MATH 651
HOMEWORK II
SPRING 2013

Problem 1. Let X be path-connected. Show that $\pi_1(X)$ is abelian if and only if the change-of-basepoint homomorphisms $\Phi_\alpha : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ do not depend on the choice of path α.

Problem 2. Show that a map $f : S^1 \longrightarrow X$ is null if and only if there exists a map $g : D^2 \longrightarrow X$ which restricts to f on $S^1 = \partial D^2$ (we say g extends the map f). (Hint: Find a homeomorphism $D^2 \cong (S^1 \times I)/(S^1 \times \{1\})$.)

Problem 3. Show that a basepoint-preserving map $f : (X, x_0) \longrightarrow (Y, y_0)$ induces a homomorphism of groups

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

by the formula $f_*([\gamma]) = [f \circ \gamma]$.

Problem 4. Given based spaces (X, x_0) and (Y, y_0), show that

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Problem 5.

(1) Let A be a set with two associative binary operations, $\cdot$ and $\star$. Suppose that $e \in A$ is a left and right unit for both $\cdot$ and $\star$. Finally, suppose that for any elements a, b, c, d in A, these operations satisfy

$$(a \cdot b) \star (c \cdot d) = (a \star c) \cdot (b \star d).$$

Show that $a \cdot b = a \star b$ and that both operations are commutative.

(2) Let G be any topological group. Use the above to show that $\pi_1(G, e)$ is necessarily abelian.

Problem 6. A **free loop** in a space X is a map $S^1 \longrightarrow X$ with no condition on the basepoint. Since $\pi_1(X, x_0) \cong [S^1, (X, x_0)]_*$, there is a natural map

$$\Lambda : \pi_1(X, x_0) \longrightarrow [S^1, X].$$

(1) Show that Λ is surjective if and only if X is path-connected.

(2) Show that $\Lambda([\gamma]) = \Lambda([\delta])$ if and only if the loops γ and δ are conjugate in $\pi_1(X, x_0)$.

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