Chapter 5
Examples: Special Graphs and Trees

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There are two main settings in which explicit calculations for random walks on large graphs can be done. One is where the graph is essentially just 1-dimensional, and the other is where the graph is highly symmetric. The main purpose of this chapter is to record some (mostly) bare-hands calculations for simple examples, in order to illuminate the general inequalities of Chapter 4. Our focus is on natural examples, but there are a few artificial examples devised to make a mathematical point. A second purpose is to set out some theory for birth-and-death chains and for trees.

Lemma 1 below is useful in various simple examples, so let's record it here. An edge (v,x) of a graph is *essential* (or a *bridge*) if its removal would disconnect the graph, into two components $A(v,x)$ and $A(x,v)$, say, containing v and x respectively. Recall that $\mathcal{E}$ is the set of (undirected) edges, and write $\mathcal{E}(v,x)$ for the set of edges of $A(v,x)$.

Lemma 1 (essential edge lemma) *For random walk on a weighted graph with essential edge* (v,x)*,*

$$E_vT_x = \frac{2\sum_{(i,j)\in\mathcal{E}(v,x)} w_{ij}}{w_{vx}} + 1 \qquad (1)$$

$$E_vT_x + E_xT_v = \frac{w}{w_{vx}}, \text{ where } w = \sum_i\sum_j w_{ij}. \qquad (2)$$

Specializing to the unweighted case,

$$E_vT_x = 2|\mathcal{E}(v,x)| + 1 \qquad (3)$$

$$E_vT_x + E_xT_v = 2|\mathcal{E}|. \qquad (4)$$

Proof. It is enough to prove (1), since (2) follows by adding the two expressions of the form (1). Because (v,x) is essential, we may delete all vertices of $A(x,v)$ except x, and this does not affect the behavior of the chain up until time T_x, because x must be the first visited vertex of $A(x,v)$. After this deletion, $\pi_x^{-1} = E_x T_x^+ = 1 + E_v T_x$ by considering the first step from x, and $\pi_x = w_{vx}/(2w_{vx} + 2\sum_{(i,j)\in\mathcal{E}(v,x)} w_{ij})$, giving (1). ■

Remarks. Of course Lemma 1 is closely related to the *edge-commute inequality* of Chapter 3 Lemma yyy. We can also regard (2), and hence (4), as consequences of the commute interpretation of resistance (Chapter 3 yyy), because the effective resistance across an essential edge (v,x) is obviously $1/w_{vx}$.

1 One-dimensional chains

1.1 Simple symmetric random walk on the integers

It is useful to record some elementary facts about simple symmetric random walk (X_t) on the (infinite) set of all integers. As we shall observe, these may be derived in several different ways.

A fundamental formula gives exit probabilities:

$$P_b(T_c < T_a) = \frac{b-a}{c-a}, \ a < b < c. \tag{5}$$

An elementary argument is that $g(i) \equiv P_i(T_c < T_a)$ satisfies the 1-step recurrence

$$g(i) = \tfrac{1}{2}g(i+1) + \tfrac{1}{2}g(i-1), \ a < i < b$$
$$g(a) = 0, \quad g(b) = 1,$$

whose solution is $g(i) = (i-a)/(b-a)$. At a more sophisticated level, (5) is a martingale result. The quantity $p \equiv P_b(T_c < T_a)$ must satisfy

$$b = E_b X(T_a \wedge T_c) = pc + (1-p)a,$$

where the first equality is the optional sampling theorem for the martingale X, and solving this equation gives (5).

For $a < c$, note that $T_a \wedge T_c$ is the "exit time" from the open interval (a,c). We can use (5) to calculate the "exit before return" probability

$$P_b(T_b^+ > T_a \wedge T_c) \quad = \quad \tfrac{1}{2}P_{b+1}(T_c < T_b) + \tfrac{1}{2}P_{b-1}(T_a < T_b)$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{1}{c-b} + \frac{1}{2}\frac{1}{b-a} \\
&= \frac{c-a}{2(c-b)(b-a)}. \qquad (6)
\end{aligned}
$$

For the walk started at b, let $m(b, x; a, c)$ be the mean number of visits to x before the exit time $T_a \wedge T_c$. (Recall from Chapter 2 our convention that "before time t" includes time 0 but excludes time t). The number of returns to b clearly has a Geometric distribution, so by (6)

$$m(b, b; a, c) = \frac{2(c-b)(b-a)}{c-a}, \quad a \le b \le c. \qquad (7)$$

To get the analog for visits to x we consider whether or not x is hit at all before exiting; this gives

$$m(b, x; a, c) = P_b(T_x < T_a \wedge T_c)\ m(x, x; a, c).$$

Appealing to (5) and (7) gives the famous mean occupation time formula

$$m(b, x; a, c) \quad = \quad \begin{cases} \frac{2(x-a)(c-b)}{c-a}, & a \le x \le b \le c \\ \frac{2(c-x)(b-a)}{c-a}, & a \le b \le x \le c. \end{cases} \qquad (8)$$

Now the (random) time to exit must equal the sum of the (random) times spent at each state. So, taking expectations,

$$E_b(T_a \wedge T_c) = \sum_{x=a}^{c} m(b, x; a, c),$$

and after a little algebra we obtain

Lemma 2 $E_b(T_a \wedge T_c) = (b-a)(c-b),\ a < b < c.$

This derivation of Lemma 2 from (8) has the advantage of giving the mean occupation time formula (8) on the way. There are two alternative ways to prove Lemma 2. An elementary proof is to set up and solve the 1-step recurrence for $h(i) \equiv E_i(T_a \wedge T_c)$:

$$h(i) = 1 + \tfrac{1}{2}h(i+1) + \tfrac{1}{2}h(i-1),\ a < i < c$$

$$h(a) = h(c) = 0.$$

The more elegant proof uses a martingale argument. Taking $b = 0$ without loss of generality, the first equality below is the optional sampling theorem for the martingale $(X^2(t) - t)$:

$$\begin{aligned} E_0(T_a \wedge T_c) &= E_0 X^2(T_a \wedge T_c) \\ &= a^2 P_0(T_a < T_c) + c^2 P_0(T_c < T_a) \\ &= a^2 \frac{c}{c-a} + c^2 \frac{-a}{c-a} \text{ by (5)} \\ &= -ac. \end{aligned}$$

The preceding discussion works in discrete or continuous time. Exact distributions at time t will of course differ in the two cases. In discrete time we appeal to the Binomial distribution for the number of $+1$ steps, to get

$$P_0(X_{2t} = 2j) = \frac{(2t)!}{(t+j)!(t-j)!} 2^{-2t}, \ -t \le j \le t \tag{9}$$

and a similar expression for odd times t. In continuous time, the numbers of $+1$ and of -1 steps in time t are independent Poisson(t) variables, so

$$P_0(X_t = -j) = P_0(X_t = j) = e^{-2t} \sum_{i=0}^{\infty} \frac{t^{2i+j}}{i!(i+j)!}, \ j \ge 0. \tag{10}$$

1.2 Weighted linear graphs

Consider the n-vertex linear graph $0 - 1 - 2 - \cdots - (n-1)$ with arbitrary edge-weights $(w_1, \ldots, w_{n-1})$, where $w_i > 0$ is the weight on edge $(i-1, i)$. Set $w_0 = w_n = 0$ to make some later formulas cleaner. The corresponding discrete-time random walk has transition probabilities

$$p_{i,i+1} = \frac{w_{i+1}}{w_i + w_{i+1}}, \ p_{i,i-1} = \frac{w_i}{w_i + w_{i+1}}, \quad 0 \le i \le n-1$$

and stationary distribution

$$\pi_i = \frac{w_i + w_{i+1}}{w}, \quad 0 \le i \le n-1$$

where $w = 2\sum_i w_i$. In probabilistic terminology, this is a *birth-and-death process*, meaning that a transition cannot alter the state by more than 1. It is elementary that such processes are automatically reversible (xxx spells out the more general result for trees), so as discussed in Chapter 3 yyy the set-up above with weighted graphs gives the general discrete-time birth-and-death process with $p_{ii} \equiv 0$. But note that the continuization does *not* give the general continuous-time birth-and-death process, which has $2(n-1)$ parameters $(q_{i,i-1}, q_{i,i+1})$ instead of just $n-1$ parameters (w_i). The formulas below could all be extended to this general case (the analog of Proposition 3 can be found in undergraduate textbooks, e.g., Karlin and Taylor [9] Chapter 4) but our focus is on the simplifications which occur in the "weighted graphs" case.

Proposition 3 *(a) For $a < b < c$,*

$$P_b(T_c < T_a) = \frac{\sum_{i=a+1}^{b} w_i^{-1}}{\sum_{i=a+1}^{c} w_i^{-1}}.$$

(b) For $b < c$,

$$E_b T_c = c - b + 2 \sum_{j=b+1}^{c} \sum_{i=1}^{j-1} w_i w_j^{-1}.$$

(c) For $b < c$,

$$E_b T_c + E_c T_b = w \sum_{i=b+1}^{c} w_i^{-1}.$$

Note that we can obtain an expression for $E_c T_b$, $b < c$, by reflecting the weighted graph about its center.

Proof. These are extensions of (5,1,2) and recycle some of the previous arguments. Writing $h(j) = \sum_{i=1}^{j} w_i^{-1}$, we have that $(h(X_t))$ is a martingale, so

$$h(b) = E_b h(X(T_a \wedge T_c)) = p h(c) + (1-p) h(a)$$

for $p \equiv P_b(T_c < T_a)$. Solving this equation gives $p = \frac{h(b)-h(a)}{h(c)-h(a)}$, which is (a).

The mean hitting time formula (b) has four different proofs! Two that we will *not* give are as described below Lemma 2: Set up and solve a recurrence equation, or use a well-chosen martingale. The slick argument is to use the essential edge lemma (Lemma 1) to show

$$E_{j-1} T_j = 1 + 2 \frac{\sum_{i=1}^{j-1} w_i}{w_j}.$$

Then

$$E_b T_c = \sum_{j=b+1}^{c} E_{j-1} T_j,$$

establishing (b). Let us also write out the non-slick argument, using mean occupation times. By considering mean time spent at i,

$$E_b T_c = \sum_{i=0}^{b-1} P_b(T_i < T_c) m(i,i,c) + \sum_{i=b}^{c-1} m(i,i,c), \tag{11}$$

where $m(i,i,c)$ is the expectation, starting at i, of the number of visits to i before T_c. But

$$\begin{aligned} m(i,i,c) &= \frac{1}{P_i(T_c < T_i^+)} \\ &= \frac{1}{p_{i.i+1} P_{i+1}(T_c < T_i)} \\ &= (w_i + w_{i+1}) \sum_{j=i+1}^{c} w_j^{-1} \text{ using (a).} \end{aligned}$$

Substituting this and (a) into (11) leads to the formula stated in (b).

Finally, (c) can be deduced from (b), but it is more elegant to use the essential edge lemma to get

$$E_{i-1} T_i + E_i T_{i-1} = w / w_i \tag{12}$$

and then use

$$E_b T_c + E_c T_b = \sum_{i=b+1}^{c} (E_{i-1} T_i + E_i T_{i-1}). \quad \blacksquare$$

We now start some little calculations relating to the parameters discussed in Chapter 4. Plainly, from Proposition 3

$$\tau^* = w \sum_{i=1}^{n-1} w_i^{-1}. \tag{13}$$

Next, consider calculating $E_\pi T_b$. We could use Proposition 3(b), but instead let us apply Theorem yyy of Chapter 3, giving $E_\pi T_b$ in terms of unit flows from b to π. In a linear graph there is only one such flow, which for $i \geq b$ has $f_{i,i+1} = \pi[i+1, n-1] = \sum_{j=i+1}^{n-1} \pi_j$, and for $i \leq b-1$ has $f_{i,i+1} = -\pi[0,i]$, and so the Proposition implies

$$E_\pi T_b = w \sum_{i=b+1}^{n-1} \frac{\pi^2[i, n-1]}{w_i} + w \sum_{i=1}^{b} \frac{\pi^2[0, i-1]}{w_i}. \tag{14}$$

There are several ways to use the preceding results to compute the average hitting time parameter τ_0. Perhaps the most elegant is

$$\begin{aligned}
\tau_0 &= \sum_i \sum_{j>i} \pi_i \pi_j (E_i T_j + E_j T_i) \\
&= \sum_{k=1}^{n-1} \pi[0,k-1]\pi[k,n-1](E_{k-1}T_k + E_k T_{k-1}) \\
&= \sum_{k=1}^{n-1} \pi[0,k-1]\pi[k,n-1] w/w_k \text{ by (12)} \\
&= w^{-1} \sum_{k=1}^{n-1} w_k^{-1} \left(w_k + 2 \sum_{j=1}^{k-1} w_j \right) \left(w_k + 2 \sum_{j=k+1}^{n-1} w_j \right). \qquad (15)
\end{aligned}$$

There are sophisticated methods (see Notes) of studying τ_1, but let us just point out that Proposition 23 later (proved in the more general context of trees) holds in the present setting, giving

$$\frac{1}{K_1} \min_x \max(E_0 T_x, E_{n-1} T_x) \leq \tau_1 \leq K_2 \min_x \max(E_0 T_x, E_{n-1} T_x). \qquad (16)$$

We do not know an explicit formula for τ_2, but we can get an upper bound easily from the "distinguished paths" result Chapter 4 yyy. For $x < y$ the path γ_{xy} has $r(\gamma_{xy}) = \sum_{u=x+1}^{y} 1/w_u$ and hence the bound is

$$\tau_2 \leq \frac{1}{w} \max_j \sum_{x=0}^{j-1} \sum_{y=j}^{n-1} \sum_{u=x+1}^{y} \frac{(w_x + w_{x+1})(w_y + w_{y+1})}{w_u}. \qquad (17)$$

jjj This uses the Diaconis–Stroock version. The Sinclair version is

$$\tau_2 \leq \frac{1}{w} \max_j \frac{1}{w_j} \sum_{x=0}^{j-1} \sum_{y=j}^{n-1} (w_x + w_{x+1})(w_y + w_{y+1})(y - x).$$

xxx literature on τ_2 (van Doorn, etc.)

