

**MATH 651  
HOMEWORK VI  
SPRING 2013**

**Problem 1.** We showed in class that if  $B$  is connected, locally path connected, and **semilocally simply connected**, then it has a universal cover  $q : X \rightarrow B$ .

- (a) Let  $B_n$  be the circle of radius  $1/n$ , centered at the point  $(1/n, 0)$  in  $\mathbb{R}^2$ . Let  $B = \bigcup_n B_n$ . Show that  $B$  is not semilocally simply connected by showing that the point  $(0, 0)$  has no relatively simply connected neighborhood.
- (b) Given any space  $Z$ , the **cone** on  $Z$  is the space  $C(Z) = Z \times I/Z \times \{1\}$ . Show that  $C(Z)$  is semilocally simply connected.

**Problem 2.** The space  $B$  from problem 1(a) looks as if it might be the infinite wedge  $\bigvee_{\mathbb{N}} S^1$ . The universal property of the wedge gives a continuous bijection

$$\bigvee_{\mathbb{N}} S^1 \rightarrow B$$

which sends the circle labelled by  $n$  to the circle  $B_n$ . Use the following argument to show this is not a homeomorphism.

Consider, for each  $n$ , the open subset  $U_n \subseteq S^1$  which is an open interval of radian length  $1/n$  centered at  $(0, 0)$ . We have not discussed infinite wedge sums, so you may assume that  $\bigvee_n U_n \subseteq \bigvee_n S^1$  is open. Let  $W_n \subseteq B_n$  be the image of  $U_n$  under the natural homeomorphism  $S^1 \cong B_n$ . Show that  $\bigcup_n W_n \subseteq B$  is not open.

**Problem 3.** A space  $X$  is locally simply connected if, given a neighborhood  $U$  of some point  $x$ , then there exists a simply connected neighborhood  $V$  of  $x$  contained in  $U$ .

- (a) Show that if  $X$  is locally simply connected, then it is semilocally simply connected.
- (b) Show that the converse is not true by showing that the space  $C(Z)$  from problem 1(b) is not locally simply connected.

**Problem 4.**

- (a) Show that in the space  $B$  from problem 1, the point  $(0, 0)$  is **not** a nondegenerate basepoint. In other words, show that no neighborhood of  $(0, 0)$  deformation retracts onto  $(0, 0)$ .
- (b) If no basepoint of  $X$  is nondegenerate, there is a way of adding in a good basepoint without changing the homotopy type. The process, known as “attaching a whisker to  $X$ ”, is to consider the space  $X \vee I$ . If we glue  $I$  to  $X$  along the point  $0 \in I$ , show that  $1 \in I$  is a nondegenerate basepoint for  $X$ .
- (c) Show that  $X \vee I$  is homotopy equivalent to  $X$ .