Chapter 14
Interacting Particles on Finite Graphs

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There is a well-established topic "interacting particle systems", treated in the books by Griffeath [14], Liggett [17], and Durrett [11], which studies different models for particles on the infinite lattice Z^d. All these models make sense, but mostly have not been systematically studied, in the context of finite graphs. Some of these models – the voter model, the antivoter model, and the exclusion process – are related (either directly or "via duality") to interacting random walks, and setting down some basic results for these models on finite graphs (sections 3 - 5) is the main purpose of this chapter. Our focus is on applying results developed earlier in the book. With the important exception of *duality*, we do not use the deeper theory developed in the infinite setting. As usual, whether the deeper theory is applicable to the type of questions we ask in the finite setting is an interesting open question. These models are most naturally presented in continuous time, so our default convention is to work with continuous-time random walk.

We have already encountered results whose natural proofs were "by coupling", and this is a convenient place to discuss couplings in general.

1 Coupling

If X and Y are random variables with Binomial (n, p_1) and (n, p_2) distributions respectively, and if $p_1 \leq p_2$, then it is intuitively obvious that

$$P(X \geq x) \leq P(Y \geq x) \text{ for all } x. \qquad (1)$$

One could verify this from the exact formulas, but there is a more elegant non-computational proof. For $1 \leq i \leq n$ define events (A_i, B_i, C_i), independent as i varies, with $P(A_i) = p_1, P(B_i) = p_2 - p_1, P(C_i) = 1 - p_2$. And define

$$X' = \sum_i 1_{A_i} \quad = \quad \text{number of A's which occur}$$
$$Y' = \sum_i 1_{A_i \cup B_i} \quad = \quad \text{number of A's and B's which occur.}$$

Then $X' \leq Y'$, so (1) holds for X' and Y', but then because $X' \stackrel{d}{=} X$ and $Y' \stackrel{d}{=} Y$ we have proved that (1) holds for X and Y. This is the prototype of a *coupling argument*, which (in its wide sense) means

> to prove some *distributional* inequality relating two random processes X, Y by constructing versions X', Y' which satisfy some *sample path* inequality.

Our first "process" example is a somewhat analogous proof of part (a) of the following result, which abstracts slightly a result stated for random walk on distance-regular graphs (Chapter 7 Proposition yyy).

Proposition 1 *Let (X_t) be an irreducible continuous-time birth-and-death chain on states $\{0, 1, \ldots, \Delta\}$.*
(a) $\frac{P_0(X_t=i)}{\pi_i}$ is non-increasing in i, for fixed t
(b) $\frac{P_0(X_t=i)}{P_0(X_t=0)}$ is non-decreasing in t, for fixed i

Proof. Fix $i_1 \leq i_2$. Suppose we can construct processes Y_t and Z_t, distributed as the given chain started at i_1 and i_2 respectively, such that

$$Y_t \leq Z_t \text{ for all } t. \tag{2}$$

Then

$$P_{i_1}(X_t = 0) = P(Y_t = 0) \geq P(Z_t = 0) = P_{i_2}(X_t = 0).$$

But by reversibility

$$P_{i_1}(X_t = 0) = \frac{\pi_0}{\pi_{i_1}} \, P_0(X_t = i_1)$$

and similarly for i_2, establishing (a).

Existence of processes satisfying (2) is a consequence of the Doeblin coupling discussed below. The proof of part (b) involves a different technique and is deferred to section 1.3.

1.1 The coupling inequality

Consider a finite-state chain in discrete or continuous time. Fix states i, j. Suppose we construct a joint process $(X_t^{(i)}, X_t^{(j)}; t \geq 0)$ such that

$$\begin{array}{c}(X_t^{(i)}, t \geq 0) \text{ is distributed as the chain started at } i \\ (X_t^{(j)}, t \geq 0) \text{ is distributed as the chain started at } j.\end{array} \quad (3)$$

And suppose there is a random time $T \leq \infty$ such that

$$X_t^{(i)} = X_t^{(j)}, \ T \leq t < \infty. \quad (4)$$

Call such a T a *coupling time*. Then the *coupling inequality* is

$$||P_i(X_t \in \cdot) - P_j(X_t \in \cdot)|| \leq P(T > t), \ 0 \leq t < \infty. \quad (5)$$

The inequality is clear once we observe $P(X_t^{(i)} \in \cdot, T \leq t) = P(X_t^{(j)} \in \cdot, T \leq t)$. The coupling inequality provides a method of bounding the variation distance $\bar{d}(t)$ of Chapter 2 section yyy.

The most common strategy for constructing a coupling satisfying (3) is via Markov couplings, as follows. Suppose the underlying chain has state space I and (to take the continuous-time case) transition rate matrix $\mathbf{Q} = (q(i,j))$. Consider a transition rate matrix $\tilde{\mathbf{Q}}$ on the product space $I \times I$. Write the entries of $\tilde{\mathbf{Q}}$ as $\tilde{q}(i,j;k,l)$ instead of the logical-but-fussy $\tilde{q}((i,j),(k,l))$. Suppose that, for each pair (i,j) with $j \neq i$,

$$\tilde{q}(i,j;\cdot,\cdot) \text{ has marginals } q(i,\cdot) \text{ and } q(j,\cdot) \quad (6)$$

in other words $\sum_l \tilde{q}(i,j;k,l) = q(i,k)$ and $\sum_k \tilde{q}(i,j;k,l) = q(j,l)$. And suppose that

$$\begin{aligned}\tilde{q}(i,i;k,k) &= q(i,k) \text{ for all } k \\ \tilde{q}(i,i;k,l) &= 0 \text{ for } l \neq k.\end{aligned}$$

Take $(X_t^{(i)}, X_t^{(j)})$ to be the chain on $I \times I$ with transition rate matrix $\tilde{\mathbf{Q}}$ and initial position (i,j), Then (3) must hold, and $T \equiv \min\{t : X_t^{(i)} = X_t^{(j)}\}$ is a coupling time. This construction gives a *Markov coupling*, and all the examples where we use the coupling inequality will be of this form. In practice it is much more understandable to define the joint process in words

xxx red and black particles.

A particular choice of $\tilde{\mathbf{Q}}$ is

$$\tilde{q}(i,j;k,l) = q(i,k)q(j,l), \ j \neq i \tag{7}$$

in which case the joint process is called to *Doeblin coupling.* In words, the Doeblin coupling consists of starting one particle at i and the other particle at j, and letting the two particles move independently until they meet, at time $M_{i,j}$ say, and thereafter letting them stick together. In the particular case of a birth-and-death process, the particles cannot cross without meeting (in continuous time), and so if $i < j$ then $X_t^{(i)} \leq X_t^{(j)}$ for all t, the property we used at (2).

1.2 Examples using the coupling inequality

Use of the coupling inequality has nothing to do with reversibility. In fact it finds more use in the irreversible setting, where fewer alternative methods are available for quantifying convergence to stationarity. In the reversible setting, coupling provides a quick way to get bounds which usually (but not always) can be improved by other methods. Here are two examples we have seen before.

Example 2 *Random walk on the d-cube (Chapter 5 Example yyy).*

For $\mathbf{i} = (i_1, \ldots, i_d)$ and $\mathbf{j} = (j_1, \ldots, j_d)$ in $I = \{0,1\}^d$, let $D(\mathbf{i},\mathbf{j})$ be the set of coordinates u where $\mathbf{i}$ and $\mathbf{j}$ differ. Write $\mathbf{i}^u$ for the state obtained by changing the i'th coordinate of $\mathbf{i}$. Recall that in continuous time the components move independently as 2-state chains with transition rates $1/d$. In words, the coupling is "run unmatched coordinates independently until they match, and then run them together". Formally, the non-zero transitions of the joint process are

$$\begin{aligned}
\tilde{q}(\mathbf{i},\mathbf{j};\mathbf{i}^u,\mathbf{j}^u) &= 1/d \text{ if } i_u = j_u \\
\tilde{q}(\mathbf{i},\mathbf{j};\mathbf{i}^u,\mathbf{j}) &= 1/d \text{ if } i_u \neq j_u \\
\tilde{q}(\mathbf{i},\mathbf{j};\mathbf{i},\mathbf{j}^u) &= 1/d \text{ if } i_u \neq j_u.
\end{aligned}$$

For each coordinate which is initially unmatched, it takes exponential (rate $2/d$) time until it is matched, and so the coupling time T satisfies

$$T \stackrel{d}{=} \max(\xi_1, \ldots, \xi_{d_0})$$

where the (ξ_u) are independent exponential (rate $2/d$) and $d_0 = d(\mathbf{i}, \mathbf{j})$ is the initial number of unmatched coordinates. So

$$P(T \le t) = (1 - \exp(-2t/d))^{d_0}$$

and the coupling inequality bounds variation distance as

$$\bar{d}(t) \le (1 - \exp(-2t/d))^d.$$

This leads to an upper bound on the variation threshold time

$$\tau_1 \le (\frac{1}{2} + o(1)) d \log d \text{ as } d \to \infty.$$

In this example we saw in Chapter 5 that in fact

$$\tau_1 \sim \frac{1}{4} d \log d \text{ as } d \to \infty$$

so the coupling bound is off by a factor of 2.

