrak{t}$ *be a short pattern whose cartoon contains a Class IV resonant episode* $\mathcal{E}$ *of order* d*. Then there exist integers* $s, \mu_1, \nu_1, \mu_2, \nu_2, \cdots, \mu_d, \nu_d$ *such that* $\mu_i + \nu_i = s$ $(i = 1, \cdots, d)$*, and the portions of the* Γ *and* Δ' *preaccordions in* $\mathcal{E}$ *are given in the following table.*



The values s, γ_L*,* γ_R *and* ξ *are constant on the type containing the pattern.*

Proof We define s, μ_i and ν_i to be the quantities that make the Γ preaccordion correct. The correctness of the second diagram may be proved using snakes as in Proposition 9. The proof that $\mu_i + \nu_i = s$ is also similar to Proposition 9. $\square$

Chapter 12

Knowability

We refer to Chapter 5 for discussion of the concept of Knowability.

Let $\mathfrak{S} = \prod \mathfrak{S}_i$ be a type. Let $\mathcal{E} = \mathcal{E}_i$ be an episode in the cartoon associated to the short Gelfand-Tsetlin pattern $\mathfrak{t} \in \mathfrak{S}$. If the episode is of Class II, let $a_0, \cdots, a_d$ and $L_1, \cdots, L_{d+1}$ be as in Proposition 9. If the class is I, III or IV, we still define the a_i and L_i analogously:

Class I	$\begin{array}{ccccccccccc} & \circ & & L_1 & & L_2 & \cdots & L_r & & \circ & \\ \circ & & a_0 & & a_1 & & \cdots & & a_r & & \circ \\ & L_0 & & L_1 & & L_2 & \cdots & L_r & & L_{r+1} & \end{array}$
Class III	$\begin{array}{ccccccccccc} & L_0 & & L_1 & & L_2 & \cdots & L_r & & \circ & \\ \circ & & a_0 & & a_1 & & \cdots & & a_r & & \circ \\ & \circ & & L_1 & & L_2 & \cdots & L_r & & L_{r+1} & \end{array}$
Class IV	$\begin{array}{ccccccccccc} & \circ & & L_1 & & L_2 & \cdots & L_r & & L_{r+1} & \\ \circ & & a_0 & & a_1 & & \cdots & & a_r & & \circ \\ & L_0 & & L_1 & & L_2 & \cdots & L_r & & \circ & \end{array}$

We say that $\mathfrak{t}$ is $\mathcal{E}$-*maximal* if $a_0 = L_0, \cdots, a_d = L_d$, and $\mathcal{E}$-*minimal* if $a_0 = L_1, \cdots, a_d = L_{d+1}$. Not every local type $\mathfrak{S}_i$ (with $\mathcal{E} = \mathcal{E}_i$) contains an $\mathcal{E}$-maximal or $\mathcal{E}$-minimal element. If it does, then $\mathfrak{S}_i$ consists of that single local pattern.

Proposition 13 *If* $\mathfrak{t}_i$ *is* $\mathcal{E}_i$*-maximal then* $\mathfrak{t}'_i$ *is* $\mathcal{E}_i$*-minimal, and*

$$G_\Delta^{\mathcal{E}_i}(\mathfrak{t}'_i) = q^{\gamma_{L_{\mathcal{E}_i}} - \gamma_{R_{\mathcal{E}_i}}} G_\Gamma^{\mathcal{E}_i}(\mathfrak{t}_i).$$

Proof Let notations be as in Proposition 9, 10, 11 or 12, depending on the class of $\mathcal{E} = \mathcal{E}_i$, and in particular $L = L_{\mathcal{E}_i}$ and $R = R_{\mathcal{E}_i}$. Each entry in the $\mathcal{E}$-portion of both $\Gamma_\mathfrak{t}$ and $\Delta_{\mathfrak{t}'}$ is boxed except γ_L and γ_R, if they happen to lie inside $\mathcal{E}$, which one or both does unless $\mathcal{E}$ is of Class II; these are neither boxed nor circled. We have, therefore

$$G_\Gamma^{\mathcal{E}_i}(\mathfrak{t}_i) = g(s-\gamma_L) \prod g(\mu_i) g(\nu_i) \times \begin{cases} h(\gamma_L) & \text{Class I or IV} \\ 1 & \text{Class II or III} \end{cases} \times \begin{cases} h(\gamma_R) & \text{Class I or III} \\ 1 & \text{Class II or IV} \end{cases}$$

and $G_\Delta^{\mathcal{E}_i}(\mathfrak{t}')$ is the same, except that $g(s-\gamma_L)$ is replaced by $g(s-\gamma_R)$. By Proposition 7 we may assume as usual that $n|\gamma_L$ and $n|\gamma_R$. It follows that $q^{\gamma_L - \gamma_R} g(s-\gamma_L) = g(s-\gamma_R)$, and the statement is proved. □

Proposition 14 (Knowability Lemma) *Let* $\mathcal{E}$ *be an episode in the cartoon associated to the short Gelfand-Tsetlin pattern* $\mathfrak{t}$*, and let* $L = L_\mathcal{E}$ *and* $R = R_\mathcal{E}$ *as in Tables 11.1 and 11.2. Let* s, μ_i *and* ν_i *be as in Proposition 9, 10, 11 or 12, depending on the class of* $\mathcal{E}$*. Assume that* $n \nmid s$*. Then either* $G_\Gamma(\mathfrak{t}) = G_\Delta(\mathfrak{t}') = 0$*, or* $\mathfrak{t}$ *is* $\mathcal{E}$*-maximal.*

The term "Knowability Lemma" should be understood as follows. It asserts that one of the following cases applies:

- Maximality: $\mathfrak{t}$ is $\mathcal{E}$-maximal, and $\mathfrak{S}_i$ consists of the single local pattern. In this case (11.4) follows from Proposition 13.
- Knowability: $n|s$ in which case all the Gauss sums that appear in all the patterns of the resotope appear in knowable combinations – $g(s)$ by itself or $g(\mu_i)g(\nu_i)$ where $\mu_i + \nu_i = s$.
- In all other cases where $n \nmid s$ we have and $G_\Gamma(\mathfrak{t}) = G_\Delta(\mathfrak{t}') = 0$ for all patterns so (11.4) is obvious.

Knowability (as explained in Chapter 5) *per se* is not important for the proof that Statement C implies Statement B, but the precise statement in Proposition 14, particularly the fact that we may assume that $n|s$, *will* be important. Theorems 11 and 12 below validate the term "knowability" by explicitly evaluating the sums that arise when $n|s$.

Proof We will discuss the cases where $\mathcal{E}$ is Class II or Class I, leaving the remaining two cases to the reader.

First assume that $\mathcal{E}$ is Class II. Let notations be as in Proposition 9. By Proposition 7 we may assume that $n|\gamma_L$ and $n|\gamma_R$. We will assume that $G_\Gamma(\mathfrak{t}) \neq 0$ and show that $s-\gamma_L$, μ_i and ν_i are all boxed. A similar argument would give the same conclusion assuming $G_\Delta(\mathfrak{t}') \neq 0$. Since $h(x)=0$ when $n \nmid x$, if such x appears in $\Gamma_\mathfrak{t}$ it is either boxed or circled. In particular $s-\gamma_L$ is either boxed or circled.

We will argue that $s-\gamma_L$ is not circled. By the Circling Lemma (Lemma 7), $\mu_1 = s-\gamma_L$, $\nu_1=\gamma_L$, and ν_1 is also circled. Now $n \nmid \mu_1 = s-\gamma_L$, so μ_1 is either circled or boxed, and it cannot be boxed, because this would imply that ν_1 is both circled and boxed, which is impossible since $G_\Gamma(\mathfrak{t}) \neq 0$. Thus $\mu_1 = s-\gamma_L$ is circled, and we may repeat the argument, showing that $s-\gamma_L = \mu_1 = \mu_2 = \ldots$ so that $\nu_1 = \nu_2 = \ldots$, and that all entries are circled. When we reach the end of the top row, μ_d is circled, which implies that $\mu_d = 0$, and so $s=\gamma_L$, which is a contradiction since we assumed that $n \nmid s$.

This proves that $s-\gamma_L$ is boxed. Now we argue by contradiction that the μ_i and ν_i are also boxed. If not, let $i \geqslant 0$ be chosen so that $\nu_1, \cdots, \nu_{i-1}$ are boxed (and therefore, so are $\mu_1, \cdots, \mu_{i-1}$) but ν_i is not boxed. We note that ν_i cannot be circled, because if ν_i is circled, then μ_{i-1} (or s if $i=0$) is both circled and boxed, which is a contradiction. Thus ν_i is neither boxed nor circled and so $n|\nu_i$. Since $\nu_i + \mu_i = s$ and $n \nmid s$, we have $n \nmid \mu_i$ and so μ_i is either boxed or circled. It cannot be boxed since this would imply that ν_i is also boxed, and our assumption is that it is not. Thus μ_i is circled. By the Circling Lemma, $\mu_i = \mu_{i+1}$, and so $n \nmid \mu_{i+1}$ which is thus either boxed or circled. It cannot be circled, since if it is, then ν_{i+1} is both circled (since μ_i is circled) and boxed (since μ_{i+1} is boxed), and we know that if a bottom row entry is both boxed and circled, then $\mathfrak{t}$ is not strict and $G_\Gamma(\mathfrak{t})=0$, which is a contradiction. Thus μ_{i+1} is circled. Repeating this argument, $\mu_i = \mu_{i+1} = \cdots$ are all circled, and when we get to the end, μ_d is circled, so by the Circling Lemma, $\mu_i = \mu_d = \gamma_R$, which is a contradiction since γ_R is divisible by n, but μ_i is not. This contradiction shows that $s-\gamma_L$ and the μ_i, ν_i are all boxed, and it follows from the definitions that $\mathfrak{t}$ is $\mathcal{E}$-maximal.

We now discuss the variant of this argument for the case that $\mathcal{E}$ is of Class I, leaving the two other cases to the reader. Let notations be as in Proposition 10. Again we assume that $G_\Gamma(\mathfrak{t}) \neq 0$, so whenever x appears in $\Gamma_\mathfrak{t}$ with $n \nmid x$ it is either boxed or circled. Due to its location in the cartoon, there is no way that $s-\gamma_L$ can be circled, so it is boxed.

Now we argue by contradiction that $\nu_1, \cdots, \nu_d$ and hence $\mu_1, \cdots, \mu_d$ are all boxed. If not, let $i \geqslant 0$ be chosen so that $\nu_1, \cdots, \nu_{i-1}$ are boxed (and therefore, so are

$\mu_1, \cdots, \mu_{i-1})$ but ν_i is not boxed. The same argument as in the Class II case shows that $n | \nu_i$ so $n \nmid \mu_i$ and that μ_i is circled, and moreover that $\mu_i = \mu_{i+1} = \cdots = \mu_d$ and that these are all circled. But now this is a contradiction since due to its location in the cartoon, μ_d cannot be circled. $\square$

Chapter 13

The Reduction to Statement D

We now switch to the language of resotopes, as defined in Chapter 5. We remind the reader that we may assume $\gamma_{L_\mathcal{E}}$ and $\gamma_{R_\mathcal{E}}$ are multiples of n for every totally resonant episode.

Proposition 15 *Statement D is equivalent to Statement C. Moreover, Statement D is true if* $n \nmid s$.

Proof The case of a totally resonant short Gelfand-Tsetlin pattern $\mathfrak{t}$ is a special case of Proposition 9, and the point is that $\Gamma_\mathfrak{t}$ is a Γ-accordion $\mathfrak{a}$, and Proposition 9 shows that $\Delta_{\mathfrak{t}'}$ is the Δ-accordion $\mathfrak{a}'$. In this case $\gamma_L = \gamma_R = 0$. Moreover as $\mathfrak{t}$ runs through its totally resonant prototype, $\mathfrak{a}$ runs through the Γ-resotope $\mathcal{A}_s(c_0, \cdots, c_d)$ with $c_i = L_i - L_{i+1}$, so Statement D boils down to Statement C. The fact that Statement D is true when $n \nmid s$ follows from the Knowability Lemma and Proposition 13. $\square$

We turn next to the proof that Statement D implies Statement B. What we will show is that for each of the four types of resonant episodes, Statement D implies (11.4); then Statement B will follow from Proposition 8. We fix an episode $\mathcal{E} = \mathcal{E}_i$, and will denote $L = L_i$, $R = R_i$. By Proposition 7 we may assume that $n|\gamma_L$ and $n|\gamma_R$. Moreover by the Knowability Lemma (Proposition 14) we may assume $n|s$, where s and other notations are as in Proposition 9, 10, 11 or 12, depending on the class of $\mathcal{E}$.

Proposition 16 *Let $\mathcal{E}$ be a Class II episode, and let notations be as in Proposition 9. If*

$$\mathfrak{a} = \left\{ \begin{array}{ccccccc} \hat{s} & & \hat{\mu}_1 & & \cdots & & \hat{\mu}_d \\ & \hat{\nu}_1 & & \cdots & & \hat{\nu}_d & \end{array} \right\}$$

where $s = \hat{s} + \gamma_R + \gamma_L$*,* $\mu_i = \hat{\mu}_i + \gamma_R$ *and* $\nu_i = \hat{\nu}_i + \gamma_L$*, then* $\mathfrak{a}$ *lies in the resotope* $\mathcal{A} = \mathcal{A}_{\hat{s}}(c_0, \cdots, c_d)$ *with* $c_i = L_i - L_{i+1}$*; let* σ *denote the signature of* $\mathfrak{a}$ *in* $\mathcal{A}$*. Then* $\mathfrak{t} \longmapsto \mathfrak{a}_\sigma$ *induces a bijection from the local type* $\mathfrak{S}_i$ *to* $\mathcal{A}$*. Assume furthermore that* $n|\gamma_L$ *and* $n|\gamma_R$*. Then*

$$q^{\gamma_L} G_\Gamma^{\mathcal{E}}(\mathfrak{t}) = q^{(d+1)(\gamma_R+\gamma_L)} \mathcal{G}_\Gamma(\mathfrak{a}_\sigma), \qquad q^{\gamma_R} G_\Delta^{\mathcal{E}}(\mathfrak{t}') = q^{(d+1)(\gamma_R+\gamma_L)} \mathcal{G}_\Delta(\mathfrak{a}'_\sigma). \tag{13.1}$$

