Integration on p-adic groups and Crystal bases

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1 Introduction

Kashiwara defined the notion of a *crystal*, and gave examples of crystal structures associated with bases of representations of quantum groups. We recommend the expository article Kashiwara [7], written a few years after the original papers, and the book of Hong and Kang [5].

One particular crystal defined by Kashiwara is denoted $\mathcal{B}(\infty)$. It is a basis of the quantized universal enveloping algebra $U_q(\mathfrak{n}_-)$ where $\mathfrak{n}_-$ is the Lie algebra of the maximal unipotent subgroup N_- of a reductive algebraic group G or more generally its n-fold metaplectic cover. Our basic philosophy is that *an integral over $N_-(F)$ where F is a nonarchimedean local field can sometimes be replaced by a sum over $\mathcal{B}(\infty)$.*

We will demonstrate this for $G = \mathrm{GL}_{r+1}$, and later for the n-fold metaplectic cover. In this introduction we will consider the "nonmetaplectic case" where $n = 1$. Let ${}^LG = \mathrm{GL}_{r+1}(\mathbb{C})$ be the (connected) Langlands dual group. Then the diagonal group $T(\mathbb{C})$ in ${}^LG$ has character group $\Lambda = X^*(T) \cong \mathbb{Z}^{r+1}$, and we may identify this with the full weight lattice.

If $\mathbf{z} = \mathrm{diag}(z_1, \cdots, z_{r+1}) \in T(\mathbb{C})$ where $z_i \in \mathbb{C}^\times$, then in this identification $\mu \in \mathbb{Z}^{r+1}$ is the character $\mathbf{z} \longmapsto \mathbf{z}^\mu = \prod z_i^{\mu_i}$. The simple positive roots are $\alpha_i = (0, \cdots, 0, 1, -1, 0, \cdots, 0)$ where the 1 is in the i-th place. The dominant weights are $\lambda = (\lambda_1, \cdots, \lambda_{r+1})$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{r+1}$. If all $\lambda_i \geqslant 0$ then we call a weight λ *effective*. Thus an effective dominant weight is a partition. We will denote by $\rho = (r, r-1, \cdots, 2, 1, 0)$. It differs from half the positive roots by a vector orthogonal to the roots, so it may substitute for $\frac{1}{2}\sum \alpha$ in many formulas such as the Weyl character formula.

The conjugacy class in ${}^LG$ parametrizes a spherical representation of $G(F)$. The induced model of this representation acts on the space of smooth functions f on

G that satisfy $f(bg) = \delta^{1/2}\chi(b)f(g)$, where b lies in the Borel subgroup $B(F)$ of upper triangular matrices, δ is the modular quasicharacter on $B(F)$ and χ is the quasicharacter of $B(F)$ defined by

$$\chi\begin{pmatrix} y_1 & * & \cdots & * \\ & y_2 & & * \\ & & \ddots & \vdots \\ & & & y_{r+1} \end{pmatrix} = \prod z_i^{\mathrm{ord}(y_i)}.$$

Various integrals that we write down will be convergent if $|z_i/z_{i+1}| < 1$, and we will assume this. Let $\mathfrak{o}$ be the ring of integers in F and let q be the cardinality of the residue field.

The standard spherical vector f° in this representation is the function such that $f^\circ(bk) = \delta^{1/2}\chi(b)$ when $b \in B(F)$ and $k \in K = \mathrm{GL}_{r+1}(\mathfrak{o})$. We mention two important integrals that illustrate the principle we stated above. The first is the formula of Gindikin and Karpelevich, which asserts that

$$\int_{N_-(F)} f^\circ(\boldsymbol{n})\, d\boldsymbol{n} = \prod_{\alpha\in\Phi^+} \frac{1 - q^{-1}\mathbf{z}^\alpha}{1 - \mathbf{z}^\alpha}. \tag{1}$$

The second is the formula of Casselman and Shalika.

The formula (1) was first proved by Langlands [10]. Another proof may be found in Casselman [2]. (The original paper of Gindikin and Karpelevich [4] is concerned with the archimedean case.) MacNamara [12] also gives a proof of a generalization of this formula, as well as the Casselman-Shalika formula, to metaplectic covers.

We will show that (1) may also be expressed as a sum over $\mathcal{B}(\infty)$. This is striking since $\mathcal{B}(\infty)$ is obtained from N_- by quantization. The work of MacNamara [12] may clarify this phenomenon by showing how to decompose $N_-(F)$ into cells parametrized by elements of $\mathcal{B}(\infty)$.

If ψ is a nondegenerate additive character of $N_-(F)$, the integral $\int_{N_-(F)} f(\boldsymbol{n})\,\psi(\boldsymbol{n})d\boldsymbol{n}$ is evaluated in the formula of Casselman and Shalika [3]. Making use of a formula of Tokuyama [14] this evaluation may be rewritten in terms of crystals. This was done by Brubaker, Bump and Friedberg [1]. We will describe a variant of their formula. The difference is that we will use the Kashiwara operators e_i where they use the f_i.

