EVENT: Start with the initial **nqthm** theory.

EVENT: For efficiency, compile those definitions not yet compiled.

EVENT: Add the shell *zn*, with recognizer function symbol *znp* and 2 accessors: *pos*, with type restriction (one-of numberp) and default value zero; *neg*, with type restriction (one-of numberp) and default value zero.

DEFINITION:
zlessp $(x,\ y) = ((\text{pos}(x) + \text{neg}(y)) < (\text{neg}(x) + \text{pos}(y)))$

DEFINITION: zlesseqp $(x,\ y) = (\neg\ \text{zlessp}(y,\ x))$

DEFINITION:
zmax $(x,\ y)$
= **if** zlessp $(x,\ y)$ **then** y
else x **endif**

DEFINITION:
zmin $(x,\ y)$
$=$ **if** zlessp $(x,\ y)$ **then** x
else y **endif**

DEFINITION: zsub1 (x) = zn (pos (x), 1 + neg (x))

DEFINITION:
pzdifference $(x,\ y)$ = ((pos (x) + neg (y)) − (neg (x) + pos (y)))

DEFINITION:
m1 $(x,\ y,\ z)$
$=$ **if** zlesseqp $(x,\ y)$ **then** `0`
else `1` **endif**

DEFINITION:
m2 $(x,\ y,\ z)$ = pzdifference (zmax $(x$, zmax $(y,\ z))$, zmin $(x$, zmin $(y,\ z)))$

DEFINITION: m3 $(x,\ y,\ z)$ = pzdifference $(x$, zmin $(x$, zmin $(y,\ z)))$

DEFINITION:
tak0 $(x,\ y,\ z)$
$=$ **if** zlesseqp $(x,\ y)$ **then** y
elseif zlesseqp $(y,\ z)$ **then** z
else x **endif**

DEFINITION:
m $(x,\ y,\ z)$ = cons (m1 $(x,\ y,\ z)$, cons (m2 $(x,\ y,\ z)$, cons (m3 $(x,\ y,\ z)$, **nil**)))

THEOREM: tak0-satisfies-tak-equation
tak0 $(x,\ y,\ z)$
$=$ **if** zlesseqp $(x,\ y)$ **then** y
else tak0 (tak0 (zsub1 (x), y, z),
tak0 (zsub1 (y), z, x),
tak0 (zsub1 (z), x, y)) **endif**

THEOREM: m1-lesseqp-0
($\neg$ zlesseqp $(x,\ y)$)
$\rightarrow$ (m1 $(x,\ y,\ z)$
$\not<$ m1 (tak0 (zsub1 (x), y, z), tak0 (zsub1 (y), z, x), tak0 (zsub1 (z), x, y)))

THEOREM: m1-lesseqp-1
($\neg$ zlesseqp $(x,\ y)$) $\rightarrow$ (m1 $(x,\ y,\ z)$ $\not<$ m1 (zsub1 (x), y, z))

THEOREM: m1-lesseqp-2
($\neg$ zlesseqp $(x,\ y)$) $\rightarrow$ (m1 $(x,\ y,\ z)$ $\not<$ m1 (zsub1 (y), z, x))

THEOREM: m1-lesseqp-3
$(\neg\ \text{zlesseqp}\,(x,\ y)) \rightarrow (\text{m1}\,(x,\ y,\ z) \not< \text{m1}\,(\text{zsub1}\,(z),\ x,\ y))$

THEOREM: m2-lesseqp-0
$$(\neg\ \text{zlesseqp}\,(x,\ y)) \rightarrow\ (\text{m2}\,(x,\ y,\ z) \not< \text{m2}\,(\text{tak0}\,(\text{zsub1}\,(x),\ y,\ z),\ \text{tak0}\,(\text{zsub1}\,(y),\ z,\ x),\ \text{tak0}\,(\text{zsub1}\,(z),\ x,\ y)))$$

THEOREM: m2-lesseqp-1
$(\neg\ \text{zlesseqp}\,(x,\ y)) \rightarrow (\text{m2}\,(x,\ y,\ z) \not< \text{m2}\,(\text{zsub1}\,(x),\ y,\ z))$