jjj Also relevant is work of N. Kahale (and others) on how optimal choice of weights in use of Cauchy–Schwarz inequality for Diaconis–Stroock–Sinclair leads to equality in case of birth-and-death chains.

jjj See also Diaconis and Saloff-Coste Metropolis paper, which mentions work of Diaconis students on Metropolizing birth-and-death chains.

xxx examples of particular **w**. **jjj** might just bring up as needed?

xxx contraction principle and lower bounds on τ_2 (relating to current Section 6 of Chapter 4)

By Chapter 4 Lemma yyy,

$$\tau_c = \max_{1 \leq i \leq n-1} \frac{\pi[0, i-1]\pi[i, n-1]}{w_i}. \tag{18}$$

1.3 Useful examples of one-dimensional chains

Example 4 *The two-state chain.*

This is the birth-and-death chain on $\{0, 1\}$ with $p_{01} = 1 - p_{00} = p$ and $p_{10} = 1 - p_{11} = q$, where $0 < p < 1$ and $0 < q < 1$ are arbitrarily specified. Since p_{00} and p_{11} are positive, this does not quite fit into the framework of Section 1.2, but everything is nonetheless easy to calculate. The stationary distribution is given by

$$\pi_0 = q/(p+q), \quad \pi_1 = p/(p+q).$$

In discrete time, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 1 - p - q$, and in the notation of Chapter 3, Section yyy for the spectral representation, the matrix S has $s_{11} = 1 - p$, $s_{22} = 1 - q$, and $s_{12} = s_{21} = (pq)^{1/2}$ with normalized right eigenvectors

$$u_1 = [(q/(p+q))^{1/2}, (p/(p+q))^{1/2}]^T, \quad u_2 = [(p/(p+q))^{1/2}, -(q/(p+q))^{1/2}]^T.$$

The transition probabilities are given by

$$\begin{aligned} P_0(X_t = 1) &= 1 - P_0(X_t = 0) = \frac{p}{p+q}[1 - (1-p-q)^n], \\ P_1(X_t = 0) &= 1 - P_1(X_t = 1) = \frac{q}{p+q}[1 - (1-p-q)^n] \end{aligned}$$

in discrete time and by

$$\begin{aligned} P_0(X_t = 1) &= 1 - P_0(X_t = 0) = \frac{p}{p+q}[1 - e^{-(p+q)t}], \\ P_1(X_t = 0) &= 1 - P_1(X_t = 1) = \frac{q}{p+q}[1 - e^{-(p+q)t}] \end{aligned}$$

in continuous time. It is routine to calculate $E_0 T_1 = 1/p$, $E_1 T_0 = 1/q$, and

$$\bar{d}(t) = e^{-(p+q)t}, \quad d(t) = \max(p/(p+q), q/(p+q))\, e^{-(p+q)t},$$

and then

$$\max_{ij} E_i T_j = \max(E_0 T_1, E_1 T_0) = \frac{1}{\min(p,q)}, \quad \tau^* = E_0 T_1 + E_1 T_0 = \frac{p+q}{pq},$$

and

$$\tau_0 = \tau_1 = \tau_2 = \tau_c = 1/(p+q).$$

Example 5 *Biased random walk with reflecting barriers.*

We consider the chain on $\{0, 1, \ldots, n-1\}$ with reflecting barriers at 0 and $n-1$ that at each unit of time moves distance 1 rightward with probability p and distance 1 leftward with probability $q = 1-p$. Formally, the setting is that of Section 1.2 with

$$w_i = \rho^{i-1}, \quad w = \frac{2(1-\rho^{n-1})}{1-\rho} \to \frac{2}{1-\rho},$$

where we assume $\rho \equiv p/q < 1$ and all asymptotics developed for this example are for fixed ρ and large n. If $\rho \neq 1$, there is by symmetry no loss of generality in assuming $\rho < 1$, and the case $\rho = 1$ will be treated later in Example 8.

Specializing the results of Section 1.2 to the present example, one can easily derive the asymptotic results

$$\max_{ij} E_i T_j \sim \tau^* \sim E_\pi T_{n-1} \sim 2\rho^{-(n-2)}/(1-\rho)^2 \tag{19}$$

and, by use of (15),

$$\tau_0 \sim \frac{1+\rho}{1-\rho} n. \tag{20}$$

For τ_c, the maximizing i in (18) equals $(1+o(1))n/2$, and this leads to

$$\tau_c \to (1+\rho)/(1-\rho). \tag{21}$$

The spectral representation can be obtained using the orthogonal polynomial techniques described in Karlin and Taylor [10] Chapter 10; see especially Section 5(b) there. The reader may verify that the eigenvalues of $\mathbf{P}$ in discrete time are 1, -1, and, for $m = 1, \ldots, n-2$,

$$\frac{2\rho^{1/2}}{1+\rho} \cos\theta_m, \text{ where } \theta_m \equiv \frac{m\pi}{n-1}$$

with (unnormalized) right eigenvector

$$\rho^{-i/2}\left[2\cos(i\theta_m) - (1-\rho)\frac{\sin((i+1)\theta_m)}{\sin(\theta_m)}\right], \quad i = 0, \ldots, n-1.$$

In particular,

$$\tau_2 = \left[1 - \frac{2\rho^{1/2}}{1+\rho}\cos\left(\frac{\pi}{n-1}\right)\right]^{-1} \to \frac{1+\rho}{(1-\rho^{1/2})^2}. \tag{22}$$

The random walk has drift $p - q = -(1-\rho)/(1+\rho) \equiv -\mu$. It is not hard to show for fixed $t > 0$ that the distances $\bar{d}_n(tn)$ and $d_n(tn)$ of Chapter 4 yyy converge to 1 if $t < \mu$ and to 0 if $t > \mu$.

jjj include details? In fact, the cutoff occurs at $\mu n + c_\rho n^{1/2}$: cf. (e.g.) Example 4.46 in [7]. Continue same paragraph:

In particular,

$$\tau_1 \sim \frac{1-\rho}{1+\rho} n \tag{23}$$

Example 6 *The $M/M/1$ queue.*

We consider the M/M/1/$(n-1)$ queue. Customers queue up at a facility to wait for a single server (hence the "1") and are handled according to a "first come, first served" queuing discipline. The first "M" specifies that the arrival point process is Markovian, i.e., a Poisson process with intensity parameter λ (say); likewise, the second "M" reflects our assumption that the service times are exponential with parameter μ (say). The parameter $n-1$ is the queue size limit; customers arriving when the queue is full are turned away.

We have described a continuous-time birth-and-death process with constant birth and death rates λ and μ, respectively. If $\lambda + \mu = 1$, this is nearly the continuized biased random walk of Example 5, the only difference being in the boundary behavior. In particular, one can check that the asymptotics in (19)–(23) remain unchanged, where $\rho \equiv \lambda/\mu$, called the *traffic intensity*, remains fixed and n becomes large. For the M/M/1/$(n-1)$ queue, the stationary distribution is the conditional distribution of $G - 1$ given $G \le n$, where G has the Geometric$(1-\rho)$ distribution. The eigenvalues are 1 and, for $m = 1, \ldots, n-1$,

$$\frac{2\rho^{1/2}}{1+\rho}\cos\theta_m, \text{ where now } \theta_m \equiv \frac{m\pi}{n}$$

with (unnormalized) right eigenvector

$$\frac{2\rho^{-i/2}}{1+\rho}\left[\cos(i\theta_m) + (\rho^{1/2}\cos\theta_m - 1)\frac{\sin((i+1)\theta_m)}{\sin(\theta_m)}\right], \quad i = 0, \ldots, n-1.$$

2 Special graphs

In this section we record results about some specific easy-to-analyze graphs. As in Section 1.3, we focus on the parameters $\tau^*, \tau_0, \tau_1, \tau_2, \tau_c$ discussed in Chapter 4; orders of magnitudes of these parameters (in the asymptotic setting discussed with each example) are summarized in terms of n, the number of vertices, in the following table. A minor theme is that some of the graphs are known or conjectured to be extremal for our parameters. In the context of extremality we ignore the parameter τ_1 since its exact definition is a little arbitrary.

jjj David: (1) Shall I add complete bipartite to table? (2) Please fill in missing entries for torus.

Orders of magnitude of parameters [$\tau = \Theta(\text{entry})$] for special graphs.

Example	τ^*	τ_0	τ_1	τ_2	τ_c
7. cycle	n^2	n^2	n^2	n^2	n
8. path	n^2	n^2	n^2	n^2	n
9. complete graph	n	n	1	1	1
10. star	n	n	1	1	1
11. barbell	n^3	n^3	n^3	n^3	n^2
12. lollipop	n^3	n^2	n^2	n^2	n
13. necklace	n^2	n^2	n^2	n^2	n
14. balanced r-tree	$n \log n$	$n \log n$	n	n	n
15. d-cube ($d = \log_2 n$)	n	n	$d \log d$	d	d
16. dense regular graphs	n	n	1	1	1
17. d-dimensional torus					
$d = 2$	jjj?	$n \log n$	$n^{2/d}$	$n^{2/d}$	jjj?$n^{1/d}$
$d \geq 3$	jjj?	n	$n^{2/d}$	$n^{2/d}$	jjj?$n^{1/d}$
19. rook's walk	n	n	1	1	1

In simpler cases we also record the t-step transition probabilities $P_i(X_t = j)$ in discrete and continuous time. In fact one could write out exact expressions for $P_i(X_t = j)$ and indeed for hitting time distributions in almost all these examples, but complicated exact expressions are seldom very illuminating.

qqq names of graphs vary—suggestions for "standard names" from readers of drafts are welcome.

Example 7 *The n-cycle.*

This is just the graph $0 - 1 - 2 - \cdots - (n-1) - 0$ on n vertices. By rotational symmetry, it is enough to give formulas for random walk started at 0. If $(\hat{X}_t)$ is random walk on (all) the integers, then $X_t = \phi(\hat{X}_t)$ is random walk on the n-cycle, for

$$\phi(i) = i \bmod n.$$

Thus results for the n-cycle can be deduced from results for the integers. For instance,

$$E_0 T_i = i(n-i) \tag{24}$$

by Lemma 2, because this is the mean exit time from $(i-n, i)$ for random walk on the integers. We can now calculate

$$\begin{aligned} \max_{ij} E_i T_j &= \lfloor n^2/4 \rfloor \\ \tau^* \equiv \max_{ij}(E_i T_j + E_j T_i) &= 2\lfloor n^2/4 \rfloor \qquad (25) \\ \tau_0 = n^{-1} \sum_j E_0 T_j &= (n^2-1)/6 \qquad (26) \end{aligned}$$

where for the final equality we used the formula

$$\sum_{m=1}^{n} m^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

As at (9) and (10) we can get an expression for the distribution at time t from the Binomial distribution (in discrete time) or the Poisson distribution (in continuous time). The former is

$$P_0(X_t = i) = \sum_{j:2j-t=i \bmod n} \frac{t!}{j!(t-j)!} 2^{-t}.$$

A more useful expression is obtained from the spectral representation. The n eigenvalues of the transition matrix are $\cos(2\pi m/n), 0 \le m \le n-1$. That is, 1 and (if n is even) -1 are simple eigenvalues, with respective normalized eigenvectors $u_{i0} = 1/\sqrt{n}$ and $u_{i,n/2} = (-1)^i/\sqrt{n}$ $(0 \le i \le n-1)$. The multiplicity of $\cos(2\pi m/n)$ is 2 for $0 < m < n/2$; the corresponding normalized eigenvectors are $u_{im} = \sqrt{2/n}\cos(2\pi im/n)$ and $u_{i,-m} = \sqrt{2/n}\sin(2\pi im/n)$ $(0 \le i \le n-1)$. Thus

$$P_0(X_t = i) = \frac{1}{n}\sum_{m=0}^{n-1}(\cos(2\pi m/n))^t \cos(2\pi im/n),$$

a fact most easily derived using Fourier analysis.

jjj Cite Diaconis book [6]? Continue same paragraph:

So the relaxation time is

$$\tau_2 = \frac{1}{1-\cos(2\pi/n)} \sim \frac{n^2}{2\pi^2}.$$

As an aside, note that the eigentime identity (Chapter 3 yyy) gives the curious identity

$$\frac{n^2-1}{6} = \sum_{m=1}^{n-1}\frac{1}{1-\cos(2\pi m/n)}$$

whose $n \to \infty$ limit is the well-known formula $\sum_{m=1}^{\infty} m^{-2} = \pi^2/6$.

If n is even, the discrete-time random walk is periodic. This parity problem vanishes in continuous time, for which we have the formula

$$P_0(X(t) = i) = \frac{1}{n}\sum_{m=0}^{n-1}\exp(-t(1-\cos(2\pi m/n)))\cos(2\pi im/n). \qquad (27)$$

Turning to total variation convergence, we remain in continuous time and consider the distance functions $\bar{d}_n(t)$ and $d_n(t)$ of Chapter 4 yyy. The reader familiar with the notion of weak convergence of random walks to Brownian motion (on the circle, in this setting) will see immediately that

$$\bar{d}_n(tn^2) \to \bar{d}_\infty(t)$$

where the limit is "$\bar{d}$ for Brownian motion on the circle", which can be written as

$$\bar{d}_\infty(t) \equiv 1 - 2P((t^{1/2}Z) \bmod 1 \ \in (1/4, 3/4))$$

where Z has the standard Normal distribution. So

$$\tau_1 \sim cn^2$$

for the constant c such that $\bar{d}_\infty(c) = e^{-1}$, whose numerical value $c \doteq 0.063$ has no real significance.

jjj David: You got 0.054. Please check. Continue same paragraph:

Similarly

$$d_n(tn^2) \to d_\infty(t) \equiv \frac{1}{2}\int_0^1 |f_t(u) - 1|\, du,$$

where f_t is the density of $(t^{1/2}Z)$ mod 1.