Let $\lambda \in \mathbb{Z}^{r+1}$. Define

$$\psi_\lambda\begin{pmatrix} 1 & & & \\ x_{2,1} & 1 & & \\ \vdots & \iddots & \ddots & \\ x_{r+1,1} & & x_{r+1,r} & 1 \end{pmatrix} = \psi_0(\varpi^{\lambda_1-\lambda_2}x_{r+1,r} + \ldots + \varpi^{\lambda_r-\lambda_{r+1}}x_{2,1})$$

where ψ_0 is a fixed additive character on F that is trivial on $\mathfrak{o}$ but not on $\mathfrak{p}^{-1}$. The integral $\int_{N_-(F)} f(\boldsymbol{n})\,\psi_\lambda(\boldsymbol{n})d\boldsymbol{n}$ is zero unless the weight λ is dominant, which we now assume. If $\rho = (r, r-1, \cdots, 2, 1, 0)$ then there is a crystal $\mathcal{B}_{\lambda+\rho}$ which we will describe, and we will express this integral as a sum over this crystal.

In order to give the relevant definitions, we recall some facts and definitions about crystals. Let Φ be a root system, which in this paper will be mainly A_r. Let α_i $(i = 1, \cdots, r)$ be the simple roots, and $\alpha_i^\vee$ their associated coroots. Let Λ be the associated weight lattice. By a *crystal* for Φ we mean a set $\mathcal{B}$ together with a map $\mathrm{wt} : \mathcal{B} \longrightarrow \Lambda$, and, for $1 \leqslant i \leqslant r$, maps $\phi_i, \varepsilon_i : \mathcal{B} \longrightarrow \mathbb{Z} \cup \{-\infty\}$ and $f_i, e_i : \mathcal{B} \longrightarrow \mathcal{B} \cup \{0\}$, where 0 is an auxiliary element. It is assumed that $\phi_i(v) = \langle \mathrm{wt}(v), \alpha_i^\vee \rangle + \varepsilon_i(v)$. If $e_i(v) \neq 0$ then it is assumed that $f_i e_i(v) = v$ and that $\mathrm{wt}(e_i(v)) = \mathrm{wt}(v) + \alpha_i$, and if $f_i(v) \neq 0$ then it is assumed that $e_i f_i(v) = v$ and that $\mathrm{wt}(f_i(v)) = \mathrm{wt}(v) - \alpha_i$.

In Kashiwara's papers the maps we have denoted e_i and f_i are denoted $\tilde{e}_i$ and $\tilde{f}_i$, because the letters e_i and f_i are already in use for a different meaning.

One may impose on $\mathcal{B}$ the structure of a directed graph with labeled edges, called the *crystal graph* in which elements are vertices, and there is an edge $x \xrightarrow{i} y$ if $f_i(x) = y$. Examples of crystal graphs may be seen in Figure 1 in the next Section.

If $\mathcal{C}$ and $\mathcal{D}$ are crystals, a *morphism* $m : \mathcal{C} \longrightarrow \mathcal{D}$ is a map $\mathcal{C} \longrightarrow \mathcal{D} \cup \{0\}$ such that if $x \in \mathcal{C}$ and $m(x) \neq 0$ then $\mathrm{wt}(m(x)) = \mathrm{wt}(x)$, $\varepsilon_i(m(x)) = \varepsilon_i(x)$ and $\phi_i(m(x)) = \phi_i(x)$, and such that if $x, y \in \mathcal{C}$ and both $m(x), m(y) \neq 0$, then $e_i(x) = y$ if and only if $e_i(m(x)) = m(y)$, and $f_i(y) = x$ if and only if $f_i(m(y)) = m(x)$. Crystals form a category.

Let G be a complex analytic group and T a maximal torus such that Φ is the root system of G with respect to T. Assuming that the derived group of G is simply connected, we may identify Λ with the group $X^*(T)$ of rational characters of T. There is defined a crystal $\mathcal{B}_\lambda$ with the property that

$$\sum_{v \in \mathcal{B}_\lambda} \boldsymbol{z}^{\mathrm{wt}(v)}$$

$(\boldsymbol{z} \in T)$ is the character of the highest weight module V_λ for λ.

By a *long word* Ω we mean a reduced expression of the long element w_0 of W as a product of simple reflections. Thus

$$\Omega = (\omega_1, \omega_2, \cdots, \omega_N)$$

where N is the number of positive roots ($N = \frac{1}{2}r(r+1)$ for $\Phi = A_r$) and $\omega_j \in \{1, 2, \cdots, r\}$ are such that $w_0 = s_{\omega_1} \cdots s_{\omega_N}$. Let $v \in \mathcal{B}_\lambda$. Let b_1 (depending on v and

Ω) be the largest integer such that $e_{\omega_1}^{b_1} v \neq 0$. Let b_2 then be the largest integer such that $e_{\omega_2}^{b_2} e_{\omega_1}^{b_1} v \neq 0$, and so forth. It is known (see Littelmann [11]) that $e_{\omega_N}^{b_N} \cdots e_{\omega_2}^{b_2} e_{\omega_1}^{b_1} v$ is the unique element v_{high} of $\mathcal{B}_\lambda$ with $\text{wt}(v_{\text{high}}) = \lambda$ the highest weight.

We decorate the pattern

$$\text{BZL}(v) = (b_1, \cdots, b_N) \tag{2}$$

by "circling" or "boxing" certain entries. We will describe the boxing rule for all Ω, but we will describe the circling rule only for $\Omega = \Omega_\Gamma$ or $\Omega = \Omega_\Delta$ where

$$\begin{aligned} \Omega_\Gamma &= (1, 2, 1, 3, 2, 1, \cdots, r, r-1, \cdots, 3, 2, 1), \\ \Omega_\Delta &= (r, r-1, r, r-2, r-1, r, \cdots, 1, 2, 3, \cdots, r). \end{aligned}$$

If $f_{\omega_i} e_{\omega_{i-1}}^{b_{i-1}} \cdots e_{\omega_1}^{b_1} v = 0$ then we decorate b_i by boxing it. In the case where $\Omega = \Omega_\Gamma$ or Ω_Δ it was proved by Littelmann [11] that

$$\begin{array}{rcl} b_1 & \geqslant & 0, \\ b_2 \geqslant b_3 & \geqslant & 0, \\ b_4 \geqslant b_5 \geqslant b_6 & \geqslant & 0, \\ \vdots & & . \end{array} \tag{3}$$

If $b_1 = 0$ then we decorate b_1 by circling it. If $b_2 = b_3$ then we decorate b_2 by circling it. If $b_3 = 0$, then we decorate b_3 by circling it, and so forth.