Example 3 *Random walk on a dense regular graph (Chapter 5 Example yyy).*

Consider a r-regular n-vertex graph. Write $\mathcal{N}(v)$ for the set of neighbors of v. For any pair v, w we can define a $1-1$ map $\theta_{v,w} : \mathcal{N}(v) \to \mathcal{N}(w)$ such that $\theta_{v,w}(x) = x$ for $x \in \mathcal{N}(v) \cap \mathcal{N}(w)$. We can now define a "greedy coupling" by

$$\tilde{q}(v, w; x, \theta_{v,w}(x) = 1/r, \quad x \in \mathcal{N}(v).$$

In general one cannot get useful bounds on the coupling time T. But consider the dense case, where $r > n/2$. As observed in Chapter 5 Example yyy, here $|\mathcal{N}(v) \cap \mathcal{N}(w)| \ge 2r - n$ and so the coupled processes (X_t, Y_t) have the property that for $w \ne v$

$$P(X_{t+dt} = Y_{t+dt} | X_t = v, Y_t = w) = \frac{|\mathcal{N}(v) \cap \mathcal{N}(w)|}{r} dt \ge \frac{2r - n}{r} dt$$

implying that T satisfies

$$P(T > t) \le \exp(-(2r - n)t/r).$$

So the coupling inequality implies $\bar{d}(t) \le \exp(-(2r-n)t/r)$, and in particular the variation threshold satisfies

$$\tau_1 \le \frac{r}{2r - n}.$$

1.3 Comparisons via couplings

We now give two examples of coupling in the wide sense, to compare different processes. The first is a technical result (inequality (8) below) which we needed in Chapter 6 yyy. The second is the proof of Proposition 1(b).

Example 4 *Exit times for constrained random walk.*

Let (X_t) be discrete-time random walk on a graph G, let A be a subset of the vertices of G and let (Y_t) be random walk on the subgraph induced by A. Given $B \subset A$, let S be the first hitting time of (Y_t) on B, and let T be the first hitting time of (X_t) on $B \cup A^c$. Then

$$E_i T \leq E_i S, \ i \in A. \tag{8}$$

This is "obvious", and the reason it's obvious is by coupling. We can construct coupled processes (X', Y') with the property that, if both particles are at the same position a in A, and if X jumps to another state b in A, then Y jumps to the same state b. This property immediately implies that, for the coupled processes started at the same state in A, we have $T' \leq S'$ and hence (8).

In words, here is the coupling (X', Y'). When the particles are at different positions they jump independently. When they are at the same position, first let X' jump; if X' jumps to a vertex in A let Y' jump to the same vertex, and otherwise let Y' jump to a uniform random neighbor in A. Formally, the coupled process moves according to the transition matrix $\tilde{\mathbf{P}}$ on $G \times A$ defined by

$$\tilde{p}(x, a; y, b) = p_G(x, y)\ p_A(a, b) \text{ if } x \notin A \text{ or } x \neq a$$

$$\tilde{p}(a, a; b, b) = p_G(a, b), \ b \in A$$

$$\tilde{p}(a, a; y, b) = p_G(a, y) p_A(a, b), \ b \in A, y \in A^c$$

where p_A and p_G refer to transition probabilities for the original random walks on A and G.

Proof of Proposition 1(b). Fix $i \geq 1$. By reversibility it is sufficient to prove

$$\frac{P_0(X_t = i)}{P_0(X_t = 0)} \text{ is non-decreasing in } t\ .$$

Consider the Doeblin coupling $(X_t^{(0)}, X_t^{(i)})$ of the processes started at 0 and at i, with coupling time T. Since $X_t^{(0)} \leq X_t^{(i)}$ we have

$$P(X_t^{(i)} = 0) = P(X_t^{(0)} = 0, T \leq t)$$

and so we have to prove

$$P(T \leq t | X_t^{(0)} = 0) \text{ is non-decreasing in } t \ .$$

It suffices to show that, for $t_1 > t$,

$$P(T \leq t | X_{t_1}^{(0)} = 0) \geq P(T \leq t | X_t^{(0)} = 0)$$

and thus, by considering the conditional distribution of $X_t^{(0)}$ given $X_{t_1}^{(0)} = 0$, it suffices to show that

$$P(T \leq t | X_t^{(0)} = j) \geq P(T \leq t | X_t^{(0)} = 0) \tag{9}$$

for $j \geq 0$. So fix j and t. Write $(X_s^{(0,j)}, 0 \leq s \leq t)$ for the process conditioned on $X_0 = 0, X_t = j$. By considering time running backwards from t to 0, the processes $X^{(0,0)}$ and $X^{(0,j)}$ are the same non-homogeneous Markov chain started at the different states 0 and j, and we can use the Doeblin coupling in this non-homogeneous setting to construct versions of these processes with

$$X_s^{(0,0)} \leq X_s^{(0,j)}, \ 0 \leq s \leq t.$$

Now introduce an independent copy of the original process, started at time 0 in state i. If this process meets $X^{(0,0)}$ before time t then it must also meet $X^{(0,j)}$ before time t, establishing (9).

2 Meeting times

Given a Markov chain, the *meeting time* $M_{i,j}$ is the time at which independent copies of the chain started at i and at j first meet. Meeting times arose in the Doeblin coupling and arise in several other contexts later, so deserve a little study. It is natural to try to relate meeting times to properties such as hitting times for a single copy of the chain. One case is rather simple. Consider a distribution dist(ξ) on a group G such that

$$\xi \ \stackrel{d}{=} \ \xi^{-1}; \qquad g\xi \ \stackrel{d}{=} \ \xi g \text{ for all } g \in G.$$

Now let X_t and Y_t be independent copies of the continuization of random flight on G with step-distribution ξ. Then if we define $Z_t = X_t^{-1} Y_t$, it is easy to check that Z is itself the continuization of the random flight, but run at twice the speed, i.e. with transition rates

$$q_Z(g,h) = 2P(g\xi = h).$$

It follows that $EM_{i,j} = \frac{1}{2} E_i T_j$. The next result shows this equality holds under less symmetry, and (more importantly) that an inequality holds without any symmetry.

Proposition 5 *For a continuous-time reversible Markov chain, let T_j be the usual first hitting time and let $M_{i,j}$ be the meeting time of independent copies of the chain started at i and j. Then $\max_{i,j} EM_{i,j} \leq \max_{i,j} E_i T_j$. If moreover the chain is symmetric (recall the definition from Chapter 7 yyy) then $EM_{i,j} = \frac{1}{2} E_i T_j$.*

Proof. This is really just a special case of the cat and mouse game of Chapter 3 section yyy, where the player is using a random strategy to decide which animal to move. Write X_t and Y_t for the chains started at i and j. Write $f(x,y) = E_x T_y - E_\pi T_y$. Follow the argument in Chapter 3 yyy to verify

$$S_t \equiv (2t + f(X_t, Y_t); 0 \leq t \leq M_{ij}) \text{ is a martingale.}$$

Then

$$\begin{aligned}
E_i T_j - E_\pi T_j &= ES_0 \\
&= ES_{M_{i,j}} \text{ by the optional sampling theorem} \\
&= 2EM_{ij} + Ef(X_{M_{ij}}, Y_{M_{ij}}) \\
&= 2EM_{i,j} - E\bar{t}(X_{M_{i,j}}), \text{ where } \bar{t}(k) = E_\pi T_j.
\end{aligned}$$

In the symmetric case we have $\bar{t}(k) = \tau_0$ for all k, establishing the desired equality. In general we have $\bar{t}(k) \leq \max_{i,j} E_i T_j$ and the stated inequality follows.

Remarks. Intuitively the bound in Proposition 5 should be reasonable for "not too asymmetric" graphs. But on the n-star (Chapter 5 yyy), for example, we have $\max_{i,j} EM_{i,j} = \Theta(1)$ while $\max_{i,j} E_i T_j = \Theta(n)$. The "$\Theta(1)$" in that example comes from concentration of the stationary distribution, and on a regular graph we can use Chapter 3 yyy to obtain

$$\sum_i \sum_j \pi_i \pi_j EM_{i,j} \geq \frac{(n-1)^2}{2n}.$$

But we can construct regular graphs which mimic the n-star in the sense that $\max_{i,j} EM_{i,j} = o(\tau_0)$. A more elaborate result, which gives the correct order of magnitude on the n-star, was given in Aldous [3].

Proposition 6 *For a continuous-time reversible chain,*

$$\max_{i,j} EM_{i,j} \leq K \left(\sum_i \frac{\pi_i}{\max(E_\pi T_i, \tau_1)} \right)^{-1}$$

for an absolute constant K.

The proof is too lengthy to reproduce, but let us observe as a corollary that we can replace the $\max_{i,j} E_i T_j$ bound in Proposition 5 by the *a priori* smaller quantity τ_0, at the expense of some multiplicative constant.

Corollary 7 *For a continuous-time reversible chain,*

$$\max_{i,j} EM_{i,j} \leq K\tau_0$$

for an absolute constant K.

Proof of Corollary 7. First recall from Chapter 4 yyy the inequality

$$\tau_1 \leq 66\tau_0. \tag{10}$$

"Harmonic mean $\leq$ arithmetic mean" gives the first inequality in

$$\begin{aligned} \left(\sum_i \frac{\pi_i}{\max(E_\pi T_i, \tau_1)} \right)^{-1} &\leq \sum_i \pi_i \max(E_\pi T_i, \tau_1) \\ &\leq \sum_i \pi_i (E_\pi T_i + \tau_1) \\ &\leq \tau_0 + \tau_1 \\ &\leq 67\tau_0 \text{ by (10)} \end{aligned}$$

and so the result is indeed a corollary of Proposition 6.

Two interesting open problems remain. First, does Proposition 6 always give the right order of magnitude, i.e.

Open Problem 8 *In the setting of Proposition 6, does there exist an absolute constant K such that*

$$K \max_{i,j} EM_{i,j} \geq \left(\sum_i \frac{\pi_i}{\max(E_\pi T_i, \tau_1)} \right)^{-1}$$

The other open problem is whether some modification of the proof of Proposition 5 would give a *small* constant K in Corollary 7. To motivate this question, note that the coupling inequality applied to the Doeblin coupling shows that for any chain $\bar{d}(t) \leq \max_{i,j} P(M_{i,j} > t)$. Then Markov's inequality shows that the variation threshold satisfies $\tau_1 \leq e \max_{i,j} EM_{i,j}$. In the reversible setting, Proposition 5 now implies $\tau_1 \leq eK\tau_0$ where K is the constant in Corollary 7. So a direct proof of Corollary 7 with small K would improve the numerical constant in inequality (10).