As for τ_c, the *sup* in its definition is attained by some A of the form $[0, i-1]$, so

$$\tau_c = \max_i \frac{\frac{i}{n}(1-\frac{i}{n})}{1/n} = \frac{1}{n}\left\lfloor \frac{n^2}{4} \right\rfloor \sim \frac{n}{2}.$$

As remarked in Chapter 4 yyy, this provides a counter-example to reversing inequalities in Theorem yyy. But if we consider $\max_A(\pi(A)E_\pi T_A)$, the *max* is attained with $A = [\frac{n}{2} - \alpha n, \frac{n}{2} + \alpha n]$, say, where $0 \le \alpha < 1/2$. By Lemma 2, for $x \in (-\frac{1}{2} + \alpha, \frac{1}{2} - \alpha)$,

$$E_{\lfloor (x \bmod 1)n \rfloor} T_A \sim \left(\tfrac{1}{2} - \alpha - x\right)\left(\tfrac{1}{2} - \alpha + x\right) n^2,$$

and so

$$E_\pi T_A \sim n^2 \int_{-\frac{1}{2}+\alpha}^{\frac{1}{2}-\alpha} \left(\frac{1}{2} - \alpha - x\right)\left(\frac{1}{2} - \alpha + x\right) dx = \frac{4(\frac{1}{2}-\alpha)^3 n^2}{3}.$$

Thus

$$\max_A(\pi(A)E_\pi T_A) \sim n^2 \sup_{0<\alpha<1/2} \frac{4(\frac{1}{2}-\alpha)^3\ 2\alpha}{3} = \frac{9n^2}{512},$$

consistent with Chapter 4 Open Problem yyy.

xxx level of detail for $\bar{d}$ results, here and later.

Remark. Parameters τ^*, τ_0, τ_1, and τ_2 are all $\Theta(n^2)$ in this example, and in Chapter 6 we'll see that they are $O(n^2)$ over the class of regular graphs. However, the exact maximum values over all n-vertex regular graphs (or the constants c in the $\sim cn^2$ asymptotics) are not known. See Chapter 6 for the natural conjectures.

Example 8 *The n-path.*

This is just the graph $0 - 1 - 2 - \cdots - (n-1)$ on n vertices. If $(\hat{X}_t)$ is random walk on (all) the integers, then $X_t = \phi(\hat{X}_t)$ is random walk on the

n-path, for the "concertina" map

$$\phi(i) = \begin{cases} i & \text{if } i \bmod 2(n-1) \le n-1 \\ 2(n-1) - (i \bmod 2(n-1)) & \text{otherwise.} \end{cases}$$

Of course the stationary distribution is not quite uniform:

$$\pi_i = \frac{1}{n-1},\ 1 \le i \le n-2;\quad \pi_0 = \pi_{n-1} = \frac{1}{2(n-1)}.$$

Again, results for the n-path can be deduced from results for the integers. Using Lemma 2,

$$E_i T_j = (j-i)(j+i),\ 0 \le i < j \le n-1. \tag{28}$$

¿From this, or from the more general results in Section 1.2, we obtain

$$\max_{ij} E_i T_j = (n-1)^2 \tag{29}$$

$$\tau^* \equiv \max_{ij}(E_i T_j + E_j T_i) = 2(n-1)^2 \tag{30}$$

$$\tau_0 = \sum_j \pi_j E_0 T_j = \frac{1}{3}(n-1)^2 + \frac{1}{6} \tag{31}$$

We can also regard X_t as being derived from random walk $\tilde{X}_t$ on the $(2n-2)$-cycle via $X_t = \min(\tilde{X}_t, 2n-2-\tilde{X}_t)$. So we can deduce the spectral representation from that in the previous example:

$$P_i(X_t = j) = \sqrt{\pi_j/\pi_i} \sum_{m=0}^{n-1} \lambda_m^t u_{im} u_{jm}$$

where, for $0 \le m \le n-1$,

$$\lambda_m = \cos(\pi m/(n-1))$$

and

$$u_{0m} = \sqrt{\pi_m};\quad u_{n-1,m} = \sqrt{\pi_m}(-1)^m;$$
$$u_{im} = \sqrt{2\pi_m}\cos(\pi i m/(n-1)),\quad 1 \le i \le n-2.$$

In particular, the relaxation time is

$$\tau_2 = \frac{1}{1-\cos(\pi/(n-1))} \sim \frac{2n^2}{\pi^2}.$$

Furthermore, $\bar{d}_n(t) = \bar{\tilde{d}}_{2n-2}(t)$ and $d_n(t) = \tilde{d}_{2n-2}(t)$ for all t, so

$$\bar{d}_n(t(2n)^2) \to \bar{d}_\infty(t)$$

$$d_n(t(2n)^2) \to d_\infty(t)$$

where the limits are those in the previous example. Thus $\tau_1 \sim cn^2$, where $c \doteq 0.25$ is 4 times the corresponding constant for the n-cycle.

xxx explain: BM on $[0, 1]$ and circle described in Chapter 16.

It is easy to see that

$$\tau_c = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is even} \\ \\ \frac{n-1}{2} - \frac{1}{2(n-1)} & \text{if } n \text{ is odd} \end{cases}$$

In Section 3.2 we will see that the n-path attains the exact maximum values of our parameters over all n-vertex trees.

Example 9 *The complete graph.*

For the complete graph on n vertices, the t-step probabilities for the chain started at i can be calculated by considering the induced 2-state chain which indicates whether or not the walk is at i. This gives, in discrete time,

$$\begin{aligned} P_i(X_t = i) &= \frac{1}{n} + \left(1 - \frac{1}{n}\right)\left(-\frac{1}{n-1}\right)^t \\ P_i(X_t = j) &= \frac{1}{n} - \frac{1}{n}\left(-\frac{1}{n-1}\right)^t, \quad j \neq i \end{aligned} \qquad (32)$$

and, in continuous time,

$$\begin{aligned} P_i(X_t = i) &= \frac{1}{n} + \left(1 - \frac{1}{n}\right)\exp\left(-\frac{nt}{n-1}\right) \\ P_i(X_t = j) &= \frac{1}{n} - \frac{1}{n}\exp\left(-\frac{nt}{n-1}\right), \quad j \neq i \end{aligned} \qquad (33)$$

It is clear that the hitting time to $j \neq i$ has Geometric$(1/(n-1))$ distribution (in continuous time, Exponential$(1/(n-1))$ distribution), and so in particular

$$E_i T_j = n - 1, \ j \neq i. \qquad (34)$$

Thus we can calculate the parameters

$$\begin{aligned} \tau^* \equiv \max_{ij}(E_jT_i + E_iT_j) &= 2(n-1) && (35) \\ \max_{ij} E_iT_j &= n-1 && (36) \\ \tau_0 \equiv n^{-1}\sum_j E_iT_j &= (n-1)^2/n. && (37) \end{aligned}$$

From (32) the discrete-time eigenvalues are $\lambda_2 = \lambda_3 = \cdots = \lambda_n = -1/(n-1)$. So the relaxation time is

$$\tau_2 = (n-1)/n. \quad (38)$$

The total variation functions are

$$\bar{d}(t) = \exp\left(-\frac{nt}{n-1}\right), \quad d(t) = \frac{n-1}{n}\exp\left(-\frac{nt}{n-1}\right),$$

so

$$\tau_1 = (n-1)/n. \quad (39)$$

It is easy to check

$$\tau_c = (n-1)/n.$$

We have already proved (Chapter 3 yyy) that the complete graph attains the exact minimum of $\tau^*, \max_{ij} E_iT_j, \tau_0$, and τ_2 over all n-vertex graphs. The same holds for τ_c, by considering (in an arbitrary graph) a vertex of minimum degree.

Example 10 *The n-star.*

This is the graph on $n \geq 3$ vertices $\{0, 1, 2, \ldots, n-1\}$ with edges $0-1$, $0-2$, $0-3$, …, $0-(n-1)$. The stationary distribution is

$$\pi_0 = 1/2, \ \pi_i = 1/(2(n-1)), \quad i \geq 1.$$

In discrete time the walk is periodic. Starting from the leaf 1, the walk at even times is simply independent and uniform on the leaves, so

$$P_1(X_{2t} = i) = 1/(n-1), \ i \geq 1$$

for $t \geq 1$. At odd times, the walk is at 0. Writing these t-step probabilities as

$$P_1(X_t = i) = \frac{1}{2(n-1)}(1+(-1)^t)1_{(i\geq 1)} + \frac{1}{2}(1+(-1)^{t+1})1_{(i=0)}, \quad t \geq 1$$

we see that the discrete-time eigenvalues are $\lambda_2 = \cdots = \lambda_{n-1} = 0,\ \lambda_n = -1$ and hence the relaxation time is

$$\tau_2 = 1.$$

The mean hitting times are

$$E_1 T_0 = 1$$
$$E_1 T_j = 2(n-1), \quad j \geq 2,$$

where the latter comes from the fact that $T_j/2$ has Geometric$(1/(n-1))$ distribution, using the "independent uniform on leaves at even times" property. Then

$$E_0 T_1 = 2n - 3.$$

We can calculate the parameters

$$\tau^* \equiv \max_{ij}(E_i T_j + E_j T_i) = 4n - 4 \tag{40}$$
$$\max_{ij} E_i T_j = 2n - 2 \tag{41}$$
$$\tau_0 = \sum_j E_0 T_j \pi_j = n - \tfrac{3}{2}. \tag{42}$$

In continuous time we find

$$\begin{aligned}
P_1(X_t = 1) &= \frac{1}{2(n-1)}(1 + e^{-2t}) + \frac{n-2}{n-1} e^{-t} \\
P_1(X_t = i) &= \frac{1}{2(n-1)}(1 + e^{-2t}) - \frac{1}{n-1} e^{-t}, \ i > 1 \\
P_1(X_t = 0) &= \tfrac{1}{2}(1 - e^{-2t}) \\
P_0(X_t = 0) &= \tfrac{1}{2}(1 + e^{-2t}) \\
P_0(X_t = 1) &= \frac{1}{2(n-1)}(1 - e^{-2t})
\end{aligned}$$

This leads to

$$\bar{d}(t) = e^{-t}, \quad d(t) = \frac{1}{2(n-1)} e^{-2t} + \frac{n-2}{n-1} e^{-t},$$

from which

$$\tau_1 = 1.$$

Clearly (isolate a leaf)

$$\tau_c = 1 - \frac{1}{2(n-1)}.$$

We shall see in Section 3.2 that the n-star attains the exact minimum of our parameters over all n-vertex trees.

Example 11 *The barbell.*

Here is a graph on $n = 2m_1 + m_2$ vertices ($m_1 \geq 2, m_2 \geq 0$). Start with two complete graphs on m_1 vertices. Distinguish vertices $v_l \neq v_L$ in one graph ("the left bell") and vertices $v_R \neq v_r$ in the other graph ("the right bell"). Then connect the graphs via a path $v_L - w_1 - w_2 - \cdots - w_{m_2} - v_R$ containing m_2 new vertices.

xxx picture

A point of the construction is that the mean time to go from a typical point v_l in the left bell to a typical point v_r in the right bell is roughly $m_1^2 m_2$. To argue this informally, it takes mean time about m_1 to hit v_L; then there is chance $1/m_1$ to hit w_1, so it takes mean time about m_1^2 to hit w_1; and from w_1 there is chance about $1/m_2$ to hit the right bell before returning into the left bell, so it takes mean time about $m_1^2 m_2$ to enter the right bell.

The exact result, argued below, is

$$\max_{ij} E_i T_j = E_{v_l} T_{v_r} = m_1(m_1-1)(m_2+1) + (m_2+1)^2 + 4(m_1-1) + 4\frac{m_2+1}{m_1}. \qquad (43)$$

It is cleaner to consider asymptotics as

$$n \to \infty,\ m_1/n \to \alpha,\ m_2/n \to 1 - 2\alpha$$

with $0 < \alpha < 1/2$. Then

$$\begin{aligned} \max_{ij} E_i T_j &\sim \alpha^2(1-2\alpha)n^3 \\ &\sim \frac{n^3}{27} \text{ for } \alpha = 1/3 \end{aligned}$$

where $\alpha = 1/3$ is the asymptotic maximizer here and for the other parameters below. Similarly

$$\begin{aligned} \tau^* &\sim 2\alpha^2(1-2\alpha)n^3 \\ &\sim \frac{2n^3}{27} \text{ for } \alpha = 1/3. \end{aligned}$$

The stationary distribution π puts mass $\to 1/2$ on each bell. Also, by (45)–(47) below, $E_{v_l} T_v = o(E_{v_l} T_{v_r})$ uniformly for vertices v in the left bell and $E_{v_l} T_v \sim E_{v_l} T_{v_r} \sim \alpha^2(1-2\alpha)n^3$ uniformly for vertices v in the right bell. Hence

$$\tau_0 \equiv \sum_v \pi_v E_{v_l} T_v \sim \frac{1}{2} E_{v_l} T_{v_r} \sim \frac{1}{2}\alpha^2(1-2\alpha)n^3$$

and so we have proved the "τ_0" part of

$$\begin{aligned} \text{each of } \{\tau_0, \tau_1, \tau_2\} &\sim \tfrac{1}{2}\alpha^2(1-2\alpha)n^3 \\ &\sim \frac{n^3}{54} \text{ for } \alpha = 1/3. \end{aligned} \tag{44}$$

Consider the relaxation time τ_2. For the function g defined to be $+1$ on the left bell, -1 on the right bell and linear on the bar, the Dirichlet form gives

$$\mathcal{E}(g,g) = \frac{2}{(m_2+1)(m_1(m_1-1)+m_2+1)} \sim \frac{2}{\alpha^2(1-2\alpha)n^3}.$$

Since the variance of g tends to 1, the extremal characterization of τ_2 shows that $\frac{1}{2}\alpha^2(1-2\alpha)n^3$ is an asymptotic lower bound for τ_2. But in general $\tau_2 \leq \tau_0$, so having already proved (44) for τ_0 we must have the same asymptotics for τ_2. Finally, without going into details, it is not hard to show that for fixed $t > 0$,

$$\bar{d}_n\left(\frac{1}{2}\alpha^2(1-2\alpha)n^3 t\right) \to e^{-t}, \ d_n\left(\frac{1}{2}\alpha^2(1-2\alpha)n^3 t\right) \to \frac{1}{2}e^{-t}$$

from which the "τ_1" assertion of (44) follows.