Proof With notations as in Proposition 9, the inequalities $L_i \geqslant a_i \geqslant L_{i+1}$ that a_i must satisfy can be written (with $\hat{\mu}_0 = \hat{s}$):

$$L_i - L_{i+1} \geqslant \hat{\mu}_{i-1} - \hat{\mu}_i \geqslant 0,$$

which are the same as the conditions that $\mathfrak{a}_\sigma$ lies in $\mathcal{A} = \mathcal{A}_s(c_0, \cdots, c_d)$, with $c_i = L_i - L_{i+1}$. Each entry in $\mathfrak{a}_\sigma$ is boxed or circled if and only if the corresponding entry in $\Gamma_\mathfrak{t}$ is, and similarly, every entry in $\mathfrak{a}'_\sigma$ is boxed or circled if and only if the corresponding entry in the (left-to-right) mirror image of $\Delta_{\mathfrak{t}'}$ is. Using the assumption that $n|\gamma_R$ and $n|\gamma_L$ and Proposition 5 we can pull a factor of q^{γ_R} from the factor of $G_\Gamma^{\mathcal{E}}(\mathfrak{t})$ corresponding to $s - \gamma_L = \hat{s} + \gamma_R$, which is

$$\begin{cases} g(\hat{s}+\gamma_R) & \text{if } s-\gamma_L \text{ is boxed;} \\ q^{\hat{s}+\gamma_R} & \text{if } s-\gamma_L \text{ is circled;} \\ h(\hat{s}+\gamma_R) & \text{otherwise,} \end{cases}$$

leaving just the corresponding contribution in $\mathcal{G}_\Gamma(\mathfrak{a}_\sigma)$; and similarly we may pull out d factors of q^{γ_R} from the contributions of $\mu_i = \widehat{\mu_i} + \gamma_R$, and d factors of q^{γ_L} from the contributions of $\nu_i = \widehat{\nu_i} + \gamma_L$. What remains is just $\mathcal{G}_\Gamma(\mathfrak{a}_\sigma)$. This gives the first identity in (13.1), and the second one is proved similarly. $\square$

Corollary. *Statement D implies (11.4) for Class II episodes.*

Although this reduction was straightforward for Class I, each of the remaining classes involves some nuances. In every case we will argue by comparing $q^{\gamma_L} G_\Gamma(\mathfrak{t})$ to $\mathcal{G}_\Gamma(\mathfrak{a}_\sigma)$, where $\mathfrak{a}_\sigma$ is the accordion associated with the totally resonant pattern

$$\begin{array}{ccccccccccc} L_0 & & L_1 & & L_2 & \cdots & L_d & & L_{d+1} \\ & a_0 & & a_1 & & \cdots & & a_d & \\ & & L_1 & & L_2 & \cdots & L_d & & \end{array} \tag{13.2}$$

Here L_i and a_i are as in Proposition 9, 10, 11 or 12. We have moved L_0 and L_d from the bottom row to the top row as needed, and discarded the rest of the top and bottom rows. The notations s, μ_i and ν_i are already in use Proposition 9, 10, 11 or 12, so we will denote

$$\mathfrak{a} = \left\{ \begin{array}{ccccccc} t & & \psi_1 & & \cdots & & \psi_d \\ & \phi_1 & & \cdots & & \phi_d & \end{array} \right\}. \tag{13.3}$$

We see that $\mathfrak{a}_\sigma$ runs through $\mathcal{A}_t(c_0, \cdots, c_d)$ by Proposition 16, with $c_i = L_i - L_{i+1}$. We will compare $G_\Gamma(\mathfrak{t})$ and $G_\Delta(\mathfrak{t}')$ with $\mathcal{G}_\Gamma(\mathfrak{a}_\sigma)$ and $\mathcal{G}_\Delta(\mathfrak{a}'_\sigma)$ respectively. A complication is that while corresponding entries of $\Gamma_\mathfrak{t}$ and $\mathfrak{a}_\sigma$ are boxed together, the circlings may not quite match; the argument will justify moving circles from one entry in $G_\Gamma(\mathfrak{t})$ to another. Specifically, if either γ_L or γ_R is within $\mathcal{E}$ and is circled, the circle needs to be moved to another location. This is justified by the following observation.

Lemma 9 (Moving Lemma) *Suppose that x and y both appear in the $\mathcal{E}$ part of $\Gamma_\mathfrak{t}$, and that y is circled, but x is neither circled nor boxed. Suppose that both x and y are both positive and $x \equiv y$ modulo n. Then we may move the circle from y to x without changing the value of $G_\Gamma^\mathcal{E}(\mathfrak{t})$.*

Proof Before moving the circle, the contribution of the two entries is $q^y h(x)$; after moving the circle, the contribution is $q^x h(y)$. These are equal by Proposition 5. (The positivity of x is needed since $h(0)$ is undefined.) □

In each case we will discuss $G_\Gamma(\mathfrak{t})$ carefully leaving $G_\Delta(\mathfrak{t}')$ more or less to the reader. The case where $\mathcal{E}$ is of Class II has already been handled in Proposition 16.

Class I episodes

We assume that the $\mathcal{E}$-portion of $\mathfrak{t}$ has the form:

$$\begin{array}{ccccccccccccc} & & \circ & & L_1 & & L_2 & \cdots & L_d & & \circ & & \\ & \diagup & & \diagup & & \diagdown \; \diagup & & & & \diagdown & & \diagdown & \\ \circ & & & a_0 & & a_1 & & \cdots & & a_d & & & \circ \\ & & \diagup & & \diagdown \; \diagup & & \diagdown & & \diagup & & \diagdown & & \\ & & L_0 & & L_1 & & L_2 & \cdots & L_d & & L_{d+1} & & \end{array} \tag{13.4}$$

We will compare $G_\Gamma(\mathfrak{t})$ with $\mathcal{G}_\Gamma(\mathfrak{a}_\sigma)$, where $\mathfrak{a}$ is the accordion (13.3) derived from the pattern in (13.2), and σ is its signature. Thus we move L_0 and L_{d+1} to the top row, which does not affect the inequalities that the a_i satisfy, and discard the rest of the pattern to obtain the totally resonant pattern (13.2), then compute its accordion. Otherwise, let $\Gamma_\mathfrak{t}$, and $\Delta_{\mathfrak{t}'}$ be as in Proposition 10, and let s, μ_i and ν_i be as defined there.

Proposition 17 *Assume $\mathcal{E}$ is a Class I episode and that $n|s, \gamma_L, \gamma_R$. As $\mathfrak{t}$ runs through its local type, $\mathfrak{a}_\sigma$ runs through $\mathcal{A}_t(c_0, \cdots, c_d)$ with $c_i = L_i - L_{i+1}$, and*

$$\begin{aligned} q^{\gamma_L} G_\Gamma^{\mathcal{E}}(\mathfrak{t}) &= h(\gamma_L) h(\gamma_R) q^{(d+1)(2s-\gamma_L-\gamma_R)} \mathcal{G}_\Gamma(\mathfrak{a}_\sigma), \\ q^{\gamma_R} G_\Delta^{\mathcal{E}}(\mathfrak{t}') &= h(\gamma_L) h(\gamma_R) q^{(d+1)(2s-\gamma_L-\gamma_R)} \mathcal{G}_\Delta(\mathfrak{a}'_\sigma). \end{aligned}$$

Proof Using (13.4) and (13.2), we have

$$t = \sum_{j=0}^{d} (a_j - L_{j+1}) = \gamma_R - (s - \gamma_L), \qquad \psi_i = \sum_{j=i}^{d} (a_j - L_{j+1}) = \gamma_R - \nu_i,$$

and $\phi_i + \psi_i = t$. If γ_L is circled, we will move the circle to $s - \gamma_L$. To justify the use of the Moving Lemma (Lemma 9) we check that γ_L and $s - \gamma_L$ are both positive and congruent to zero modulo n. Positivity of γ_L follows since $\gamma_L \geqslant \mu_d$, and $\mu_d > 0$ since if $\mu_d = 0$ then it is circled, which it cannot be due to its location in the cartoon. To see that $s - \gamma_L > 0$, if it is zero then both $s - \gamma_L$ and γ_L are circled, which implies that $L_1 = a_1 = L_2$, but $L_1 > L_2$. Both $s - \gamma_L$ and γ_L are multiples of n by assumption.

If γ_R is circled, we will move the circle to μ_d. To see that this is justified, we must check that γ_R and μ_d are positive and multiples of n. We are assuming $n|\gamma_R$, and it is positive since $\gamma_R \geqslant s - \gamma_L$ which cannot be zero; if it were, it would be circled, which it cannot be due to its position in the cartoon. Also by the Circling Lemma, since γ_R is circled it equals ν_d; thus $\mu_d = s - \nu_d = s - \gamma_R \equiv 0$ modulo n. And μ_d cannot be zero since it is not circled, due to its location in the cartoon.

With these circling modifications, γ_L and γ_R are neither circled nor boxed, hence produce factors in $G_\Gamma^{\mathcal{E}}(\mathfrak{t})$ of $h(\gamma_L)$ and $h(\gamma_R)$. The remaining factors in $G_\Gamma^{\mathcal{E}}(\mathfrak{t})$ can be handled as follows. Let $F(x) = q^x$ if x is a boxed entry in $\Gamma_\mathfrak{t}$ or $\mathfrak{a}_\sigma$, $g(x)$ if it is circled, and $h(x)$ if it is neither boxed nor circled. We have

$$\begin{aligned} q^{\gamma_L} F(s - \gamma_L) &= q^{2s-\gamma_L-\gamma_R} F(t), \\ F(\mu_i) &= q^{s-\gamma_R} F(\psi_i), \\ F(\nu_i) &= q^{s-\gamma_L} F(\phi_i), \end{aligned}$$

and multiplying these identities together gives the stated identity for $q^{\gamma_L} G_\Gamma^{\mathcal{E}}(\mathfrak{t})$. (There are two entries $h(\gamma_L)$ and $h(\gamma_R)$ that have to be taken out.) The Δ' preaccordion is handled similarly. □

Class III episodes

Now we assume that the $\mathcal{E}$-portion of $\mathfrak{t}$ has the form:

$$\begin{array}{ccccccccccccccccccc}
 & & L_0 & & & & L_1 & & & & L_2 & \cdots & L_d & & & & \circ & & \\
 & & & \diagdown & & \diagup & & \diagdown & & \diagup & & & & \diagdown & & & & \diagdown & \\
\circ & & & & a_0 & & & & a_1 & & & \cdots & & & a_d & & & & \circ \\
 & \diagdown & & & & \diagdown & & \diagup & & \diagdown & & & & \diagup & & \diagdown & & & \\
 & & \circ & & & & L_1 & & & & L_2 & \cdots & L_r & & & & L_{d+1} & &
\end{array} \qquad (13.5)$$

Proposition 18 *Assume $\mathcal{E}$ is a Class III episode and that $n|s, \gamma_L, \gamma_R$. If $a_0 = L_1$, $a_1 = L_2, \cdots, a_d = L_{d+1}$ then the local type consists of a single pattern $\mathfrak{t}$, for which (11.4) is satisfied. Assume that this is not the case. Then as $\mathfrak{t}$ runs through its local type, $\mathfrak{a}_\sigma$ runs through $\mathcal{A}_t(c_0, \cdots, c_d)$ with $c_i = L_i - L_{i+1}$, and*

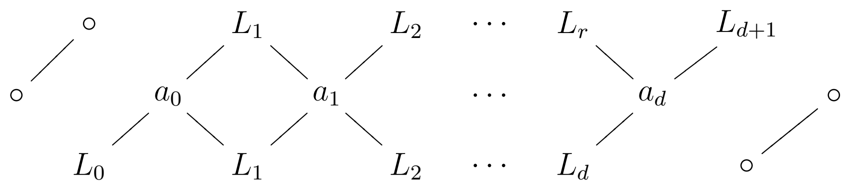
$$q^{\gamma_L} G_\Gamma^{\mathcal{E}}(\mathfrak{t}) = q^{(d+1)(s+\gamma_L-\gamma_R)} h(\gamma_R) \mathcal{G}_\Gamma(\mathfrak{a}_\sigma), \qquad q^{\gamma_R} G_\Delta^{\mathcal{E}}(\mathfrak{t}') = q^{(d+1)(s+\gamma_L-\gamma_R)} h(\gamma_R) \mathcal{G}_\Delta(\mathfrak{a}'_\sigma).$$

Proof If $a_0 = L_1$, $a_1 = L_2, \cdots, a_d = L_{d+1}$ then the local type consists of a single element $\mathfrak{t}$. We will handle this case separately. For this $\mathfrak{t}$ it is easy to see that all entries except μ_d are circled in $\Gamma_\mathfrak{t}$, while in $\Delta_{\mathfrak{t}'}$ all entries except $s - \gamma_R$ are circled. But by the Circling Lemma $s - \gamma_L = \mu_1 = \cdots = \mu_d$ and $\mu_d > 0$ since it cannot be circled due to its location in the cartoon. Thus we may move the circle from $s - \gamma_L$ to μ_d and then compare $G_\Gamma^{\mathcal{E}}(\mathfrak{t})$ and $G_\Delta^{\mathcal{E}}(\mathfrak{t}')$ to see directly that (11.4) is true.

We exclude this case and assume that at least one of the inequalities $a_i \geqslant L_{i+1}$ is strict. Using (13.3) and (13.5) we have $t = \gamma_R - \gamma_L$, $\phi_i = \gamma_R - \gamma_L$, $\psi_i = \nu_i - \gamma_L$ and $\psi_i = \mu_i + \gamma_R - s$, where $\gamma_R, \gamma_L, s, \mu_i$ and ν_i are as in Proposition 11, and σ is the signature of $\mathfrak{a}$ in $\mathcal{A}$. If γ_L is circled then we move the circle from γ_L to μ_d in $\Gamma_\mathfrak{t}$. This is justified as in the Class I case, except that the justification we gave there for the claim that $\gamma_R > 0$ is no longer valid. It follows now from our assumption that one of the inequalities $a_i \geqslant L_{i+1}$ is strict. After moving the circle from γ_L to μ_d in $\Gamma_\mathfrak{t}$, each factor $s - \gamma_L$, μ_i, ν_i is circled or boxed in the (circling-modified) $\Gamma_\mathfrak{t}$ if and only if the corresponding factor t, ψ_i or ϕ_i is circled or boxed in $\mathfrak{a}_\sigma$. Moreover $s + \gamma_L - \gamma_R \equiv 0$ modulo n so we can pull out a factor of $q^{s+\gamma_L-\gamma_R}$ from the contributions of $s - \gamma_L$ and each pair μ_i, ν_i, to $q^{\gamma_L} G_\Gamma^{\mathcal{E}}(\mathfrak{t})$, and what remains is $h(\gamma_R)\mathcal{G}_\Gamma(\mathfrak{a}_\sigma)$. A similar treatment gives the other identity. □

Class IV episodes

Now we assume that the $\mathcal{E}$-portion of $\mathfrak{t}$ has the form:



(13.6)

If $a_0 = L_1,\, a_1 = L_2, \cdots, a_d = L_{d+1}$ then the local type consists of a single element $\mathfrak{t}$. In this case γ_L is circled in both $\Gamma_{\mathfrak{t}}$ and $\Delta_{\mathfrak{t}'}$ and we don't try to move it. We have

$$q^{\gamma_L} G_\Gamma^{\mathcal{E}}(\mathfrak{t}) = h(s) q^{\gamma_L} \prod_{i=1}^{d} q^{\mu_i} \prod_{i=1}^{d} q^{\nu_i} = h(s) q^{\gamma_L} q^{ds} = q^{\gamma_R} G_\Delta^{\mathcal{E}}(\mathfrak{t}').$$

We exclude this case and assume that at least one of the inequalities $a_i \geqslant L_{i+1}$ is strict. Using (13.3) and (13.6) we have $t = \gamma_L - \gamma_R$, $\psi_i = \mu_i - \gamma_R$ and $\phi_i = \nu_i + \gamma_L - s$.

