Great Circle Distances and Bearings Between Two Locations

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If we're given two points A and B on the Earth (assumed spherical with radius R), then how can we calculate the distance between the points and the direction (bearing) of one point from the other?

Before we get much into this, let's say a few things about our conventions. We'll denote latitude by ϕ. North latitude will be considered positive and south latitude negative. Longitude will be denoted by L. West longitude is considered positive and east longitude negative. Point A will have latitude ϕ_A and longitude L_A. Similarly, point B will have latitude ϕ_B and longitude L_B.

Distance will be denoted by D, and bearing by β. Bearings will be measured eastward from north. For example, a bearing of $0°$ is north, while a bearing of $90°$ is east. South is represented by a bearing of $180°$, and west is indicated by a bearing of $270°$ (or $-90°$).

To keep this writeup short, we won't be deriving the formulas we use. However, we will give some numerical examples.

First, let's discuss distance. Distance on the Earth is obtained by first calculating an angle θ from the following formula:

$$\cos\theta \;=\; \sin\phi_A\,\sin\phi_B + \cos\phi_A\,\cos\phi_B\,\cos\Delta L$$

where $\Delta L = L_A - L_B$. Once the value of θ (in radians) is known, then the distance D between the two points is $D = R\theta$. The units of distance will be those of R.

Example. Suppose we choose point A to be Boulder, Colorado. Then $\phi_A = 40°\,01'$ N $= 40.0167°$, and $L_A = 105°\,17'$ W $= 105.2833°$. We'll choose point B to be Wallaroo, Australia. Then $\phi_B = 33°\,56'$ S $= -33.9333°$, and $L_B = 137°\,39'$ E $= -137.65°$. Also, we have $\Delta L = 242.9333°$. We'll take the radius R of the Earth to be 6378.14 km.

First, we calculate θ:

$$\begin{aligned}
\cos\theta &= \sin\phi_A\,\sin\phi_B + \cos\phi_A\,\cos\phi_B\,\cos\Delta L \\
&= (\sin 40.0167°)(\sin(-33.9333°)) + (\cos 40.0167°)(\cos(-33.9333°))(\cos 242.9333°) \\
&= (0.643)(-0.5582) + (0.7659)(0.8297)(-0.455) \\
&= -0.6481
\end{aligned}$$

This gives $\theta = \cos^{-1}(-0.6481) = 130.4° = 2.2759$ radians. Then finally we have

$$D = R\theta = (6378.14)(2.2759) \approx 14{,}520 \text{ km.}$$

We should note that when the points A and B are very close together, say within a few kilometers of each other (a situation that arises frequently in practice), the above method for calculating distance becomes numerically unstable. The reason for this is that the angle θ in such cases is nearly zero, and taking the inverse cosine of a number very close to one is what gives rise to the numerical instability. The same trouble appears when A and B are nearly antipodal (that is, when A and B are almost exactly opposite points on the Earth), though that situation hardly ever arises in practice.

Because of this, navigators for centuries have known of an alternative formula involving the *haversine*. The haversine is a trig function that isn't taught in schools anymore, but it still exists and can be put to good use here. We can regard it as being defined by

$$\operatorname{hav}\alpha \;=\; \sin^2\frac{\alpha}{2}$$

The angle θ can be calculated via the haversine as follows:

$$\operatorname{hav}\theta \;=\; \operatorname{hav}\Delta\phi + \cos\phi_A\,\cos\phi_B\,\operatorname{hav}\Delta L$$

where $\Delta\phi$ is the difference in latitudes. Once θ is known, we have $D \;=\; R\,\theta$ as before. The numerical problems due to nearby points are eliminated by this approach. (The problems due to nearly antipodal points are not helped by using the haversine, but as stated earlier, this situation hardly ever arises in practice.)

Now let's look at bearings. To calculate the bearing of the point B as seen from the point A, calculate two quantities S and C as follows:

$$\begin{aligned} S &= \cos\phi_B\,\sin\Delta L \\ C &= \cos\phi_A\,\sin\phi_B - \sin\phi_A\,\cos\phi_B\,\cos\Delta L \end{aligned}$$

Then the bearing β of B from A is given by $\beta \;=\; \tan^{-1}\dfrac{S}{C}$. Caution: don't *ever* use this formula! It won't resolve the quadrant of β correctly, and you'll have big troubles if C is zero. Instead of taking an arctangent then, we'll use the `atan2` function, which is available in most programming languages. In your code you can write

```
beta = atan2(S, C);
```

(or whatever is equivalent in your programming language) and it'll work just fine.

Example. We'll use the same two locations as in our earlier example—Boulder and Wallaroo. We have

$$\begin{aligned} S &= \cos\phi_B\,\sin\Delta L \\ &= (\cos(-33.9333^\circ))(\sin 242.9333^\circ) \\ &= (0.8297)(-0.8905) \\ &= -0.7388 \end{aligned}$$

and also

$$\begin{aligned} C &= \cos\phi_A\,\sin\phi_B - \sin\phi_A\,\cos\phi_B\,\cos\Delta L \\ &= (\cos 40.0167^\circ)(\sin(-33.9333^\circ)) - (\sin 40.0167^\circ)(\cos(-33.9333^\circ))(\cos 242.9333^\circ) \\ &= (0.7659)(-0.5582) - (0.643)(0.8297)(-0.455) \\ &= -0.1848 \end{aligned}$$

Thus, finally,

$$\begin{aligned} \beta &= \texttt{atan2(-0.7388, -0.1848)} \\ &= -104.04^\circ \\ &= 255.96^\circ \end{aligned}$$

Notice that if we had used $\beta = \tan^{-1}\dfrac{S}{C}$, we would have gotten $\beta = 75.96^\circ$, which is not even in the right quadrant.

One final point—we want to caution the reader that there is in general no simple relationship between the bearing of B as seen from A and the bearing of A as seen from B. In particular, it is almost *never* the case that the two bearings differ by 180°, as one might naively expect. This means that if both bearings are needed, you really have no other option than to slog through the formula twice.

For example, we just calculated that the bearing of Wallaroo as seen from Boulder is 255.96°. Working through the formula with the two locations reversed, we find that the bearing of Boulder as seen from Wallaroo is 63.57°, which gives a difference in the two bearings of 167.61°.