jjj Proof? (It's not so terrifically easy, either! How much do we want to include?) I've (prior to writing this) carefully written out an argument similar to the present one, also involving approximate exponentiality of a hitting time distribution, for the balanced r-tree below. Here is a rough sketch for the argument for $\bar{d}$ here; note that it uses results about the next example (the lollipop). (The argument for d is similar.) The pair (v_l, v_r) of initial states achieves $\bar{d}(t)$ for every t (although the following can be made to work without knowing this "obvious fact" a priori). Couple chains starting in these states by having them move symmetrically in the obvious fashion. Certainly these copies will couple by the time T the copy started at v_l has reached the center vertex $w_{m_2/2}$ of the bar. We claim that the distribution of T is approximately exponential, and its expected value is $\sim \frac{1}{2}m_1^2 m_2 \sim \frac{1}{2}\alpha^2(1-2\alpha)n^3$ by the first displayed result for the lollipop example, with m_2 changed to $m_2/2$ there. (In keeping with this observation, I'll refer to the "half-stick" lollipop in the next paragraph.)

jjj (cont.) To get approximate exponentiality for the distribution of T, first argue easily that it's approximately the same as that of $T_{w_{m/2}}$ for the half-stick lollipop started in stationarity. But that distribution is, in turn, approximately exponential by Chapter 3 Proposition yyy, since $\tau_2 = \Theta(n^2) = o(n^3)$ for the half-stick lollipop. ∎

Proof of (43). The mean time in question is the sum of the following mean times:

$$E_{v_l}T_{v_L} = m_1 - 1 \tag{45}$$

$$E_{v_L}T_{v_R} = m_1(m_1-1)(m_2+1) + (m_2+1)^2 \tag{46}$$

$$E_{v_R}T_{v_r} = 3(m_1-1) + 4\frac{m_2+1}{m_1}. \tag{47}$$

Here (45) is just the result (34) for the complete graph. And (46) is obtained by summing over the edges of the "bar" the expression

$$E_{w_i}T_{w_{i+1}} = m_1(m_1-1) + 2i + 1, \quad i = 0, \ldots, m_2 \tag{48}$$

obtained from the general formula for mean hitting time across an essential edge of a graph (Lemma 1), where $w_0 = v_L$ and $w_{m_2+1} = v_R$. To argue (47), we start with the 1-step recurrence

$$E_{v_R}T_{v_r} = 1 + \frac{1}{m_1}E_{w_{m_2}}T_{v_r} + \frac{m_1-2}{m_1}E_xT_{v_r}$$

where x denotes a vertex of the right bell distinct from v_R and v_r. Now

$$E_{w_{m_2}}T_{v_r} = m_1(m_1-1) + 2m_2 + 1 + E_{v_R}T_{v_r}$$

using the formula (48) for the mean passage time from w_{m_2} to v_R. Starting from x, the time until a hit on either v_R or v_r has Geometric$(2/(m_1-1))$ distribution, and the two vertices are equally likely to be hit first. So

$$E_xT_{v_r} = (m_1-1)/2 + \tfrac{1}{2}E_{v_R}T_{v_r}.$$

The last three expressions give an equation for $E_{v_R}T_{v_r}$ whose solution is (47). And it is straightforward to check that $E_{v_l}T_{v_r}$ does achieve the maximum, using (45)–(47) to bound the general E_iT_j. ■

It is straightforward to check

$$\tau_c \sim \frac{\alpha^2 n^2}{2}.$$

Example 12 *The lollipop.*

xxx picture

This is just the barbell without the right bell. That is, we start with a complete graph on m_1 vertices and add m_2 new vertices in a path. So there are $n = m_1 + m_2$ vertices, and w_{m_2} is now a leaf. In this example, by (45) and (46), with m_2 in place of $m_2 + 1$, we have

$$\max_{ij} E_i T_j = E_{v_l} T_{w_{m_2}} = m_1(m_1 - 1)m_2 + (m_1 - 1) + m_2^2.$$

In the asymptotic setting with

$$n \to \infty, m_1/n \to \alpha, m_2/n \to 1 - \alpha$$

where $0 < \alpha < 1$, we have

$$\begin{aligned} \max_{ij} E_i T_j &\sim \alpha^2(1-\alpha)n^3 \\ &\sim \frac{4n^3}{27} \text{ for } \alpha = 2/3, \end{aligned} \tag{49}$$

where $\alpha = 2/3$ gives the asymptotic maximum.

Let us discuss the other parameters only briefly, in the asymptotic setting. Clearly $E_{w_{m_2}} T_{v_L} = m_2^2 \sim (1-\alpha)^2 n^2$ and it is not hard to check

$$E_{w_{m_2}} T_{v_l} = \max_v E_v T_{v_l} \sim (1-\alpha)^2 n^2, \tag{50}$$

whence

$$\tau^* = \max_{ij}(E_i T_j + E_j T_i) = E_{v_l} T_{w_{m_2}} + E_{w_{m_2}} T_{v_l} \sim \alpha^2(1-\alpha)n^3.$$

Because the stationary distribution puts mass $\Theta(1/n)$ on the "bar", (50) is also enough to show that $\tau_0 = O(n^2)$. So by the general inequalities between our parameters, to show

$$\text{each of } \{\tau_0, \tau_1, \tau_2\} = \Theta(n^2) \tag{51}$$

it is enough to show that $\tau_2 = \Omega(n^2)$. But for the function g defined to be 0 on the "bell", 1 at the end w_{m_2} of the "bar," and linear along the bar, a brief calculation gives

$$\mathcal{E}(g, g) = \Theta(n^{-3}), \quad \text{var}_{\pi} g = \Theta(n^{-1})$$

so that $\tau_2 \geq (\text{var}_{\pi} g)/\mathcal{E}(g,g) = \Omega(n^2)$, as required.

Finally, in the asymptotic setting it is straightforward to check that τ_c is achieved by $A = \{w_1, \ldots, w_{m_2}\}$, giving

$$\tau_c \sim 2(1-\alpha)n.$$

Remark. The barbell and lollipop are the natural candidates for the n-vertex graphs which maximize our parameters. The precise conjectures and known results will be discussed in Chapter 6.

jjj We need to put somewhere—Chapter 4 on τ_c? Chapter 6 on **max** parameters over n-vertex graphs? in the barbell example?—the fact that $\mathbf{max}\,\tau_c$ is attained, when n is even, by the barbell with $m_2 = 0$, the **max** value being $(n^2-2n+2)/8 \sim n^2/8$. Similarly, when n is odd, the **max**imizing graph is formed by joining complete graphs on $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ vertices respectively by a single edge, and the **max** value is easy to write down (I've kept a record) but not so pretty; however, this value too is $\sim n^2/8$, which is probably all we want to say anyway. Here is the first draft of a proof:

For random walk on an unweighted graph, τ_c is the maximum over nonempty proper subsets A of the ratio

$$\frac{(\deg A)(\deg A^c)}{2|\mathcal{E}|(A, A^c)}, \tag{52}$$

where $\deg A$ is defined to be the sum of the degrees of vertices in A and (A, A^c) is the number of directed cut edges from A to A^c.

jjj Perhaps it would be better for exposition to stick with *un*directed edges and introduce factor $1/2$?

Maximizing now over choice of graphs, the **max** in question is no larger than the maximum M, over all choices of $n_1 > 0$, $n_2 > 0$, e_1, e_2, and e' satisfying $n_1 + n_2 = n$ and $0 \le e_i \le \binom{n_i}{2}$ for $i = 1, 2$ and $1 \le e' \le n_1 n_2$, of the ratio

$$\frac{(2e_1 + e')(2e_2 + e')}{2(e_1 + e_2 + e')e'}. \tag{53}$$

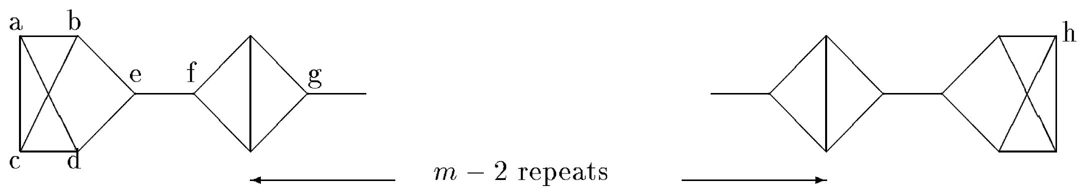
(We don't claim equality because we don't check that each n_i-graph is connected. But we'll show that M is in fact achieved by the connected graph claimed above.)

Simple calculus shows that the ratio (53) is (as one would expect) increasing in e_1 and e_2 and decreasing in e'. Thus, for given n_1, (53) is maximized by considering complete graphs of size n_1 and $n_2 = n - n_1$ joined by a single edge. Call the maximum value $M(n_1)$. If n is even, it is then easy to see that M_{n_1} is maximized by $n_1 = n/2$, giving $M = (n^2 - 2n + 2)/8$, as desired.

For the record, here are the slightly tricky details if n is odd. Write $\nu = n/2$ and $n_1 = \nu - y$ and put $x = y^2$. A short calculation gives $M(n_1) = 1 + g(x)$, where $g(x) \equiv [(a - x)(b - x) - 1]/(2x + c)$ with $a = \nu^2$, $b = (\nu - 1)^2$, and $c = 2\nu(\nu - 1) + 2$. Easy calculus shows that g is U-shaped over $[0, \nu]$ and then that $g(1/4) \geq g(\nu^2)$. Thus $M(n_1)$ is maximized when $n_1 = \nu - \frac{1}{2} = \lfloor n/2 \rfloor$. ■

Example 13 *The necklace.*

This graph, pictured below, is 3-regular with $n = 4m + 2$ vertices, consisting of m subgraphs linked in a line, the two end subgraphs being different from the intervening ones. This is an artificial graph designed to mimic the n-path while being regular, and this construction (or some similar one) is the natural candidate for the n-vertex graph which asymptotically maximizes our parameters over regular graphs.



This example affords a nice illustration of use of the commute interpretation of resistance. Applying voltage 1 at vertex a and voltage 0 at e, a brief calculation gives the potentials at intervening vertices as

$$g(b) = 4/7, \ g(c) = 5/7, \ g(d) = 4/7$$

and gives the effective resistance $r_{ae} = 7/8$. Since the effective resistance between f and g equals 1, we see the maximal effective resistance is

$$r_{ah} = \tfrac{7}{8} + (2m - 3) + \tfrac{7}{8} = 2m - \tfrac{5}{4}.$$

So

$$\tau^* = E_a T_h + E_h T_a = 3 \times (4m + 2) \times \left(2m - \frac{5}{4}\right) \sim \frac{3n^2}{2}.$$

One could do elementary exact calculations of other parameters, but it is simpler to get asymptotics from the Brownian motion limit, which implies that the asymptotic ratio of each parameter (excluding τ_c) in this example and the n-path is the same for each parameter. So

$$\tau_0 \sim \frac{n^2}{4}, \ \tau_2 \sim \frac{3n^2}{2\pi^2}.$$

jjj I haven't checked this carefully, and I also have abstained from writing anything further about τ_1.

Finally, it is clear that $\tau_c \sim 3n/4$, achieved by breaking a "link" between "beads" in the middle of the necklace.

Example 14 *The balanced r-tree.*

Take $r \geq 2$ and $h \geq 1$. The *balanced r-tree* is the rooted tree where all leaves are at distance h from the root, where the root has degree r, and where the other internal vertices have degree $r+1$. Call h the *height* of the tree. For $h = 1$ we have the $(r+1)$-star, and for $r = 2$ we have the balanced *binary* tree. The number of vertices is

$$n = 1 + r + r^2 + \cdots + r^h = (r^{h+1} - 1)/(r-1).$$

The chain $\hat{X}$ induced (in the sense of Chapter 4 Section yyy) by the function

$$f(i) = h - (\text{distance from } i \text{ to the root})$$

is random walk on $\{0, \ldots, h\}$, biased towards 0, with reflecting barriers, as in Example 5 with

$$\rho = 1/r.$$

In fact, the transition probabilities for X can be expressed in terms of $\hat{X}$ as follows. Given vertices v_1 and v_2 with $f(v_1) = f_1$ and $f(v_2) = f_2$, the paths $[\text{root}, v_1]$ and $[\text{root}, v_2]$ intersect in the path $[\text{root}, v_3]$, say, with $f(v_3) = f_3 \geq f_1 \vee f_2$. Then

$$P_{v_1}(X_t = v_2) = \sum_{m=f_3}^{h} P_{f_1}\left(\max_{0 \leq s \leq t} \hat{X}_s = m, \hat{X}_t = f_2\right) r^{-(m-f_2)}.$$

As a special case, suppose that v_1 is on the path from the root to v_2; in this case $v_3 = v_1$. Using the essential edge lemma (or Theorem 20 below) we can calculate

$$E_{v_2}T_{v_1} = 2(r-1)^{-2}(r^{f_1+1} - r^{f_2+1}) - 2(r-1)^{-1}(f_1 - f_2) - (f_1 - f_2),$$

$$E_{v_1}T_{v_2} = 2(n-1)(f_1 - f_2) - E_{v_2}T_{v_1}. \qquad (54)$$

Using this special case we can deduce the general formula for mean hitting times. Indeed, $E_{v_1}T_{v_2} = E_{v_1}T_{v_3} + E_{v_3}T_{v_2}$, which leads to

$$\begin{aligned} E_{v_1}T_{v_2} \quad = \quad & 2(n-1)(f_3 - f_2) + 2(r-1)^{-2}(r^{f_2+1} - r^{f_1+1}) \\ & -2(r-1)^{-1}(f_2 - f_1) - (f_2 - f_1). \qquad (55) \end{aligned}$$

The maximum value $2h(n-1)$ is attained when v_1 and v_2 are leaves and v_3 is the root. So

$$\tfrac{1}{2}\tau^* = \max_{v,x} E_v T_x = 2(n-1)h. \tag{56}$$

(The τ^* part is simpler via (88) below.) Another special case is that, for a leaf v,

$$E_v T_{\text{root}} = 2(r-1)^{-2}(r^{h+1}-r) - 2h(r-1)^{-1} - h \sim 2n/(r-1), \tag{57}$$

$$E_{\text{root}} T_v = 2(n-1)h - E_v T_{\text{root}} \sim 2nh \tag{58}$$

where asymptotics are as $h \to \infty$ for fixed r. Since $E_{\text{root}} T_w$ is decreasing in $f(w)$, it follows that

$$\tau_0 = \sum_w \pi_w E_{\text{root}} T_w \le (1+o(1))2nh.$$

On the other hand, we claim $\tau_0 \ge (1+o(1))2nh$, so that

$$\tau_0 \sim 2nh.$$

To sketch the proof, given a vertex w, let v be a leaf such that w lies on the path from v to the root. Then

$$E_{\text{root}} T_w = E_{\text{root}} T_v - E_w T_v,$$

and $E_w T_v \le 2(n-1)f(w)$ by (54). But the stationary distribution puts nearly all its mass on vertices w with $f(w)$ of constant order, and $n = o(nh)$.