Proposition 19 *Assume $\mathcal{E}$ is a Class IV episode and that $n|s, \gamma_L, \gamma_R$. As $\mathfrak{t}$ runs through its local type, $\mathfrak{a}_\sigma$ runs through $\mathcal{A}_t(c_0, \cdots, c_d)$ with $c_i = L_i - L_{i+1}$, and*

$$q^{\gamma_L} G_\Gamma^{\mathcal{E}}(\mathfrak{t}) = h(\gamma_L) q^{(d+1)(\gamma_R - \gamma_L + s)} \mathcal{G}_\Gamma(\mathfrak{a}_\sigma), \qquad q^{\gamma_R} G_\Delta^{\mathcal{E}}(\mathfrak{t}') = h(\gamma_L) q^{(d+1)(\gamma_R - \gamma_L + s)} \mathcal{G}_\Delta(\mathfrak{a}'_\sigma).$$

Proof If γ_L is circled, we must move the circle from γ_L to $s - \gamma_L$. This is justified the same way as in the Class I case, except that the positivity of γ_L must be justified differently. In this case, it follows from our assumption that one of the inequalities $a_i \geqslant L_{i+1}$ is strict. Now we can pull out a factor of $q^{s+\gamma_R-\gamma_L}$ from the contributions of $s - \gamma_L$ and eacch pair μ_i, ν_i, and the statement follows as in our previous cases. □

Theorem 9 *Statement D (or, equivalently, Statement C) implies Statement B.*

Proof The equivalence of Statements D and C is the Corollary to Proposition 16. By Proposition 8 we must show (11.4) for every episode $\mathcal{E}$. By Proposition 7 we may assume that $n|\gamma_L$ and $n|\gamma_R$. Moreover by the Knowability Lemma (Proposition 14) we may assume $n|s$ because if $n \nmid s$ then Proposition 13 is applicable. We may then apply Proposition 16, 17, 18 or 19 depending on the class of $\mathcal{E}$. □

Chapter 14

Statement E implies Statement D

We fix a nodal signature. Let $B(\eta) = \{i|\eta_i = \square\}$. Let $\mathcal{CP}_\eta(c_0, \cdots, c_d) \in \mathfrak{Z}_\Gamma$ be the following "cut and paste" virtual resotope

$$\mathcal{CP}_\eta(c_0, \cdots, c_d) = \sum_{T \subseteq B(\eta)} (-1)^{|T|} \mathcal{A}_s(c_0^T, \cdots, c_d^T), \tag{14.1}$$

where

$$c_i^T = \begin{cases} c_i & \text{if } i \in T, \\ \infty & \text{if } i \notin T. \end{cases}$$

We recall that $\mathrm{CP}_\eta(c_0, \cdots, c_d)$ is the set of Γ-accordions

$$\mathfrak{a} = \left\{ \begin{array}{ccccccc} s & & \mu_1 & & \cdots & & \mu_d \\ & \nu_1 & & \cdots & & \nu_d & \end{array} \right\}$$

that satisfy the inequalities (5.21), with the convention that $\mu_0 = s$ and $\mu_{d+1} = 0$. Geometrically, this set is a simplex, and we will show that it is the support of $\mathcal{CP}_\eta(c_0, \cdots, c_d)$, though the latter virtual resotope is a superposition of resotopes whose supports include elements that are outside of $\mathrm{CP}_\eta(c_0, \cdots, c_d)$; it will be shown that the alternating sum causes such terms to cancel.

Finally, if $\mathfrak{a} \in \mathrm{CP}_\eta(c_0, \cdots, c_d)$ let $\theta(\mathfrak{a}, \eta)$ be the signature obtained from η by changing η_i to $*$ when the inequality

$$\mu_i - \mu_{i+1} \geqslant \begin{cases} c_i & \text{if } \eta_i = \square, \\ 0 & \text{if } \eta_i = \circ, \end{cases}$$

is strict. Note that these are the inequalities defining $\mathfrak{a} \in \mathrm{CP}_\eta(c_0, \cdots, c_d)$. Strictly speaking $\mathfrak{a}$ and η do not quite determine $\theta(\mathfrak{a}, \eta)$ because it also depends on the c_i. We omit these data since they are fixed, while $\mathfrak{a}$ and η will vary.

Proposition 20 *The support of* $\mathcal{CP}_\eta(c_0, \cdots, c_d)$ *is the simplex* $\mathrm{CP}_\eta(c_0, \cdots, c_d)$*. Suppose that* $\mathfrak{a} \in \mathrm{CP}_\eta(c_0, \cdots, c_d)$*. If* τ *is any signature, then the coefficient of* $\mathfrak{a}_\tau$ *in* $\mathcal{CP}_\eta(c_0, \cdots, c_d)$ *is zero unless* τ *is obtained from* $\theta(\mathfrak{a}, \eta)$ *by changing some* $\square$*'s to* $*$*. If it is so obtained, the coefficient is* $(-1)^\varepsilon$*, where* ε *is the number of* $\square$*'s in* τ*.*

Proof Suppose the Γ-accordion $\mathfrak{a}$ does not satisfy (5.21). We will show that it does not appear in the support of $\mathcal{CP}_\eta(c_0, \cdots, c_d)$. By assumption $\mu_i - \mu_{i+1} < c_i$ for some $i \in B(\eta)$. We group the subsets of $B(\eta)$ into pairs T, T' where $T = T' \cup \{i\}$. It is clear that $\mathfrak{a}$ occurs in $\mathcal{A}_s(c_0^T, \cdots, c_d^T)$ if and only if it occurs in $\mathcal{A}_s(c_0^{T'}, \cdots, c_d^{T'})$, and with the same signature. Since these have opposite signs, their contributions cancel. This proves that the support of $\mathcal{CP}_\eta(c_0, \cdots, c_d)$ is contained in the simplex $\mathcal{C}$. The opposite inclusion will be clear from the precise description of the coefficients, which is our next step to prove.

We note that $\theta(\mathfrak{a}, \eta) = \theta_0 \cdots \theta_d$ where

$$\theta_i = \begin{cases} \square & \text{if } \mu_i - \mu_{i+1} = c_i, \\ \circ & \text{if } \mu_i - \mu_{i+1} = 0, \\ * & \text{otherwise.} \end{cases}$$

We emphasize that if $\theta_i = \square$ then $i \in B(\eta)$, while if $\theta_i = \circ$ then $i \notin B(\eta)$. (The case $\theta_i = *$ can arise whether or not $i \in B(\eta)$.)

Suppose that $\mathfrak{a} \in \mathrm{CP}_\eta(c_0, \cdots, c_d)$. In order for $\mathfrak{a}_\tau$ to have a nonzero coefficient, it must appear as the coefficient of $\mathfrak{a}$ in $\mathcal{A}_s(c_0^T, \cdots, c_d^T)$ for some subset T of $B(\eta)$. We will prove that if τ is the signature of $\mathfrak{a}$ in this resotope we have

$$\tau_i = \begin{cases} \circ & \text{if } \mu_i - \mu_{i+1} = 0, \text{ in which case } i \notin B(\eta); \\ \square & \text{if } i \in T; \\ * & \text{otherwise.} \end{cases} \tag{14.2}$$

First, if $i \in T$ then $c_i^T = c_i$ so $\mu_i - \mu_{i+1} \leqslant c_i$, for we have already stipulated (by assuming $\mathfrak{a} \in \mathrm{CP}_\eta(c_0, \cdots, c_d)$) that $\mu_i - \mu_{i+1} \geqslant c_i$. Therefore $\mu_i - \mu_{i+1} = c_i$ when $i \in T$, and so $\tau_i = \square$ when $i \in T$. And if $i \notin T$, the signature of τ is definitely not $\square$ since $c_i^T = \infty$; if $i \in B(\eta) - T$ it also cannot be $\circ$ since $\mu_i - \mu_{i+1} \geqslant c_i > 0$. This proves (14.2).

It is clear from (14.2) that τ is obtained from $\theta(\mathfrak{a})$ by changing some $\square$'s to $*$'s, and which ones are changed determines T. This point is important since it shows that (unlike the case where $\mathfrak{a} \notin \mathrm{CP}_\eta(c_0, \cdots, c_d)$) a given $\mathfrak{a}_\tau$ can only appear in only one term in (14.1), so there cannot be any cancellation. If τ is obtained from $\theta(\mathfrak{a})$ by changing some $\square$'s to $*$'s then it does appear in $\mathcal{A}_s(c_0^T, \cdots, c_d^T)$ for a unique T and so $\mathfrak{a}_\tau$ appears in $\mathcal{CP}_\eta(c_0, \cdots, c_d)$ with a nonzero coefficient. The sign with which it appears is $(-1)^{|T|}$, and T we have noted is the set of i for which $\tau_i = \square$. $\square$

Theorem 10 *Statement E implies Statement D.*

Proof Let $\mathfrak{a} \in \mathrm{CP}_\eta(c_0, \cdots, c_d)$ and let $\sigma = \theta(\mathfrak{a}, \eta)$. What we must show is that (5.22) implies (5.19). We extend the function G_Γ from the set of decorated Γ-accordions to the free abelian group $\mathfrak{Z}_\Gamma$ by linearity. Also the involution $\mathfrak{a}_\eta \longmapsto \mathfrak{a}'_\eta$ on decorated accordions induces an isomorphism $\mathfrak{Z}_\Gamma \longrightarrow \mathfrak{Z}_\Delta$ that we will denote $\mathcal{A} \longmapsto \mathcal{A}'$.

Then (5.19) can be written $\mathcal{G}_\Gamma(\mathcal{A}) = \mathcal{G}_\Delta(\mathcal{A}')$. By the principle of inclusion-exclusion (Stanley [40], page 64), we have

$$\mathcal{A}_s(c_0, \cdots, c_d) = \sum_{T \subseteq B(\eta)} \mathcal{CP}_{\eta^T}(c_0^T, \cdots, c_d^T),$$

Where if T is a subset of $B(\eta)$ then η^T is the signature obtained by changing η_i from $\Box$ to $\circ$ for all $i \in T$. This means that if we show $\mathcal{G}_\Gamma(\mathcal{C}) = \mathcal{G}_\Delta(\mathcal{C}')$ when $\mathcal{C} = \mathcal{CP}_\eta(c_0, \cdots, c_d)$ then (5.19) will follow. The left-hand side in this identity is a sum of $\mathcal{G}_\Gamma(\mathfrak{a}_\tau)$ with $\mathfrak{a}$ in $\mathrm{CP}_\eta(c_0, \cdots, c_d)$, and the coefficient of $\mathfrak{a}_\tau$ in this sum is the same as its coefficient in $\Lambda_\Gamma(\mathfrak{a}, \sigma)$ by Proposition 20. $\Box$

Chapter 15

Evaluation of Λ_Γ and Λ_Δ, and Statement G

Let η be a nodal signature, and let σ be a subsignature. Let

$$\mathfrak{a} = \left\{ \begin{array}{ccccccccc} s & & \alpha_1 & & \alpha_2 & & \cdots & & \alpha_d \\ & \beta_1 & & \beta_2 & & & \cdots & \beta_d & \end{array} \right\}$$

be an accordion belonging to the open facet $\mathcal{S}_\sigma$ of $\mathrm{CP}_\eta(c_0, \cdots, c_d)$. Assuming that $n|s$ we will evaluate $\Lambda_\Gamma(\mathfrak{a}, \sigma)$.

We will denote

$$V(a,b) = (q-1)^a q^{(d+1)s-b}, \qquad V(a) = V(a,a).$$

Let

$$\begin{aligned} \varepsilon_\Gamma(\sigma) = \varepsilon_\Gamma &= \left\{ \begin{array}{ll} 1 & \text{if } \sigma_0 = \Box, \\ 0 & \text{otherwise,} \end{array} \right. \\ \mathcal{K}_\Gamma(\sigma) = \mathcal{K}_\Gamma &= \{i | 1 \leqslant i \leqslant d, \sigma_i = \Box, \sigma_{i-1} \neq \circ\}, \qquad k_\Gamma = |\mathcal{K}_\Gamma|, \\ \mathcal{N}_\Gamma(\sigma) = \mathcal{N}_\Gamma &= \{i | 1 \leqslant i \leqslant d, \sigma_i = \Box, \sigma_{i-1} = \circ\}, \qquad n_\Gamma = |\mathcal{N}_\Gamma|, \end{aligned}$$

and

$$\mathcal{C}_\Gamma(\sigma) = \mathcal{C}_\Gamma = \{i | 1 \leqslant i \leqslant d, \sigma_0, \sigma_1, \cdots, \sigma_{i-1} \text{ not all } \circ \text{ and either } i \in \mathcal{N}_\Gamma \text{ or } \sigma_i = *\}. \tag{15.1}$$

Let $c_\Gamma = |\mathcal{C}_\Gamma|$, and let t_Γ be the number of i with $1 \leqslant i \leqslant d$ and $\sigma_i \neq *$. Given a set of indices $\Sigma = \{i_1, \cdots, i_k\} \subseteq \{1, 2, \cdots, d\}$, let

$$\delta_n(i_1, \cdots, i_k) = \delta_n(\Sigma) = \delta_n(\Sigma; \mathfrak{a}) = \left\{ \begin{array}{ll} 1 & \text{if } n \text{ divides } \alpha_{i_1}, \cdots, \alpha_{i_k}, \\ 0 & \text{otherwise.} \end{array} \right. \tag{15.2}$$

Let

$$\chi_\Gamma(\mathfrak{a},\sigma) = \chi_\Gamma = \prod_{i\in\mathcal{C}_\Gamma(\sigma)} \delta_n(i).$$

Finally, let

$$a_\Gamma(\sigma) = a_\Gamma = 2(d - t_\Gamma + n_\Gamma) + \left\{ \begin{array}{cc} -1 & \text{if } \sigma_0 = \circ \\ 0 & \text{if } \sigma_0 = \Box \\ 1 & \text{if } \sigma_0 = * \end{array} \right\} + \left\{ \begin{array}{cc} 1 & \text{if } \sigma_d = \circ \\ 0 & \text{if } \sigma_d \neq \circ \end{array} \right\}.$$

Proposition 21 *Assume that $n|s$. Given a Γ-accordion*

$$\mathfrak{a} = \left\{ \begin{array}{ccccccccc} s & & \alpha_1 & & \alpha_2 & & \cdots & & \alpha_d \\ & \beta_1 & & \beta_2 & & & \cdots & \beta_d & \end{array} \right\}$$

and an associated signature $\sigma \subseteq \sigma_{\mathfrak{T}}$ not containing the substring $\circ, \Box$, then

$$\mathcal{G}_\Gamma(\mathfrak{a},\sigma) = (-1)^{\varepsilon_\Gamma} \chi_\Gamma \cdot V(a_\Gamma, a_\Gamma + d_\Gamma), \tag{15.3}$$

where

$$d_\Gamma = \left(\sum_{\substack{1\leqslant i \leqslant d \\ \sigma_i = \Box}} (1 + \delta_n(i)) \right) + \left\{ \begin{array}{cc} 1 & \textit{if } \sigma_0 = \Box \\ 0 & \textit{if } \sigma_0 \neq \Box \end{array} \right\}.$$

Recall that any subsignature σ containing the string $\circ\Box$ has $\mathcal{G}_\Gamma(\mathfrak{a},\sigma) = 0$. We will abuse notation and rewrite the definition of $\mathcal{G}_\Gamma(\mathfrak{a},\sigma)$ as

$$\mathcal{G}_\Gamma(\mathfrak{a},\sigma) = \mathcal{G}_\Gamma(\mathfrak{a}_\sigma) = \prod_{x\in\mathfrak{a}} f_\sigma(x),$$

where

$$f_\sigma(x) = \begin{cases} g(x) & \text{if } x \text{ is boxed in } \mathfrak{a}_\sigma \text{ (but not circled)}, \\ q^x & \text{if } x \text{ is circled (but not boxed)}, \\ h(x) & \text{if } x \text{ is neither boxed nor circled}, \\ 0 & \text{if } x \text{ is both boxed and circled}. \end{cases}$$

This is an abuse of notation, since f_σ is not a function; it depends not only on the numerical value x but also its location in the decorated accordion $\mathfrak{a}_\sigma$. However this should cause no confusion.