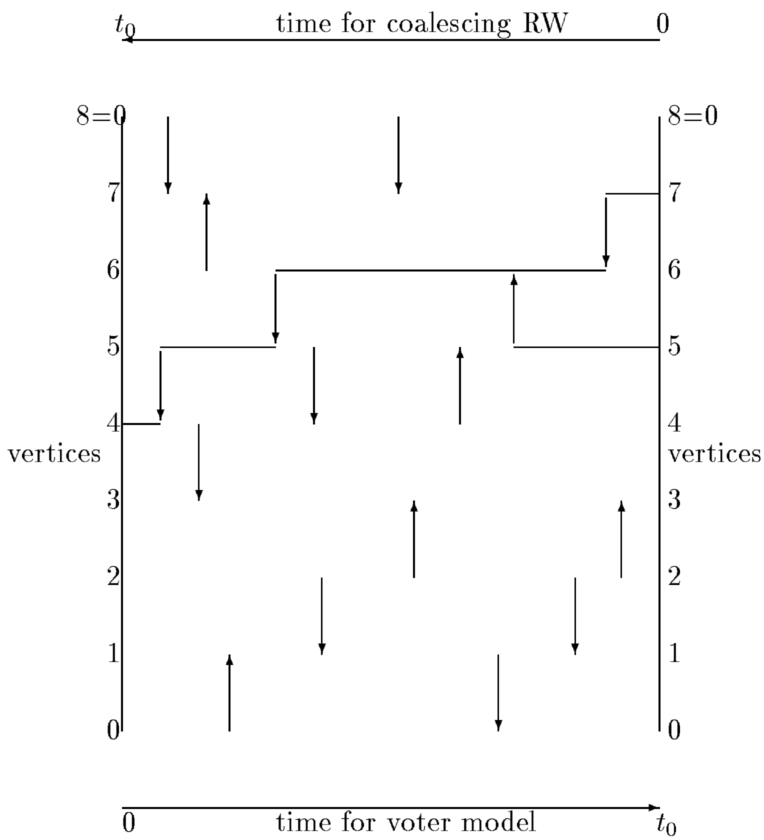
3 Coalescing random walks and the voter model

Sections 3 and 4 treat some models whose behavior relates "by duality" to random-walk-type processes. It is possible (see Notes) to fit all our examples into an abstract duality framework, but for the sake of concreteness I haven't done so. Note that for simplicity we work in the setting of *regular* graphs, though the structural results go over to general graphs and indeed to weighted graphs.

Fix a r-regular n-vertex graph G. In the *voter model* we envisage a person at each vertex. Initially each person has a different opinion (person i has opinion i, say). As time passes, opinions change according to the following rule. For each person i and each time interval $[t, t + dt]$, with chance dt the person chooses uniformly at random a neighbor (j, say) and changes (if necessary) their opinion to the current opinion of person j. Note that the total number of existing opinions can only decrease with time, and at some random time $C_{\mathbf{vm}}$ there will be only one "consensus" opinion.

In the *coalescing random walk* process, at time 0 there is one particle at each vertex. These particles perform independent continuous-time random walks on the graph, but when particles meet they coalesce into clusters and the cluster thereafter sticks together and moves as a single random walk. So at time t there are clusters, composed of one or more particles, at distinct vertices, and during $[t, t+dt]$ each cluster has chance dt to move to a random neighbor and (if that neighbor is occupied by another cluster) to coalesce with that other cluster. Note that the total number of clusters can only decrease with time, and at some random time $C_{\mathbf{crw}}$ the particles will have all coalesced into a single cluster.

Remarkably, the two random variables $C_{\mathbf{vm}}$ and $C_{\mathbf{crw}}$ associated with the two models turn out to have the same distribution, depending only on the graph G. The explanation is that the two processes can be obtained by looking at the same picture in two different ways. Here's the picture. For each edge e and each direction on e, create a Poisson process of rate $1/r$. In the figure, G is the 8-cycle, "time" is horizontal and an event of the Poisson process for edge (i, j) at time t is indicated by a vertical arrow $i \to j$ at time t.



In the voter model, we interpret time as increasing left-to-right from 0 to t_0, and we interpret an arrow $j \to i$ at time t as meaning that person j adopts i's opinion a time t. In the coalescing random walk model, we interpret time as increasing right-to-left from 0 to t_0, and we interpret an arrow $j \to i$ at time t as meaning that the cluster (if any) at state j at time t jumps to state i, and coalesces with the cluster at i (if any).

So for fixed t_0, we can regard both processes as constructed from the same Poisson process of "arrows". For any vertices i, j, k the event (for the voter model)

> The opinions of persons i and j at time t_0 are both the opinion initially held by k

is exactly the same as the event (for the coalescing random walk process)

> The particles starting at i and at j have coalesced before time t_0 and their cluster is at vertex k at time t_0.

The horizontal lines in the figure indicate part of the trajectories. In terms of the coalescing random walks, the particles starting at 5 and 7 coalesce, and the cluster is at 4 at time t_0. In terms of the voter model, the opinion initially held by person 4 is held by persons 5 and 7 at time t_0. The reader may (provided this is not a library book) draw in the remaining trajectories, and will find that exactly 3 of the initial opinions survive, i.e. that the random walks coalesce into 3 clusters.

In particular, the event (for the voter model)

> By time t_0 everyone's opinion is the opinion initially held by person k

is exactly the same as the event (for the coalescing random walk process)

> All particles have coalesced by time t_0, and the cluster is at k at time t_0.

So $P(C_{\mathbf{vm}} \leq t_0) = P(C_{\mathbf{crw}} \leq t_0)$, and these two times (which we shall now call just C) do indeed have the same distribution.

We now discuss bounds on EC. It is interesting that the two models give us quite different ways to prove bounds. Bounding EC here is somewhat analogous to the problem of bounding mean cover time, discussed in Chapter 6.

3.1 A bound using the voter model

Recall from Chapter 4 yyy the definition of the Cheeger time constant τ_c. In the present setting of a r-regular graph, the definition implies that for any subset A of vertices

$$\text{number of edges linking } A \text{ and } A^c \geq \frac{r|A|(n-|A|)}{n\tau_c}. \tag{11}$$

Proposition 9 *(a) If G is s-edge-connected then $EC \leq \frac{rn^2}{4s}$.*
(b) $EC \leq 2 \log 2 \ \tau_c n$.

Proof. The proof uses two ideas. The first is a straightforward comparison lemma.

Lemma 10 *Let (X_t) be a continuous-time chain on states I. Let $f : I \to \{0, 1, \ldots, n\}$ be such that $f(X_t)$ never jumps by more than 1, and such that there exist strictly positive constants $\gamma, a(1), \ldots, a(n-1)$ such that, for each $1 \leq i \leq n-1$ and each state x with $f(x) = i$,*

$$\frac{P(f(X_{t+dt}) = i+1 | X_t = x)}{dt} = \frac{P(f(X_{t+dt}) = i-1 | X_t = x)}{dt} \geq \gamma a(i).$$

Then

$$E_x T_{\{f^{-1}(0), f^{-1}(n)\}} \leq \gamma^{-1} E^*_{f(x)} T^*_{\{0,n\}}$$

where $E^* T^*$ refers to mean hitting time for the chain X^* on states $\{0, 1, \ldots, n\}$ with transition rates

$$q_{i,i+1} = q_{i,i-1} = a(i).$$

The second idea is that our voter model can be used to define a less-informative "two-party" model. Fix an initial set B of vertices, and group the opinions of the individuals in B into one political party ("Blues") and group the remaining opinions into a second party ("Reds"). Let N^B_t be the number of Blues at time t and let $C^B \leq C$ be the first time at which everyone belongs to the same party. Then

$$\begin{aligned} & P(N^B_{t+dt} = N^B_t + 1 | \text{ configuration at time } t) \\ & = P(N^B_{t+dt} = N^B_t - 1 | \text{ configuration at time } t) \\ = & \ \frac{\text{number of edges linking Blue - Red vertices at time } t}{r} \ dt. \end{aligned} \tag{12}$$

Cases (a) and (b) now use Lemma 10 with different comparison chains. For (a), while both parties coexist, the number of edges being counted in (12) is at least s. To see this, fix two vertices v, x of different parties, and consider (c.f. Chapter 6 yyy) a collection of s edge-disjoint paths from v to x. Each path must contain at least one edge linking Blue to Red. Thus the quantity (12) is at least $\frac{s}{r}\ dt$. If that quantity were $\frac{1}{2}\ dt$ then N^B_t would be continuous time random walk on $\{0, \ldots, n\}$ and the quantity EC^B would be the mean time, starting at $|B|$, for simple random walk to hit 0 or n, which by Chapter 5 yyy we know equals $|B|(n-|B|)$. So using Lemma 10

$$EC^B \leq \frac{r}{2s}|B|(n-|B|) \leq \frac{rn^2}{8s}. \tag{13}$$

For (b), use (11) to see that the quantity (12) must be at least $\frac{N_t^B(n-N_t^B)}{n\tau_c}\,dt$. Consider for comparison the chain on $\{0,\ldots,n\}$ with transition rates $q_{i,i+1} = q_{i,i-1} = i(n-i)/n$. For this chain

$$\begin{aligned} E_i^* T^*_{\{0,n\}} &= \sum_{j=1}^{n-1} E_i(\text{ time spent in } j \text{ before } T^*_{\{0,n\}} \\ &= \sum_{j=1}^{n-1} m_i(j) \frac{\frac{1}{2}}{j(n-j)/n} \end{aligned}$$

where $m_i(j)$ is the mean occupation time for simple symmetric random walk and the second term is the speed-up factor for the comparison chain under consideration. Using the formula for $m_i(j)$ from Chapter 5 yyy,

$$E_i^* T^*_{\{0,n\}} = i \sum_{j=i}^{n-1} \frac{1}{j} + (n-i) \sum_{j=1}^{i-1} \frac{1}{n-j} \leq n \log 2.$$

So using Lemma 10

$$EC^B \leq \tau_c n \log 2. \tag{14}$$

Finally, imagine choosing $\mathbf{B}$ at random by letting each individual initially be Blue or Red with probability 1/2 each, independently for different vertices. Then by considering some two individuals with different opinions at time t,

$$P(C^{\mathbf{B}} > t) \geq \frac{1}{2} P(C > t).$$

Integrating over t gives $EC \leq 2EC^{\mathbf{B}}$. But $EC^{\mathbf{B}} \leq \max_B EC^B$, so the Proposition follows from (13) and (14).