Now let us recall from [1] the definition

$$G_\Omega(v) = G_\Omega^{(e)}(v) = \prod_{i=1}^N \begin{cases} h(b_i) & \text{if } b_i \text{ is neither circled nor boxed,} \\ g(b_i) & \text{if } b_i \text{ is boxed but not circled,} \\ q^{b_i} & \text{if } b_i \text{ is circled but not boxed,} \\ 0 & \text{if } b_i \text{ is both circled and boxed.} \end{cases} \tag{4}$$

In [1] (and in the final Section below), h and g are n-th order Gauss sums, where n is an integer prime to the residue characteristic such that the ground field contains the n-th roots of unity. In the case at hand, $n = 1$ and they can be made explicit:

$$g(a) = -q^{a-1}, \qquad h(a) = (q-1)q^{a-1}. \tag{5}$$

We may also dualize these definitions by interchanging the roles of the e_i and f_i. Thus we would alternatively let b_1 be the largest integer such that $f_{\omega_1}^{b_1} v \neq 0$. Let b_2 then be the largest integer such that $f_{\omega_2}^{b_2} f_{\omega_1}^{b_1} v \neq 0$, and so forth. It is known (see Littelmann [11]) that $f_{\omega_N}^{b_N} \cdots f_{\omega_2}^{b_2} f_{\omega_1}^{b_1} v$ is the unique element v_{low} of $\mathcal{B}_\lambda$ with $\text{wt}(v_{\text{low}}) =$

$w_0\lambda$ the lowest weight. In this scheme, we box b_i if $e_{\omega_i} f_{\omega_{i-1}}^{b_{i-1}} \cdots f_{\omega_1}^{b_1} v = 0$. The inequalities (3) are again satisfied, and as before $b_1 = 0$ then we decorate b_1 by circling it, and so forth. Then we may define

$$G_\Omega^{(f)}(v) = \prod_{i=1}^{N} \begin{cases} h(b_i) & \text{if } b_i \text{ is neither circled nor boxed,} \\ g(b_i) & \text{if } b_i \text{ is boxed but not circled,} \\ q^{b_i} & \text{if } b_i \text{ is circled but not boxed,} \\ 0 & \text{if } b_i \text{ is both circled and boxed.} \end{cases}$$

We can make exactly the same definitions for $v \in \mathcal{B}(\infty)$. However only the definition of $G_\Omega^{(e)}(v)$ makes sense, since there is no largest integer such that $f_1^{b_1} v \neq 0$. Indeed, if $w \in \mathcal{B}(\infty)$ then $f_i^k w \neq 0$ for all k. Therefore we may define $G_\Omega^{(e)}(v)$ but not $G_\Omega^{(f)}(v)$. Also circling can occur but not boxing; indeed $f_{\omega_i} e_{\omega_{i-1}}^{b_{i-1}} \cdots e_{\omega_1}^{b_1} v \neq 0$ for the same reason.

If λ is any weight, there is a crystal $\mathcal{T}_\lambda$ having one element t_λ with weight λ. It has the properties that $e_i(t_\lambda) = f_i(t_\lambda) = 0$ and $\phi_i(t_\lambda) = \varepsilon_i(t_\lambda) = -\infty$. We have $\mathcal{T}_\lambda \otimes \mathcal{T}_\mu \cong \mathcal{T}_{\lambda+\mu}$. Tensoring any crystal $\mathcal{B}$ with $\mathcal{T}_\lambda$ produces an a crystal that is isomorphic to $\mathcal{B}$ as a directed graph, but in which the weights are shifted: $\mathrm{wt}(x \otimes t_\lambda) = \mathrm{wt}(x) + \lambda$ for $x \in \mathcal{B}$.

If λ is a dominant weight, let χ_λ be the irreducible character of ${}^L G = \mathrm{GL}_{r+1}(\mathbb{C})$ with highest weight λ.

Theorem 1 *If λ is a dominant weight and $\Omega = \Omega_\Gamma$ or Ω_Δ then*

$$\begin{aligned} \int_{N_-(F)} f^\circ(\boldsymbol{n}) \psi_\lambda(\boldsymbol{n})\, d\boldsymbol{n} &= \prod_{\alpha \in \Phi^+} (1 - q^{-1}\mathbf{z}^\alpha) \chi_\lambda(\mathbf{z}) \\ &= \sum_{\mathcal{B}_{\lambda+\rho} \otimes \mathcal{T}_{-\lambda-\rho}} G_\Omega(v) q^{-\langle w_0(\mathrm{wt}(v)), \rho \rangle} \mathbf{z}^{w_0(\mathrm{wt}(v))}. \end{aligned}$$

The first equality is the Casselman-Shalika formula. We will also rewrite the formula of Gindikin and Karpelevich in the following similar way.

