

HYPERBOLIC MAXWELL VARIATIONAL INEQUALITIES FOR BEAN'S CRITICAL-STATE MODEL IN TYPE-II SUPERCONDUCTIVITY

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Abstract. This article is focused on the numerical analysis for three-dimensional Bean's critical-state model in type-II superconductivity. We derive hyperbolic mixed variational inequalities of the second kind for the evolution Maxwell equations with Bean's constitutive law between the electric field and the current density. On the basis of the variational inequality in the magnetic induction formulation, a semi-discrete Ritz-Galerkin approximation problem is rigorously analyzed, and a strong convergence result is proven. Thereafter, we propose a concrete realization of the Ritz-Galerkin approximation through a mixed finite element method based on edge elements of Nédélec's first family, Raviart-Thomas face elements, divergence-free Raviart-Thomas face elements, and piecewise constant elements. As a final result, we prove error estimates for the proposed mixed finite element method.

Key words. hyperbolic Maxwell variational inequality, Bean's critical-state model, type-II superconductivity, mixed finite element, convergence analysis, error analysis.

AMS subject classifications. 35L87, 35Q60, 65M60

1. Introduction. The physical nature of superconductivity was discovered a century ago by Heike Kamerlingh Onnes (1853–1926). He observed that the electrical resistance in mercury drops completely to zero, if the temperature is cooled down below the critical temperature (4.15 K for mercury). This is the first fundamental property of superconductors, which in particular allows electric currents to flow in a superconductor without energy dissipation. The second fundamental nature of superconductivity was discovered by Fritz Walther Meissner (1882–1974). He found out that, being in the superconducting state at an extremely cold temperature, a superconductor does not allow any penetration of a *weak* magnetic field (Meissner effect). Today, superconductivity makes many new applications and key technologies possible. They include applications in Magnetic Resonance Imaging (MRI), magnetic confinement fusion technologies, high-energy particle accelerators, magnetic levitation technologies, magnetic energy storage, and many more.

Superconductors are classified into two different types. In type-I superconductors, the Meissner effect occurs under the condition that the temperature is below the critical one T_c and the applied magnetic field is below some critical level H_c . Above this threshold, the Meissner effect instantly breaks down (sharp transition to normal state). Typical examples for type-I superconductors are pure metals such as aluminium, mercury, and gallium. Type-II superconductors behave completely differently from the first type. More precisely, they admit two different critical levels $H_{c1} < H_{c2}$. If the applied magnetic field is below the lower critical value H_{c1} , then Meissner effect occurs. If the magnetic field is stronger than H_{c1} but weaker than H_{c2} , then the magnetic field partially enters the material, but the superconducting state is not completely destroyed. This kind of physical state is called Shubnikov phase or mixed state. Finally, the superconducting state completely breaks down, if the applied magnetic field is stronger than H_{c2} . Type-II superconductors admit greater critical temperatures and critical values of magnetic field than those of the first kind. These properties enable them to preserve their superconducting effects in the presence of a strong magnetic field at higher temperatures. Examples for type-II superconductors are alloys and oxide ceramic materials.

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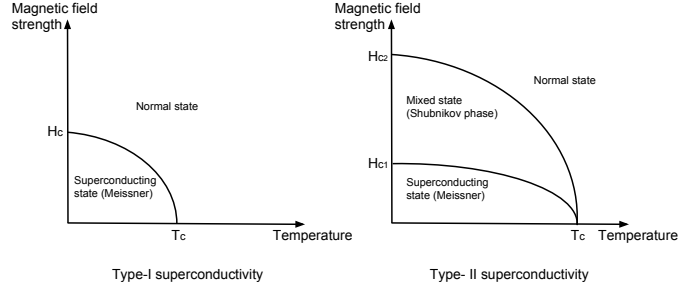


Fig. 1.1. Sharp transition to the normal state in typ-I superconductors (left plot) and the mixed state in type-II superconductors (right plot).