3.2 A bound using the coalescing random walk model

The following result bounds the mean coalescing time in terms of mean hitting times of a single random walk.

Proposition 11 $EC \leq e(\log n + 2) \max_{i,j} E_i T_j$

Proof. We can construct the coalescing random walk process in two steps. Order the vertices arbitrarily as $i_1, \ldots, i_n$. First let the n particles perform independent random walks for ever, with the particles starting at i, j first meeting at time $M_{i,j}$, say. Then when two particles meet, let them cluster and follow the future path of the lower-labeled particle. Similarly, when two clusters meet, let them cluster and follow the future path of the lowest-labeled particle in the combined cluster. Using this construction, we see

$$C_{\mathbf{crw}} \leq \max_j M_{i_1,j}. \tag{15}$$

Now let $m^* \equiv \max_{i,j} EM_{i,j}$. Using subexponentiality as in Chapter 2 section yyy,

$$P(M_{i,j} > t) \leq \exp(-\lfloor \frac{t}{em^*} \rfloor). \tag{16}$$

and so

$$\begin{aligned} EC &= \int_0^\infty P(C > t)dt \\ &\leq \int_0^\infty \min(1, \sum_j P(M_{i_1,j} > t)) \ dt \text{ by (15)} \\ &\leq \int_0^\infty \min(1, ne \exp(-\frac{t}{em^*})) \ dt \text{ by (16)} \\ &= em^*(2 + \log n) \end{aligned}$$

where the final equality is the calculus fact

$$\int_0^\infty \min(1, Ae^{-at}) \ dt = a^{-1}(1 + \log A), \quad A \geq 1.$$

The result now follows from Proposition 5.

3.3 Conjectures and examples

The complete graph. On the complete graph, the number K_t of clusters at time t in the coalescing random walk model is itself the continuous-time chain with transition rates

$$q_{k,k-1} = k(k-1)/(n-1); \quad n \geq k \geq 2.$$

Since $C_{\mathbf{crw}}$ is the time taken for K_t to reach state 1,

$$EC = \sum_{k=2}^{m} \frac{n-1}{k(k-1)} = \frac{(n-1)^2}{n} \sim n.$$

Recall from Chapter 7 yyy that in a vertex-transitive graph with τ_2/τ_0 small, the first hitting time to a typical vertex has approximately exponential distribution with mean τ_0. Similarly, the meeting time $M_{i,j}$ for typical i, j has approximately exponential distribution with mean $\tau_0/2$. It seems intuitively clear that, for fixed small k, when the number of clusters first reaches k these clusters should be approximately uniformly distributed, so that the mean further time until one of the $k(k-1)/2$ pairs coalesce should be about $\frac{\tau_0}{k(k-1)}$. Repeating the analysis of the complete graph suggests

Open Problem 12 *Prove that for a sequence of vertex-transitive graphs with $\tau_2/\tau_0 \to 0$, we have $EC \sim \tau_0$.*

In the general setting, there is good reason to believe that the *log* term in Proposition 11 can be removed.

Open Problem 13 *Prove there exists an absolute constant K such that on any graph*

$$EC \leq K \max_{v,w} E_v T_w.$$

The assertion of Open Problem 12 in the case of the torus Z_m^d for $d \geq 2$ was proved by Cox [5]. A detailed outline is given in [11] Chapter 10b, so we will not repeat it here, but see the remark in section 3.5 below.

xxx discuss $d = 1$?

3.4 Voter model with new opinions

For a simple variation of the voter model, fix a parameter $0 < \lambda < \infty$ and suppose that each individual independently decides at rate λ (i.e. with chance λdt in each time interval $[t, t + dt]$) to adopt a new opinion, not previously held by anyone. For this process we may take as state space the set of partitions $\mathbf{A} = \{A_1, A_2, \ldots\}$ of the vertex-set of the underlying graph G, where two individuals have the same opinion iff they are in the same component A of $\mathbf{A}$. The duality relationship holds with the following modification. In the dual process of coalescing random walks, each cluster "dies" at rate λ. Thus in the dual process run forever, each "death" of a cluster involves particles started at some set A of vertices, and this partition $\mathbf{A} = \{A_i\}$ of vertices into components is (by duality) distributed as the stationary distribution of the voter model with new opinions. This is the *unique* stationary distribution, even though (e.g. on the n-cycle) the Markov chain may not be irreducible because of the existence of transient states.

The time to approach stationarity in this model is controlled by the time $\tilde{C}$ for the dual process to die out completely. Clearly $E\tilde{C} \leq EC + 1/\lambda$, where C is the coalescing time discussed in previous sections, and we do not have anything new to say beyond what is implied by previous results. Instead, we study properties of the stationary distribution $\mathbf{A} = \{A_i\}$. A natural parameter is the chance, γ say, that two random individuals have the same opinion, i.e.

$$\gamma \equiv E \sum_i \frac{|A_i|^2}{n^2}. \tag{17}$$

Lemma 14

$$\gamma = \frac{2E\mathcal{E}}{\lambda r n^2} + \frac{1}{n},$$

where $\mathcal{E}$ is the number of edges with endpoints in different components, under the stationary distribution.

Proof. Run the stationary process, and let $\mathbf{A}(t)$ and $\mathcal{E}(t)$ be the partition and the number of edges linking distinct components, at time t, and let $S(t) = \sum_i |A_i(t)|^2$. Then

$$\frac{E(S^2(t+dt) - S^2(t)|\text{ configuration at time } t)}{dt}$$

$$= \frac{4}{r}\mathcal{E}(t) + 2\lambda \sum_i |A_i(t)|(1 - |A_i(t)|). \tag{18}$$

The first term arises from the "voter" dynamics. If an opinion change involves an edge linking components of sizes a and b, then the change in S^2 has expectation

$$\frac{(a+1)^2 + (a-1)^2 + (b+1)^2 - (b-1)^2}{2} - (a^2 + b^2) = 2$$

and for each of the $\mathcal{E}(t)$ edges linking distinct components, opinion changes occur at rate $2/r$. The second term arises from new opinions. A new opinion occurs in a component of size a at rate λa, and the resulting change in S^2 is

$$(a-1)^2 + 1^2 - a^2 = 2(1-a).$$

Stationarity implies that the expectation of (18) equals zero, and so

$$\frac{4}{r}E\mathcal{E} = 2\lambda \sum_i E|A_i|(|A_i| - 1) = 2\lambda(n^2\gamma - n)$$

and the lemma follows.

Corollary 15 $\frac{1+\lambda\tau_c}{1+\lambda\tau_c n} \leq \gamma \leq \frac{\lambda+1}{\lambda n}$.

Proof. Clearly $E\mathcal{E}$ is at most the total number of edges, $nr/2$, so the upper bound follows from the lemma. For the lower bound, (11) implies

$$\xi \geq \frac{r\sum_i |A_i|(n-|A_i|)}{2n\tau_c}$$

and hence

$$E\mathcal{E} \geq \frac{r}{2n\tau_c}(n^2 - n^2\gamma)$$

and the bound follows from the lemma after brief manipulation.

We now consider bounds on γ obtainable by working with the dual process. Consider the meeting time M of two independent random walks started with the stationary distribution. Then by duality (xxx explain)

$$\gamma = P(M < \xi_{(2\lambda)})$$

where $\xi_{(2\lambda)}$ denotes a random variable with exponential (2λ) distribution independent of the random walks. Now M is the hitting time of the stationary "product chain" (i.e. two independent continuous-time random walks) on the diagonal $A = \{(v,v)\}$, so by Chapter 3 yyy M has completely monotone distribution, and we shall use properties of complete monotonicity to get

Corollary 16

$$\frac{1}{1+2\lambda EM} \leq \gamma \leq \frac{1}{1+2\lambda EM} + \frac{\tau_2}{EM}.$$

Proof. We can write $M \stackrel{d}{=} R\xi_{(1)}$, where $\xi_{(1)}$ has exponential(1) distribution and R is independent of $\xi_{(1)}$. Then

$$\begin{aligned}
\gamma &= P(R\xi_{(1)} < \xi_{(2\lambda)}) \\
&= E\ P(R\xi_{(1)} < \xi_{(2\lambda)}|R) \\
&= E\frac{1}{1+2\lambda R} \\
&\geq \frac{1}{1+2\lambda ER} \text{ by Jensen's inequality} \\
&= \frac{1}{1+2\lambda EM}.
\end{aligned}$$

For the upper bound, apply Chapter 3 yyy to the product chain to obtain

$$P(M > t) \geq \exp(-t/EM) - \tau_2/EM$$

(recall that τ_2 is the same for the product chain as for the underlying random walk). So

$$\begin{aligned} 1-\gamma &= P(M \geq \xi_{(2\lambda)}) \\ &= \int_0^\infty P(M \geq t)\ 2\lambda e^{-2\lambda t} dt \\ &\geq \frac{2\lambda EM}{1 = 2\lambda EM} - \frac{\tau_2}{EM} \end{aligned}$$

and the upper bound follows after rearrangement.