Proof Using the signature σ to determine the rules for boxing and circling in $\mathfrak{a}$ we see that if $\sigma_0 = \Box$, then $f_\sigma(s) = g(s) = (-1) \cdot q^{s-1}$. The (-1) here accounts for the $(-1)^{\varepsilon_\Gamma}$ in (15.3). If $\sigma_i = \Box$ for $i > 0$, we have

$$f_{\sigma\sigma}(\alpha_i) f_\sigma(\beta_i) = g(\alpha_i) g(\beta_i) = \begin{cases} q^{s-1} & \text{if } n \nmid \alpha_i, \\ q^{s-2} & \text{if } n \nmid \alpha_i. \end{cases}$$

If $\sigma_0 = \circ$, then $s = \alpha_1$, $\beta_1 = 0$, and $f_\sigma(s) = q^s$. If $\sigma_i = \circ$, $0 < i < d$, then $\alpha_i = \alpha_{i+1}$ and $\beta_i = \beta_{i+1}$ so that while the circling in the accordion strictly speaking occurs at α_i and β_{i+1}, we may equivalently consider it to occur at α_i and β_i for bookkeeping purposes and

$$f_\sigma(\alpha_i) f_\sigma(\beta_{i+1}) = f_\sigma(\alpha_i) f_\sigma(\beta_i) = q^s.$$

And if $\sigma_d = \circ$, then $\alpha_d = 0$ and $\beta_d = s$, so that

$$f_\sigma(\alpha_d) f_\sigma(\beta_d) = h(s) = (q-1) q^{s-1}.$$

Finally if $\sigma_0 = *$, then $f_\sigma(s) = (q-1)q^{s-1}$. If $\sigma_i = *$, $1 \leqslant i \leqslant d$ then

$$f_\sigma(\alpha_i) f_\sigma(\beta_i) = h(\alpha_i) h(\beta_i) = \begin{cases} (q-1)^2 q^{s-2} & \text{if } n \nmid \alpha_i, \\ 0 & \text{if } n \nmid \alpha_i. \end{cases}$$

Now note that the assumption that σ does not contain the string $\circ, \Box$ implies that $n_\Gamma = 0$, simplifying the definitions of χ_Γ and a_Γ above. The case of $\sigma_i = \Box$ is seen to account for the d_Γ defined above, the $\sigma_i = *$ account for both the χ_Γ and the a_Γ. However, one does need to count somewhat carefully at the ends of the accordion according to the above cases. In particular, we see that $\sigma_0 = \circ$ implies $\alpha_1 = s$, so that if j is the first index with $\sigma_j \neq \circ$, then $\sigma_j = *$ by assumption. But then $\alpha_j = s$ and the divisibility condition $n | \alpha_j$ is automatic, hence redundant and omitted from the definition. $\Box$

Lemma 10 *Given a signature σ which does not contain the sequence $\circ \Box$ and $d_\Gamma(\sigma)$ as defined in Proposition 21, we may write $d_\Gamma = k_\Gamma + \varepsilon_\Gamma + \sum_{i \in \mathcal{K}_\Gamma(\sigma)} \delta_n(i)$. Then for any m with $0 \leqslant m < d_\Gamma$*

$$\begin{aligned} \binom{d_\Gamma}{m} &= k_\Gamma + \binom{\varepsilon_\Gamma}{m} + \sum_{i \in \mathcal{K}_\Gamma(\sigma)} \delta_n(i) \binom{k_\Gamma + \varepsilon_\Gamma}{m-1} + \\ &\quad \cdots + \sum_{\{i_1, \ldots, i_l\} \subseteq \mathcal{K}_\Gamma(\sigma)} \delta_n(i_1, \ldots, i_l) \binom{k_\Gamma + \varepsilon_\Gamma}{m-l} + \cdots \end{aligned}$$

where we understand each of the binomial coefficients to be 0 if the lower entry is either negative or larger than the upper entry.

Proof The result follows from repeated application of the identity

$$\binom{c+\delta_n(i)}{m} = \binom{c}{m} + \delta_n(i)\binom{c}{m-1},$$

valid for any constants c and m and index i, While d_Γ contains divisibility conditions and hence depends on $\mathfrak{a}$, $k_\Gamma + \varepsilon_\Gamma$ is an absolute constant depending only on the signature σ. □

Theorem 11 *Fix a nodal signature η, and assume that $n|s$. Given an accordion $\mathfrak{a} \in \mathcal{S}_\sigma$ with subsigature $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_d) \subset \eta$,*

$$\Lambda_\Gamma(\mathfrak{a}, \sigma) = (-1)^{n_\Gamma+\varepsilon_\Gamma}\chi_\Gamma \sum_{x=0}^{k_\Gamma} \sum_{\substack{\Sigma \subseteq \mathcal{K}_\Gamma(\sigma) \\ |\Sigma| = x}} \delta_n(\Sigma)(-1)^x V(a_\Gamma + x, a_\Gamma + k_\Gamma), \qquad (15.4)$$

where the inner sum ranges over all possible subsets of cardinality x in $\mathcal{K}_\Gamma$.

Before giving the proof, let's do an example. Let

$$\sigma = (\square, *, \circ, \square, *, \square, *) \quad \subseteq \quad \eta = (\square, \circ, \circ, \square, \square, \square, \circ)$$

then one can read off the following data from the signature:

$$n_\Gamma = 1, \quad \varepsilon_\Gamma = 1, \quad \chi_\Gamma = \delta_n(1,3,4,6), \quad t_\Gamma = 3, \quad \mathcal{K}_\Gamma = \{5\}, \quad k_\Gamma = 1, \quad a_\Gamma = 8$$

so

$$\Lambda_\Gamma(\sigma) = \delta_n(1,3,4,6)\,(V(8,9) - \delta_n(5)V(9,9))$$

Proof We may express

$$V(a, a+b) = \sum_{u=0}^{b} (-1)^u \binom{b}{u} V(a+u, a+u) \qquad (15.5)$$

from the binomial theorem, given the definition $V(a,b) = (q-1)^a q^{(d+1)s-b}$.

From the definition we have

$$\Lambda_\Gamma(\mathfrak{a}, \sigma) = \mathcal{G}_\Gamma(\mathfrak{a}, \sigma) - \sum_{\sigma^{(1)}} \mathcal{G}_\Gamma(\mathfrak{a}, \sigma^{(1)}) + \ldots + (-1)^i \sum_{\sigma^{(i)}} \mathcal{G}_\Gamma(\mathfrak{a}, \sigma^{(i)}) + \ldots,$$

where the sums run over $\sigma^{(i)} \subseteq \sigma$ obtained from σ by replacing exactly i occurences of $\Box$ by $*$. We will apply Proposition 21 to evaluate these terms, then simplify.

If the sequence $\circ\Box$ appears within σ, then we call signature Γ-*non-strict*, as any corresponding short pattern $\mathfrak{T}$ with $\sigma \subseteq \sigma_{\mathfrak{T}}$ is non-strict, and by definition $\mathcal{G}_\Gamma(\mathfrak{T}, \sigma) = 0$. Thus the alternating sum for Λ_Γ will only contain non-zero contributions from subsignatures when all such $\Box$'s occurring as part of a $\circ\Box$ string in σ have been removed (i.e. changed to an $*$). Upon doing this, the signature will no longer possess any subwords of the form $\circ\Box$, and we may again apply the above formula for G_Γ to these subsignatures. This is reflected in the definition of a_Γ and in the statement of the Theorem.

We first assume that $\sigma_0 \neq \Box$ and that $\circ\Box$ does not occur within σ. Then

$$\begin{aligned}
G_\Gamma(\mathfrak{a}, \sigma) &= \chi_\Gamma V(a_\Gamma, a_\Gamma + d_\Gamma) = \chi_\Gamma \sum_{u=0}^{d_\Gamma} (-1)^u \binom{d_\Gamma}{u} V(a_\Gamma + u) \\
&= \chi_\Gamma \sum_{u=0}^{d_\Gamma} (-1)^u V(a_\Gamma + u) \sum_{l=0}^{k_\Gamma} \sum_{\{i_1,\dots,i_l\}:\sigma_{i_j}=\Box} \binom{k_\Gamma}{u-l} \delta_n(i_1, \dots, i_l) \\
&= \chi_\Gamma \sum_{l=0}^{k_\Gamma} \sum_{\{i_1,\dots,i_l\}:\sigma_{i_j}=\Box} \delta_n(i_1, \dots, i_l) \sum_{u=l}^{d_\Gamma} (-1)^u \binom{k_\Gamma}{u-l} V(a_\Gamma + u)
\end{aligned}$$

where we have used Lemmas 21 and 10, resp., in the first two steps, and in the last step have simply interchanged the order of summation.

By similar calculation (still assuming that $\sigma_0 \neq \Box$ for simplicity of exposition) we have

$$\begin{aligned}
\sum_{\sigma^{(m)} \subseteq \sigma} G_\Gamma(\mathfrak{a}, \sigma^{(m)}) &= \chi_\Gamma \sum_{\{i_1,\dots,i_m\}:\sigma_{i_j}=\Box} \delta_n(i_1, \dots, i_m) \sum_{l=0}^{k_\Gamma - m} \sum_{\{i'_1,\dots,i'_l\}:\sigma_{i'_j} \neq \sigma_{i_j}} \delta_n(i_1, \dots, i_l) \\
&\quad \times \sum_{u=l}^{d_\Gamma - 2m} (-1)^u \binom{k_\Gamma - m}{u-l} V(a_\Gamma + 2m + u)
\end{aligned}$$

where we can write the upper bound on the sum over u as an absolute constant, since either the divisibility conditions are satisfied and the upper bound (equal to $d_\Gamma(\sigma^{(m)})$) is indeed $d_\Gamma(\sigma) - 2m$ or else the term is 0. Simplifying by combining the two sums with divisibility conditions, we have

$$\sum_{\sigma^{(m)}\subseteq\sigma} G_\Gamma(\mathfrak{a},\sigma^{(m)}) = \chi_\Gamma \sum_{l=0}^{k_\Gamma-m} \sum_{\{i_1,\ldots,i_{m+l}\}:\sigma_{i_j}=\Box} \binom{m+l}{m}\delta_n(i_1,\ldots,i_{m+l}) (-1)^l \sum_{v=0}^{d_\Gamma-2m-l}(-1)^v\binom{k_\Gamma-m}{v}V(a_\Gamma+2m+l+v)$$

Hence

$$\begin{aligned}
\Lambda_\Gamma(\mathfrak{a},\sigma) &= \chi_\Gamma\sum_{m=0}^{k_\Gamma}(-1)^m\sum_{l=0}^{k_\Gamma-m}\sum_{\{i_1,\ldots,i_{m+l}\}:\sigma_{i_j}=\Box}\binom{m+l}{m}\delta_n(i_1,\ldots,i_{m+l})\\
&\quad (-1)^l\sum_{v=0}^{d_\Gamma-2m-l}(-1)^v\binom{k_\Gamma-m}{v}V(a_\Gamma+2m+l+v)\\
&= \chi_\Gamma\sum_{m=0}^{k_\Gamma}\sum_{x=m}^{k_\Gamma}\sum_{\{i_1,\ldots,i_x\}:\sigma_{i_j}=\Box}\binom{x}{m}\delta_n(i_1,\ldots,i_x)(-1)^x\\
&\quad \sum_{v=0}^{d_\Gamma-x-m}(-1)^v\binom{k_\Gamma-m}{v}V(a_\Gamma+m+x+v)\\
&= \chi_\Gamma\sum_{x=0}^{k_\Gamma}\sum_{\{i_1,\ldots,i_x\}:\sigma_{i_j}=\Box}\delta_n(i_1,\ldots,i_x)(-1)^x\\
&\quad \sum_{m=0}^{x}\binom{x}{m}\sum_{v=0}^{d_\Gamma-x-m}(-1)^v\binom{k_\Gamma-m}{v}V(a_\Gamma+m+x+v) \qquad (15.6)
\end{aligned}$$

where in the first step we changed the sum over l to a sum over $x = m + l$ and interchanged the order of summation in the second step. Now let $w = m + v$, so (15.6) equals

$$\begin{aligned}
&\sum_{m=0}^{x}(-1)^m\sum_{w=m}^{d_\Gamma-x}(-1)^w\binom{x}{m}\binom{k_\Gamma-m}{w-m}V(a_\Gamma+x+w) =\\
&\sum_{w=0}^{d_\Gamma-x}(-1)^wV(a_\Gamma+x+w)\sum_{m=0}^{w}(-1)^m\binom{x}{m}\binom{k_\Gamma-m}{w-m} \qquad (15.7)
\end{aligned}$$

But

$$\sum_{m=0}^{w}(-1)^m\binom{x}{m}\binom{k_\Gamma-m}{w-m} = \binom{k_\Gamma-x}{w} \qquad (15.8)$$

so combining (15.6) and (15.7) and applying (15.8)

$$\Lambda_\Gamma(\mathfrak{a},\sigma) =$$
$$\chi_\Gamma \sum_{m=0}^{k_\Gamma} (-1)^m \sum_{\{i_1,\ldots,i_{m+l}\}:\sigma_{i_j}=\Box} \delta_n(i_1,\cdots,i_x)(-1)^x \sum_{w=0}^{d_\Gamma - x} (-1)^w \binom{k_\Gamma - x}{w} V(a_\Gamma + x + w)$$

The cases where $\sigma_0 = \Box$ or where $\bigcirc\,\Box$ appears in the signature follow by a straightforward generalization. $\Box$

We turn now to the evaluation of $\Lambda_\Delta(\mathfrak{a}',\sigma)$, where σ is unchanged and

$$\mathfrak{a}' = \left\{ \begin{array}{ccccccccc} \beta_1 & & \beta_2 & & \cdots & & \beta_d & & s \\ & \alpha_1 & & \alpha_2 & & \cdots & & \alpha_d & \end{array} \right\}.$$

Let

$$\begin{array}{rcl}
\varepsilon_\Delta(\sigma) & = & \varepsilon_\Delta = \left\{ \begin{array}{ll} 1 & \text{if } \sigma_d = \Box, \\ 0 & \text{otherwise,} \end{array} \right. \\
\mathcal{K}_\Delta & = & \{0 < i \leqslant d \ \mid \ \sigma_{i-1} = \Box, \sigma_i \neq \bigcirc\}, \quad k_\Delta(\sigma) = k_\Delta = |\mathcal{K}_\Delta|, \\
\mathcal{N}_\Delta & = & \{0 < i \leqslant d \ \mid \ \sigma_{i-1} = \Box, \sigma_i = \bigcirc\}, \quad n_\Delta(\sigma) = n_\Delta = |\mathcal{N}_\Delta|, \\
\mathcal{C}_\Delta & = & \{1 \leqslant i \leqslant d \ |\sigma_i, \sigma_{i+1}, \cdots, \sigma_d \text{ not all } \bigcirc \text{ and either } \sigma_{i-1} = * \text{ or } i \in \mathcal{N}_\Delta\}, \\
\chi_\Delta(\mathfrak{a},\sigma) & = & \chi_\Delta = \displaystyle\prod_{i \in \mathcal{C}_\Delta(\sigma)} \delta_n(i), \qquad t_\Delta = |\{0 \leqslant i < d \ \mid \ \sigma_i = \eta_i\}| \\
a_\Delta(\sigma) & = & a_\Delta = 2(d - (t_\Delta - n_\Delta)) + \left\{ \begin{array}{cl} -1 & \text{if } \sigma_d = \bigcirc \\ 0 & \text{if } \sigma_d = \Box \\ 1 & \text{if } \sigma_d = * \end{array} \right\} + \left\{ \begin{array}{ll} 1 & \text{if } \sigma_0 = \bigcirc \\ 0 & \text{if } \sigma_0 \neq \bigcirc \end{array} \right\}.
\end{array}$$

We give $\delta_n(i_1,\cdots,i_n)$ the same meaning as before: it is $\delta_n(i_1,\cdots,i_k;\mathfrak{a})$. But since the top row of $\mathfrak{a}'$ is in terms of the β's, it is worth noting that it can also be described as 1 if n divides $\beta_{i_1},\cdots,\beta_{i_k}$ and 0 otherwise. Indeed, $n|s$ so $n|\alpha_i$ if and only if $n|\beta_i = s - \alpha_i$.