Being in the mixed state, a type-II superconductor allows partial penetration of the applied magnetic field in the form of flux tubes. Every tube carries exactly one single magnetic flux quantum and is surrounded by a supercurrent vortex. If we modify the applied magnetic field, then the density of the flux tubes and the supercurrents will change. This dynamic magnetization process is not reversible and exhibits hysteresis. A well-known critical-state model describing such a complex irreversible magnetization process was proposed by Bean [7, 8]. His model postulates a non-smooth constitutive relation between the electric field and the current density as follows:

- (A1) the current density strength cannot exceed some critical value $j_c \in \mathbb{R}^+$;
- (A2) the electric field vanishes, if the current density strength is strictly less than j_c ;
- (A3) the electric field is parallel to the current density.

We note that Bean made a simplifying assumption of a constant critical current density $j_c \in \mathbb{R}^+$, which is physically reasonable in the case of a not so strong magnetic field. According to experiments, however, the critical current density can depend on the magnetic field $j_c = j_c(|\mathbf{H}|)$ in the case of strong external fields. This physical phenomenon was observed by Kim et. al [21]. We refer to [10] for a comprehensive review on the derivation of the Bean critical state constitutive relation from different mathematical models, including Ginzburg-Landau and London equations. See also [11, 12] for mathematical and numerical results on Ginzburg-Landau equations.

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let Ω_{sc} be an open set satisfying $\bar{\Omega}_{sc} \subset \Omega$. Here, the subset Ω_{sc} represents a type-II superconductor. Assuming that the temperature of the superconductor Ω_{sc} is below the critical one, the evolution of the electromagnetic waves in Ω is described by the Maxwell equations

$$(1.1a) \quad \begin{cases} \epsilon \mathbf{E}_t - \mathbf{curl} \mathbf{H} + \mathbf{J} = \mathbf{f} & \text{in } \Omega \times (0, T), \\ \mu \mathbf{H}_t + \mathbf{curl} \mathbf{E} = 0 & \text{in } \Omega \times (0, T), \\ \mathbf{E} \times \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, T), \\ \mathbf{E}(\cdot, 0) = \mathbf{E}_0 & \text{in } \Omega, \\ \mathbf{H}(\cdot, 0) = \mathbf{H}_0 & \text{in } \Omega, \end{cases}$$

along with the Bean constitutive law (A1)-(A3) for the electric field and the current density:

$$(1.1b) \quad \begin{cases} \mathbf{J}(x, t) \cdot \mathbf{E}(x, t) = g(x)|\mathbf{E}(x, t)| & \text{a.e. in } \Omega \times (0, T), \\ |\mathbf{J}(x, t)| \leq g(x) & \text{a.e. in } \Omega \times (0, T). \end{cases}$$

In the setting of (1.1a), $\mathbf{E} : \Omega \times (0, T) \rightarrow \mathbb{R}^3$ denotes the electric field, $\mathbf{H} : \Omega \times (0, T) \rightarrow \mathbb{R}^3$ the

magnetic field, $\mathbf{J} : \Omega \times (0, T) \rightarrow \mathbb{R}^3$ the current density, $\mathbf{f} : \Omega \times (0, T) \rightarrow \mathbb{R}^3$ applied current source, and $\mathbf{E}_0, \mathbf{H}_0 : \Omega \rightarrow \mathbb{R}^3$ the initial electric and magnetic fields. Furthermore, the scalar functions $\epsilon, \mu : \Omega \rightarrow \mathbb{R}$ stand for the electric permittivity and the magnetic permeability, respectively. They are of class $L^\infty(\Omega)$ and satisfy

$$\underline{\epsilon} \leq \epsilon(x) \leq \bar{\epsilon} \quad \text{a.e. in } \Omega \quad \text{and} \quad \underline{\mu} \leq \mu(x) \leq \bar{\mu} \quad \text{a.e. in } \Omega,$$

for some constants $0 < \underline{\epsilon} < \bar{\epsilon}$ and $0 < \underline{\mu} < \bar{\mu}$. Moreover, the scalar function $g : \Omega \rightarrow \mathbb{R}$ is assumed to be of class $L^\infty(\Omega)$ and nonnegative. In the context of Bean's critical-state model, it is given by

$$g(x) = j_c \chi_{\Omega_{sc}}(x)$$

where $j_c \in \mathbb{R}^+$ is the critical current density of the type-II superconductor Ω_{sc} , and $\chi_{\Omega_{sc}}$ denotes the characteristic function of Ω_{sc} .