Note that on a vertex-transitive graph Proposition 5 implies $EM = \tau_0/2$. So on a sequence of vertex-transitive graphs with $\tau_2/\tau_0 \to 0$ and with $\lambda\tau_0 \to \theta$, say, Corollary 16 implies $\gamma \to \frac{1}{1+\theta}$. But in this setting we can say much more, as the next section will show.

3.5 Large component sizes in the voter model with new opinions

xxx discuss coalescent, GEM and population genetics.

xxx genetics already implicit in xxx

Fix $0 < \theta < \infty$. take independent random variables (ξ_i) with distribution

$$P(\xi > x) = (1-x)^\theta, \quad 0 < x < 1$$

and define

$$(X_1^{(\theta)}, X_2^{(\theta)}, X_3^{(\theta)}, \ldots) = (\xi_1, (1-\xi_1)\xi_2, (1-\xi_1)(1-\xi_2)\xi_3, \ldots)$$

so that $\sum_i X_i^{(\theta)} = 1$.

Proposition 17 *Consider a sequence of vertex-transitive graphs for which $\tau_2/\tau_0 \to 0$. Consider the stationary distribution* $\mathbf{A}$ *of the voter model with new opinions, presented in size-biased order. If $\lambda\tau_0 \to \theta$ then*

$$\left(\frac{|A_1|}{n}, \ldots, \frac{|A_k|}{n}\right) \stackrel{d}{\to} (X_1^{(\theta)}, \ldots, X_k^{(\theta)}) \text{ for all fixed } k.$$

xxx proof

Remark. The same argument goes halfway to proving Open Problem 12, by showing

Corollary 18 *Consider a sequence of vertex-transitive graphs for which $\tau_2/\tau_0 \to 0$. Let $C^{(k)}$ be the coalescing time for k walks started at independent uniform positions. Then, for fixed k, $EC^{(k)} \sim \tau_0(1-k^{-1})$.*

xxx argument similar (?) to part of the proof in Cox [5] for the torus.

3.6 Number of components in the voter model with new opinions

xxx τ_c result

4 The antivoter model

Recall from section 3 the definition of the voter model on a r-regular n-vertex graph. We now change this in two ways. First, we suppose there are only two different opinions, which it is convenient to call ± 1. Second, the evolution rule is

> For each person i and each time interval $[t, t+dt]$, with chance dt the person chooses uniformly at random a neighbor (j, say) and changes (if necessary) their opinion to the opposite of the opinion of person j.

The essential difference from the voter model is that opinions don't disappear. Writing $\eta_v(t)$ for the opinion of individual v at time t, the process $\eta(t) = (\eta_v(t), v \in G)$ is a continuous-time Markov chain on state-space $\{-1,1\}^G$. So, provided this chain is irreducible, there is a unique stationary distribution $(\eta_v, v \in G)$ for the antivoter model.

This model on infinite lattices was studied in the "interacting particle systems" literature [14, 17], and again the key idea is duality. In this model the dual process consists of *annihilating* random walks. We will not go into details about the duality relation, beyond the following definition we need later. For vertices v, w, consider independent continuous-time random walks started at v and at w. We have previously studied $M_{v,w}$, the time at which the two walks first meet, but now we define $N_{v,w}$ to be the total number of jumps made by the two walks, up to and including the time $M_{v,w}$. Set $N_{v,v} = 0$.

Donnelly and Welsh [10] considered our setting of a finite graph, and showed that Proposition 19 is a simple consequence of the duality relation.

Proposition 19 *The antivoter process has a unique stationary distribution* (η_v)*, which satisfies*
(i) $E\eta_v = 0$
(ii) $c(v,w) \equiv E\eta_v\eta_w = P(N_{v,w} \text{ is even }) - P(N_{v,w} \text{ is odd })$.
If G is neither bipartite nor the n-cycle, then the set of all $2^n - 2$ non-unanimous configurations is irreducible, and the support of the stationary distribution is that set.

In particular, defining

$$S \equiv \sum_v \eta_v$$

so that S or $-S$ is the "margin of victory" in an election, we have $ES = 0$ and

$$\text{var}\ \ S = \sum_v \sum_w c(v,w). \tag{19}$$

On a bipartite graph with bipartition (A, A^c) the stationary distribution is

$$P(\eta_v = 1 \forall v \in A, \eta_v = -1 \forall v \in A^c) = P(\eta_v = -1 \forall v \in A, \eta_v = 1 \forall v \in A^c) = 1/2$$

and $c(v,w) = -1$ for each edge. Otherwise $c(v,w) > -1$ for every edge.

The antivoter process is in general a non-reversible Markov chain, because it can transition from a configuration in which v has the same opinion as all its neighbors to the configuration where v has the opposite opinion, but the reverse transition is impossible. Nevertheless we could use duality to discuss convergence time. But, following [10], the spatial structure of the stationary distribution is a more novel and hence more interesting question. Intuitively we expect neighboring vertices to be negatively correlated and the variance of S to be smaller than n (the variance if opinions were independent). In the case of the complete graph on n vertices, $N_{v,w}$ has (for $w \neq v$) the geometric distribution

$$P(N_{v,w} > m) = \left(1 - \frac{1}{n-1}\right)^m ; m \geq 0$$

from which we calculate $c(v,w) = -1/(2n-3)$ and var $S = \frac{n(n-2)}{2n-3} < n/2$. We next investigate var S in general.

4.1 Variances in the antivoter model

Write $\xi = (\xi_v)$ for a configuration of the antivoter process and write

$$S(\xi) = \sum_v \xi_v$$

$$a(\xi) = \text{ number of edges } (v,w) \text{ with } \xi_v = \xi_w = 1$$

$$b(\xi) = \text{ number of edges } (v,w) \text{ with } \xi_v = \xi_w = -1.$$

A simple counting argument gives

$$2(a(\xi) - b(\xi)) = rS(\xi). \tag{20}$$

Lemma 20 var $S = \frac{2}{r}E(a(\eta)+b(\eta))$, *where* η *is the stationary distribution.*

Proof. Writing (η_t) for the stationary process and $dS_t = S(\eta_{t+dt}) - S(\eta_t)$, we have

$$P(dS_t = +2|\eta_t) = b(\eta_t)dt$$
$$P(dS_t = -2|\eta_t) = a(\eta_t)dt$$

and so

$$\begin{aligned} 0 &= ES^2(\eta_{t+dt}) - ES^2(\eta_t) \text{ by stationarity} \\ &= 2ES(\eta_t)dS_t \; + E(dS_t)^2 \\ &= 4ES(\eta_t)(b(\eta_t) - a(\eta_t))dt \; + 4E(a(\eta_t) + b(\eta_t))dt \\ &= -2rES^2(\eta_t)dt + 4E(a(\eta_t) + b(\eta_t))dt \text{ by (20)} \end{aligned}$$

establishing the Lemma.

Since the total number of edges is $nr/2$, Lemma 20 gives the upper bound which follows, and the lower bound is also clear.

Corollary 21 *Let* $\kappa = \kappa(G)$ *be the largest integer such that, for any subset* A *of vertices, the number of edges with both ends in* A *or both ends in* A^c *is at least* κ. *Then*

$$\frac{2\kappa}{r} \le \text{var} \quad S \le n.$$

Here κ is a natural measure of "non-bipartiteness" of G. We now show how to improve the upper bound by exploiting duality. One might expect some better upper bound for "almost-bipartite" graphs, but Examples 27 and 28 indicate this may be difficult.

Proposition 22 var $\;S < n/2$.

Proof. Take two independent stationary continuous-time random walks on the underlying graph G, and let $(X_t^{(1)}, X_t^{(2)}; t = \ldots, -1, 0, 1, 2, \ldots)$ be the jump chain, i.e. at each time we choose at random one component to make a step of the random walk on the graph. Say an "event' happens at t if $X_t^{(1)} = X_t^{(2)}$, and consider the inter-event time distribution L:

$$P(L = l) = P(\min\{t > 0 : X_t^{(1)} = X_t^{(2)}\} = l | X_0^{(1)} = X_0^{(2)}).$$

In the special case where G is vertex-transitive the events form a renewal process, but we use only stationarity properties (c.f. Chapter 2 yyy) which hold in the general case. Write

$$T = \min\{t \geq 0 : X_t^{(1)} = X_t^{(2)}\}$$

where the stationary chain is used. Then

$$P_{v,w}(T = t) \equiv P(T = t | X_0^{(1)} = v, X_0^{(2)} = w) = P(N_{v,w} = t)$$

and so by (19) and Proposition 19(ii),

$$\begin{aligned} \text{var} \ \ S &= \sum_v \sum_w (P_{v,w}(T \text{ is even}) - P_{v,w}(T \text{ is odd})) \\ &= n^2(P(T \text{ is even}) - P(T \text{ is odd})). \end{aligned}$$

If successive events occur at times t_0 and t_1, then

$$\begin{aligned} |\{s : t_0 < s \leq t_1 : t_1 - s \text{ is even } \}| - \{s : t_0 < s \leq t_1 : t_1 - s \text{ is odd}\}| &= 0 \text{ if } |t_1 - t_0| \text{ is even} \\ &= 1 \text{ if } |t_1 - t_0| \text{ is odd} \end{aligned}$$

and an ergodic argument gives

$$P(T \text{ is even}) - P(T \text{ is odd}) = P(L \text{ is odd})/EL.$$

But $EL = 1/P(\text{event}) = n$, so we have established

Lemma 23 $n^{-1}\text{var} \ \ S = P(L \textit{ is odd})$.