Theorem 12 *With notation as above we have*

$$\Lambda_\Delta(\mathfrak{a}',\sigma) \ =$$
$$(-1)^{n_\Delta + \varepsilon_\Delta} \chi_\Delta \sum_{x=0}^{k_\Delta} \sum_{\substack{\Sigma \subseteq \mathcal{K}_\Delta \\ |\Sigma| = x}} \delta_n(\Sigma)(-1)^x V(a_\Delta + x, a_\Delta + k_\Delta).$$

where (as defined above) the inner sums range over subsets of $\mathcal{K}_\Delta$.

Proof We can reuse our previous work by noting that

$$\Lambda_\Delta(\mathfrak{a}', \sigma) = \Lambda_\Gamma(\widetilde{\mathfrak{a}}, \tilde{\sigma}),$$

where

$$\widetilde{\mathfrak{a}} = \left\{ \begin{matrix} s & & \beta_d & & \beta_{d-1} & & \cdots & & \beta_1 \\ & \alpha_d & & \alpha_{d-1} & & \cdots & & \alpha_1 & \end{matrix} \right\}, \qquad \tilde{\sigma} = \sigma_d \sigma_{d-1} \cdots \sigma_0.$$

Roughly speaking we can just take the mirror image of our previous formula. But there is one point of caution: in going from $\mathfrak{a}$ to $\widetilde{\mathfrak{a}}$ we reflected σ in the range 0 to d, while we reflected α in the range 1 to d (and changed it to β, which has no effect on δ). This means the $\mathcal{C}_\Delta(\sigma)$, if it is to be the set of locations where the congruences are taken in evaluating δ, is *not* the mirror image of $\mathcal{C}_\Gamma(\widetilde{\mathfrak{a}})$ in the range 0 to d, but the shift of that mirror image to the right by 1, which makes $\mathcal{C}_\Delta(\sigma)$, like $\mathcal{C}_\Gamma(\sigma)$, a subset of the range from 1 to d. There are corresponding adjustments in the definitions of $\mathcal{K}_\Delta$ and $\mathcal{N}_\Delta$. □

Let $\mathcal{A}_\Gamma(\sigma)$ denote the set of Γ-admissible sets for σ, and we let $\mathcal{A}_\Delta(\sigma)$ denote the set of Δ-admissible sets. Let Π be an f-packet (as defined in Chapter 5 before Statement F). By Theorems 11 and 12 we may reformulate Statement F in the following way.

Statement G. *With notation as in Chapter 15, we have*

$$\sum_\sigma (-1)^{n_\Gamma(\sigma)+\varepsilon_\Gamma(\sigma)} \sum_{\substack{0 \leqslant x \leqslant k_\Gamma(\sigma) \\ \Sigma \in \mathcal{A}_\Gamma(\sigma) \\ |\Sigma - \mathcal{C}_\Gamma(\sigma)| = x}} (-1)^x V(a_\Gamma + x, a_\Gamma + k_\Gamma)\, \delta_n(\Sigma, \mathfrak{a}) \quad =$$

$$\sum_\sigma (-1)^{n_\Delta(\sigma)+\varepsilon_\Delta(\sigma)} \sum_{\substack{0 \leqslant x \leqslant k_\Gamma(\sigma) \\ \Sigma \in \mathcal{A}_\Gamma(\sigma) \\ |\Sigma - \mathcal{C}_\Gamma(\sigma)| = x}} (-1)^x V(a_\Delta + x, a_\Delta + k_\Delta)\, \delta_n(\Sigma, \mathfrak{a}) \qquad (15.9)$$

The outer sum is over f-subsignatures σ of η, since in (5.23) each such signature appears exactly once on each side. We recall that the packet Π in Statement F intersects each open f-facet $\mathcal{S}_\sigma$ in a unique element $\mathfrak{a}$, and so $\mathfrak{a}$ is determined by σ. We have restored $\mathfrak{a}$ to the notation $\delta_n(\Sigma; \mathfrak{a})$ from which it was suppressed in Theorems 11 and 12, because the dependence of these terms on $\mathfrak{a}$ – or, equivalently, on σ – will now become our most important issue.

Chapter 16

Concurrence

This chapter contains purely combinatorial results that are needed for the proof. The motivation of these results comes from the appearance of divisibility conditions through the factor $\delta_n(\Sigma; \mathfrak{a})$ defined in (15.2) that appears in Theorems 11 and 12. We refer to the discussion of Statement G in Chapter 5 for the context of the results of this Chapter.

Let $0 \leqslant f \leqslant d$. In Chapter 5 we defined bijections $\phi_{\sigma,\tau} : \mathcal{S}_\sigma \longrightarrow \mathcal{S}_\tau$ between the open f-facets, and a related equivalence relation, whose classes we call f-packets. According to Statement F, the sum of $\Lambda_\Gamma(\mathfrak{a}, \sigma)$ over an f-packet is equal to the corresponding sum of $\Lambda_\Delta(\mathfrak{a}', \sigma)$. Moreover in Theorems 11 and 12, we have rewritten Λ_Γ and Λ_Δ as sums over ordered subsets of $\mathcal{K}_\Gamma$ and $\mathcal{K}_\Delta$. In order to prove Statement F, we will proceed by identifying terms in the resulting double sum that can be matched, and that is the aim of the results of this chapter.

Definition 1 (Concurrence) *Let σ and τ be subsignatures of η that have the same number of $*$'s. Fix two subsets $\Sigma = \{j_1, \cdots, j_l\}$ and $\Sigma' = \{j'_1, \ldots, j'_l\}$ of $\{1, 2, \cdots, d\}$ of equal cardinality, and arranged in ascending order:*

$$0 \leqslant j_1 < j_2 < \cdots < j_l \leqslant d, \qquad 0 \leqslant j'_1 < j'_2 < \cdots < j'_l \leqslant d.$$

We say that the pairs (σ, Σ) and (τ, Σ') concur *if the following conditions are satisfied. We require that for $1 \leqslant m \leqslant l$ the two sets*

$$\{t \mid j_m \leqslant t \leqslant d, \sigma_t = *\}, \qquad \{t \mid j'_m \leqslant t \leqslant d, \tau_t = *\} \tag{16.1}$$

have the same cardinality, and that $\eta_i = \circ$ for

$$\min(j_m, j'_m) \leqslant i < \max(j_m, j'_m). \tag{16.2}$$

Concurrence is an equivalence relation.

Example 1 Let $\eta = (\eta_0, \eta_1, \ldots, \eta_5) = (\circ, \square, \circ, \circ, \square, \square)$. The pairs

$$((*, \square, *, \circ, *, \square), \{2, 4, 5\}); \qquad ((\circ, *, *, *, \square, \square), \{2, 3, 5\})$$

concur. However

$$((*, \square, *, \circ, *, \square), \{2, 4, 5\}); \qquad ((\circ, *, *, *, \square, \square), \{2, 4, 5\})$$

do not, as the number of $*$'s to the right of σ_4, τ_4 differ.

Proposition 22 *Suppose that the pairs* (σ, Σ) *and* (τ, Σ') *concur. Then if* $\phi_{\sigma,\tau}(\mathfrak{a}) = \mathfrak{b}$*, where*

$$\mathfrak{a} = \left\{ \begin{array}{ccccccc} s & & \alpha_1 & & \cdots & & \alpha_d \\ & \beta_1 & & \cdots & & \beta_d & \end{array} \right\}, \qquad \mathfrak{b} = \left\{ \begin{array}{ccccccc} s & & \mu_1 & & \cdots & & \mu_d \\ & \nu_1 & & \cdots & & \nu_d & \end{array} \right\},$$

we have $\alpha_{j_m} = \mu_{j'_m}$ $(1 \leqslant m \leqslant l)$.

This implies that

$$\delta_n(\Sigma; \mathfrak{a}) = \delta_n(\Sigma'; \mathfrak{b}),$$

which can be used to compare the contributions of these ordered subsets to $\Lambda_\Gamma(\mathfrak{a}, \sigma)$ and $\Lambda_\Delta(\mathfrak{a}', \sigma)$ with the corresponding contributions to $\Lambda_\Gamma(\mathfrak{b}, \tau)$ and $\Lambda_\Delta(\mathfrak{b}', \tau)$ in the formulas of Theorems 11 and 12.

Proof It is sufficient to check this when $\mathfrak{a}$ is a vertex of $\overline{\mathcal{S}_\sigma}$. Indeed, both α_{j_m} and $\mu_{j'_m}$ are affine-linear functions of $\alpha_1, \cdots, \alpha_d$, so if they are the same when $\mathfrak{a} = \mathfrak{a}_k$ is a vertex, they will be the same for convex combinations of the vertices, that is, for all elements of $\overline{\mathcal{S}_\sigma}$. Because $\mathfrak{a}_k$ is a vertex of $\overline{\mathcal{S}_\sigma}$, $\sigma_k = *$; if σ_k is the r-th $*$ in σ, then by definition $\phi_{\sigma,\tau}(\mathfrak{a}) = \mathfrak{a}_l$ where τ_l is the r-th $*$ in τ. This is a consequence of the definition of $\phi_{\sigma,\tau}$. Now our assumption on the cardinality of the two sets (16.1) implies that $k \leqslant j_m$ if and only if $l \leqslant j'_m$.

Now we prove that $\alpha_{j_m} = \mu_{j'_m}$. There are now two cases, depending on whether $j_m \leqslant k$ (and so $j'_m \leqslant l$) or not. First suppose that $j_m \leqslant k$ and $j'_m \leqslant l$. Then we have $\alpha_i - \alpha_{i+1} = c'_i$ for all i except k and $\mu_i = \mu_{i+1} = c'_i$ for all i except l, and $\alpha_0 = s = \mu_0$. This means that $\alpha_i = \mu_i$ when $i \leqslant \min(k, l)$, *a fortiori* when $i \leqslant \min(j_m, j'_m)$. Suppose for definiteness that $j_m \leqslant j'_m$, so $\min(j_m, j'_m) = j_m$. Thus we have proved that $\alpha_{j_m} = \mu_{j_m}$. Since by hypothesis $\eta_{j_m} = \eta_{j_m+1} = \cdots = \eta_{j'_m-1} = \circ$ we also have $\mu_{j_m} = \mu_{j_m+1} = \cdots \mu_{j'_m}$ and herefore $\alpha_{j_m} = \mu_{j'_m}$. The case where $j_m \geqslant j'_m$ is similar, and the case where $j_m \leqslant k$ and $j'_m \leqslant l$ is settled.

Next suppose that $j_m > k$ and so $j'_m > l$. Then $\alpha_i - \alpha_{i+1} = c'_i$ for all i except k and $\mu_i = \mu_{i+1} = c'_i$ for all i except l, and $\alpha_{d+1} = 0 = \mu_{d+1}$, we get $\alpha_i = \mu_i$ for $i > \max(k, l)$, *a fortiori* for $i > \max(j_m, j'_m)$. Suppose for definiteness that $j_m \leqslant j'_m$, so that $\max(j_m, j'_m) = j'_m$. We have prove that $\alpha_{j'_m} = \mu_{j'_m}$. Our hypothesis that $\eta_{j_m} = \eta_{j_m+1} = \cdots = \eta_{j'_m-1} = \bigcirc$ implies that $\alpha_{j_m} = \alpha_{j_m+1} = \cdots \alpha_{j'_m}$, and so we get $\alpha_{j_m} = \mu_{j'_m}$. The case $j_m \geqslant j'_m$ is again similar. □

We now introduce certain operations on signatures that give rise to concurrences.

Definition 2 (Γ- and Δ-swaps) *Let σ and τ be subsignatures of η. We say that τ is obtained from σ by a* Γ-swap *at $i-1, i$ if*

$$\sigma_j = \tau_j \quad \textit{for all } j \neq i-1, i, \quad \sigma_{i-1} = *, \sigma_i = \Box, \quad \tau_{i-1} = \bigcirc, \tau_i = *,$$

and by a Δ-swap *at $i-1, i$ if*

$$\sigma_j = \tau_j \quad \textit{for all } j \neq i-1, i, \quad \sigma_{i-1} = \Box, \sigma_i = *, \quad \tau_{i-1} = *, \tau_i = \bigcirc.$$

Definition 3 (Γ- and Δ-admissibility) *We say that a subset $\Sigma = \{j_1, j_2, \cdots, j_m\}$ of $\{1, 2, 3, \cdots, d\}$ is* Γ-admissible *for σ if*

$$\mathcal{C}_\Gamma(\sigma) \subset \Sigma \subset \mathcal{C}_\Gamma(\sigma) \cup \mathcal{K}_\Gamma(\sigma),$$

and similarly it is Δ-admissible *if $\mathcal{C}_\Delta(\sigma) \subset \Sigma \subset \mathcal{C}_\Delta(\sigma) \cup \mathcal{K}_\Delta(\sigma)$.*

Proposition 23 (Swapped data concur) *(a) Suppose τ is obtained from a Γ-swap at $i-1, i$. Assume that $i \notin \Sigma$. Let $0 < j_1 < j_2 < \cdots < j_l \leqslant d$ be a sequence such that $j_m \neq i$ for all m. Let*

$$j'_m = \begin{cases} j_m & \textit{if } j_m \neq i-1; \\ i & \textit{if } j_m = i-1. \end{cases}$$

Then (σ, Σ) and (τ, Σ') concur, where $\Sigma = \{j_1, \cdots, j_m\}$ and $\Sigma' = \{j'_1, \cdots, j'_m\}$. Moreover Σ is Γ-admissible for σ if and only if Σ' is Γ-admissible for τ.