Theorem 2 *We have*

$$\int_{N_-(F)} f^\circ(\boldsymbol{n})\, d\boldsymbol{n} = \prod_{\alpha \in \Phi^+} \frac{1 - q^{-1}\mathbf{z}^\alpha}{1 - \mathbf{z}^\alpha} = \sum_{\mathcal{B}(\infty)} G_\Omega(v) q^{\langle \mathrm{wt}(v), \rho \rangle} \mathbf{z}^{-\mathrm{wt}(v)}.$$

In fact in both these Theorems, the final sum may be written as a sum over $\mathcal{B}(\infty)$. Indeed, there is a morphism $M_{\lambda+\rho} : \mathcal{B}(\infty) \longrightarrow \mathcal{B}_{\lambda+\rho} \otimes T_{-\lambda-\rho}$ due to Kashiwara that we will make use of in the next Section, and the sum over $\mathcal{B}_{\lambda+\rho} \otimes T_{-\lambda-\rho}$ may therefore be interpreted as a sum over $\mathcal{B}(\infty)$, with only finitely many nonzero terms (those that do not map to zero in the morphism).

Thus both Theorems illustrate the philosophy that we can sometimes replace integrals over $N_-(F)$ by sums over $B(\infty)$, which is a basis of quantized enveloping algebra of $N_-(F)$.

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2 Proofs of the theorems

The paper of Hong and Lee [6] describes $\mathcal{B}(\infty)$ in explicit terms by means of tableaux. We will not review their work but it was useful in the preparation of this paper.

We have already mentioned the crystal $\mathcal{T}_\lambda$ having just one element t_λ of weight λ, such that $e_i(t_\lambda) = f_i(t_\lambda) = 0$ and $\phi_i(t_\lambda) = \varepsilon_i(t_\lambda) = -\infty$. There is a morphism $M_\lambda : \mathcal{B}(\infty) \longrightarrow \mathcal{B}_\lambda \otimes \mathcal{T}_{-\lambda}$ that was introduced by Kashiwara (see [7], Theorem 8.1), which we will make use of. Let u_0 and b_λ be the highest weight vectors in $\mathcal{B}(\infty)$ and $\mathcal{B}_\lambda$, so $\mathrm{wt}(u_0) = 0$ and $\mathrm{wt}(b_\lambda) = \lambda$. The morphism maps u_0 to $b_\lambda \otimes t_{-\lambda}$. It maps all but a finite number of elements to 0. Those elements u of $\mathcal{B}(\infty)$ that do not map to zero form a directed subgraph of the crystal graph of $\mathcal{B}(\infty)$ that is a copy of $\mathcal{B}_\lambda$ as a colored directed graph. To illustrate this morphism, Figure 1 shows $\mathcal{B}_\lambda$ (using Kashiwara's notation for the crystal elements as tableaux) in the case $\lambda = (2, 1, 0)$; tensoring this with $\mathcal{T}_{-\lambda}$ so that the highest weight vector has weight 0, this is embedded in $\mathcal{B}(\infty)$, where the labeling is a modification of the notation in Hong and Lee [6]. (From the partial tableaux in Figure 1, one obtains representatives of the crystal T_∞ in [6] by adding sufficiently many 1's at the beginning of the first row, 2's at the beginning of the second row, etc.)

We will prove Theorem 1. If ψ_λ is an additive character of N_- as defined in the introduction, the Casselman-Shalika formula for GL_{r+1} is written as follows

$$\int_{N_-} f^\circ(\boldsymbol{n})\psi_\lambda(\boldsymbol{n})d\boldsymbol{n} = \mathbf{z}^{-w_0\lambda} \left[\prod_{\alpha\in\Phi^+} (1 - q^{-1}\mathbf{z}^\alpha) \right] s_\lambda(z_1, \cdots, z_{r+1}),$$

where the integral is absolutely convergent if $|\mathbf{z}^\alpha| < 1$, and $s_\lambda(z_1, \cdots, z_{r+1})$ is the standard Schur polynomial.

On the other hand, Brubaker, Bump and Friedberg show the following Tokuyama's deformation of the Weyl character formula for crystals.

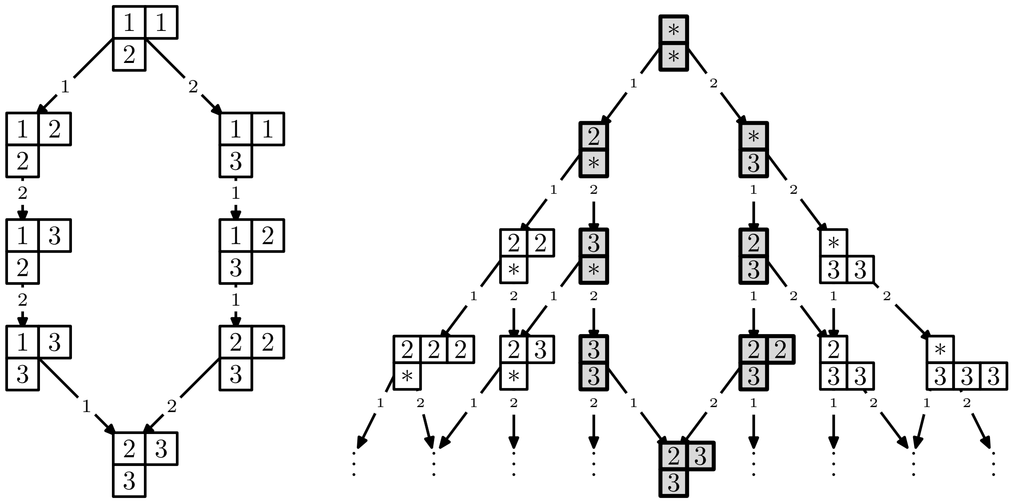

Figure 1: The crystal $\mathcal{B}_\lambda \otimes \mathcal{T}_{-\lambda}$, with $\lambda = (2,1,0)$, and its image in $\mathcal{B}(\infty)$.