Now consider

$$T^- = \min\{t \geq 0 : X_{-t}^{(1)} = X_{-t}^{(2)}\}.$$

If successive events occur at t_0 and t_1, then there are $t_1 - t_0 - 1$ times s with $t_0 < s < t_1$, and another ergodic argument shows

$$P(T + T^- = l) = \frac{(l-1)P(L = l)}{EL}, \ l \geq 2.$$

So

$$\begin{aligned} n^{-1}(P(L \text{ is even}) - P(L \text{ is odd})) &= \frac{1}{EL}\sum_{l \geq 2}(-1)^l P(L = l) \text{ since } EL = n \\ &= \sum_{l \geq 2} \frac{(-1)^l}{l-1} P(T + T^- = l). \qquad (21) \end{aligned}$$

Now let $\phi(z)$ be the generating function of a distribution on $\{1,2,3,\ldots\}$ and let Z, Z^- be independent random variables with that distribution. Then

$$\sum_{l\geq 2} \frac{(-1)^l}{l-1} P(Z+Z^- = l) = \int_{-1}^{0} \frac{\phi^2(z)}{z^2}\, dz > 0. \tag{22}$$

Conditional on $(X_0^{(1)}, X_0^{(2)}) = (v,w)$ with $w \neq v$, we have that T and T^- are independent and identically distributed. So the sum in (21) is positive, implying $P(L \text{ is odd}) < 1/2$, so the Proposition follows from the Lemma.

Implicit in the proof are a corollary and an open problem. The open problem is to show that var S is in fact maximized on the complete graph. This might perhaps be provable by sharpening the inequality in (22).

Corollary 24 *On an edge-transitive graph, write* $c_{edge} = c(v,w) = E\eta_v\eta_w$ *for an arbitrary edge* (v,w)*. Then*

$$\text{var}\ \ S = n(1+c_{edge})/2$$

$$c_{edge} < 0.$$

Proof. In an edge-transitive graph, conditioning on the first jump from (v,v) gives

$$P(L \text{ is odd}) = P(N_{v,w} \text{ is even})$$

for an edge (v,w). But $P(N_{v,w} \text{ is even }) = (1+c_{\text{edge}})/2$ by Proposition 19(ii), so the result follows from Lemma 23 and Proposition 22.

4.2 Examples and Open Problems

In the case of the complete graph, the number of $+1$ opinions evolves as the birth-and-death chain on states $\{1,2,\ldots,n-1\}$ with transition rates

$$\begin{aligned} i \to i+1 \quad & \text{rate } \tfrac{(n-i)(n-1-i)}{n(n-1)} \\ \to i-1 \quad & \text{rate } \tfrac{i(i-1)}{n(n-1)} \end{aligned}$$

From the explicit form of the stationary distribution we can deduce that as $n \to \infty$ the asymptotic distribution of S is Normal. As an exercise in technique (see Notes) we ask

Open Problem 25 *Find sufficient conditions on a sequence of graphs which imply S has asymptotic Normal distribution.*

Example 26 *Distance-regular graphs.*

On a distance-regular graph of diameter Δ, define $1 = f(0), f(1), \ldots, f(\Delta)$ by

$$f(i) = c(v,w) = P(N_{v,w} \text{ is even }) - P(N_{v,w} \text{ is odd }), \text{ where } d(x,y) = i.$$

Conditioning on the first step of the random walks,

$$f(i) = -\left(p_{i,i+1}f(i+1) + p_{i,i}f(i) + p_{i,i-1}f(i-1)\right),\ 1 \leq i \leq \Delta \qquad (23)$$

where (c.f. Chapter 7 yyy) the $p_{i,j}$ are the transition probabilities for the birth-and-death chain associated with the discrete-time random walk. In principle we can solve these equations to determine $f(1) = c_{\text{edge}}$. Note that the bipartite case is the case where $p_{i,i} \equiv 0$, which is the case where $f(i) \equiv (-1)^i$ and $c_{\text{edge}} = -1$. A simple example of a non-bipartite distance-regular graph is the "2-subsets of a d-set" example (Chapter 7 yyy) for $d \geq 4$. Here $\Delta = 2$ and

$$p_{1,0} = \frac{1}{2(d-2)} \quad p_{1,1} = \tfrac{d-3}{2(d-2)} \quad p_{1,2} = \frac{d-2}{2(d-2)}$$
$$p_{2,1} = \tfrac{4}{2(d-2)} \quad p_{2,2} = \frac{2d-8}{2(d-2)}.$$

Solving equations (23) gives $c_{\text{edge}} = -1/(3d-7)$.

Corollary 24 said that in an edge-transitive graph, $c_{\text{edge}} < 0$. On a vertex-transitive graph this need not be true for every edge, as the next example shows.

Example 27 *An almost bipartite vertex-transitive graph.*

Consider the $m+2$-regular graph on $2m$ vertices, made by taking m-cycles $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_m)$ and adding all edges (v_i, w_j) between the two "classes". One might guess that, under the stationary distribution, almost all individuals in a class would have the same opinion, different for the two classes. But in fact the tendency for agreement between individuals in the same class is bounded: as $m \to \infty$

$$\begin{aligned} c(v_i, w_j) &\to -\frac{1}{9} \\ c(v_i, v_j) &\to \frac{1}{9},\ j \neq i. \end{aligned} \qquad (24)$$

To prove this, consider two independent continuous-time random walks, started from opposite classes. Let N be the number of jumps before meeting and let $M \geq 1$ be the number of jumps before they are again in opposite classes. Then

$$P(M \text{ is odd }) = \frac{4}{m} + O(m^{-2}); \quad P(N < M) = \frac{1}{m} + O(m^{-2}).$$

So writing $M_1 = M, M_2, M_3, \ldots$ for the cumulative numbers of jumps each time the two walks are in opposite components, and writing

$$Q \equiv \min\{j : M_j \text{ is odd}\},$$

we have

$$P(\text{ walks meet before } Q) = \frac{1}{5} + O(m^{-1}).$$

Writing $Q_1 = Q, Q_2, Q_3, \ldots$ for the sucessive j's at which M_j changes parity, and

$$L \equiv \max\{k : M_{Q_k} < \text{ meeting time}\}$$

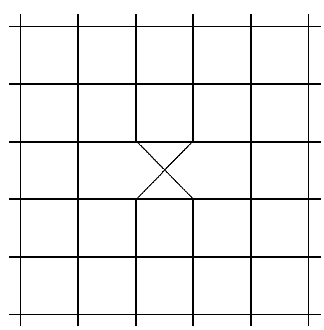
for the number of parity changes before meeting,

$$P(L = l) = \frac{1}{5}\left(\frac{4}{5}\right)^l + O(m^{-1}), \quad l \geq 0$$

So $P(\eta_{v_i,w_j} \text{ is odd}) = P(L \text{ is even}) \to \frac{5}{9}$ and (24) follows easily.

Example 28 *Another almost-bipartite graph.*

Consider the torus Z_m^d with $d \geq 2$ and with even $m \geq 4$, and make the graph non-bipartite by moving two edges as shown.



Let $m \to \infty$ and consider the covariance $c(v_m, w_m)$ across edges (v_m, w_m) whose distance from the modified edges tends to infinity. One might suspect that the modification had only "local" effect, in that $c(v_m, w_m) \to -1$. In fact,

$$\begin{aligned} c(v_m, w_m) &\to -1, \quad d = 2 \\ &\to \beta(d) > -1, \ d \geq 3. \end{aligned}$$

We don't give details, but the key observation is that in $d \geq 3$ there is a bounded-below chance that independent random walks started from v_m and w_m will traverse one of the modified edges before meeting.

5 The interchange process

xxx notation: $\tilde{X}$ for process or underlying RW?

Fix a graph on n vertices. Given n distinguishable particles, there are $n!$ "configurations" with one particle at each vertex. The *interchange process* is the following continuous-time reversible Markov chain on configurations.

> On each edge there is a Poisson, rate 1, process of "switch times", at which times the particles at the two ends of the edge are interchanged.

The stationary distribution is uniform on the $n!$ configurations. We want to study the time taken to approach the uniform distribution, as measured by the parameters τ_2 and τ_1.

As with the voter model, there is an induced process obtained by declaring some subset of particles to be "visible", regarding the visible particles as indistinguishable, and ignoring the invisible particles. Interchanging two visible particles has no effect, so the dynamics of the induced process are as follows.

> On each edge there is a Poisson, rate 1, process of "switch times". At a switch time, if one endpoint is unoccupied and the other endpoint is occupied by a (visible) particle, then the particle moves to the other endpoint.

This is the finite analog of the *exclusion process* studied in the interacting particle systems literature. But in the finite setting, the interchange process seems more fundamental.

If we follow an individual particle, we see a certain continuous-time Markov chain $\tilde{X}_t$, say, with transition rate 1 along each edge. In the terminology of Chapter 3 yyy this is the *fluid model* random walk, rather than the usual continuized random walk. Write $\tilde{\tau}_2$ for the relaxation time of $\tilde{X}$. The contraction principle (Chapter 4 yyy) implies $\tau_2 \geq \tilde{\tau}_2$.

Open Problem 29 *Does $\tau_2 = \tilde{\tau}_2$ in general?*

If the answer is "yes", then the general bound of Chapter 4 yyy will give

$$\tau_1 \leq \tilde{\tau}_2(1 + \frac{1}{2}\log n!) = O(\tilde{\tau}_2 n \log n)$$

but the following bound is typically better.

Proposition 30 $\tau_1 \leq (2 + \log n)e \ \max_{v,w} \tilde{E}_v \tilde{T}_w$.

Proof. We use a coupling argument. Start two versions of the interchange process in arbitrary initial configurations. Set up independent Poisson processes $\mathcal{N}_e$ and $\mathcal{N}_e^*$ for each edge e. Say edge e is *special* at time t if the particles at the end-vertices in process 1 are the same two particles as in process 2, but in transposed position. The evolution rule for the coupled processes is

> Use the same Poisson process $\mathcal{N}_e$ to define simultaneous switch times for both interchange processes, except for special edges where we use $\mathcal{N}_e$ for process 1 and $\mathcal{N}_e^*$ for process 2.