(b) Suppose that τ is obtained from a Δ-swap at $i-1, i$. Assume that $i \notin \Sigma$. Let $0 < j_1 < j_2 < \cdots < j_l \leqslant d$ be a sequence such that $j_m \neq i$ for all m. Let

$$j'_m = \begin{cases} j_m & \textit{if } j_m \neq i+1; \\ i & \textit{if } j_m = i+1. \end{cases}$$

Then (σ, Σ) and (τ, Σ') concur, where $\Sigma = \{j_1, \cdots, j_m\}$ and $\Sigma' = \{j'_1, \cdots, j'_m\}$. Moreover Σ is Δ-admissible for σ if and only if Σ' is Δ-admissible for τ.

Proof This is straightforward to check from the definitions of concurrence and admissibility. One point merits further discussion. Suppose we are in case (a) for definiteness. If Σ is Γ-admissible, then according to the definition (15.1), $i-1 \in \mathcal{C}_\Gamma(\sigma) \subseteq \Sigma$ if $\sigma_0, \ldots, \sigma_{i-2}$ are not all $\circ$. In this case, $i \in \mathcal{C}_\Gamma(\tau) \subseteq \Sigma'$. If instead, $\sigma_0 = \cdots = \sigma_{i-2} = \circ$, then $i-1 \notin \Sigma$ but then under the Γ-swap, $\tau_0 = \cdots = \tau_{i-1} = \circ$ and so $i \notin \Sigma'$. $\square$

If the hypotheses of Proposition 23 are satisfied we say that (τ, Σ') is obtained from (σ, Σ) by a Γ-*swap* (or Δ-*swap*).

Let us define an equivalence relation on the set of pairs (σ, Σ), where σ is a subsignature of η and Σ is a Γ-admissible subset of $\{1, 2, \cdots, d\}$.

Definition 4 (Γ- and Δ-packs) *We write* $(\sigma, \Sigma) \sim_\Gamma (\tau, \Sigma)$ *if* (τ, Σ) *can be obtained by a sequence of* Γ*-swaps or inverse* Γ*-swaps. We call an equivalence class a* Γ-pack*; and* Δ-packs *are defined similarly.*

Lemma 11 *Each* Γ*-pack or* Δ*-pack contains a unique element with maximal number of* $\circ$ *'s. Within the pack, this unique element* (σ, Σ) *is characterized as follows.*

Γ-*pack:* *Whenever* $\eta_{i-1}\eta_i = \circ\square$ *and* $\sigma_{i-1}\sigma_i = *\square$ *we have* $i \in \Sigma$,

Δ-*pack:* *Whenever* $\eta_{i-1}\eta_i = \square\circ$ *and* $\sigma_{i-1}\sigma_i = \square *$ *we have* $i \in \Sigma$.

Proof If $\eta_{i-1}\eta_i = \circ\square$ and $\sigma_{i-1}\sigma_i = *\square$ then a Γ-swap is possible at $i-1, i$ if and only if $i \notin \Sigma$. Indeed, the fact that Σ is Γ-admissible for σ means that $i-1 \in \Sigma$. This assertion therefore follows from Proposition 23.

Clearly the element maximizing the number of $\circ$'s is obtained by making all possible swaps. The statements are now clear for the Γ-pack, and for the Δ pack they are similar. $\square$

Definition 5 (Origins) *We call the unique element with the greatest number of* $\circ$ *'s the* origin *of the pack. We say that* (σ, Σ) *is a* Γ-origin *if it is the origin of its* Γ*-pack, and* Δ-origins *are defined the same way.*

As we have explained, our goal is to exhibit a bijection ψ between the Γ-packs and the Δ-packs. It will be sufficient to exhibit a bijection between their origins. Let (σ, Σ) be the origin of a Γ-pack; we will denote $\psi(\sigma, \Sigma) = (\sigma', \Sigma')$. We can define σ' immediately. To obtain σ', we break η (which involves only $\square$'s and $\circ$'s) into maximal strings of the form $\circ \cdots \circ$ and $\square \cdots \square$, and we prescribe σ' on these ranges.

- **($\circ$'s in σ reflect across the midpoint of the string of $\circ$'s in η)** Let $\eta_h, \ldots, \eta_k$ be a maximal consecutive string of $\circ$'s in η (so $\eta_{h-1}, \eta_{k+1} \neq \circ$). If $h \leqslant i \leqslant k$ then $\sigma'_i = \sigma_{h+k-i}$.
- **(Distinguished $\Box$'s in σ slide one index leftward)** Let $\sigma_h \ldots \sigma_k$ be a maximal consecutive string of $\Box$'s in σ (so $\sigma_{h-1}, \sigma_{k+1} \neq \Box$). Let $h \leqslant i \leqslant k$ be the smallest element of Σ in this range, or if none exists, let $i = k+1$. Then if $\eta_{h-1} = \Box$ and $\sigma_{h-1} = *$ then $\psi(\sigma) = \sigma'$ has $\sigma'_{h-1} = \cdots \sigma'_{i-2} = \Box$, $\sigma'_{i-1} = *$, and $\sigma'_i = \cdots = \sigma'_k = \Box$. If either $\eta_{h-1} = \circ$ or $\sigma_{h-1} \neq *$, then ψ leaves the string of $\Box$'s in σ unchanged.

The last rule merits further explanation. Since σ is a subsignature of η, the maximal chain $\sigma_h \cdots \sigma_k$ of boxes in σ is contained in a (usually longer) maximal chain of boxes $\eta_l \eta_{l+1} \cdots \eta_m$ within η; thus $l \leqslant h$ and $m \geqslant k$ and the range from l to m is thus broken up into smaller ranges of which $\sigma_h \ldots \sigma_k = \Box \cdots \Box$ is one. We assume that $\sigma_{h-1} = *$ and that $\eta_{h-1} = \Box$. In this case we will modify $\sigma_h \cdots \sigma_k$. But if the condition that $\sigma_{h-1} = *$ and that $\eta_{h-1} = \Box$ is not met, we leave it unchanged – and the condition will be met if and only if $h > l$. Then with i as in the second rule above, we make the following shift:

$$\left\{ \begin{array}{lllllll} \sigma_{h-1} & \sigma_h & \cdots & \sigma_{i-1} & \cdots & \sigma_k \\ * & \Box & \cdots & \Box & \cdots & \Box \end{array} \right\} \longrightarrow \left\{ \begin{array}{lllllll} \sigma'_{h-1} & \sigma'_h & \cdots & \sigma'_{i-1} & \cdots & \sigma'_k \\ \Box & \Box & \cdots & * & \cdots & \Box \end{array} \right\}. \quad (16.3)$$

It is useful to divide up the nodal signature η into blocks of consecutive $\Box$'s alternating with blocks of consecutive $\circ$'s (where a block might consist of just one of these characters), e.g.

$$\eta = (\eta_0, \eta_1, \ldots, \eta_7) = (\Box, \circ, \circ, \underbrace{\Box, \Box, \Box}_{\text{block of } \Box\text{'s}}, \circ, \circ,).$$

Formally, a $\Box$-*block* is a maximal consecutive set $B = \{h, h+1, \cdots, k\}$ such that η_i are all $\Box$'s, and $\circ$-*blocks* are defined similarly. The map ψ can be understood according to what it does to the indices of σ contained within each of these blocks (and no two indices from different blocks interact under ψ). In particular, the number of $*$'s in σ contained within a block of η is preserved under ψ. We use this fact repeatedly in the proofs, as it often implies that it is enough to work locally within a block of $\Box$'s or $\circ$'s.

We have not yet described what ψ does to Σ. The next result will make this possible. Define

$$\begin{array}{rcl} P_\sigma(u) & = & |\{j \geqslant u \mid \sigma_j = *\}|, \\ Q_\sigma(u) & = & |\{j \geqslant u \mid \sigma_j = \Box\}|. \end{array}$$

If $u, v \in \{1, 2, \cdots, d\}$ then we say that the pair (u, v) is *equalized* for σ and σ' if

$$P_\sigma(u) = P_{\sigma'}(v), \qquad Q_\sigma(u) = Q_{\sigma'}(v). \tag{16.4}$$

Lemma 12 *Let σ, σ' be signatures with $\psi(\sigma') = \sigma$.*

(i) If $1 \leqslant u \leqslant d$ and $\eta_u \neq \eta_{u-1}$ then (u, u) is equalized.

(ii) Suppose that B is a $\circ$-block and that $u \in B$ such that $\sigma_u = *$. Assume that $\sigma_j \neq \circ$ for some $j < u$. Then there exists $0 < v \in B$ such that $\sigma'_v = *$ and (u, v) is equalized.

(iii) Suppose that B is a $\circ$-block and that $v \in B$ such that $\sigma'_v = *$ but $\sigma'_{v-1} \neq \circ$. Then there exists $0 < u \in B$ such that (u, v) is equalized, $\sigma_u = *$ and $\sigma_j \neq \circ$ for some $j < u$.

(iv) Given i as in the second rule for ψ on signatures, the pair (i, i) is equalized.

The condition in (ii) and (iii) that $\sigma_u = *$ and $\sigma_j \neq \circ$ for some $j < u$ means that $u \in \mathcal{C}_\Gamma(\sigma)$.

Proof Part (i) follows from the fact that u is at the left edge of a block when $\eta_u \neq \eta_{u-1}$. Indeed, if B is a $\square$- or $\circ$-block then σ and σ' have the same number of $*$'s and $\square$'s in B. Since u is at the left edge of a block, then the accumulated numbers of $*$ and $\square$ in that block and those to the right are the same for σ and σ' and so (u, u) is equalized.

To prove (ii), observe that the number of $*$ in the $\circ$-block $B = \{h, h+1, \cdots, k\}$ are the same, and $(k+1, k+1)$ are equalized (or else $k = d$), so counting from the right, if σ_u is the r-th $*$ within the block, we can take σ'_v to be the r-th $*$ for σ' in the block, and we have equalization. The hypothesis that $\sigma_u \neq \circ$ for some $j < u$ guarantees that either B is not the first block, or that σ_u is not the leftmost $*$ in the block, so $v > 0$.

To prove (iii), we argue the same way, and the only thing to be checked is that $j > 0$ and that $\sigma_j \neq \circ$ for some $j < u$. This follows from the assumption that $\sigma'_{v-1} \neq \circ$, since if $\sigma'_{v-1} = \square$ then B is not the first block, while if $\sigma'_{v-1} = *$ then σ'_v is not the first $*$ in B for σ', hence also not the first $*$ in B for σ.

For (iv), the $\square$-block containing i can be broken up into segments of the form $*\square \cdots \square$ as in the left-hand side of (16.3) and possibly an initial string consisting entirely of $\square$'s. According to the second rule for ψ on signatures, the image of each such segment under ψ also contains exactly one $*$ (excluding the possible initial string of $\square$'s without $*$'s) and the same number of $\square$'s. As i occurs to the right of both the $*$ in σ and σ' in the respective segments as depicted in (16.3), it is thus clear that (i, i) is equalized. $\square$

Proposition 24 (Concurrence of origins) *Let (σ, Σ) be the origin of a Γ-pack, and let $\sigma' = \psi(\sigma)$. Given any $j \in \Sigma$ we can associate a corresponding index $j' \in \Sigma'$ as follows. There exists a unique $1 \leqslant t = t(j) \leqslant d$ such that (j, t) are equalized, and such that $\sigma'_t \neq \circ$. Define $j' = \psi(j)$ so that $j' - 1$ is the largest index $< t(j)$ such that $\sigma'_{j'-1} \neq \circ$. Then the $\Sigma' = \{\psi(j) | j \in \Sigma\}$ is Δ-admissible for σ', and in fact (σ', Σ') is a Δ-origin. Moreover the pairs (σ, Σ') and (σ', Σ') concur. The map $\psi : (\sigma, \Sigma) \longmapsto (\sigma', \Sigma')$ is a bijection from the set of Γ-origins to the set of Δ-origins.*

Before proving this, we give several examples.

1. If $\eta = (\circ, \circ, \circ, \circ, \circ, \circ, \Box)$,

$$\psi((\circ, \circ, *, \circ, *, *, *), \{4, 5, 6\}) \mapsto ((*, *, \circ, *, \circ, \circ, *), \{1, 2, 4\})$$

   Indeed, ψ reflects all entries in the initial block of 6 $\circ$'s in η. In the block consisting of a single $\Box$ at the end of η, σ contains no $\Box$'s and so σ' agrees with σ on this block. The reader will check that that $t(6) = 4, t(5) = 2$, and $t(4) = 1$. Thus Σ' is as defined in the Proposition. To check that Σ is Γ-admissible for σ, note that $\mathcal{C}_\Gamma(\sigma) = \{4, 5, 6\}$ and $\mathcal{K}_\Gamma(\sigma) = \varnothing$ so indeed Σ is to be of form $\mathcal{C}_\Gamma(\sigma) \cup \Phi$ where Φ is a (possibly empty) subset of $\mathcal{K}_\Gamma(\sigma)$. Moreover, we wanted to ensure that Σ' is of the form $\Sigma' = \mathcal{C}_\Delta(\sigma') \cup \Phi'$ where $\Phi' \subseteq \mathcal{K}_\Delta(\sigma')$. Referring back to the definitions of these sets in (15.9) and (15.9), we see that $\mathcal{C}_\Delta(\sigma') = \{1, 2, 4\}$ so we satisfy the necessary condition. (For the record, $\mathcal{K}_\Delta(\sigma') = \varnothing$ in this case.) Finally, no Γ-swaps or Δ-swaps are possible so (σ, Σ) is a Γ-origin and (σ', Σ') is a Δ-origin.

2. If $\eta = (\Box, \Box, \circ, \circ, \Box, \Box, \circ, \Box, \circ)$,

$$\psi((\Box, *, \circ, \circ, \Box, \Box, *, *, \circ), \{1, 4, 5, 6, 7\}) \mapsto$$

$$((\Box, *, \circ, \circ, \Box, \Box, *, *, \circ), \{1, 2, 5, 6, 7\})$$

   Note that there is no change in the signature from σ to σ' as no $\Box$'s can move left in the strings of $\Box$'s contained in η, and reflection in strings of $\circ$'s leaves these strings unchanged. The index sets are more interesting. From the definitions, we compute that $\mathcal{C}_\Gamma(\sigma) = \{1, 4, 6, 7\}$, $\mathcal{K}_\Gamma(\sigma) = \{5\}$, $\mathcal{C}_\Delta(\sigma') = \{2, 7\}$, and $\mathcal{K}_\Delta(\sigma') = \{1, 5, 6\}$. This illustrates that these sets may have very different cardinalities. We see that the sets Σ and Σ' are admissible.