We claim next that

$$\tau_1 \sim \tau_2 \sim 2n/(r-1).$$

Since $\tau_2 \le \tau_1$, it is enough to show

$$\tau_1 \le (1+o(1))\frac{2n}{r-1} \tag{59}$$

and

$$\tau_2 \ge (1+o(1))\frac{2n}{r-1}. \tag{60}$$

Proof of (59). Put

$$t_n \equiv \frac{2n}{r-1}$$

for brevity. We begin the proof by recalling the results (22) and (19) for the induced walk $\hat{X}$:

$$\hat{\tau}_2 \to \frac{(r+1)}{(r^{1/2}-1)^2},$$

$$E_{\hat{\pi}}\hat{T}_h \sim \frac{2r^{h+1}}{(r-1)^2} \sim t_n. \qquad (61)$$

By Proposition yyy of Chapter 3,

$$\sup_t \left| P_{\hat{\pi}}(\hat{T}_h > t) - \exp\left(-\frac{t}{E_{\hat{\pi}}\hat{T}_h}\right)\right| \le \frac{\hat{\tau}_2}{E_{\hat{\pi}}\hat{T}_h} = \Theta(n^{-1}) = o(1). \qquad (62)$$

For $\hat{X}$ started at 0, let $\hat{S}$ be a stopping time at which the chain has exactly the stationary distribution. Then, for $0 \le s \le t$,

$$P_0(\hat{T}_h > t) \le P_0(\hat{S} > s) + P_{\hat{\pi}}(\hat{T}_h > t - s).$$

Since $\hat{\tau}_1^{(2)} = O(h) = O(\log n)$ by (23), we can arrange to have $E_0\hat{S} = O(\log n)$. Fixing $\epsilon > 0$ and choosing $t = (1+\epsilon)t_n$ and (say) $s = (\log n)^2$, (62) and (61) in conjunction with Markov's inequality yield

$$\begin{aligned} P_0(\hat{T}_h > (1+\epsilon)t_n) &= \exp\left[-\frac{(1+\epsilon)t_n - (\log n)^2}{E_{\hat{\pi}}\hat{T}_h}\right] \\ &\quad + O((\log n)^{-1}) + O(n^{-1}) \\ &\to e^{-(1+\epsilon)}. \end{aligned}$$

Returning to the continuous-time walk on the tree, for n sufficiently large we have

$$P_v(T_{\text{root}} > (1+\epsilon)t_n) \le P_0(\hat{T}_h > (1+\epsilon)t_n) \le e^{-1}$$

for every vertex v. Now a simple coupling argument (**jjj** spell out details?: Couple the induced walks and the tree-walks will agree when the induced walk starting farther from the origin has reached the origin) shows that

$$\bar{d}_n((1+\epsilon)t_n) \le e^{-1}$$

for all large n. Hence $\tau_1 \le (1+\epsilon)t_n$ for all large n, and (59) follows. ■

Proof of (60).

jjj [This requires further exposition in both Chapters 3 and 4-1. In Chapter 3, it needs to be made clear that one of the inequalities having to do with

CM hitting time distributions says precisely that $E_{\alpha_A} T_A \geq E_\pi T_A / \pi(A^c) \geq E_\pi T_A$. In Chapter 4-1 (2/96 version), it needs to be noted that Lemma 2(a) (concerning τ_2 for the joining of two copies of a graph) extends to the joining of any finite number of copies.]

Let G denote a balanced r-tree of height h. Let G'' denote a balanced r-tree of height $h-1$ with root y and construct a tree G' from G'' by adding an edge from y to an additional vertex z. We can construct G by joining r copies of G' at the vertex z, which becomes the root of G. Let π' and π'' denote the respective stationary distributions for the random walks on G' and G'', and use the notation T' and T'', respectively, for hitting times on these graphs. By Chapter 4 jjj,

$$\tau_2 = E_{\alpha'} T'_z \tag{63}$$

where α' is the quasistationary distribution on G' associated with the hitting time T'_z. By Chapter 3 jjj, the expectation (63) is no smaller than $E_{\pi'} T'_z$, which by the collapsing principle equals

$$\pi'(G'') \left(E_{\pi''} T''_y + E_y T'_z \right) = \pi'(G'') \left(E_{\pi''} T''_y + E_y T_z \right).$$

But it is easy to see that this last quantity equals $(1+o(1)) E_\pi T_z$, which is asymptotically equivalent to $2n/(r-1)$ by (61). ■

¿From the discussion at the beginning of Section 3.1, it follows that τ_c is achieved at any of the r subtrees of the root. This gives

$$\tau_c = \frac{(2r^h - r - 1)(2r^h - 1)}{2r(r^h - 1)} \sim \frac{2n}{r}.$$

An extension of the balanced r-tree example is treated in Section 2.1 below.

Example 15 *The d-cube.*

This is a graph with vertex-set $\mathbf{I} = \{0,1\}^d$ and hence with $n = 2^d$ vertices. Write $\mathbf{i} = (i_1, \ldots, i_d)$ for a vertex, and write $|\mathbf{i} - \mathbf{j}| = \sum_u |i_u - j_u|$. Then $(\mathbf{i}, \mathbf{j})$ is an edge if and only if $|\mathbf{i} - \mathbf{j}| = 1$, and in general $|\mathbf{i} - \mathbf{j}|$ is the graph-distance between $\mathbf{i}$ and $\mathbf{j}$. Thus discrete-time random walk proceeds at each step by choosing a coordinate at random and changing its parity.

It is easier to use the continuized walk $\mathbf{X}(t) = (X_1(t), \ldots, X_d(t))$, because the component processes $(X_u(t))$ are independent as u varies, and each is in fact just the continuous-time random walk on the 2-path with transition rate $1/d$. This follows from an elementary fact about the superposition of (marked) Poisson processes.

Thus, in continuous time,

$$\begin{aligned} P_{\mathbf{i}}(\mathbf{X}(t)=\mathbf{j}) &= \prod_{u=1}^{d}\left[\frac{1}{2}\left(1+(-1)^{|i_u-j_u|}e^{-2t/d}\right)\right] \\ &= 2^{-d}\left(1-e^{-2t/d}\right)^{|\mathbf{i}-\mathbf{j}|}\left(1+e^{-2t/d}\right)^{d-|\mathbf{i}-\mathbf{j}|}. \qquad (64) \end{aligned}$$

By expanding the right side, we see that the continuous-time eigenvalues are

$$\lambda_k = 2k/d \text{ with multiplicity } \binom{d}{k}, \quad k=0,1,\ldots,d. \qquad (65)$$

(Of course, this is just the general fact that the eigenvalues of a d-fold product of continuous-time chains are

$$(\lambda_{i_1}+\cdots+\lambda_{i_d}; 1\le i_1,\ldots,i_d\le n) \qquad (66)$$

where $(\lambda_i; 1\le i\le n)$ are the eigenvalues of the marginal chain.)

In particular,

$$\tau_2 = d/2. \qquad (67)$$

By the eigentime identity (Chapter 3 yyy)

$$\begin{aligned} \tau_0 &= \sum_{m\ge 2}\frac{1}{\lambda_m} = \frac{d}{2}\sum_{k=1}^{d}k^{-1}\binom{d}{k} \\ &= 2^d(1+d^{-1}+O(d^{-2})), \qquad (68) \end{aligned}$$

the asymptotics being easy analysis.

From (64) it is also straightforward to derive the discrete-time t-step transition probabilities:

$$P_{\mathbf{i}}(\mathbf{X}_t=j) = 2^{-d}\sum_{m=0}^{d}\left(1-\frac{2m}{d}\right)^t\sum_r(-1)^r\binom{|\mathbf{i}-\mathbf{j}|}{r}\binom{d-|\mathbf{i}-\mathbf{j}|}{m-r}.$$

Starting the walk at $\mathbf{0}$, let $Y_t=|\mathbf{X}(t)|$. Then Y is the birth-and-death chain on states $\{0,1,\ldots,d\}$ with transition rates (transition probabilities, in discrete time)

$$q_{i,i+1}=\frac{d-i}{d},\ q_{i,i-1}=\frac{i}{d},\quad 0\le i\le d.$$

xxx box picture

This is the *Ehrenfest urn model* mentioned in many textbooks. In our terminology we may regard Y as random walk on the weighted linear graph (Section 1.2) with weights

$$w_i = \binom{d-1}{i-1}, \quad w = 2^d.$$

In particular, writing T^Y for hitting times for Y, symmetry and (13) give

$$\frac{1}{2}\tau^{*Y} = \frac{1}{2}(E_0 T_d^Y + E_d T_0^Y) = E_0 T_d^Y = 2^{d-1} \sum_{i=1}^{d} \frac{1}{\binom{d-1}{i-1}}.$$

On the d-cube, it is "obvious" that $E_{\mathbf{0}} T_{\mathbf{j}}$ is maximized by $\mathbf{j} = \mathbf{1}$, and this can be verified by observing in (64) that $P_{\mathbf{0}}(\mathbf{X}(t) = \mathbf{j})$ is minimized by $\mathbf{j} = \mathbf{1}$, and hence $Z_{\mathbf{0j}}$ is minimized by $\mathbf{j} = \mathbf{1}$, so we can apply Chapter 2 yyy. Thus

$$\frac{1}{2}\tau^* = \max_{\mathbf{ij}} E_{\mathbf{i}} T_{\mathbf{j}} = E_{\mathbf{0}} T_{\mathbf{1}} = 2^{d-1} \sum_{i=1}^{d} \frac{1}{\binom{d-1}{i-1}} \sim 2^d(1 + 1/d + O(1/d^2)). \quad (69)$$

The asymptotics are the same as in (68). In fact it is easy to use (64) to show

$$Z_{\mathbf{ii}} = 2^{-d}\tau_0 = 1 + d^{-1} + O(d^{-2})$$

$$Z_{\mathbf{ij}} = O(d^{-2}) \text{ uniformly over } |\mathbf{i} - \mathbf{j}| \geq 2$$

and then by Chapter 2 yyy

$$E_{\mathbf{i}} T_{\mathbf{j}} = 2^d(1 + d^{-1} + O(d^{-2})) \text{ uniformly over } |\mathbf{i} - \mathbf{j}| \geq 2.$$

Since

$$1 + E_1 T_0^Y = E_0 T_1^Y + E_1 T_0^Y = w/w_1 = 2^d,$$

it follows that

$$E_{\mathbf{i}} T_{\mathbf{j}} = 2^d - 1 \text{ if } |\mathbf{i} - \mathbf{j}| = 1.$$

xxx refrain from write out exact $E_{\mathbf{i}} T_{\mathbf{j}}$—refs

To discuss total variation convergence, we have by symmetry (and writing $\mathbf{d}$ to distinguish from dimension d)

$$\bar{\mathbf{d}}(t) = \|P_{\mathbf{0}}(\mathbf{X}(t) \in \cdot) - P_{\mathbf{1}}(\mathbf{X}(t) \in \cdot)\|$$

$$\mathbf{d}(t) = \|P_{\mathbf{0}}(\mathbf{X}(t) \in \cdot) - \pi(\cdot)\|.$$

Following Diaconis et al [8] we shall sketch an argument leading to

$$\mathbf{d}\left(\tfrac{1}{4}d\log d + sd\right) \to L(s) \equiv P\left(|Z| \le \tfrac{1}{2}e^{-2s}\right) \ , -\infty < s < \infty \tag{70}$$

where Z has the standard Normal distribution. This implies

$$\tau_1 \sim \tfrac{1}{4}d\log d. \tag{71}$$

For the discrete-time walk made aperiodic by incorporating chance $1/(d+1)$ of holding, (70) and (71) remain true, though rigorous proof seems complicated: see [8].

Fix u, and consider $\mathbf{j} = \mathbf{j}(u)$ such that $|\mathbf{j}| - d/2 \sim ud^{1/2}/2$. Using $1 - \exp(-\delta) \approx \delta - \frac{1}{2}\delta^2$ as $\delta \to 0$ in (64), we can calculate for $t = t(d) = \frac{1}{4}d\log d + sd$ with s fixed that

$$2^d P_{\mathbf{0}}(\mathbf{X}(t) = \mathbf{j}) \to \exp\left(-\frac{e^{-4s}}{2} - ue^{-2s}\right).$$

Note the limit is > 1 when $u < u_0(s) \equiv -e^{-2s}/2$. Now

$$\mathbf{d}(t) = \frac{1}{2}\sum_{\mathbf{j}} |P_{\mathbf{0}}(\mathbf{X}(t) = \mathbf{j}) - 2^{-d}| \sim \sum (P_{\mathbf{0}}(\mathbf{X}(t) = \mathbf{j}) - 2^{-d})$$

where the second sum is over $\mathbf{j}(u)$ with $u < u_0(s)$. But from (64) we can write this sum as

$$P\left(B\left(\tfrac{1}{2}(1 - d^{-1/2}e^{-2s})\right) \le |\mathbf{j}(u_0(s))|\right) - P\left(B\left(\tfrac{1}{2}\right) \le |\mathbf{j}(u_0(s))|\right)$$

where $B(p)$ denotes a Binomial(d,p) random variable. By the Normal approximation to Binomial, this converges to

$$P(Z \le -u_0(s)) - P(Z \le u_0(s))$$

as stated.