If the displacement current $\epsilon \mathbf{E}_t$ is significantly smaller in comparison with $-\mathbf{curl} \mathbf{H} + \mathbf{J}$, then Maxwell's equations (1.1a) can be approximated by neglecting $\epsilon \mathbf{E}_t$. This approximation is called eddy current approximation (see [1]), which leads to a magnetic field formulation in form of a parabolic variational inequality of the first kind. Prizhagin [25, 26] was the first, who introduced and analyzed this formulation. The finite element analysis in a 2D setting was investigated in [14]. Some years later, Elliott and Kashima [13] investigated the numerical analysis of the associated 3D parabolic variational inequality, where the finite element approximations of the extended Bean model by Bossavit [9] and the \mathbf{E} - \mathbf{J} power law were analyzed. In the case of a nonlinear critical current density $j_c = j_c(|\mathbf{H}|)$, the eddy current approximation of (1.1) leads to a parabolic quasi-variational inequality. Barrett and Prigozhin [4] analyzed the associated parabolic quasi-variational inequality problem in a scalar 2D setting and its dual formulation. Recently, they [6] introduced a nonconforming finite element method. They proved the convergence of their nonconforming method and illustrated its efficiency numerically. See also [5] concerning the mathematical and numerical analysis for a mixed formulation of a thin film magnetization problem in type-II superconductivity.

All the previously mentioned contributions are devoted to the eddy current approximation, which simplifies the Maxwell equations (1.1a) by eliminating the displacement current $\epsilon \mathbf{E}_t$. However, in many important physical phenomena such as high-frequency physics including radio frequency and microwave physics, the displacement current $\epsilon \mathbf{E}_t$ is of significance and in general cannot be neglected. Jochmann [18, 19] was the first, who introduced (1.1) and proved its existence and uniqueness of solutions. In [20], he extended the existence and uniqueness result to the nonlinear case $j_c = j_c(|\mathbf{H}|)$. The optimization of (1.1) has also been recently analyzed in [29] (see also [27, 28]).

This article is focused on the numerical analysis for (1.1). To the best of the author's knowledge, there is no earlier contribution to the numerical analysis of (1.1). In this article, we introduce a hyperbolic mixed variational inequality of the second kind and prove the equivalence between the proposed variational inequality and the non-smooth Maxwell system (1.1). In particular, we derive a hyperbolic mixed variational inequality in the magnetic induction formulation, which serves as the key tool for our numerical analysis. Based on this formulation, a semi-discrete Ritz-Galerkin approximation is proposed and rigorously analyzed. We prove a strong convergence result through the use of a discrete mixed variational problem and the Hilbert projection theorem applied to the range of the rotation operator acting on a curl-conforming finite-dimensional subspace. Hereafter, we discuss a concrete realization of the Ritz-Galerkin approximation through mixed finite elements, including edge elements of Nédélec's first family, Raviart-Thomas face elements, divergence-free Raviart-Thomas face elements, and piecewise constant elements. For the proposed mixed finite element method, we are able to prove error estimates yielding a convergence rate of the method.