Theorem 3 ([1], Theorem 5) *If λ is a dominant weight, and if $z_1, \cdots, z_{r+1}$ are the eigenvalues of $g \in \mathrm{GL}_{r+1}(\mathbb{C})$, then*

$$\prod_{\alpha \in \Phi^+} (1 - q^{-1}\mathbf{z}^{\alpha})\chi_\lambda(g) = \sum_{v \in \mathcal{B}_{\rho+\lambda}} G^{(f)}_{\Omega_\Gamma}(v) q^{-\langle \mathrm{wt}(v) - w_0(\lambda+\rho), \rho\rangle} \mathbf{z}^{\mathrm{wt}(v) - w_0\rho},$$

where χ_λ is the character of the irreducible representation with highest weight λ.

When z_i are the eigenvalues of $g \in \mathrm{GL}_{r+1}(\mathbb{C})$, we have $s_\lambda(z_1, \cdots, z_{r+1}) = \chi_\lambda(g)$. Therefore, by this theorem, the integral $\int_{N_-} f^\circ(\boldsymbol{n})\psi_\lambda(\boldsymbol{n})d\boldsymbol{n}$ in the formula of Casselman and Shalika is evaluated in terms of crystal graphs. ([1, (3.7)])

$$\int_{N_-} f^\circ(\boldsymbol{n})\psi_\lambda(\boldsymbol{n})d\boldsymbol{n} = \sum_{v \in \mathcal{B}_{\rho+\lambda}} G^{(f)}_{\Omega_\Gamma}(v) q^{-\langle \mathrm{wt}(v) - w_0(\lambda+\rho), \rho\rangle} \mathbf{z}^{\mathrm{wt}(v) - w_0(\rho+\lambda)}. \tag{6}$$

Now we will replace the right hand side with the equation using $G^{(e)}_{\Omega_\Gamma}$. The following equivalence of two descriptions is obtained in [1].

Theorem 4 ([1], Statement A′)

$$\sum_{v \in \mathcal{B}_{\lambda+\rho}} G^{(f)}_{\Omega_\Gamma}(v) = \sum_{v \in \mathcal{B}_{\lambda+\rho}} G^{(f)}_{\Omega_\Delta}(v).$$

By this Theorem, the right hand side of (6) is written as

$$\sum_{v\in\mathcal{B}_{\lambda+\rho}} G^{(f)}_{\Omega_\Delta}(v)q^{-\langle \mathrm{wt}(v)-w_0(\lambda+\rho),\rho\rangle}\mathbf{z}^{\mathrm{wt}(v)-w_0(\rho+\lambda)}.$$

There is a map Sch : $\mathcal{B}_{\lambda+\rho}\to\mathcal{B}_{\lambda+\rho}$ called the Schützenberger involution such that $\mathrm{Sch}\circ e_i = f_{r+1-i}\circ\mathrm{Sch}$ and $\mathrm{Sch}\circ f_i = e_{r+1-i}\circ\mathrm{Sch}$. Let $v' = \mathrm{Sch}(v)$ for $v\in\mathcal{B}_{\lambda+\rho}$. Since $\mathrm{wt}(v') = w_0\mathrm{wt}(v)$ and $G^{(f)}_{\Omega_\Delta}(v) = G^{(e)}_{\Omega_\Gamma}(\mathrm{Sch}(v)) = G^{(e)}_{\Omega_\Gamma}(v')$, it becomes

$$\sum_{v\in\mathcal{B}_{\rho+\lambda}} G^{(e)}_{\Omega_\Gamma}(v')q^{-\langle w_0(\mathrm{wt}(v')-\rho-\lambda),\rho\rangle}\mathbf{z}^{w_0(\mathrm{wt}(v')-\rho-\lambda)}.$$

Let $v'' := v'\otimes t_{-\lambda-\rho}$ with $v'\in\mathcal{B}_{\lambda+\rho}$ and $t_{-\lambda-\rho}\in\mathcal{T}_{-\lambda-\rho}$. Since $\mathrm{wt}(v'') = \mathrm{wt}(v')-\lambda-\rho$ and $G^{(e)}_{\Omega_\Gamma}(v'') = G^{(e)}_{\Omega_\Gamma}(v')$, with the morphism $M_{\lambda+\rho} : \mathcal{B}(\infty)\to\mathcal{B}_{\lambda+\rho}\otimes\mathcal{T}_{-\lambda-\rho}$ we obtain

$$\sum_{v''\in\mathcal{B}_{\lambda+\rho}\otimes\mathcal{T}_{-\lambda-\rho}} G^{(e)}_{\Omega_\Gamma}(v'')q^{-\langle w_0(\mathrm{wt}(v'')),\rho\rangle}\mathbf{z}^{w_0(\mathrm{wt}(v''))}.$$

This proves Theorem 1.

In order to prove Theorem 2, we need to discuss the limiting argument at first.