3. If $\eta = (\circ, \Box, \circ, \Box, \Box, \Box, \Box, \circ)$,

$$\psi((\circ, \Box, \circ, *, \Box, \Box, \Box, *), \{3, 6, 7\}) \mapsto ((\circ, \Box, \circ, \Box, \Box, *, \Box, *), \{2, 6, 7\})$$

Here, the blocks of circles are all of length 1, so $*$'s and $\circ$'s in σ within these blocks do not change under ψ in σ'. We have $\sigma_2 = \square$, but $2 \in \mathcal{N}_\Gamma(\sigma) \subseteq \Sigma$ so this $\square$ remains fixed in σ'. In the block of 4 $\square$'s, we see σ contains 3 $\square$'s. The smallest index from this string which is in Σ is 6, corresponding to the last $\square$. So the first two $\square$'s move left, and the third remains fixed. Again, from the definitions, we compute that $\mathcal{C}_\Gamma(\sigma) = \{3,7\}$, $\mathcal{K}_\Gamma(\sigma) = \{4,5,6\}$, $\mathcal{C}_\Delta(\sigma') = \{2,6\}$, and $\mathcal{K}_\Delta(\sigma') = \{4,5,7\}$, so Σ and Σ' are admissible.

Proof The first thing to check is that if $j \in \Sigma$ we may find v such that (j,v) are equalized. (If $j \notin \Sigma$ this may not be true.) There may be several possible v's, if σ' has $\circ$'s in the vicinity, and t will be the smallest. So the existence of v is all that needs to be proved – the condition that $\sigma'_t \neq \circ$ has the effect of selecting the smallest, so that t will be uniquely determined.

If B is a $\circ$-block, then the existence of v is guaranteed by Lemma 12. If B is a $\square$-block, then B can be broken into segments in which σ and σ' are as follows. There is an initial segment (possibly empty) of $\square$'s that is common to both σ and σ', and the remaining segments look like this:

$$\left\{\begin{array}{cccccc} \sigma_l & \sigma_{l+1} & \sigma_{l+1} & \cdots & \sigma_{m-1} & \sigma_m \\ * & \square & \square & \cdots & \square & \square \end{array}\right\} \xrightarrow{\psi} \left\{\begin{array}{cccccc} \sigma'_l & \sigma'_{l+1} & \cdots & \sigma'_{m-2} & \sigma'_{m-1} & \sigma'_m \\ \square & \square & \cdots & \square & \square & * \end{array}\right\}.$$

We claim that the only possible element of Σ in $\{l, l+1, \cdots, m\}$ is l. The reason is that if there was an element i of Σ in the range $\{l+1, \cdots, m\}$ the prescription for σ' would move the $*$ to $i-1$, and this is not the case. Now $(m+1, m+1)$ are equalized (or $m = d$) and it follows that (l,l) are equalized. So we have the case $j = l$, and then we can take $v = l$ also. It is easy to see that if j lies in the initial segment (if nonempty) that consists entirely of $\square$'s that we may take $u = j$ in this case also.

This proves that t satisfying (16.4) exists.

We will make use of the following observation.

$$\text{If } \eta_{l'-1}\eta_{l'} = \circ\square \text{ and } \sigma'_{j'-1} = * \text{ for some } j' \leqslant l' \text{ then } l' \in \Sigma \text{ and } t(l') = l'. \qquad (16.5)$$

To prove this, note that if $\sigma_{l'} = *$ or if $\sigma_{l'} = \square$ and $\sigma_{l'-1} = \circ$ then $l' \in \mathcal{C}_\Gamma(\sigma) \subseteq \Sigma$. The fact that $\sigma_i \neq \circ$ for some $i < l'$, needed here for the definition of $\mathcal{C}_\Gamma(\sigma)$, may be deduced from the fact that $\sigma'_{j'-1} \neq \circ$ since it means that the $\circ$-block containing j' either is not the first block, or else contains some $*$'s for σ' and hence also for σ. Since $\eta_{l'-1}\eta_{l'} = \circ\square$ this leaves only the case $\sigma_{l'-1}\sigma_{l'} = *\square$, and in this case $l' \in \Sigma$ follows from the fact that σ is a Γ-origin by Lemma 13. Now (l', l') is equalized by Lemma 12 (i), and so $t(l') = l'$. This proves (16.5).

Now we need to check that $\Sigma' = \{j'_1, j'_2, \cdots\}$ is Δ-admissible, that is, $\mathcal{C}_\Delta(\sigma') \subseteq \Sigma' \subseteq \mathcal{C}_\Delta(\sigma') \cup \mathcal{K}_\Delta(\sigma')$. That $\Sigma' \subseteq \mathcal{C}_\Delta(\sigma') \cup \mathcal{K}_\Delta(\sigma')$ is almost immediate, as the set $\mathcal{C}_\Delta(\sigma') \cup \mathcal{K}_\Delta(\sigma')$ contains every index j' with $\sigma'_{j'-1} = \square$ or $*$, so long as $\sigma'_{i'} \neq \circ$ for some $i' \geqslant j'$. Since each element $\psi(j) = j' \in \Sigma'$ with $j \in \Sigma$ has $\sigma'_{j'-1} = \square$ or $*$, we need only check that $\sigma'_{i'} \neq \circ$ for some $i' \geqslant j'$. This is clear since $j \in \mathcal{C}_\Gamma(\sigma) \cup \mathcal{K}_\Gamma(\sigma)$ so $P_\sigma(j)$ or $Q_\sigma(j)$ is positive, and hence $P_{\sigma'}(t)$ or $Q_{\sigma'}(t)$ is positive, and $j' \leqslant t$.

We next show that Σ' contains $\mathcal{C}_\Delta(\sigma')$. Thus to each $j' \in \mathcal{C}_\Delta(\sigma')$ we must find $j \in \Sigma$ such that $\psi(j) = j'$.

First assume that $\sigma'_{j'} = \circ$. By definition of $\mathcal{C}_\Delta(\sigma')$ we have $\sigma'_{j'-1} \neq \circ$. Also by definition of $\mathcal{C}_\Delta(\sigma')$ there will be some $l' > j'$ such that $\sigma'_{l'} \neq \circ$. Let l' be the smallest such value. Suppose that $\eta_{l'} = \circ$. Then $\sigma'_{l'} = *$. Since l' is the smallest value $> j'$ such that $\sigma'_{l'} \neq \circ$ we have $\sigma'_i = \circ$ and hence $\eta'_i = \circ$ for $j' \leqslant i < l'$ and so the entire range $j' \leqslant i \leqslant l'$ is contained within the same $\circ$-block B. By Lemma 12 (iii) there exists $j \in B$ such that $\sigma_j = *$ and j, j' are equalized, and moreover, $\sigma_i \neq \circ$ for some $i < j$. Thus $j \in \mathcal{C}_\Gamma(\sigma)$ so $j \in \Sigma$ and $t(j) = j'$, so $\psi(j) = j'$ (because $\sigma'_{j'-1} \neq \circ$). Thus we may assume that $\eta_{l'} = \square$. We note that $\eta_{l'-1} = \circ$ since $\sigma'_{l'-1} = \circ$. Thus $l' \in \Sigma'$ and $t(l') = l'$ by (16.5). Since $\sigma'_{j'} = \sigma'_{j'-1} = \ldots = \sigma'_{l'-1} = \circ$ but $\sigma'_{j'-1} \neq \circ$ we have $\psi(l') = j'$. This finishes the case $\sigma'_{j'} = \circ$.

Next suppose that $\sigma'_{j'} \neq \circ$. Then $\sigma'_{j'-1} = *$ since $j' \in \mathcal{C}_\Delta(\sigma')$. If $\eta_{j'} = \circ$ then $\sigma'_{j'}$ must be $*$. The assumption that $j' \in \mathcal{C}_\Delta(\sigma')$ then implies that $\sigma'_{j'-1} = *$ also and so Lemma 12 (iii) implies that $t(j) = j'$ for some j in the same $\circ$-block as j', with $j \in \mathcal{C}_\Gamma(\sigma) \subseteq \Sigma$. Then since $\sigma'_{j'-1} \neq \circ$ we have $\psi(j) = j'$. Thus we may assume that $\eta_{j'} = \square$. In this case we will show that $j' \in \Sigma$ and $\psi(j') = j'$ If $\eta_{j'-1}\eta_{j'} = \square\square$ then since $\sigma'_{j'-1} = *$ it follows from the description of σ' in $\square$-blocks (see (16.3)) that $j' \in \Sigma$, and by Lemma 12 (iii), (j', j') is equalized, so $t(j') = j'$ and so since $\sigma'_{j'-1} = *$ it follows that $\psi(j') = j'$. On the other hand if $\eta_{j'-1}\eta_{j'} = \circ\square$ then we still have $j' \in \Sigma$ by (16.5), and since $\sigma'_{j'-1} = *$ we have $\psi(j') = j'$. This completes the proof that Σ' contains $\mathcal{C}_\Delta(\sigma')$.

Now we know that Σ' is Δ-admissible for σ'. Next, we show that (σ', Σ') is a Δ-origin. We must show that if $\eta_{j'-1}\eta_{j'} = \square\circ$ and $\sigma'_{j'-1}\sigma'_{j'} = \square *$ then $j' \in \Sigma'$. It follows from Lemma 12 that there exists j in the same $\circ$-block as j' such that $\sigma_j = *$ and $j \in \mathcal{C}_\Gamma(\sigma)$, and (j, j') are equalized. Then $t(j) = j'$ and since $\sigma_{j'-1} \neq \circ$ we have $\psi(j) = j'$. Thus $j' \in \Sigma'$.

Next we observe that (σ, Σ) and (σ', Σ') concur. To see this, observe first that if $j \in \Sigma$ and $j' = \psi(j) \in \Sigma'$, then (j, j') is equalized. This implies that the two sets (16.1) have the same cardinality (with $\tau = \sigma'$). Moreover, if j is in a $\circ$-block, then j' is in the same block, while if j is in a $\square$-block then $j' = j$ with the exception that if j is the left-most element of a $\square$-block, then j' can lie in the $\circ$-block to the left.

These considerations show that $\eta_i = \circ$ when (16.2) is satisfied. Therefore σ and σ' concur.

We see that ψ maps Γ-origins to Δ-origins. To establish that it is a bijection between Γ-origins and Δ-origins, we first note show the map ψ from Γ-origins to Δ-origins is injective. Indeed, we can reconstruct σ and Σ from σ' and Σ' as follows. On $\circ$-blocks, the reconstruction is straightforward – the signature is just reversed on each Δ-block, and the elements of Σ within a Δ-block are just the values where $\sigma_j = *$, except that if $\sigma_i = \circ$ for all $i < j$ then j is omitted from σ. On $\Box$-blocks, the reconstruction of Σ must preceed the reconstruction of σ. It follows from the preceding discussion that on the intersection of Σ with a $\Box$-block, ψ is the identity map except that if the very first element of the block is in Σ, ψ can move it into the preceding $\circ$-block. Thus if $j \in \{1, 2, \cdots, d\}$ and $\eta_j = \Box$ we can tell if j is in Σ as follows. If j not the first element on its block then $j \in \Sigma$ if and only if $j \in \Sigma'$. If j is the first element, then $j \in \Sigma$ if and only if $j \in \Sigma'$ or (from the definition of $\mathcal{C}_\Gamma$) if $\sigma_{j-1} = \circ$ – and we recall that the signature is already known on $\circ$-blocks. Once Σ is known on $\Box$-blocks, σ can be reconstructed by reversing the process that gave us σ'.

Since the map ψ is injective, we need only check that the number of Δ-origins equals the number of Γ-origins. We extend ψ to a larger set by including η as part of the data: let Ω_Γ be the set of all triples (η, σ, Σ) such that η is a nodal signature, σ a subsignature, and Σ a Γ-origin for σ; and similarly we define Ω_Δ. Then ψ gives an injection $\Omega_\Gamma \longrightarrow \Omega_\Delta$. It will follow that ψ is a bijection if we show that the two sets have the same cardinality. A naive bijection between the two sets can be exhibited as follows. Let (η, σ, Σ) be given. Define $(\tilde{\eta}, \tilde{\sigma}, \tilde{\Sigma})$ by $\tilde{\eta}_i = \eta_{d-i}$, $\tilde{\sigma}_i = \sigma_{d-i}$, and $\tilde{\Sigma} = \{d+1-j | j \in \Sigma\}$. Note that η and σ are reversed in the range from 0 to d, while Σ is reversed in the range from 1 to d. Then $(\tilde{\sigma}, \tilde{\Sigma})$ is a Δ-origin if and only if (σ, Σ) is a Γ-origin, and so $|\Omega_\Gamma| = |\Omega_\Delta|$. $\Box$

Chapter 17

Proof of Statement G

In Chapter 15 we reduced the proof to Statement G, given at the end of that Chapter, and we now have the tools to prove it.

Lemma 13 *The cardinality of each Γ-pack or Δ-pack is a power of 2.*

Proof In a Γ-swap $*\Box$ is replaced by $\circ *$ in the signature. Since both signatures are subsignatures of η, this means that η has $\circ\Box$ at this location. From this it is clear that if a Γ-swap is possible at $i-1, i$ then no swap is possible at $i-2, i-1$ or $i, i+1$, and so the swaps are independent. Thus the cardinality of the pack is a power of 2. $\Box$

Given an origin (σ, Σ) for a Γ-equivalence class, define $p_\Gamma(\sigma, \Sigma) = k$ where 2^k is the cardinality of the Γ-pack to which the representative (σ, Σ) belongs. We may similarly define p_Δ for Δ-packs.

Proposition 25 *Let (σ, Σ) be an origin for a Γ-equivalence class. Then $p_\Gamma(\sigma, \Sigma)$, as defined above, can be given explicitly by*

$$p_\Gamma(\sigma, \Sigma) = \Big| \{ i \in \{1, 2, \cdots, d\} \mid (\sigma_{i-1}, \sigma_i) = (\circ, *), \eta_i = \Box \} \Big|. \tag{17.1}$$

Proof Recall that elements of a Γ-pack differ by a series of Γ-swaps from (τ, T) to (σ, Σ), which change $\tau_{i-1}, \tau_i = *, \Box$ to $\sigma_{i-1}, \sigma_i = \circ, *$ provided $i \notin T$. Hence $p_\Gamma(\sigma, \Sigma)$ is clearly at most the number of indices satisfying the condition on the right-hand side of (17.1).

Given an origin (σ, Σ), let $\tau \subseteq \eta$ be any subsignature possessing an $(*, \Box)$ at (τ_{i-1}, τ_i) where σ has a $(\circ, *)$ at (σ_{i-1}, σ_i). Let T be the set of indices obtained from Σ by replacing each such $i \in \Sigma$ by $i-1$ (and leaving all other indices unchanged). We claim that $(\tau, T) \sim_\Gamma (\sigma, \Sigma)$. By our discussion above, it suffices to show that

$i \notin T$. Indeed, by assumption, $(\sigma_i, \sigma_{i+1}) \neq (\circ, *)$ so $i+1$ is not changed to i from Σ to T according to our rule. Moreover, $i \in \Sigma$ is sent to $i-1 \in T$. Hence (17.1) follows. □

Lemma 14 *Let E_Γ be a Γ-pack with origin (σ, Σ). Let $\Sigma = \mathcal{C}_\Gamma(\sigma) \cup \Phi$ with $\Phi \subseteq \mathcal{K}_\Gamma(\sigma)$ and let $x = |\Phi|$. Then*

$$\sum_{(\sigma,\Sigma)\in E_\Gamma} (-1)^\epsilon\, V(a_\Gamma(\sigma)+x, a_\Gamma(\sigma)+k_\Gamma(\sigma)) \;=\; (-1)^\epsilon V(a_\Gamma(\sigma)+x+p_\Gamma(\sigma,\Sigma), a_\Gamma(\sigma)+k_\Gamma(\sigma)+p_\Gamma(\sigma,\Sigma)),$$

where $\epsilon = \epsilon(\sigma)$ is the number of $\circ$ in σ and ϵ is the number of $\circ$ in σ.