As an aside, symmetry and Chapter 4 yyy give

$$\tau_0 \le E_{\mathbf{0}}T_{\mathbf{1}} \le \tau_1^{(2)} + \tau_0$$

and so the difference $E_{\mathbf{0}}T_{\mathbf{1}} - \tau_0$ is $O(d\log d)$, which is much smaller than what the series expansions (68) and (69) imply.

The fact that the "half-cube" $A = \{\mathbf{i} \in \mathbf{I} : i_d = 0\}$, yielding

$$\tau_c = d/2,$$

achieves the *sup* in the definition of τ_c can be proved using a slightly tricky induction argument. However, the result follows immediately from (67) together with the general inequality $\tau_2 \geq \tau_c$.

Example 16 *Dense regular graphs.*

Consider an r-regular n-vertex graph with $r > n/2$. Of course here we are considering a class of graphs rather than a specific example. The calculations below show that these graphs necessarily mimic the complete graph (as far as smallness of the random walk parameters is concerned) in the asymptotic setting $r/n \to c > 1/2$.

The basic fact is that, for any pair i, j of vertices, there must be at least $2r - n$ other vertices k such that $i - k - j$ is a path. To prove this, let a_1 (resp., a_2) be the number of vertices $k \neq i, j$ such that exactly 1 (resp., 2) of the edges $(k, i), (k, j)$ exist. Then $a_1 + a_2 \leq n - 2$ by counting vertices, and $a_1 + 2a_2 \geq 2(r - 1)$ by counting edges, and these inequalities imply $a_2 \geq 2r - n$.

Thus, by Thompson's principle (Chapter 3, yyy) the effective resistance $r_{ij} \leq \frac{2}{2r-n}$ and so the commute interpretation of resistance implies

$$\tau^* \leq \frac{2rn}{2r - n} \sim \frac{2cn}{2c - 1}. \tag{72}$$

A simple "greedy coupling" argument (Chapter 14, Example yyy) shows

$$\tau_1 \leq \frac{r}{2r - n} \sim \frac{c}{2c - 1}. \tag{73}$$

This is also a bound on τ_2 and on τ_c, because $\tau_c \leq \tau_2 \leq \tau_1$ always, and special case 2 below shows that this bound on τ_c cannot be improved asymptotically (nor hence can the bound on τ_1 or τ_2). Because $E_\pi T_j \leq n\tau_2$ for regular graphs (Chapter 3 yyy), we get

$$E_\pi T_j \leq \frac{nr}{2r - n}.$$

This implies

$$\tau_0 \leq \frac{nr}{2r - n} \sim \frac{cn}{2c - 1}$$

which also follows from (72) and $\tau_0 \leq \tau^*/2$. We can also argue, in the notation of Chapter 4 yyy, that

$$\max_{ij} E_i T_j \leq \tau_1^{(2)} + \max_j E_\pi T_j \leq \frac{4e}{e-1}\tau_1 + n\tau_1 \leq (1+o(1))\frac{nr}{2r-n} \sim \frac{cn}{2c-1}.$$

Special case 1. The orders of magnitude may change for $c = 1/2$. Take two complete $(n/2)$-graphs, break one edge in each (say edges (v_1, v_2) and (w_1, w_2)) and add edges (v_1, w_1) and (v_2, w_2). This gives an n-vertex $((n/2)-1)$-regular graph for which all our parameters are $\Theta(n^2)$.

jjj I haven't checked this.

Special case 2. Can the bound $\tau_c \leq r/(2r-n) \sim c/(2c-1)$ be asymptotically improved? Eric Ordentlicht has provided the following natural counter-example. Again start with two $(n/2)$-complete graphs on vertices (v_i) and (w_i). Now add the edges (v_i, w_j) for which $0 \leq (j-i) \bmod (n/2) \leq r-(n/2)$. This gives an n-vertex r-regular graph. By considering the set A consisting of the vertices v_i, a brief calculation gives

$$\tau_c \geq \frac{r}{2r-n+2} \sim \frac{c}{2c-1}.$$

Example 17 *The d-dimensional torus Z_m^d.*

The torus is the set of d-dimensional integers $\mathbf{i} = (i_1, \ldots, i_d)$ modulo m, considered in the natural way as a $2d$-regular graph on $n = m^d$ vertices. It is much simpler to work with the random walk in *continuous* time, $\mathbf{X}(t) = (X_1(t), \ldots, X_d(t))$, because its component processes $(X_u(t))$ are independent as u varies; and each is just continuous-time random walk on the m-cycle, slowed down by a factor $1/d$. Thus we can immediately write the time-t transition probabilities for $\mathbf{X}$ in terms of the corresponding probabilities $p_{0,j}(t)$ for continuous-time random walk on the m-cycle (see Example 7 above) as

$$p_{\mathbf{0},\mathbf{j}}(t) = \prod_{u=1}^{d} p_{0,j_u}(t/d).$$

Since the eigenvalues on the m-cycle are $(1-\cos(2\pi k/m), 0 \leq k \leq m-1)$, by (66) the eigenvalues of $\mathbf{X}$ are

$$\lambda_{(k_1 \ldots k_d)} = \frac{1}{d}\sum_{u=1}^{d}(1-\cos(2\pi k_u/m)),\ 0 \leq k_u \leq m-1.$$

In particular, we see that the relaxation time satisfies

$$\tau_2 \sim \frac{dm^2}{2\pi^2} = \frac{dn^{2/d}}{2\pi^2}$$

where here and below asymptotics are as $m \to \infty$ for fixed d. This relaxation time could more simply be derived from the N-cycle result via the general "product chain" result of Chapter 4 yyy. But writing out all the eigenvalues enables us to use the eigentime identity.

$$\tau_0 = \sum_{k_1} \cdots \sum_{k_d} 1/\lambda_{(k_1,\ldots,k_d)}$$

(the sum excluding $(0,\ldots,0)$), and hence

$$\tau_0 \sim m^d R_d \tag{74}$$

where

$$R_d \equiv \int_0^1 \cdots \int_0^1 \frac{1}{\frac{1}{d}\sum_{u=1}^d (1-\cos(2\pi x_u))}\, dx_1 \cdots dx_d \tag{75}$$

provided the integral converges. The reader who is a calculus whiz will see that in fact $R_d < \infty$ for $d \geq 3$ only, but this is seen more easily in the alternative approach of Chapter 15, Section yyy.

xxx more stuff: connection to transience, recurrent potential, etc

xxx new copy from lectures

xxx τ_1, τ_c

jjj David: I will let you develop the rest of this example. Note that τ_1 is considered very briefly in Chapter 15, eq. (17) in 3/6/96 version. Here are a few comments for τ_c. First suppose that $m > 2$ is even and $d \geq 2$. Presumably, τ_c is achieved by the following half-torus:

$$A := \{\mathbf{i} = (i_1,\ldots,i_d) \in Z_m^d : 0 \leq i_d < m/2\}.$$

In the notation of (52) observe

$$|\mathcal{E}| = dn, \quad \deg A = dn, \quad \deg A^c = dn, \quad (A, A^c) = 2m^{d-1} = 2n/m,$$

whence

$$\tau(A) = \tfrac{d}{4} n^{1/d}.$$

[By Example 15 (the d-cube) this last result is also true for $m = 2$, and (for even $m \geq 2$) it is by Example 7 (the n-cycle) also true for $d = 1$.] If we have correctly conjectured the maximizing A, then

$$\tau_c = \tfrac{d}{4} n^{1/d} \text{ if } m \text{ is even,}$$

and presumably(??)

$$\tau_c \sim \tfrac{d}{4} n^{1/d}$$

in any case.

Example 18 *Chess moves.*

Here is a classic homework problem for an undergraduate Markov chains course.

> Start a knight at a corner square of an otherwise-empty chessboard. Move the knight at random, by choosing uniformly from the legal knight-moves at each step. What is the mean number of moves until the knight returns to the starting square?

It's a good question, because if you don't know Markov chain theory it looks too messy to do by hand, whereas using Markov chain theory it becomes very simple. The knight is performing random walk on a graph (the 64 squares are the vertices, and the possible knight-moves are the edges). It is not hard to check that the graph is connected, so by the elementary Chapter 3 yyy for a corner square v the mean return time is

$$E_v T_v^+ = \frac{1}{\pi_v} = \frac{2|\mathcal{E}|}{d_v} = |\mathcal{E}|,$$

and by drawing a sketch in the margin the reader can count the number of edges $|\mathcal{E}|$ to be 168.

Other chess pieces—queen, king, rook—define different graphs (the bishop's is of course not connected, and the pawn's not undirected). One might expect that the conventional ordering of the "strength" of the pieces as (queen, rook, knight, king) is reflected in parameters τ_0 and τ_2 (**jjj** how about the other taus?) being increasing in this ordering. The reader is invited to perform the computations. (**jjj**: an undergraduate project?) We have done so only for the rook's move, treated in the next example.

The computations for the queen, knight, and king are simplified if the walks are made on a toroidal chessboard. (There is no difference for the rook.)

jjj Chess on a bagel, anyone? Continue same paragraph:

Then Fourier analysis (see Diaconis [6]) on the abelian group Z_m^2 (with $m = 8$) can be brought to bear, and the eigenvalues are easy to compute.

We omit the details, but the results for (queen, rook, knight, king) are asymptotically

$$\begin{aligned}\tau_0 &= (m^2 + \tfrac{7}{12}m + O(1),\ m^2 + m + O(1),\\ &\quad \textbf{jjj}?(1+o(1))c_{\text{knight}}m^2\log m,\ \textbf{jjj}?(1+o(1))c_{\text{king}}m^2\log m)\\ \tau_2 &\sim \left(\tfrac{4}{3},\ 2,\ \tfrac{1}{5\pi^2}m^2,\ \tfrac{2}{3\pi^2}m^2\right)\end{aligned}$$

as $m \to \infty$, in conformance with our expectations, and numerically

$$\begin{aligned}\tau_0 &= (65.04, 67.38, 69.74, 79.36)\\ \tau_2 &= (1.29, 1.75, 1.55, 4.55)\end{aligned}$$

for $m = 8$. The only surprise is the inverted τ_2 ordering for (rook, knight).

Example 19 *Rook's random walk on an m-by-m chessboard.*

jjj Do we want to do this also on a d-dimensional grid? We need to mention how this is a serious example, used with Metropolis for sampling from log concave distributions; reference is [2]? [3]?

Number the rows and columns of the chessboard each 0 through $m-1$ in arbitrary fashion, and denote the square of the chessboard at row i_1 and column i_2 by $\mathbf{i} = (i_1, i_2)$. In continuous time, the rook's random walk $(\mathbf{X}(t))$ is the product of two continuous-time random walks on the complete graph K_m on m vertices, each run at rate $1/2$. Thus (cf. Example 9)

$$P_{\mathbf{i}}(\mathbf{X}(t) = \mathbf{j}) = \prod_{u=1}^{2}\left[\frac{1}{m} + \left(\delta_{i_u,j_u} - \frac{1}{m}\right)\exp\left(-\frac{mt}{2(m-1)}\right)\right], \tag{76}$$

which can be expanded to get the discrete-time multistep transition probabilities, if desired. We recall that the eigenvalues for discrete-time random walk on K_m are 1 with multiplicity 1 and $-1/(m-1)$ with multiplicity $m-1$. It follows [recall (66)] that the eigenvalues for the continuous-time rook's walk are

$$0,\ \frac{m}{2(m-1)},\ \frac{m}{m-1}\quad \text{with resp. multiplicities}\quad 1,\ 2(m-1),\ (m-1)^2.$$

In particular,

$$\tau_2 = \frac{2(m-1)}{m}, \tag{77}$$

which equals 1.75 for $m = 8$ and converges to 2 as m grows. Applying the eigentime identity, a brief calculation gives

$$\tau_0 = \frac{(m-1)^2(m+3)}{m}, \qquad (78)$$

which equals 67.375 for $m = 8$ and $m^2 + m + O(1)$ for m large.

Starting the walk $\mathbf{X}$ at $\mathbf{0} = (0,0)$, let $Y(t)$ denote the Hamming distance $H(\mathbf{X}(t), \mathbf{0})$ of $\mathbf{X}(t)$ from $\mathbf{0}$, i.e., the number of coordinates (0, 1, or 2) in which $\mathbf{X}(t)$ differs from $\mathbf{0}$. Then Y is a birth-and-death chain with transition rates

$$q_{01} = 1, \quad q_{10} = \frac{1}{2(m-1)}, \quad q_{12} = \frac{1}{2}, \quad q_{21} = \frac{1}{m-1}.$$

This is useful for computing mean hitting times. Of course

$$E_{\mathbf{i}} T_{\mathbf{j}} = 0 \text{ if } H(\mathbf{i},\mathbf{j}) = 0.$$

Since

$$1 + E_1 T_0^Y = E_0 T_1^Y + E_1 T_0^Y = m^2,$$

it follows that

$$E_{\mathbf{i}} T_{\mathbf{j}} = m^2 - 1 \text{ if } H(\mathbf{i},\mathbf{j}) = 1.$$

Finally, it is clear that $E_2 T_1^Y = m - 1$, so that

$$E_2 T_0^Y = E_2 T_1^Y + E_1 T_0^Y = m^2 + m - 2,$$

whence

$$E_{\mathbf{i}} T_{\mathbf{j}} = m^2 + m - 2 \text{ if } H(\mathbf{i},\mathbf{j}) = 2.$$

These calculations show

$$\tfrac{1}{2}\tau^* = \max_{\mathbf{ij}} E_{\mathbf{i}} T_{\mathbf{j}} = m^2 + m - 2,$$

which equals 70 for $m = 8$, and they provide another proof of (78).