Given $\boldsymbol{n}\in N_-$ we may write $\boldsymbol{n} = t\boldsymbol{n}_+k$ where $t\in T$, $\boldsymbol{n}_+\in N$ and $k\in\mathrm{GL}_{r+1}(\mathfrak{o})$. The element t is not uniquely determined but its image $\bar{t}$ in $T/T(\mathfrak{o})$ is uniquely determined. The group $T/T(\mathfrak{o})$ is discrete, and $v : T/T(\mathfrak{o})\longrightarrow\mathbb{Z}^{r+1}$ defined by

$$v\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_{r+1}\end{pmatrix} = (\mathrm{ord}(t_1),\cdots,\mathrm{ord}(t_{r+1}))$$

is an isomorphism. Define a map $\beta : N_-\longrightarrow\mathbb{Z}^{r+1}$ by $\beta(\boldsymbol{n}) = v(\bar{t})$.

Proposition 1 *The map β is proper.*

We recall that if X and Y are Hausdorff topological spaces then a map $f : X\longrightarrow Y$ is *proper* if the inverse image of a compact set is compact. Since $\mathbb{Z}^{r+1}$ is discrete, this means that the inverse image of a finite set is compact in N_-.

Proof Write $\boldsymbol{n} = t\boldsymbol{n}_+k$ with $t\in T$, $\boldsymbol{n}_+\in N$ and $k\in K$. Let S be a subset of $\{1,\cdots,r+1\}$ with $k = |S|$. If $A = (a_{ij})$ is an $(r+1)\times(r+1)$ matrix, denote by $M_S(A)$ the minor

$$\det(a_{i,j}|i\in\{r+2-k,r+3-k,\cdots,r+1\},j\in S)$$

formed with the bottom k rows of A and columns in j. We call $M_S(A)$ a *bottom minor*. Since $\boldsymbol{n}_+$ is upper triangular and unipotent, $M_S(\boldsymbol{n}_+k) = M_S(k)$, and since t is diagonal,

$$M_S(\boldsymbol{n}) = \left[\prod_{j=r+2-k}^{r+1} t_j\right] M_S(k).$$

Since the entries in $M_S(k)$ are in $\mathfrak{o}$, this means that

$$|M_S(\boldsymbol{n})| \leq \left|\prod_{j=r+2-k}^{r+1} t_j\right|.$$

Now since $\boldsymbol{n}$ is lower triangular and unipotent it is easy to see that each entry n_{ij} in $\boldsymbol{n}$ (with $i > j$) equals $M_S(\boldsymbol{n})$ where $S = \{j, i+1, i+2, \cdots, r+1\}$. For example if $r + 1 = 4$ and

$$\boldsymbol{n} = \begin{pmatrix} 1 & & & \\ n_{21} & 1 & & \\ n_{31} & n_{32} & 1 & \\ n_{41} & n_{42} & n_{43} & 1 \end{pmatrix}$$

then $n_{31} = M_S(\boldsymbol{n})$ where $S = \{1, 4\}$. It is now clear that if t is confined to a compact subset of T then the entries of $\boldsymbol{n}$ are bounded, and it follows that β is a proper map. □

Let $R = \mathbb{C}[q][[z^{\alpha_1}, \cdots, z^{\alpha_r}]]$ and $\mathcal{P} := \{\sum k_i\alpha_i | 1 \leq i \leq r, k_i \geq 0\}$. If $v \in \mathcal{B}_{\lambda+\rho}$, $\mathrm{wt}(v) - w_0(\lambda + \rho) \in \mathcal{P}$. It follows by (6), that $\int_{N_-} f^\circ(\boldsymbol{n})\psi_\lambda(\boldsymbol{n})d\boldsymbol{n} \in R$. Applying Proposition 1, we have following

Proposition 2 *$\int_{N_-} f^\circ(\boldsymbol{n})\psi_\lambda(\boldsymbol{n})\,d\boldsymbol{n}$ converges $\int_{N_-} f^\circ(\boldsymbol{n})\,d\boldsymbol{n}$ in the topology of the ring R when λ goes to ∞.*

Proof Let S be a finite subset of Λ contained in $\mathcal{P}$. By Proposition 1, there is a compact subset C of N_- such that, for $\boldsymbol{n} \in N_- - C$, $\beta(\boldsymbol{n}) = \sum k_i\alpha_i \notin S$. Assume $\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \cdots > N$ for some integer N. The difference $\int_{N_-} f^\circ(\boldsymbol{n})\,d\boldsymbol{n} - \int_{N_-} f^\circ(\boldsymbol{n})\psi_\lambda(\boldsymbol{n})\,d\boldsymbol{n}$ is written into 2 parts

$$\int_C f^\circ(\boldsymbol{n})(1 - \psi_\lambda(\boldsymbol{n}))d\boldsymbol{n} + \int_{N_- - C} f^\circ(\boldsymbol{n})(1 - \psi_\lambda(\boldsymbol{n}))d\boldsymbol{n}.$$

Choose N so large that $\psi_\lambda = 1$ on C. Then the first term vanishes. Let E_S be the additive subgroup of R consisting of $\sum c_{k_1\cdots k_r}(q)\mathbf{z}^{k_1\alpha_1+\ldots+k_r\alpha_r}$, such that $c_{k_1\cdots k_r}(q) = 0$

if $\sum k_i \alpha_i \in S$. These form a base of neighborhoods of the identity in R. Since $f^\circ(\boldsymbol{n}) \in R$, it means the second term converges in R. $\square$

We will prove Theorem 2.