2. Preliminaries. Throughout this paper, c denotes a generic positive constant that can take different values on different occasions. For a given Hilbert space V , we use the notation $\|\cdot\|_V$ and $(\cdot, \cdot)_V$ for the norm and the scalar product in V . Furthermore, a bold typeface is employed to indicate a three-dimensional vector function or a Hilbert space of three-dimensional vector functions. Our main Hilbert spaces are

$$\mathbf{H}(\mathbf{curl}) := \{\mathbf{q} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \mathbf{q} \in \mathbf{L}^2(\Omega)\} \quad \text{and} \quad \mathbf{H}(\text{div}) := \{\mathbf{q} \in \mathbf{L}^2(\Omega) \mid \text{div} \mathbf{q} \in L^2(\Omega)\},$$

where the \mathbf{curl} - and div -operators are understood in the sense of distributions. As usual, $\mathbf{C}_0^\infty(\Omega)$ stands for the space of all infinitely differentiable vector functions with compact support contained in Ω . We denote the closure of $\mathbf{C}_0^\infty(\Omega)$ with respect to the $\mathbf{H}(\mathbf{curl})$ -topology and the $\mathbf{H}(\text{div})$ -topology, respectively, by $\mathbf{H}_0(\mathbf{curl}) := \overline{\mathbf{C}_0^\infty(\Omega)}^{\|\cdot\|_{\mathbf{H}(\mathbf{curl})}}$ and $\mathbf{H}_0(\text{div}) := \overline{\mathbf{C}_0^\infty(\Omega)}^{\|\cdot\|_{\mathbf{H}(\text{div})}}$. Furthermore,

$$\mathbf{H}_0(\text{div}=0) := \{\mathbf{q} \in \mathbf{L}^2(\Omega) \mid (\mathbf{q}, \nabla \phi)_{\mathbf{L}^2(\Omega)} = 0, \quad \forall \phi \in H^1(\Omega)\}.$$

Finally, for every positive function $\alpha \in L^\infty(\Omega)$, we use the notation $\mathbf{L}_\alpha^2(\Omega)$ for the weighted $\mathbf{L}^2(\Omega)$ -space endowed with the weighted scalar product $(\alpha \cdot, \cdot)_{\mathbf{L}^2(\Omega)}$.

The existence of unique mild and strong solutions to the hyperbolic system (1.1) has been proved by Jochmann in [18, Theorem 1] and [19, Lemma 4.3]. We summarize the existence and uniqueness result for the strong solution in the following lemma:

LEMMA 2.1. *Let $\mathbf{f} \in W^{1,\infty}((0, T), \mathbf{L}^2(\Omega))$ and $(\mathbf{E}_0, \mathbf{H}_0) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}(\mathbf{curl})$. Then, there exist a unique pair $(\mathbf{E}, \mathbf{H}) \in L^\infty((0, T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}(\mathbf{curl})) \cap W^{1,\infty}((0, T), \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_\mu^2(\Omega))$ and a unique $\mathbf{J} \in L^\infty((0, T), \mathbf{L}^\infty(\Omega))$ satisfying (1.1b) and*

$$(2.1) \quad \begin{cases} \epsilon \frac{d}{dt} \mathbf{E}(t) - \mathbf{curl} \mathbf{H}(t) + \mathbf{J}(t) = \mathbf{f}(t) & \text{for a.e. } t \in (0, T), \\ \mu \frac{d}{dt} \mathbf{H}(t) + \mathbf{curl} \mathbf{E}(t) = 0 & \text{for a.e. } t \in (0, T), \\ (\mathbf{E}, \mathbf{H})(0) = (\mathbf{E}_0, \mathbf{H}_0). \end{cases}$$

In other words, $(\mathbf{E}, \mathbf{H}, \mathbf{J})$ is the strong solution of (1.1).

3. Hyperbolic Variational Inequalities. We start by introducing the functional

$$\varphi : \mathbf{L}^1(\Omega) \rightarrow \mathbb{R}, \quad \varphi(\mathbf{v}) = \int_{\Omega} g(x) |\mathbf{v}(x)| dx.$$

Since $g \in L^\infty(\Omega)$ is nonnegative, φ defines a seminorm on $\mathbf{L}^1(\Omega)$. Taking the functional φ into account, we introduce a hyperbolic mixed variational inequality of the second kind and prove the equivalence between (1.1) and the proposed variational inequality.

