Least-Squares Circle Fit

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Given a finite set of points in $\mathbb{R}^2$, say $\{\,(x_i, y_i)\,|\,0\,\leq\,i\,<\,N\,\}$, we want to find the circle that "best" (in a least-squares sense) fits the points. Define

$$\overline{x}\;=\;\frac{1}{N}\,\sum_i\,x_i \quad \text{and} \quad \overline{y}\;=\;\frac{1}{N}\,\sum_i\,y_i$$

and let $u_i\;=\;x_i - \overline{x},\quad v_i\;=\;y_i - \overline{y}$ for $0\,\leq\,i\,<\,N$. We solve the problem first in (u, v) coordinates, and then transform back to (x, y).

Let the circle have center (u_c, v_c) and radius R. We want to minimize $S\;=\;\sum_i\,(\,g(u_i, v_i)\,)^2$, where $g(u, v)\;=\;(u - u_c)^2 + (v - v_c)^2 - \alpha$, and where $\alpha\;=\;R^2$. To do that, we differentiate $S(\alpha, u_c, v_c)$.

$$\frac{\partial S}{\partial \alpha}\;=\;2\,\sum_i\,g(u_i, v_i)\,\frac{\partial g}{\partial \alpha}(u_i, v_i)$$

$$=\;-2\,\sum_i\,g(u_i, v_i)$$

Thus $\partial S/\partial \alpha\;=\;0$ iff

$$\boxed{\sum_i\,g(u_i, v_i)\;=\;0} \qquad\qquad \textbf{Eq. 1}$$

Continuing, we have

$$\frac{\partial S}{\partial u_c}\;=\;2\,\sum_i\,g(u_i, v_i)\,\frac{\partial g}{\partial u_c}(u_i, v_i)$$

$$=\;2\,\sum_i\,g(u_i, v_i)\,2(u_i - u_c)(-1)$$

$$=\;-4\,\sum_i\,(u_i - u_c)\,g(u_i, v_i)$$

$$=\;-4\,\sum_i\,u_i\,g(u_i, v_i) + 4\,u_c\,\underbrace{\sum_i\,g(u_i, v_i)}_{=\;0\text{ by }\textbf{Eq. 1}}$$

Thus, in the presence of **Eq. 1**, $\partial S/\partial u_c\;=\;0$ holds iff

$$\boxed{\sum_i\,u_i\,g(u_i, v_i)\;=\;0} \qquad\qquad \textbf{Eq. 2}$$

Similarly, requiring $\partial S/\partial v_c\;=\;0$ gives

$$\boxed{\sum_i\,v_i\,g(u_i, v_i)\;=\;0} \qquad\qquad \textbf{Eq. 3}$$

Expanding **Eq. 2** gives

$$\sum_i u_i \left[u_i^2 - 2\,u_i\,u_c + u_c^2 + v_i^2 - 2\,v_i\,v_c + v_c^2 - \alpha \right] \;=\; 0$$

Defining $S_u \;=\; \sum_i u_i, \; S_{uu} \;=\; \sum_i u_i^2, \; etc.$, we can rewrite this as

$$S_{uuu} - 2\,u_c\,S_{uu} + u_c^2\,S_u + S_{uvv} - 2\,v_c\,S_{uv} + v_c^2\,S_u - \alpha\,S_u \;=\; 0$$

Since $S_u \;=\; 0$, this simplifies to

$$u_c\,S_{uu} + v_c\,S_{uv} \;=\; \frac{1}{2}\left(S_{uuu} + S_{uvv} \right) \qquad\qquad \textbf{Eq. 4}$$

In a similar fashion, expanding **Eq. 3** and using $S_v \;=\; 0$ gives

$$u_c\,S_{uv} + v_c\,S_{vv} \;=\; \frac{1}{2}\left(S_{vvv} + S_{vuu} \right) \qquad\qquad \textbf{Eq. 5}$$

Solving **Eq. 4** and **Eq. 5** simultaneously gives (u_c, v_c). Then the center (x_c, y_c) of the circle in the original coordinate system is $(x_c, y_c) \;=\; (u_c, v_c) + (\overline{x}, \overline{y})$.

To find the radius R, expand **Eq. 1**:

$$\sum_i \left[u_i^2 - 2\,u_i\,u_c + u_c^2 + v_i^2 - 2\,v_i\,v_c + v_c^2 - \alpha \right] \;=\; 0$$

Using $S_u \;=\; S_v \;=\; 0$ again, we get

$$N\left(u_c^2 + v_c^2 - \alpha \right) + S_{uu} + S_{vv} \;=\; 0$$

Thus

$$\alpha \;=\; u_c^2 + v_c^2 + \frac{S_{uu} + S_{vv}}{N} \qquad\qquad \textbf{Eq. 6}$$

and, of course, $R \;=\; \sqrt{\alpha}$.

See the next page for an example!

Example : Let's take a few points from the parabola $y = x^2$ and fit a circle to them. Here's a table giving the points used:

i	x_i	y_i	u_i	v_i
0	0.000	0.000	-1.500	-3.250
1	0.500	0.250	-1.000	-3.000
2	1.000	1.000	-0.500	-2.250
3	1.500	2.250	0.000	-1.000
4	2.000	4.000	0.500	0.750
5	2.500	6.250	1.000	3.000
6	3.000	9.000	1.500	5.750

Here we have $N = 7$, $\overline{x} = 1.5$, and $\overline{y} = 3.25$. Also, $S_{uu} = 7$, $S_{uv} = 21$, $S_{vv} = 68.25$, $S_{uuu} = 0$, $S_{vvv} = 143.81$, $S_{uvv} = 31.5$, $S_{vuu} = 5.25$. Thus (using **Eq. 4** and **Eq. 5**) we have the following 2×2 linear system for (u_c, v_c):

$$\begin{bmatrix} 7 & 21 \\ 21 & 68.25 \end{bmatrix} \begin{bmatrix} u_c \\ v_c \end{bmatrix} = \begin{bmatrix} 15.75 \\ 74.531 \end{bmatrix}$$

Solving this system gives $(u_c, v_c) = (-13.339, 5.1964)$, and thus $(x_c, y_c) = (-11.839, 8.4464)$. Substituting these values into **Eq. 6** gives $\alpha = 215.69$, and hence $R = 14.686$. A plot of this example appears below.