When λ goes to ∞, then the limiting argument as above and Theorem 1 lead to

$$\int_{N_-} f^\circ(\boldsymbol{n}) d\boldsymbol{n} = \sum_{v \in \mathcal{B}(\infty)} G^{(e)}_{\Omega_\Gamma}(v) q^{-\langle w_0(\mathrm{wt}(v), \rho\rangle} \mathbf{z}^{w_0(\mathrm{wt}(v))}.$$

There is a map $\iota_\lambda : \mathcal{B}_\lambda \to \mathcal{B}_{-w_0\lambda}$, which satisfies $\iota_\lambda \circ f_i = f_{r+1-i} \circ \iota_\lambda$ and $\iota_\lambda \circ e_{r+1-i} = e_i \circ \iota_\lambda$. There is a corresponding bijection $\iota : \mathcal{B}(\infty) \to \mathcal{B}(\infty)$:

$$\begin{array}{ccc} \mathcal{B}(\infty) & \xrightarrow{\iota} & \mathcal{B}(\infty) \\ {\scriptstyle M_{\lambda+\rho}} \big\downarrow & & \big\downarrow {\scriptstyle M_{-w_0(\lambda+\rho)}} \\ \mathcal{B}_{\lambda+\rho} \otimes \mathcal{T}_{-\lambda-\rho} & \xrightarrow{\iota_{\lambda+\rho}} & \mathcal{B}_{-w_0(\lambda+\rho)} \otimes \mathcal{T}_{w_0(\lambda+\rho)} \end{array}$$

Let $\tilde{v} = \iota(v)$ for $v \in \mathcal{B}(\infty)$. Then since $G^{(e)}_{\Omega_\Delta}(\tilde{v}) = G^{(e)}_{\Omega_\Gamma}(v)$ and $\mathrm{wt}(\tilde{v}) = -w_0 \mathrm{wt}(v)$, we have

$$\int_{N_-} f^\circ(\boldsymbol{n}) d\boldsymbol{n} = \sum_{\tilde{v} \in \mathcal{B}(\infty)} G^{(e)}_{\Omega_\Delta}(\tilde{v}) q^{\langle \mathrm{wt}(\tilde{v}), \rho\rangle} \mathbf{z}^{-\mathrm{wt}(\tilde{v})}.$$

This concludes Theorem 2.

3 The metaplectic case

Finally, we have metaplectic analogs of these formulas. We assume that the ground field F has residue characteristic prime to n and contains the group μ_n of n-th roots of unity in the algebraic closure of F. We fix an isomorphism of μ_n with the group of n-th roots of unity in $\mathbb{C}^\times$. To avoid unnecessary minor complications we will take $G = \mathrm{SL}_{r+1}$ rather than GL_{r+1} in this section.

Let $\tilde{G}(F)$ be the n-fold metaplectic cover of $\mathrm{SL}_{r+1}(F)$, constructed first by Matsumoto [13] that splits over $K = \mathrm{SL}_{r+1}(\mathfrak{o})$. Let K^* be the image of K in $\tilde{G}(F)$ under the splitting. It is a central extension

$$1 \longrightarrow \mu_n \longrightarrow \tilde{G}(F) \longrightarrow \mathrm{SL}_{r+1}(F) \longrightarrow 1.$$

We choose a section $\boldsymbol{s} : \mathrm{SL}_{r+1}(F) \longrightarrow \tilde{G}(F)$ and a cocycle $\sigma : \mathrm{SL}_{r+1}(F) \times \mathrm{SL}_{r+1}(F) \longrightarrow \mu_n$ whose class in $H^2(\tilde{G}(F), \mu_n)$ determines the extension, so that, identifying μ_n with its image in $\tilde{G}(F)$, we have $\boldsymbol{s}(g)\boldsymbol{s}(g') = \sigma(g,g')\boldsymbol{s}(gg')$. We may choose $\boldsymbol{s}$ and σ so that

$$\sigma\left(\boldsymbol{s}\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_{r+1}\end{pmatrix}, \boldsymbol{s}\begin{pmatrix} u_1 & & \\ & \ddots & \\ & & u_{r+1}\end{pmatrix}\right) = \prod_{i<j}(t_i,u_j)^{-1},$$

where (t,u) is the n-th order Hilbert symbol, and so that $\sigma(n,g) = \sigma(g,n) = 1$ when n is in the group $N(F)$ of upper triangular unipotent matrices in $\mathrm{SL}_{r+1}(F)$.

Identifying μ_n both with its image in $\tilde{G}(F)$ and with its image in $\mathbb{C}$, we call a function $f : \tilde{G}(F) \longrightarrow \mathbb{C}$ *genuine* if $f(\varepsilon g) = \varepsilon f(g)$ for $\varepsilon \in \mu_n$. There exists a unique genuine function $\tilde{f}^\circ$ on $\tilde{G}(F)$ that satisfies

$$\tilde{f}^\circ\left(\boldsymbol{s}\begin{pmatrix} t_1 & * & \cdots & * \\ & t_2 & & \vdots \\ & & \ddots & * \\ & & & t_{r+1}\end{pmatrix} k\right) = \begin{cases} \prod z_i^{\operatorname{ord}(t_i)} & \text{if } n|\operatorname{ord}(t_i) \text{ for } 1 \leqslant i \leqslant r+1, \\ 0 & \text{otherwise,}\end{cases}$$

when $k \in K^*$. Let $i : N_-(F) \longrightarrow \tilde{G}(F)$ be the canonical splitting homomorphism, which satisfies $\boldsymbol{s}(w_0)i(\boldsymbol{n})\boldsymbol{s}(w_0)^{-1} = \boldsymbol{s}(w_0\boldsymbol{n}w_0^{-1})$ when $\boldsymbol{n} \in N_-$, where w_0 is a representative of the long Weyl group element.