From (76) it is easy to derive

$$\bar{d}_m(t) = \left(2 - \frac{2}{m}\right) \exp\left(-\frac{mt}{2(m-1)}\right) - \left(1 - \frac{2}{m}\right) \exp\left(-\frac{mt}{m-1}\right)$$

and thence

$$\tau_1 = -2\frac{m-1}{m}\left[\ln\left(1 - \left(1 - e^{-1}\frac{m(m-2)}{(m-1)^2}\right)^{1/2}\right) + \ln\left(\frac{m-1}{m-2}\right)\right],$$

which rounds to 2.54 for $m = 8$ and converges to $-2\ln(1-(1-e^{-1})^{1/2}) \doteq 3.17$ as m becomes large.

Any set A of the form $\{(i_1, i_2) : i_u \in J\}$ with either $u = 1$ or $u = 2$ and J a nonempty proper subset of $\{0, \ldots, m-1\}$ achieves the value

$$\tau_c = 2\frac{m-1}{m}.$$

A direct proof is messy, but this follows immediately from the general inequality $\tau_c \leq \tau_2$, (77), and a brief calculation that the indicated A indeed gives the indicated value.

xxx other examples left to reader? complete bipartite; ladders

jjj Note: I've worked these out and have handwritten notes. How much do we want to include, if at all? (I could at least put the results in the table.)

2.1 Biased walk on a balanced tree

Consider again the balanced r-tree setup of Example 14. Fix a parameter $0 < \lambda < \infty$. We now consider biased random walk (X_t) on the tree, where from each non-leaf vertex other than the root the transition goes to the parent with probability $\lambda/(\lambda+r)$ and to each child with probability $1/(\lambda+r)$. As in Example 14 (the case $\lambda = 1$), the chain $\hat{X}$ induced by the function

$$f(i) = h - (\text{distance from } i \text{ to the root})$$

is (biased) reflecting random walk on $\{0, \ldots, h\}$ with respective probabilities $\lambda/(\lambda+r)$ and $r/(\lambda+r)$ of moving to the right and left from any $i \neq 0, h$; the ratio of these two transition probabilities is

$$\rho = \lambda/r.$$

The stationary distribution $\hat{\pi}$ for $\hat{X}$ is a modified geometric:

$$\hat{\pi}_m = \frac{1}{\hat{w}} \times \begin{cases} 1 & \text{if } m = 0 \\ (1+\rho)\rho^{m-1} & \text{if } 1 \leq m \leq h-1 \\ \rho^{h-1} & \text{if } m = h \end{cases}$$

where

$$\hat{w} = 2\sum_{m=0}^{h-1} \rho^m = \begin{cases} 2(1-\rho^h)/(1-\rho) & \text{if } \rho \neq 1 \\ 2h & \text{if } \rho = 1. \end{cases}$$

Since the stationary distribution π for X is assigns the same probability to each of the $r^{h-f(v)}$ vertices v with a given value of $f(v)$, a brief calculation shows that $\pi_v p_{vx} = \lambda^{f(v)}/\hat{w}r^h$ for any edge ($v =$ child, $x =$ parent) in the tree. In the same notation, it follows that X is random walk on the balanced r-tree with edge weights $w_{vx} = \lambda^{f(v)}$ and total weight $w = \sum_{v,x} w_{vx} = \hat{w}r^h$.

The distribution $\hat{\pi}$ concentrates near the root-level if $\rho < 1$ and near the leaves-level if $\rho > 1$; it is nearly uniform on the h levels if $\rho = 1$. On the other hand, the weight assigned by the distribution π to an individual vertex v is a decreasing function of $f(v)$ (thus favoring vertices near the leaves) if $\lambda < 1$ (i.e., $\rho < 1/r$) and is an increasing function (thus favoring vertices near the root) if $\lambda > 1$; it is uniform on the vertices in the unbiased case $\lambda = 1$.

The mean hitting time calculations of Example 14 can all be extended to the biased case. For example, for $\lambda \neq 1$ the general formula (55) becomes [using the same notation as at (55)]

$$\begin{aligned} E_{v_1}T_{v_2} &= \hat{w}r^h \frac{\lambda^{-f_3} - \lambda^{-f_2}}{\lambda^{-1} - 1} + 2(\rho^{-1} - 1)^{-2}\left(\rho^{-(f_2+1)} - \rho^{-(f_1+1)}\right) \\ &\quad -2(\rho^{-1} - 1)^{-1}(f_2 - f_1) - (f_2 - f_1) \end{aligned} \tag{79}$$

if $\rho \neq 1$ and

$$E_{v_1}T_{v_2} = \hat{w}r^h \frac{\lambda^{-f_3} - \lambda^{-f_2}}{\lambda^{-1} - 1} + f_2^2 - f_1^2$$

if $\rho = 1$. The maximum value is attained when v_1 and v_2 are leaves and v_3 is the root. So if $\lambda \neq 1$,

$$\tfrac{1}{2}\tau^* = \max_{v,x} E_v T_x = \hat{w}r^h \frac{\lambda^{-h} - 1}{\lambda^{-1} - 1}. \tag{80}$$

The orders of magnitude for all of the τ-parameters (with r and λ, and hence ρ, fixed as h, and hence n, becomes large) are summarized on a case-by-case basis in the next table. Following are some of the highlights in deriving these results; the details, and derivation of exact formulas and more detailed asymptotic results, are left to the reader.

Orders of magnitude of parameters [$\tau = \Theta(\text{entry})$]
for λ-biased walk on a balanced r-tree of height h ($\rho = \lambda/r$).

Value of ρ	τ^*	τ_0	τ_1	τ_2	τ_c
$\rho < 1/r$	ρ^{-h}	ρ^{-h}	ρ^{-h}	ρ^{-h}	ρ^{-h}
$\rho = 1/r$ ($\equiv$ Example 14)	nh	nh	n	n	n
$1/r < \rho < 1$	n	n	ρ^{-h}	ρ^{-h}	ρ^{-h}
$\rho = 1$	nh	n	h	h	h
$\rho > 1$	n	n	h	1	1

For $\tau_0 = \sum_x \pi_x E_{\text{root}} T_x$ we have $\tau_0 \leq E_{\text{ root}} T_{\text{leaf}}$. If $\rho < 1/r$, this bound is tight:

$$\tau_0 \sim E_{\text{ root}} T_{\text{leaf}} \sim \frac{2\rho^{-h}}{(1-\rho)^2(1-\lambda)}(\lambda - \rho);$$

for $\rho > 1/r$ a more careful calculation is required.

If $\rho < 1$, then the same arguments as for the unbiased case ($\rho = 1/r$) show

$$\tau_1 \sim \tau_2 \sim 2\rho^{-(h-1)}/(1-\rho)^2.$$

In this case it is not hard to show that

$$\tau_c = \Theta(\rho^{-h})$$

as well. If $\rho = 1$, then it is not hard to show that

$$\tau_1 = \Theta(h), \quad \tau_c \sim 2(1 - \tfrac{1}{r})h$$

with τ_c achieved at a branch of the root (excluding the root), and so

$$\tau_2 = \Theta(h)$$

as well. If $\rho > 1$, then since $\hat{X}$ has positive drift equal to $(\rho - 1)/(\rho + 1)$, it follows that

$$\tau_1 \sim \frac{\rho + 1}{\rho - 1} h.$$

The value τ_c is achieved by isolating a leaf, giving

$$\tau_c \to 1,$$

and so, by the inequalities $\tau_c \leq \tau_2 \leq 8\tau_c^2$ of Chapter 4, Section yyy,

$$\tau_2 = \Theta(1)$$

as well.

jjj Limiting value of τ_2 when $\rho > 1$ is that of τ_2 for biased infinite tree? Namely?

3 Trees

For random walk on a finite tree, we can develop explicit formulas for means and variances of first passage times, and for distributions of first hitting places. We shall only treat the unweighted case, but the formulas can be extended to the weighted case without difficulty.

xxx notation below —change w to x ? Used i, j, v, w, x haphazardly for vertices.

In this section we'll write r_v for the degree of a vertex v, and $d(v, x)$ for the distance between v and x. On a tree we may unambiguously write $[v, x]$ for the path from v to x. Given vertices $j, v_1, v_2, \ldots$ in a tree, the intersection of the paths $[j, v_1]$, $[j, v_2], \ldots$ is a (maybe trivial) path; write $d(j, v_1 \wedge v_2 \wedge \cdots) \geq 0$ for the length of this intersection path.

On an n-vertex tree, the random walk's stationary distribution is

$$\pi_v = \frac{r_v}{2(n-1)}.$$

Recall from the beginning of this chapter that an edge (v, x) of a graph is *essential* if its removal would disconnect the graph into two components $A(v, x)$ and $A(x, v)$, say, containing v and x respectively. Obviously, in a tree every edge is essential, so we get a lot of mileage out of the essential edge lemma (Lemma 1).

Theorem 20 *Consider discrete-time random walk on an n-vertex tree.*
For each edge (i, j),

$$E_i T_j = 2|A(i, j)| - 1 \tag{81}$$

$$E_i T_j + E_j T_i = 2(n-1). \tag{82}$$

For arbitrary i, j,

$$E_i T_j = -d(i, j) + 2 \sum_v d(j, i \wedge v) = \sum_v r_v d(j, i \wedge v) \tag{83}$$

$$E_i T_j + E_j T_i = 2(n-1) d(i, j). \tag{84}$$

For each edge (i, j),

$$\mathrm{var}_i T_j = -E_i T_j + \sum_{v \in A(i,j)} \sum_{w \in A(i,j)} r_v r_w (2d(j, v \wedge w) - 1). \tag{85}$$

For arbitrary i, j,

$$\mathrm{var}_i T_j = -E_i T_j + \sum_v \sum_w r_v r_w d(j, i \wedge v \wedge w) \left[2d(j, v \wedge w) - d(j, i \wedge v \wedge w)\right]. \tag{86}$$

Remarks. 1. There are several equivalent expressions for the sums above: we chose the most symmetric-looking ones. We've written sums over vertices, but one could rephrase in terms of sums over edges.

2. In continuous time, the terms "$-E_i T_j$" disappear from the variance formulas—see xxx.

Proof of Theorem 20. Equations (81) and (82) are rephrasings of (3) and (4) from the essential edge lemma. Equation (84) and the first equality in (83) follow from (82) and (81) by summing over the edges in the path $[i, j]$. Note alternatively that (84) can be regarded as a consequence of the commute interpretation of resistance, since the effective resistance between i and j is $d(i, j)$. To get the second equality in (83),consider the following deterministic identity (whose proof is obvious), relating sums over vertices to sums over edges.

Lemma 21 *Let* f *be a function on the vertices of a tree, and let* j *be a distinguished vertex. Then*

$$\sum_v r_v f(v) = \sum_{v \neq j} (f(v) + f(v^*))$$

where v^* *is the first vertex (other than* v*) in the path* $[v, j]$.

To apply to (83), note

$$\begin{aligned} d(j, i \wedge v^*) &= d(j, i \wedge v) \text{ if } v \notin [i, j] \\ &= d(j, i \wedge v) - 1 \text{ if } v \in [i, j],\ v \neq j. \end{aligned}$$

The equality in Lemma 21 now becomes the equality in (83).

We prove (85) below. To derive (86) from it, sum over the edges in the path $[i, j] = (i = i_0, i_1, \ldots, i_m = j)$ to obtain

$$\mathrm{var}_i T_j = -E_i T_j + \sum_v \sum_w \sum_l (2d(i_{l+1}, v \wedge w) - 1) \tag{87}$$

where $\sum_l$ denotes the sum over all $0 \leq l \leq m - 1$ for which $A(i_l, i_{l+1})$ contains both v and w. Given vertices v and w, there exist unique smallest values of p and q so that $v \in A(i_p, i_{p+1})$ and $w \in A(i_q, i_{q+1})$. If $p \neq q$, then the sum $\sum_l$ in (87) equals

$$\begin{aligned}\sum_{l=p\vee q}^{m-1} (2d(i_{l+1}, i_{p\vee q}) - 1) &= \sum_{l=p\vee q}^{m-1} (2((l+1) - (p \vee q)) - 1) \\ &= (m - (p \vee q))^2 = d^2(j, v \wedge w) \\ &= d(j, i \wedge v \wedge w) \left[2d(j, v \wedge w) - d(j, i \wedge v \wedge w)\right],\end{aligned}$$

as required by (86). If $p = q$, then the sum $\sum_l$ in (87) equals

$$\sum_{l=p}^{m-1} (2d(i_{l+1}, i_p) + 2d(i_p, v \wedge w) - 1)$$

which again equals $d(j, i \wedge v \wedge w) \left[2d(j, v \wedge w) - d(j, i \wedge v \wedge w)\right]$ by a similar calculation.

So it remains to prove (85), for which we may suppose, as in the proof of Lemma 1, that j is a leaf. By considering the first step from j to i we have

$$\mathrm{var}_j T_j^+ = \mathrm{var}_i T_j.$$

Now yyy of Chapter 2 gives a general expression for $\mathrm{var}_j T_j^+$ in terms of $E_\pi T_j$, and in the present setting this becomes

$$\mathrm{var}_j T_j^+ = 2(n-1) - (2(n-1))^2 + \sum_v 2r_v E_v T_j.$$

Using the second equality in (83), we may rewrite the sum as

$$\sum_{v\neq j} \sum_{w\neq j} r_v r_w 2d(j, v \wedge w).$$

Also,

$$\sum_{v\neq j} r_v = 2(n-1) - 1.$$

Combining these expressions gives

$$\mathrm{var}_i T_j = -(2n-3) + \sum_{v\neq j} \sum_{w\neq j} r_v r_w (2d(j, v \wedge w) - 1).$$

But by (81), $E_i T_j = 2n - 3$. ■

3.1 Parameters for trees

Here we discuss the five parameters of Chapter 4. Obviously by (84)

$$\tau^* = 2(n-1)\Delta \tag{88}$$

where Δ is the diameter of the tree. As for τ_c, it is clear that the *sup* in its definition is attained by $A(v,w)$ for some edge (v,w). Note that

$$\pi(A(v,w)) = \frac{2|A(v,w)|-1}{2(n-1)}. \tag{89}$$

This leads to

$$\begin{aligned}\tau_c &= \max_{(v,w)} \frac{\frac{2|A(v,w)|-1}{2(n-1)}\frac{2|A(w,v)|-1}{2(n-1)}}{\frac{1}{2(n-1)}} \\ &= \max_{(v,w)} \frac{4|A(v,w)||A(w,v)|-2n+1}{2(n-1)}. \end{aligned} \tag{90}$$

Obviously the *max* is attained by an edge for which $|A(v,w)|$ is as close as possible to $n/2$. This is one of several notions of "centrality" of vertices and edges which arise in our discussion—see Buckley and Harary [5] for a treatment of centrality in the general graph context, and for the standard graph-theoretic terminology.