Clearly, once an individual particle is matched (i.e. at the same vertex in both processes), it remains matched thereafter. And if we watch the process (X_t, Y_t) recording the positions of particle i in each process, it is easy to check this process is the same as watching two independent copies of the continuous-time random walk, run until they meet, at time U_i, say. Thus $\max_i U_i$ is a coupling time and the coupling inequality (5) implies

$$\bar{d}(t) \leq P(\max_i U_i > t).$$

Now U_i is distributed as $M_{v(i),w(i)}$, where $v(i)$ and $w(i)$ are the initial positions of particle i in the two versions and where $M_{v,w}$ denotes meeting time for independent copies of the underlying random walk $\tilde{X}_t$. Writing $m^* = \max_{v,w} EM_{v,w}$, we have by subexponentiality (as at (16))

$$P(M_{v,w} > t) \leq \exp\left(1 - \frac{t}{em^*}\right)$$

and so

$$\bar{d}(t) \leq n \exp\left(1 - \frac{t}{em^*}\right).$$

This leads to $\tau_1 \leq (2 + \log n)em^*$ and the result follows from Proposition 5.

5.1 Card-shuffling interpretation

Taking the underlying graph to be the complete graph on n vertices, the discrete-time jump chain of the interchange process is just the "card shuffling by random transpositions" model from Chapter 7 yyy. On any graph G, the jump chain can be viewed as a card-shuffling model, but note that parameters τ are multiplied by $|\mathcal{E}|$ (the number of edges in G) when passing from the interchange process to the card-shuffling model. On the complete graph we have $\max_{v,w} \tilde{E}_v \tilde{T}_w = \Theta(1)$ and $|\mathcal{E}| = \Theta(n^2)$, and so Proposition 30 gives the bound $\tau_1 = O(n^2 \log n)$ for card shuffling by random transpositions, which is crude in view of the exact result $\tau_1 = \Theta(n \log n)$. In contrast, consider the n-cycle, where $\max_{v,w} \tilde{E}_v \tilde{T}_w = \Theta(n^2)$ and $|\mathcal{E}| = n$. Here the jump process is the "card shuffling by random adjacent transpositions" model from Chapter 7 yyy. In this model, Proposition 30 gives the bound $\tau_1 = O(n^3 \log n)$ which as mentioned in Chapter 7 yyy is the correct order of magnitude.

Diaconis and Saloff-Coste [8] studied the card-shuffling model as an application of more sophisticated techniques of comparison of Dirichlet forms.

xxx talk about their results.

6 Other interacting particle models

As mentioned at the start of the chapter, the models discussed in sections 3 - 5 are special in that their behavior relates to the behavior of processes built up from independent random walks on the underlying graph. In other models this is not necessarily true, and the results in this book have little application.

xxx mention Ising model and contact process.

6.1 Product-form stationary distributions

Consider a continuous-time particle process whose state space is the collection of subsets of vertices of a finite graph (representing the subset of vertices occupied by particles), and where only one state can change occupancy at a time. The simplest stationary distribution would be of the form

$$\text{each vertex } v \text{ is occupied independently with probability } \theta/(1+\theta) \quad (25)$$

where $0 < \theta < \infty$ is a parameter. By considering the detailed balance equations (Chapter 3 yyy), such a process will be reversible with stationary distribution (25) iff its transition rates satisfy

> For configurations $\mathbf{x}^0, \mathbf{x}^1$ which coincide except that vertex v is unoccupied in $\mathbf{x}^0$ and occupied in $\mathbf{x}^1$, we have $\frac{q(\mathbf{x}^0,\mathbf{x}^1)}{q(\mathbf{x}^1,\mathbf{x}^0)} = \theta$.

There are many ways to set up such transition rates. Here is one way, observed by Neuhauser and Sudbury [20]. For each edge (w,v) at time t with w occupied,

if v is occupied at time t, then with chance dt it becomes unoccupied by time $t+dt$

if v is unoccupied at time t, then with chance θdt it becomes occupied by time $t+dt$.

If we exclude the empty configuration (which cannot be reached from other configurations) the state space is irreducible and the stationary distribution is given by (25) conditioned on being non-empty.

Convergence times for this model have not been studied, so we ask

Open Problem 31 *Give bounds on the relaxation time τ_2 in this model.*

6.2 Gaussian families of occupation measures

We mentioned in Chapter 3 yyy that, in the setting of a finite irreducible reversible chain (X_t), the fundamental matrix $\mathbf{Z}$ has the property

$$\pi_i Z_{ij} \text{ is symmetric and positive-definite .}$$

So by a classical result (e.g. [12] Theorem 3.6.4) there exists a mean-zero Gaussian family (γ_i) such that

$$E\gamma_i\gamma_j = \pi_i Z_{ij} \text{ for all } i,j. \tag{26}$$

What do such Gaussian random variables represent? It turns out there is a simple interpretation involving occupation measures of "charged particles". Take two independent copies $(X^+_t : -\infty < t < \infty)$ and $(X^-_t : -\infty < t < \infty)$ of the stationary chain, in continuous time for simplicity. For fixed $u > 0$ consider the random variables

$$\gamma_i^{(u)} \equiv \frac{1}{2}\int_{-u}^{0} \left(1_{(X^+_t = i)} - 1_{(X^-_t = i)} \right) dt.$$

Picture one particle with charge $+1/2$ and the other particle with charge $-1/2$, and then $\gamma_i^{(u)}$ has units "charge $\times$ time". Clearly $E\gamma_i^{(u)} = 0$ and it is easy to calculate

$$\begin{aligned} E\gamma_i^{(u)}\gamma_j^{(u)} &= \frac{1}{2}E\int_{-u}^0\int_{-u}^0 \left(1_{(X_s=i,X_t=j)} - \pi_i\pi_j\right)\ ds\ dt \\ &= \frac{1}{2}\pi_i\int_{-u}^0\int_{-u}^0 (P(X_t=j|X_s=i)-\pi_j)\ ds\ dt \\ &= \pi_i\int_0^u \left(1-\frac{r}{u}\right)(p_{ij}(r)-\pi_j)\ dr \end{aligned}$$

and hence

$$u^{-1}E\gamma_i^{(u)}\gamma_j^{(u)} \to \pi_i Z_{ij} \text{ as } u\to\infty. \tag{27}$$

The central limit theorem for Markov chains (Chapter 2 yyy) implies that the $u\to\infty$ distributional limit of $(u^{-1/2}\gamma_i^{(u)})$ is some mean-zero Gaussian family (γ_i), and so (27) identifies the limit as the family with covariances (26).

As presented here the construction may seem an isolated curiousity, but in fact it relates to deep ideas developed in the context of continuous-time-and-space reversible Markov processes. In that context, the *Dynkin isomorphism theorem* relates continuity of local times to continuity of sample paths of a certain Gaussian process. See [19] for a detailed account. And various interesting Gaussian processes can be constructed via "charged particle" models – see [1] for a readable account of such constructions. Whether these sophisticated ideas can be brought to bear upon the kinds of finite-state problems in this book is a fascinating open problem.

7 Other coupling examples

Example 32 *An m-particle process on the circle.*

Fix $m<K$. Consider m indistinguishable balls distributed amongst K boxes, at most one ball to a box, and picture the boxes arranged in a circle. At each step, pick uniformly at random a box, say box i. If box i is occupied, do nothing. Otherwise, pick uniformly at random a direction (clockwise or counterclockwise) search from i in that direction until encountering a ball, and move that ball to box i. This specifies a Markov chain on the $\binom{K}{m}$ possible configurations of balls. The chain is reversible and the stationary distribution is uniform. Can we estimate the "mixing time" parameters τ_1

and τ_2? Note that as $K \to \infty$ there is a limit process involving m particles on the continuous circle, so we seek bounds which do not depend on K.

There is a simple-to-describe coupling, where for each of the two versions we pick at each time the same box and the same direction. The coupling has the usual property (c.f. the proof of Proposition 30) that the number of "matched" balls (i.e. balls in the same box in both processes) can only increase. But analyzing the coupling time seems very difficult. Cuellar-Montoya [7] carries through a lengthy analysis to show that $\tau_1 = O(m^{10})$. In the other direction, the bound

$$\tau_2 \geq \frac{m^3}{8\pi^2}$$

is easily established, by applying the extremal characterization (Chapter 3 yyy) to the function

$$g(\mathbf{x}) = \sum_{i=1}^{m} \sin(2\pi x_i/m)$$

where $\mathbf{x} = (x_1, \ldots, x_m)$ denotes the configuration with occupied boxes $\{x_1, \ldots, x_m\}$. It is natural to conjecture $\tau_2 = \Theta(m^3)$ and $\tau_1 = O(m^3 \log m)$.

The next example, from Jerrum [15] (xxx cite final version), uses a coupling whose construction is not quite obvious.

Example 33 *Permutations and words.*

Fix a finite alphabet A of size $|A|$. Fix m, and consider the set A^m of "words" $\mathbf{x} = (x_1, \ldots, x_m)$ with each $x_i \in A$. Consider the Markov chain on A^m in which a step $\mathbf{x} \to \mathbf{y}$ is specified by the following two-stage procedure.

Stage 1. Pick a permutation σ of $\{1, 2, \ldots, m\}$ uniformly at random from the set of permutations σ satisfying $x_{\sigma(i)} = x_i \forall i$.

Stage 2. Let $(c_j(\sigma); j \geq 1)$ be the cycles of σ. For each j, and independently as j varies, pick uniformly an element α_j of A, and define $y_i = \alpha_j$ for every $i \in c_j(\sigma)$.

Here is an alternative description. Write Π for the set of permutations of $\{1, \ldots, m\}$. Consider the bipartite graph on vertices $A^m \cup \Pi$ with edge-set $\{(\mathbf{x}, \sigma) : x_{\sigma(i)} = x_i \forall i\}$. Then the chain is random walk on this bipartite graph, watched every second step when it is in A^m.