In the Introduction, G_Ω was defined when $n = 1$. In [1], the definition (4) is given for general n. It is the same, except that (5) is generalized. We make use of the n-th order Gauss sum define, with ψ_0 as in the Introduction, by

$$g(m,c) = \sum_{\substack{d \bmod c \\ \gcd(d,c)=1}} (d,c)\psi_0\left(\frac{md}{c}\right).$$

Then with ϖ a fixed prime element $g(a) = g(\varpi^{a-1},\varpi^a)$ and $h(a) = g(\varpi^a,\varpi^a)$. Since boxing does not occur for $\mathcal{B}(\infty)$, the function h is most relevant here, and it can be made explicit:

$$h(a) = \begin{cases} (q-1)q^{a-1} & \text{if } n|a, \\ 0 & \text{otherwise.}\end{cases} \tag{7}$$

We may now generalize Theorem 2 as follows.

Theorem 5 *We have*

$$\int_{N_-(F)} \tilde{f}^\circ(\boldsymbol{n})\,d\boldsymbol{n} = \prod_{\alpha\in\Phi^+}\frac{1-q^{-1}\mathbf{z}^{n\alpha}}{1-\mathbf{z}^{n\alpha}} = \sum_{\mathcal{B}(\infty)} G_\Omega(v)q^{\langle \mathrm{wt}(v),\rho\rangle}\mathbf{z}^{-\mathrm{wt}(v)}. \tag{8}$$

Proof The formula of Gindikin and Karpelevich in this context is the formula

$$\int_{N_-(F)} \tilde{f}^\circ(\boldsymbol{n})\, d\boldsymbol{n} = \prod_{\alpha\in\Phi^+} \frac{1-q^{-1}\mathbf{z}^{n\alpha}}{1-\mathbf{z}^{n\alpha}},$$

and it is Proposition I.2.4 of Kazhdan and Patterson [9]. Another proof, closely related to our point of view in this paper, is in MacNamara [12].

We will prove the second equality. With $v \in \mathcal{B}(\infty)$ and with b_i as in (2) we have $\langle \mathrm{wt}(v), \rho\rangle = -\sum b_i$. Thus

$$\sum_{\mathcal{B}(\infty)} G_\Omega(v) q^{\langle \mathrm{wt}(v),\rho\rangle} \mathbf{z}^{-\mathrm{wt}(v)} = \sum_{\mathcal{B}(\infty)} G'_\Omega(v)\mathbf{z}^{-\mathrm{wt}(v)}$$

where (since boxing does not occur for $\mathcal{B}(\infty)$) we have

$$G'_\Omega(v) = \prod_{i=1}^N \begin{cases} q^{-b_i} h(b_i) & \text{if } b_i \text{ is not circled,} \\ 1 & \text{if } b_i \text{ is circled.}\end{cases}$$

Using (7), $G'_\Omega(v) = (1-q^{-1})^{s(v)}$, where $s(v)$ is the number of b_i that are not circled, provided that these uncircled b_i are all multiples of n; while $G'_\Omega(v) = 0$ if any b_i that is not circled is a multiple of n. Thus we must show that

$$\prod_{\alpha\in\Phi^+} \frac{1-q^{-1}\mathbf{z}^{n\alpha}}{1-\mathbf{z}^{n\alpha}} = \sum_{\substack{v \in \mathcal{B}(\infty) \\ \mathrm{BZL}(v) = (b_1,\cdots,b_N) \\ \text{if } b_i \text{ is uncircled then } n|b_i}} (1-q^{-1})^{s(v)} \mathbf{z}^{-\mathrm{wt}(v)}.$$

Now we argue that this may actually be written

$$\prod_{\alpha\in\Phi^+} \frac{1-q^{-1}\mathbf{z}^{n\alpha}}{1-\mathbf{z}^{n\alpha}} = \sum_{\substack{v \in \mathcal{B}(\infty) \\ \mathrm{BZL}(v) = (b_1,\cdots,b_N) \\ n|b_i \text{ for all } i}} (1-q^{-1})^{s(v)} \mathbf{z}^{-\mathrm{wt}(v)}. \tag{9}$$

Thus we claim that if $n|b_i$ for all uncircled b_i then n divides all b_i, whether circled or not. Indeed, if b_i is circled, then either it is zero (hence a multiple of n) or, $b_i = b_{i+1}$. If b_{i+1} is circled, then $n|b_{i+1}$ so $n|b_i$, and the claim is proved; otherwise, we may repeat the argument. We have $b_i = b_{i+1} = \ldots = b_j$ and the last b_j is uncircled, so $n|b_j$ and therefore $n|b_i$. (This is observation also appears as the "Circling Lemma" in [1].) Thus we are reduced to proving (9).

Now Kashiwara [8] proved a similarity property of crystals: let λ be a dominant weight. Then there exists a similarity map that we will denote $n\cdot : \mathcal{B}_\lambda \longrightarrow \mathcal{B}_{n\lambda}$ such that $\mathrm{wt}(n \cdot v) = n\ \mathrm{wt}(v)$ and $f_i^n(n \cdot v) = n \cdot (f_i v)$. It follows from the description of $\mathcal{B}(\infty)$ that there exists a corresponding similarity map $n\cdot : \mathcal{B}(\infty) \longrightarrow \mathcal{B}(\infty)$, and we may summarize what we have learned by saying that the right-hand side of (8) is the sum over v in the image of the similarity map. Pulling the sum back to $\mathcal{B}(\infty)$ through the similarity map, we may now apply Theorem 2 (with $\mathbf{z}^n$ replacing $\mathbf{z}$), since that Theorem proves (9) in the $n = 1$ case. $\square$

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