THEOREM: m2-lesseqp-2
$$((\neg\ \text{zlesseqp}\,(x,\ y)) \land (\text{m1}\,(\text{zsub1}\,(y),\ z,\ x) = \text{m1}\,(x,\ y,\ z))) \rightarrow\ (\text{m2}\,(x,\ y,\ z) \not< \text{m2}\,(\text{zsub1}\,(y),\ z,\ x))$$

THEOREM: m2-lesseqp-3
$$((\neg\ \text{zlesseqp}\,(x,\ y)) \land (\text{m1}\,(\text{zsub1}\,(z),\ x,\ y) = \text{m1}\,(x,\ y,\ z))) \rightarrow\ (\text{m2}\,(x,\ y,\ z) \not< \text{m2}\,(\text{zsub1}\,(z),\ x,\ y))$$

THEOREM: m3-lessp-0
$$((\neg\ \text{zlesseqp}\,(x,\ y)) \land\ (\text{m1}\,(\text{tak0}\,(\text{zsub1}\,(x),\ y,\ z),\ \text{tak0}\,(\text{zsub1}\,(y),\ z,\ x),\ \text{tak0}\,(\text{zsub1}\,(z),\ x,\ y)) = \text{m1}\,(x,\ y,\ z))) \rightarrow\ (\text{m3}\,(\text{tak0}\,(\text{zsub1}\,(x),\ y,\ z),\ \text{tak0}\,(\text{zsub1}\,(y),\ z,\ x),\ \text{tak0}\,(\text{zsub1}\,(z),\ x,\ y)) < \text{m3}\,(x,\ y,\ z))$$

THEOREM: m3-lessp-1
$$((\neg\ \text{zlesseqp}\,(x,\ y)) \land (\text{m1}\,(\text{zsub1}\,(x),\ y,\ z) = \text{m1}\,(x,\ y,\ z))) \rightarrow\ (\text{m3}\,(\text{zsub1}\,(x),\ y,\ z) < \text{m3}\,(x,\ y,\ z))$$

THEOREM: m3-lessp-2
$$((\neg\ \text{zlesseqp}\,(x,\ y)) \land (\text{m1}\,(\text{zsub1}\,(y),\ z,\ x) = \text{m1}\,(x,\ y,\ z))) \rightarrow\ (\text{m3}\,(\text{zsub1}\,(y),\ z,\ x) < \text{m3}\,(x,\ y,\ z))$$

THEOREM: m3-lessp-3
$$((\neg\ \text{zlesseqp}\,(x,\ y)) \land (\text{m1}\,(\text{zsub1}\,(z),\ x,\ y) = \text{m1}\,(x,\ y,\ z))) \rightarrow\ (\text{m2}\,(\text{zsub1}\,(z),\ x,\ y) < \text{m2}\,(x,\ y,\ z))$$

EVENT: Disable zlessp.

EVENT: Disable m1.

EVENT: Disable m2.

EVENT: Disable m3.

Event: Disable tak0.

Event: Disable zsub1.

Definition:
make-ordinal3 (x)
$=$ cons (cons (1 + car (x), 0), cons (1 + cadr (x), fix (caddr (x))))

Theorem: ordinalp-make-ordinal3
ordinalp (make-ordinal3 (x))

Definition:
lex3 (x, y) = ord-lessp (make-ordinal3 (x), make-ordinal3 (y))

Theorem: m-goes-down-1
$(\neg$ zlesseqp $(x, y)) \rightarrow$ lex3 (m (zsub1 (x), y, z), m (x, y, z))

Theorem: m-goes-down-2
$(\neg$ zlesseqp $(x, y)) \rightarrow$ lex3 (m (zsub1 (y), z, x), m (x, y, z))

Theorem: m-goes-down-3
$(\neg$ zlesseqp $(x, y)) \rightarrow$ lex3 (m (zsub1 (z), x, y), m (x, y, z))

Theorem: m-goes-down-0
$(\neg$ zlesseqp $(x, y))$
$\rightarrow$ lex3 (m (tak0 (zsub1 (x), y, z), tak0 (zsub1 (y), z, x), tak0 (zsub1 (z), x, y)),
m (x, y, z))

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