Proposition 22 *On an n-vertex tree,*

$$\tau_0 = \frac{1}{2} + \frac{2}{n}\sum_{(v,w)} \left[|A(v,w)||A(w,v)| - \frac{1}{2(n-1)}\left(|A(v,w)|^2 + |A(w,v)|^2 \right) \right]$$

where $\sum_{(v,w)}$ denotes the sum over all undirected edges (v,w).

Proof. Using the formula for the stationary distribution, for each i

$$\tau_0 = \frac{1}{2(n-1)} \sum_j r_j E_i T_j.$$

Appealing to Lemma 21 (with i as the distinguished vertex)

$$\tau_0 = \frac{1}{2(n-1)} \sum_j (2E_iT_j - a(i,j))$$

where $a(i,i)=0$ and $a(i,j)=E_xT_j$, where (j,x) is the first edge of the path $[j,i]$. Taking the (unweighted) average over i,

$$\tau_0 = \frac{1}{2n(n-1)}\sum_i\sum_j(2E_iT_j - a(i,j)).$$

Each term E_iT_j is the sum of terms E_vT_w along the edges (v,w) of the path $[i,j]$. Counting how many times a directed edge (v,w) appears,

$$\tau_0 = \frac{1}{2n(n-1)}\sum\left(2|A(v,w)||A(w,v)| - |A(v,w)|\right)E_vT_w,$$

where we sum over *directed* edges (v,w). Changing to a sum over undirected edges, using $E_vT_w + E_wT_v = 2(n-1)$ and $E_vT_w = 2|A(v,w)|-1$, gives

$$\begin{aligned} 2n(n-1)\tau_0 &= \sum_{(v,w)} [2|A(v,w)||A(w,v)|2(n-1) \\ &\quad -|A(v,w)|(2|A(v,w)|-1) \\ &\quad -|A(w,v)|(2|A(w,v)|-1)]. \end{aligned}$$

This simplifies to the assertion of the Proposition. ∎

For τ_1 we content ourselves with a result "up to equivalence".

Proposition 23 *There exist constants $K_1, K_2 < \infty$ such that*

$$\frac{1}{K_1}\min_i\max_j E_jT_i \le \tau_1 \le K_2\min_i\max_j E_jT_i.$$

Of course the expectations can be computed by (83).

Proof. We work with the parameter

$$\tau_1^{(3)} \equiv \max_{i,j}\sum_k \pi_k|E_jT_k - E_iT_k|$$

which we know is equivalent to τ_1. Write

$$\sigma = \min_i\max_j E_jT_i.$$

Fix an i attaining the minimum. For arbitrary j we have (the first equality uses the random target lemma, cf. the proof of Chapter 4 Lemma yyy)

$$\begin{aligned} \sum_k \pi_k|E_jT_k - E_iT_k| &= 2\sum_k \pi_k(E_jT_k - E_iT_k)^+ \\ &\le 2\sum_k \pi_k E_jT_i \text{ because } E_jT_k \le E_jT_i + E_iT_k \\ &\le 2\sigma \end{aligned}$$

and so $\tau_1^{(3)} \leq 4\sigma$.

For the converse, it is elementary that we can find a vertex i such that the size (n^*, say) of the largest branch from i satisfies $n^* \leq n/2$. (This is another notion of "centrality". To be precise, we are excluding i itself from the branch.) Fix this i, and consider the j which maximizes $E_j T_i$, so that $E_j T_i \geq \sigma$ by definition. Let B denote the set of vertices in the branch from i which contains j. Then

$$E_j T_k = E_j T_i + E_i T_k, \ k \in B^c$$

and so

$$\tau_1^{(3)} \geq \sum_k \pi_k |E_j T_k - E_i T_k| \geq \pi(B^c) E_j T_i \geq \pi(B^c)\sigma.$$

But by (89) $\pi(B) = \frac{2n^*-1}{2(n-1)} \leq \frac{1}{2}$, so we have shown $\tau_1^{(3)} \geq \sigma/2$. ■

We do not know whether τ_2 has a simple expression "up to equivalence" analogous to Proposition 23. It is natural to apply the "distinguished paths" bound (Chapter 4 yyy). This gives the inequality

$$\begin{aligned} \tau_2 &\leq 2(n-1) \max_{(v,w)} \sum_{x \in A(v,w)} \sum_{y \in A(w,v)} \pi_x \pi_y d(x,y) \\ &= 2(n-1) \max_{(v,w)} \Big(\pi(A(v,w)) E\left[d(v,V) 1_{(V \in A(w,v))} \right] \\ &\qquad + \pi(A(w,v)) E\left[d(v,V) 1_{(V \in A(v,w))} \right] \Big) \end{aligned}$$

where V has the stationary distribution π and where we got the equality by writing $d(x,y) = d(v,y) + d(v,x)$. The edge attaining the *max* gives yet another notion of "centrality."

xxx further remarks on τ_2.

3.2 Extremal trees

It is natural to think of the n-path (Example 8) and the n-star (Example 10) as being "extremal" amongst all n-vertex trees. The proposition below confirms that the values of τ^*, $\max_{i,j} E_i T_j$, τ_0, τ_2, and τ_c in those examples are the exact extremal values (minimal for the star, maximal for the path).

Proposition 24 *For any n-vertex tree with $n \geq 3$,*

(a) $4(n-1) \leq \tau^* \leq 2(n-1)^2$

(b) $2(n-1) \leq \max_{i,j} E_i T_j \leq (n-1)^2$

(c) $n - \frac{3}{2} \leq \tau_0 \leq (2n^2 - 4n + 3)/6$.
(d) $1 \leq \tau_2 \leq (1 - \cos(\pi/(n-1)))^{-1}$.
(e) $1 - \frac{1}{2(n-1)} \leq \tau_c \leq \frac{4\lfloor n^2/4 \rfloor - 2n + 1}{2(n-1)}$.

Proof. (a) is obvious from (88), because Δ varies between 2 for the n-star and $(n-1)$ for the n-path. The lower bound in (b) follows from the lower bound in (a). For the upper bound in (b), consider some path $i = v_0, v_1, \ldots, v_d = j$ in the tree, where plainly $d \leq (n-1)$. Now $|A(v_{d-1}, v_d)| \leq n - 1$ and so

$$|A(v_{d-i}, v_{d-i+1})| \leq n - i \text{ for all } i$$

because the left side decreases by at least 1 as i increases. So

$$\begin{aligned} E_i T_j &= \sum_{m=0}^{d-1} E_{v_m} T_{v_{m+1}} \\ &= \sum_{m=0}^{d-1} (2|A(v_m, v_{m+1})| - 1) \text{ by (81)} \\ &\leq \sum_{m=0}^{d-1} (2(m + n - d) - 1) \\ &\leq \sum_{l=1}^{n-1} (2l - 1) \\ &= (n-1)^2. \end{aligned}$$

To prove (c), it is enough to show that the sum in Proposition 22 is minimized by the n-star and maximized by the n-path. For each undirected edge (v, w), let

$$b(v, w) = \min(|A(v, w)|, |A(w, v)|) \leq n/2.$$

Let $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1})$ be the non-decreasing rearrangement of these values $b(v, w)$. The summands in Proposition 22 are of the form

$$a(n - a) - \frac{1}{2(n-1)}(a^2 + (n-a)^2)$$

with a ranging over the b_i.

One can check that this quantity is an increasing function of $a \leq n/2$. Thus it is enough to show that the vector $\mathbf{b}$ on an arbitrary n-tree dominates coordinatewise the vector **b** for the n-star and is dominated by the vector **b** for the n-path. The former is obvious, since on the n-star $\mathbf{b} = (1, 1, \ldots, 1)$. The latter needs a little work. On the n-path $\mathbf{b} = (1, 1, 2, 2, 3, 3, \ldots)$. So we must prove that in any n-tree

$$b_i \leq \left\lfloor \frac{i+1}{2} \right\rfloor \text{ for all } i. \tag{91}$$

Consider a *rooted* tree on m vertices. Breaking an edge e gives two components; let $a(e)$ be the size of the component not containing the root. Let $(a_1, a_2, \ldots)$ be the non-decreasing rearrangement of $(a(e))$. For an m-path rooted at one leaf, $(a_1, a_2, \ldots) = (1, 2, 3, \ldots)$. We assert this is extremal, in that for any rooted tree

$$a_i \leq i \text{ for all } i. \tag{92}$$

This fact can be proved by an obvious induction on m, growing trees by adding leaves.

Now consider an unrooted tree, and let **b** be as above. There exists some vertex v, of degree $r \geq 2$, such that each of the r branches from v has size (excluding v) at most $n/2$. Consider these branches as trees rooted at v, apply (92), and it is easy to deduce (91).

For (d), the lower bound is easy. Fix a leaf v and let w be its neighbor. We want to apply the extremal characterization (Chapter 3 yyy) of τ_2 to the function

$$g(v) = 1 - \pi_v - \pi_w, g(w) = 0, g(\cdot) = -\pi_v \text{ elsewhere.}$$

For this function, $\sum \pi_x g(x) = 0$,

$$[g, g] = \pi_v(1 - \pi_v - \pi_w)^2 + (1 - \pi_v - \pi_w)\pi_v^2,$$

and by considering transitions out of w

$$\mathcal{E}(g, g) = \pi_v(1 - \pi_v - \pi_w)^2 + (\pi_w - \pi_v)\pi_v^2.$$

Since $\pi_w \leq 1/2$ we have $[g, g] \geq \mathcal{E}(g, g)$ and hence $\tau_2 \geq [g, g]/\mathcal{E}(g, g) \geq 1$.

qqq Anyone have a short proof of upper bound in (d)?

Finally, (e) is clear from (90). ■

Other extremal questions. Several other extremal questions have been studied. Results on cover time are given in Chapter 6. Yaron [20] shows that for leaves l the mean hitting time $E_\pi T_l$ is maximal on the n-path and minimal on the n-star. (He actually studies the variance of return times, but Chapter 2 yyy permits the rephrasing.) Finally, if we are interested in the mean hitting time $E_x T_A$ or the hitting place distribution, we can reduce to the case where A is the set L of leaves, and then set up recursively-solvable equations for $h(i) \equiv E_i T_L$ or for $f(i) = P_i(T_A = T_l)$ for fixed $l \in L$. An elementary treatment of such ideas is in Pearce [18], who essentially proved that (on n-vertex trees) $\max_x E_x T_L$ is minimized by the n-star and maximized by the n-path.

4 Notes on Chapter 5

Most of the material seems pretty straightforward, so we will give references sparingly.

Introduction. The essential edge lemma is one of those oft-rediscovered results which defies attribution.

Section 1.2. One can of course use the essential edge lemma to derive the formula for mean hitting times in the general birth-and-death process. This approach seems more elegant than the usual textbook derivation. Although we are fans of martingale methods, we didn't use them in Proposition 3(b), because to define the right martingale requires one to know the answer beforehand!

For a birth-and-death chain the spectral representation involves orthogonal polynomials. This theory was developed by Karlin and McGregor in the 1950s, and is summarized in Chapter 8 of Anderson [1]. It enables one to write down explicit formulas for $P_i(X_t = j)$ in special cases. But it is less clear how to gain qualitative insight, or inequalities valid over all birth-and-death chains, from this approach.

An alternative approach which is more useful for our purposes is based on *Siegmund duality* (see e.g. [1] Section 7.4). Associated with a birth-and-death process (X_t) is another birth-and-death process (Y_t) which is "dual" in the sense that

$$P_i(X_t \leq j) = P_j(Y_t \geq i) \text{ for all } i, j, t$$

and whose transition rates have a simple specification in terms of those of (X_t). It is easy to see that τ_1 for (X_t) is equivalent to $\max_j E_j T_{0,n}$ for (Y_t), for which there is an explicit formula. This gives an alternative to (16).

Section 2.

That the barbell is a good candidate for an "extremal" graph with respect to random walk properties was realized by Landau and Odlyzko [12], who computed the asymptotics of τ_2, and by Mazo [14], who computed the asymptotics of the unweighted average of $(E_i T_j; i, j \in I)$, which in this example is asymptotically our τ_0. Note we were able to give a one-line argument for the asymptotics of τ_2 by relying on the general fact $\tau_2 \leq \tau_0$.

Formulas for quantities associated with random walk on the d-cube and with the Ehrenfest urn model have been repeatedly rediscovered, and we certainly haven't given all the known results. Bingham [4] has an extensive bibliography. Palacios [17] uses the simple "resistance" argument used in the text, and notes that the same argument can be used on the Platonic graphs. Different methods of computing $E_{\mathbf{0}} T_{\mathbf{1}}$ lead to formulas looking different from our (69), for instance

$$E_{\mathbf{0}} T_{\mathbf{1}} = d \sum_{i=1}^{d} 2^{i-1}/i \qquad \text{[11], eq. (4.27)}$$

$$= d \sum_{1 \leq j \leq d,\ j \text{ odd}} j^{-1} \binom{d}{j} \qquad \text{[4].}$$

Similarly, one can get different-looking expressions for τ_0. Wilf [19] lists 54 identities involving binomial coefficients—it would be amusing to see how many could be derived by calculating a random walk on the d-cube quantity in two different ways!

Comparing our treatment of dense regular graphs (Example 16) with that in [16] should convince the reader of the value of general theory.

Section 3. An early reference to formulas for the mean and variance of hitting times on a tree (Theorem 20) is Moon [15], who used less intuitive generating function arguments. The formulas for the mean have been repeatedly rediscovered.

Of course there are many other questions we can ask about random walk on trees. Some issues treated later are

xxx list.

xxx more sophisticated ideas in Lyons [13].

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