From the second description, it is clear that the stationary probabilities $\pi(\mathbf{x})$ are proportional to the degree of $\mathbf{x}$ in the bipartite graph, giving

$$\pi(\mathbf{x}) \propto \prod_a n_a(\mathbf{x})!$$

where $n_a(\mathbf{x}) = |\{i : x_i = a\}|$. We shall use a coupling argument to establish the following bound on variation distance:

$$\bar{d}(t) \leq m \left(1 - \frac{1}{|A|}\right)^t \tag{28}$$

implying that the variation threshold satisfies

$$\tau_1 \leq 1 + \frac{1 + \log m}{-\log(1 - \frac{1}{|A|})} \leq 1 + (1 + \log m)|A|.$$

The construction of the coupling depends on the following lemma, whose proof is deferred.

Lemma 34 *Givcen finite sets F^1, F^2 we can construct (for $u = 1, 2$) a uniform random permutation σ^u of F^u with cycles $(C_j^u; j \geq 1)$, where the cycles are labeled such that*

$$C_j^1 \cap F^1 \cap F^2 = C_j^2 \cap F^1 \cap F^2 \textit{ for all } j.$$

We construct a step $(\mathbf{x}^1, \mathbf{x}^2) \to (\mathbf{Y}^1, \mathbf{Y}^2)$ of the coupled processes as follows. For each $a \in A$, set $F^{1,a} = \{i : x_i^1 = a\}$, $F^{2,a} = \{i : x_i^2 = a\}$. Take random permutations $\sigma^{1,a}$, $\sigma^{2,a}$ as in the lemma, with cycles $C_j^{1,a}, C_j^{2,a}$. Then $(\sigma^{1,a}, a \in A)$ define a uniform random permutation σ^1 of $\{1, \ldots, m\}$, and similarly for σ^2. This completes stage 1. For stage 2, for each pair (a, j) pick a uniform random element α_j^a of A and set

$$Y_i^1 = \alpha_j^a \text{ for every } i \in C_j^{1,a}$$

$$Y_i^2 = \alpha_j^a \text{ for every } i \in C_j^{2,a}.$$

This specifies a Markov coupling. By construction

$$\begin{array}{ll} \text{if } x_i^1 = x_i^2 & \text{then } Y_i^1 = Y_i^2 \\ \text{if } x_i^1 \neq x_i^2 & \text{then } P(Y_i^1 = Y_i^2) = 1/|A|. \end{array}$$

So the coupled processes $(\mathbf{X}^1(t), \mathbf{X}^2(t))$ satisfy

$$P(X_i^1(t) \neq X_i^2(t)) = \left(1 - \frac{1}{|A|}\right)^t 1_{(X_i^1(0) \neq X_i^2(0))}.$$

In particular $P(\mathbf{X}^1(t) \neq \mathbf{X}^2(t)) \leq m(1 - 1/|A|)^t$ and the coupling inequality (5) gives (28).

xxx proof of Lemma – tie up with earlier discussion.

7.1 Markov coupling may be inadequate

Recall the discussion of the coupling inequality in section 1.1. Given a Markov chain and states i, j, theory (e.g. [18] section 3.3) says there exists a *maximal coupling* $X_t^{(i)}, X_t^{(j)}$ with a coupling time T for which the coupling inequality (5) holds with *equality*. But this need not be a *Markov* coupling, i.e. of form (6), as the next result implies. The point is that there exist fixed-degree expander graphs with $\tau_2 = O(1)$ and so $\tau_1 = O(\log n)$, but whose girth (minimal cycle length) is $\Omega(\log n)$. On such a graph, the upper bound on τ_1 obtained by a Markov coupling argument would be $\Theta(ET)$, which the Proposition shows is $n^{\Omega(1)}$.

Proposition 35 *Fix vertices i, j in a r-regular graph ($r \geq 3$) with girth g. Let $(X_t^{(i)}, X_t^{(j)})$ be any Markov coupling of discrete-time random walks started at i and j. Then the coupling time T satisfies*

$$ET \geq \frac{1 - (r-1)^{-d(i,j)/2}}{r-2}\ (r-1)^{\frac{g}{4}-\frac{1}{2}}.$$

Proof. We quote a simple lemma, whose proof is left to the reader.

Lemma 36 *Let ξ_1, ξ_2 be (dependent) random variables with $P(\xi_u = 1) = \frac{r-1}{r}$, $P(\xi_u = -1) = \frac{1}{r}$. Then*

$$E\theta^{\xi_1+\xi_2} \leq \frac{r-1}{r}\theta^2 + \frac{1}{r}\theta^{-2},\ 0 < \theta < 1.$$

In particular, setting $\theta = (r-1)^{-1/2}$, we have

$$E\theta^{\xi_1+\xi_2} \leq 1.$$

Now consider the distance $D_t \equiv d(X_t^{(i)}, X_t^{(j)})$ between the two particles. The key idea is

$$\begin{aligned} E(\theta^{D_{t+1}} - \theta^{D_t} | X_t^{(i)}, X_t^{(j)}) &\leq 0 \text{ if } D_t \leq \lfloor g/2 \rfloor - 1 \\ &\leq (\theta^{-2} - 1)\theta^{\lfloor g/2 \rfloor} \text{ else.} \end{aligned} \tag{29}$$

The second inequality follows from the fact $D_{t+1} - D_t \geq -2$. For the first inequality, if $D_t \leq \lfloor g/2 \rfloor - 1$ then the incremental distance $D_{t+1} - D_t$ is distributed as $\xi_1 + \xi_2$ in the lemma, so the conditional expectation of $\theta^{D_{t+1} - D_t}$ is ≤ 1. Now define a martingale (M_t) via $M_0 = 0$ and

$$M_{t+1} - M_t = \theta^{D_{t+1}} - \theta^{D_t} - E(\theta^{D_{t+1}} - \theta^{D_t} | X_t^{(i)}, X_t^{(j)}).$$

Rearranging,

$$\begin{aligned} \theta^{D_t} - \theta^{D_0} &= M_t + \sum_{s=0}^{t-1} E(\theta^{D_{s+1}} - \theta^{D_s} | X_s^{(i)}, X_s^{(j)}) \\ &\leq M_t + (\theta^{-2} - 1)\theta^{\lfloor g/2 \rfloor} t \text{ by (29).} \end{aligned}$$

Apply this inequality at the coupling time T and take expectations: we have $EM_T = 0$ by the optional sampling theorem (Chapter 2 yyy) and $D_T = 0$, so

$$1 - \theta^{d(i,j)} \leq (\theta^{-2} - 1)\theta^{\lfloor g/2 \rfloor} ET$$

and the Proposition follows.

8 Notes on Chapter 14

Section 1. Coupling has become a standard tool in probability theory. Lindvall [18] contains an extensive treatment, emphasizing its use to prove limit theorems. Stoyan [21] emphasizes comparison results in the context of queueing systems.

Birth-and-death chains have more monotonicity properties than stated in Proposition 1 – see van Doorn [22] for an extensive treatment. The coupling (2) of a birth-and-death process is better viewed as a specialization of couplings of stochastically monotone processes, c.f. [18] Chapter 4.3.

Section 1.1. Using the coupling inequality to prove convergence to stationarity (i.e. the convergence theorem, Chapter 2 yyy) and the analogs for continuous-space processes is called the *coupling method.* See [18] p. 233 for some history. Systematic use to bound variation distance in finite-state chains goes back to Aldous [2]. repeated here. The coupling inequality is often presented as involving the chain started from an arbitrary point and the stationary chain, leading to a bound on $d(t)$ instead of $\bar{d}(t)$.

Section 3. The voter model on Z^d, and its duality with coalescing random walk, has been extensively studied – see [11, 17] for textbook treatments. The general notion of duality is discussed in [17] section 2.3. The voter model on general finite graphs has apparently been studied only once, by Donnelly and Welsh [9]. They studied the two-party model, and obtained the analog of Proposition 9(a) and some variations.

In the context of Open Problem 13 one can seek to use the randomization idea in Matthews' method, and the problem reduces to proving that, in the coalescing of k randomly-started particles, the chance that the final join is between a $(k-1)$-cluster and a 1-cluster is small.

Section 3.5. On the *infinite* two-dimensional lattice, the meeting time M of independent random walks is such that $\log M$ has approximately an exponential distribution. Rather surprisingly, with a logarthmic time transformation one can get a analog of Proposition 17 on the infinite lattice – see Cox and Griffeath [6].

Section 4. Donnelly and Welsh [10] obtained Proposition 19 and a few other results, e.g. that, over edge-transitive graphs, c_{edge} is uniquely maximized on the complete graph.

In the context of Open Problem 25, there are many known Normal limits in the context of interacting particle systems on the infinite lattice, but it is not clear how well those techniques extend to general finite graphs. It would be interesting to know whether Stein's method could be used here (see Baldi and Rinott [4] for different uses of Stein's method on graphs).

Section 5. The name "interchange process" is my coinage: the process, in the card-shuffling interpretation, was introduced by Diaconis and Saloff-Coste [8].

The interchange process can of course be constructed from a Poisson process of directed edges, as was the voter model in section 3. On the n-path, this "graphical representation" has an interpretation as a method to create a pseudo-random permutation with paper and pencil – see Lange and Miller [16] for an entertaining elementary exposition.

Miscellaneous. One can define a wide variety of "growth and coverage" models on a finite graph, where there is some prescribed rule for growing a random subset $\mathcal{S}_t$ of vertices, starting with a single vertex, and the quantity of interest is the time T until the subset has grown to be the complete graph. Such processes have been studied as models for rumors, broadcast information and percolation – see e.g. Weber [23] and Fill and Pemantle [13].

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9 Other stuff

xx interacting SA with 2 particles

xx exclusion process via distinguished paths, after Diaconis S-C.

xx J-S matchings via distinguished paths

xx coupling proof of differentiation of stat dists – notes.

xx interacting version of approximate counting