Proof It is easy to see that a Γ-swap does not change $a_\Gamma(\sigma)$, while it decreases $k_\Gamma(\sigma)$ by 1. Thus repeatedly applying the identity

$$V(a,b) - V(a,b+1) = V(a+1,b+1)$$

gives this result. □

Lemma 15 *Let E_Δ be a Δ-pack with origin (σ, Σ). Let $\Sigma = \mathcal{C}_\Delta(\sigma) \cup \Phi$ with $\Phi \subseteq \mathcal{K}_\Gamma(\sigma)$ and let $x = |\Phi|$. Then*

$$\sum_{(\sigma,\Sigma)\in E_\Delta} (-1)^\epsilon\, V(a_\Delta(\sigma)+x, a_\Delta(\sigma)+k_\Delta(\sigma)) \;=\; (-1)^\epsilon V(a_\Delta(\sigma)+x+p_\Delta(\sigma,\Sigma), a_\Delta(\sigma)+k_\Delta(\sigma)+p_\Delta(\sigma,\Sigma)),$$

where $\epsilon = \epsilon(\sigma)$ is the number of $\circ$ in σ and ϵ is the number of $\circ$ in σ.

Proof Similar to the proof of Lemma 14. □

Theorem 13 *Let ψ be the bijection on equivalence classes given above, let (σ, Σ) be a Γ-origin and let $\psi(\sigma,\Sigma) = (\sigma', \Sigma')$ be the corresponding Δ-origin. Write $\Sigma = \mathcal{C}_\Gamma(\sigma) \cup \Phi$ with $\Phi \subseteq \mathcal{K}_\Gamma(\sigma)$ and similarly, $\Sigma' = \mathcal{C}_\Delta(\sigma') \cup \Phi'$ with $\Phi' \subseteq \mathcal{K}_\Delta(\sigma')$. Then*

$$V(a_\Gamma(\sigma)+|\Phi|+p_\Gamma(\sigma,\Sigma), a_\Gamma(\sigma)+k_\Gamma(\sigma)+p_\Gamma(\sigma,\Sigma)) \;=\; V(a_\Delta(\sigma')+|\Phi'|+p_\Delta(\sigma',\Sigma'), a_\Delta(\sigma')+k_\Delta(\sigma')+p_\Delta(\sigma',\Sigma')). \quad (17.2)$$

Proof First we will prove the equality of the second components

$$a_\Gamma(\sigma) + k_\Gamma(\sigma) + p_\Gamma(\sigma, \Sigma) = a_\Delta(\sigma') + k_\Delta(\sigma') + p_\Delta(\sigma', \Sigma'). \tag{17.3}$$

Consider the left-hand side of (17.3). The quantities $k_\Gamma(\sigma)$ and $a_\Gamma(\sigma)$ are defined in Chapter 15 before Proposition 21, where the latter is further defined in terms of $n_\Gamma(\sigma)$ and $t_\Gamma(\sigma)$. Hence, the left-hand side of (17.3) is given by

$$2(d - (t_\Gamma - n_\Gamma)) + \left\{ \begin{array}{cc} -1 & \sigma_0 = \bigcirc \\ 0 & \sigma_0 = \Box \\ 1 & \sigma_0 = * \end{array} \right\} + \left\{ \begin{array}{cc} 1 & \sigma_d = \bigcirc \\ 0 & \sigma_d \neq \bigcirc \end{array} \right\}$$
$$+ \Big| \{ i \in [1, d] \mid \sigma_i = \Box, \sigma_{i-1} \neq \bigcirc \} \Big| + \Big| \{ i \in [1, d] \mid (\sigma_{i-1}, \sigma_i) = (\bigcirc, *), \eta_i = \Box \} \Big|,$$

where $[a, b]$ denotes $\{x \in \mathbb{Z} \mid a \leqslant x \leqslant b\}$. Now

$$2d - 2t_\Gamma + \left\{ \begin{array}{cc} -1 & \sigma_0 = \bigcirc \\ 0 & \sigma_0 = \Box \\ 1 & \sigma_0 = * \end{array} \right\} = 2d + 1 - 2BC(\sigma) + \left\{ \begin{array}{cc} 1 & \sigma_0 = \Box \\ 0 & \sigma_0 \neq \Box \end{array} \right\}$$

where $BC(\sigma) = |\{i \in [0, d] \mid \sigma_i \neq *\}|$ is the total number of boxes and circles in σ. Also the quantity $2n_\Gamma$ contributes a 2 for each $i \in [1, d]$ with $(\sigma_{i-1}, \sigma_i) = (\bigcirc, \Box)$. We may regard this 2 as contributing 1 for each $\Box$ preceded by a $\bigcirc$ and 1 for each $\bigcirc$ followed by a $\Box$. From this it follows that

$$\begin{aligned} 2n_\Gamma + \Big| \{ i \in [1, d] \mid \sigma_i = \Box, \sigma_{i-1} \neq \bigcirc \} \Big| + \Big| \{ i \in [1, d] \mid (\sigma_{i-1}, \sigma_i) = (\bigcirc, *), \eta_i = \Box \} \Big| \\ = \Big| \{ i \in [1, d] \mid \sigma_i = \Box \} \Big| + \Big| \{ i \in [0, d-1] \mid \sigma_i = \bigcirc, \eta_{i+1} = \Box \} \Big|. \end{aligned}$$

Combining terms, we see that the left hand side of (17.3) is the sum of the two terms

$$2d + 1 - 2BC(\sigma) + \Big| \{ i \in [0, d] \mid \sigma_i = \Box \} \Big| \tag{17.4}$$

and

$$\Big| \{ i \in [0, d-1] \mid \sigma_i = \bigcirc, \eta_{i+1} = \Box \} \Big| + \left\{ \begin{array}{cc} 1 & \sigma_d = \bigcirc \\ 0 & \sigma_d \neq \bigcirc \end{array} \right\}. \tag{17.5}$$

Similarly, the right hand side of (17.3) is the sum of the two terms

$$2d + 1 - 2BC(\sigma') + \Big| \{ i \in [0, d] \mid \sigma'_i = \Box \} \Big| \tag{17.6}$$

and

$$\Big| \{ i \in [1, d] \mid \sigma'_i = \bigcirc, \eta_{i-1} = \Box \} \Big| + \left\{ \begin{array}{cc} 1 & \sigma'_0 = \bigcirc \\ 0 & \sigma'_0 \neq \bigcirc \end{array} \right\}. \tag{17.7}$$

Now, since the map ψ preserves the number of boxes and the number of circles in the signature, we have

$$BC(\sigma) = BC(\sigma')$$

and

$$\big|\,\{i \in [0,d] \mid \sigma_i = \square\}\,\big| = \big|\,\{i \in [0,d] \mid \sigma'_i = \square\}\,\big|.$$

Hence the quantities (17.4) and (17.6) are equal. The quantity (17.5) counts the number of $i \in [0,d]$ such that $\sigma_i = \circ$, $\eta_{i+1} \neq \circ$ (this includes the possibility that $i = d$ and η_{i+1} is not defined). But ψ reflects the entries of σ lying over strings of $\circ$'s in η. After doing so, each such i reflects to a $\circ$ in σ' that is preceeded by a $\square$ (or is initial) in η. These are exactly the indices counted by (17.7). Hence they are equal.

This completes the proof of (17.3). To finish the proof of the theorem, we must show that

$$a_\Gamma(\sigma) + |\Phi| + p_\Gamma(\sigma, \Sigma) = a_\Delta(\sigma') + |\Phi'| + p_\Delta(\sigma', \Sigma').$$

By the construction of the bijection ψ, we have

$$c_\Gamma(\sigma) + |\Phi| = c_\Delta(\sigma') + |\Phi'|,$$

since these count the number of divisibility conditions, and this number is necessarily constant when the bijection obtains. In view of (17.3), it thus suffices to establish

$$c_\Gamma(\sigma) + k_\Gamma(\sigma) = c_\Delta(\sigma') + k_\Delta(\sigma'). \tag{17.8}$$

The case $\eta_i = \sigma_i = \circ$ for all $0 \leqslant i \leqslant d$ is trivial and we exclude it henceforth.

The quantity $c_\Gamma(\sigma) + k_\Gamma(\sigma)$ counts the number of $i \in [1,d]$ such that $\sigma_i = \square$ or $\sigma_i = *$ but $\sigma_0, \dots, \sigma_{i-1}$ are not all $\circ$. We claim that (excluding the trivial case above)

$$c_\Gamma(\sigma) + k_\Gamma(\sigma) = \big|\,\{i \in [0,d] \mid \sigma_i = \square \text{ or } \sigma_i = *\}\,\big| - 1. \tag{17.9}$$

To check this, there are two cases. First, suppose $\sigma_0 = \square$ or $\sigma_0 = *$. Then the index 0 is counted in the first term on the right hand side of (17.9) even though it is not in the range $1 \leqslant i \leqslant d$, but this is accounted for by subtracting 1 there. The indices $i \in [1,d]$ with $\sigma_i = \square$ or $\sigma_i = *$ are counted on both sides. Hence (17.9) holds. The other possibility is $\eta_0 = \sigma_0 = \bigcirc$. The index $i = 0$ is not counted in the first term on the right hand side of (17.9). However, σ begins with a $\bigcirc$, and the first index i_0 such that $\sigma_{i_0} \neq \bigcirc$ *is* counted in the first term on the right hand side of (17.9). Subtracting 1 there makes up for the exclusion of the index i_0 on the left hand side as it corresponds to a $\square$ or $*$ preceeded by a nonempty initial string of $\bigcirc$'s. The remaining indices $i > i_0$ such that $\sigma_i = \Box$ or $\sigma_i = *$ are counted on both sides. Hence (17.9) is also true in this case.

Similarly, we have (again excluding the case that all $\sigma'_i = \bigcirc$)

$$c_\Delta(\sigma') + k_\Delta(\sigma') = |\{i \in [0, d] \mid \sigma'_i = \Box \text{ or } \sigma'_i = *\}| - 1.$$

But since the map ψ preserves the number of boxes and the number of stars in the signature, we conclude that (17.8) holds, and the Theorem is proved. $\Box$

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Notation

F	an algebraic number field	1
μ_n	n-th roots of unity	1
S	a finite set of places of F	1
$\mathfrak{o}_S$	ring of S integers	1
F_S	$\prod_{v \in S} F_v$	1
$(\ ,\ )_S$	S-Hilbert (norm residue) symbol	1
Ψ	auxiliary function in Z_Ψ	1
$\mathcal{M}$	vector space of auxiliary functions Ψ	2
Z_Ψ	Weyl group multiple Dirichlet Series	2
H	Coefficients of Z_Ψ	2
$\left(\frac{c}{d}\right)$	power residue symbol	2
α_i	simple roots	2
p	a fixed prime of $\mathfrak{o}_S$	3
$\mathfrak{T}$	Gelfand-Tsetlin pattern	3
k_Γ, k_Δ	integer vectors of weight functions	4
H_Γ, H_Δ	two particular definitions for H	4
$\Gamma(\mathfrak{T})$, $\Delta(\mathfrak{T})$	decorated integer arrays	4
$\textcircled{4}$, $\boxed{7}$	decorated integers	5
ψ	additive character of $F_S/\mathfrak{o}_S$	5
$g(m,c)$	Gauss sum $\sum_{a \bmod c} \left(\frac{a}{c}\right) \psi\left(\frac{am}{c}\right)$	5
$g(a)$	$g(p^{a-1}, p^a)$	5
$h(a)$	$g(p^a, p^a)$	5
q	residue field cardinality	5
$G_\Gamma(\mathfrak{T}), G_\Delta(\mathfrak{T})$	products of Gauss sums	6
q_i	Schützenberger involution	7
t_i	reflection of $(r+1-i)$-th row	7

Λ	$\mathbb{Z}^{r+1}$, the weight lattice	9
λ	a weight, often dominant	9
$\mathcal{B}_\lambda$	crystal graph with highest weight λ	9
$m(\mu, \lambda)$	multiplicity of μ in highest weight module for λ	9
$\mathrm{wt} : \mathcal{B}_\lambda \to \Lambda$	weight function	9
α_i	simple roots	9
e_i, f_i	Kashiwara operators	9
$\mathrm{Sch} : \mathcal{B}_\lambda \to \mathcal{B}_\lambda$	Schützenberger involution	10
$\psi_\lambda, \phi_\lambda : \mathcal{B}_\lambda \to \mathcal{B}_{-w_0\lambda}$	involutions	10
w_0	long Weyl group element	10
rev	mirror image array	11
ϕ_i, ϵ_i	number of times f_i or e_i applies	13
Ω	a reduced word	13
$v \left[\begin{array}{ccc} b_1 & \cdots & b_N \\ \Omega_1 & \cdots & \Omega_N \end{array} \right] v'$	path from v to v'	14
v_{high}	highest weight vector	14
v_{low}	lowest weight vector	14
$\mathrm{BZL}_\Omega(v)$	string vector of v with respect to long word Ω	14
Ω_Γ	the word $(1, 2, 1, 3, 2, 1, \cdots)$	14
Ω_Γ	the word $(r, r-1, r, r-2, r-1, r, \cdots)$	14
$G_\Gamma(v)$, $G_\Delta(v)$	products of Gauss sums	20
s_λ	Schur polynomial	23
$GT(\lambda), GT_\lambda$	Gelfand-Tsetlin patterns with top row λ	23
σ_i	simple reflections	32
$\mathfrak{t}$	short Gelfand-Tsetlin pattern	34
$\mathfrak{t}'$	involute of $\mathfrak{t}$	34
$\mathfrak{S}$	short pattern prototype	35
$\Gamma_\mathfrak{t}$, $\Delta_\mathfrak{t}$	preaccordions	35
$G_\Gamma(\mathfrak{t})$, $G_\Delta(\mathfrak{t}')$	products of Gauss sums	37
$\mathcal{E}$	episode	38
$\mathfrak{a}$	accordion	39
$\mathfrak{a}'$	involute accordion	40
$\bigcirc, \square, ast$	signature-runes	41
$\mathfrak{a}_\sigma$	a decorated accordion	42
$\mathcal{G}_\Gamma(\mathfrak{a}_\sigma), G_\Delta(\mathfrak{a}'_\sigma)$	products of Gauss sums	42
$\mathcal{A}$, $\mathcal{A}^\Gamma_s(c_0, \cdots, c_d)$	Γ-resotope	42
$\mathcal{A}'$	Δ-resotope (involute of $\mathcal{A}$	42
$\mathfrak{Z}_\Gamma$	free abelian group on the decorated accordions	42
$\mathrm{CP}_\eta(c_0, \cdots, c_d)$	cut and paste simplex of Γ-accordions	45
$\mathrm{sgn}(\tau)$	$(-1)^\varepsilon$, $\varepsilon =$ number of $\square$	45
$\Lambda_\Gamma(\mathfrak{a}, \sigma)$	alternating sum of $\mathcal{G}_\Gamma$	45
$\mathcal{S}_\sigma$	open simplex	47
$\overline{\mathcal{S}_\sigma}$	closed simplex	47

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