THEOREM 3.1. *Let $\mathbf{f} \in W^{1,\infty}((0, T), \mathbf{L}^2(\Omega))$ and $(\mathbf{E}_0, \mathbf{H}_0) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}(\mathbf{curl})$. Then, the hyperbolic mixed variational inequality*

$$(3.1) \quad \left\{ \begin{array}{l} \int_{\Omega} \epsilon \frac{d}{dt} \mathbf{E}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) + \mu \frac{d}{dt} \mathbf{H}(t) \cdot (\mathbf{w} - \mathbf{H}(t)) dx \\ + \int_{\Omega} \mathbf{curl} \mathbf{E}(t) \cdot (\mathbf{w} - \mathbf{H}(t)) - \mathbf{H}(t) \cdot \mathbf{curl} (\mathbf{v} - \mathbf{E}(t)) dx + \varphi(\mathbf{v}) - \varphi(\mathbf{E}(t)) \\ \geq \int_{\Omega} \mathbf{f}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) dx, \\ \text{for a.e. } t \in (0, T) \text{ and all } (\mathbf{v}, \mathbf{w}) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{L}^2(\Omega), \\ (\mathbf{E}, \mathbf{H})(0) = (\mathbf{E}_0, \mathbf{H}_0) \end{array} \right.$$

admits a unique solution $(\mathbf{E}, \mathbf{H}) \in L^\infty((0, T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}(\mathbf{curl})) \cap W^{1,\infty}((0, T), \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_\mu^2(\Omega))$. The unique solution of (3.1) is exactly the strong solution of (1.1).

Proof. Uniqueness: Inserting $(\mathbf{v}, \mathbf{w}) = 0$ and $(\mathbf{v}, \mathbf{w}) = (2\mathbf{E}(t), 2\mathbf{H}(t))$ in (3.1), it follows that every solution $(\mathbf{E}, \mathbf{H}) \in L^\infty((0, T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}(\mathbf{curl})) \cap W^{1,\infty}((0, T), \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_\mu^2(\Omega))$ of (3.1) satisfies

$$\int_{\Omega} \epsilon \frac{d}{dt} \mathbf{E}(t) \cdot \mathbf{E}(t) + \mu \frac{d}{dt} \mathbf{H}(t) \cdot \mathbf{H}(t) dx + \varphi(\mathbf{E}(t)) = \int_{\Omega} \mathbf{f}(t) \cdot \mathbf{E}(t) dx \quad \text{for a.e. } t \in (0, T),$$

such that every solution of (3.1) satisfies

$$\int_{\Omega} \epsilon \frac{d}{dt} \mathbf{E}(t) \cdot \mathbf{v} + \mu \frac{d}{dt} \mathbf{H}(t) \cdot \mathbf{w} dx + \int_{\Omega} \mathbf{curl} \mathbf{E}(t) \cdot \mathbf{w} - \mathbf{H}(t) \cdot \mathbf{curl} \mathbf{v} dx + \varphi(\mathbf{v}) \geq \int_{\Omega} \mathbf{f}(t) \cdot \mathbf{v} dx$$

for a.e. $t \in (0, T)$ and all $(\mathbf{v}, \mathbf{w}) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{L}^2(\Omega)$.

Suppose now that $(\mathbf{E}^1, \mathbf{H}^1), (\mathbf{E}^2, \mathbf{H}^2) \in L^\infty((0, T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}(\mathbf{curl})) \cap W^{1,\infty}((0, T), \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_\mu^2(\Omega))$ are solutions to (3.1). Then, the difference $(\mathbf{e}, \mathbf{h}) := (\mathbf{E}^1 - \mathbf{E}^2, \mathbf{H}^1 - \mathbf{H}^2)$ fulfils

$$(3.2) \quad \int_{\Omega} \epsilon \frac{d}{dt} \mathbf{e}(t) \cdot \mathbf{v} + \mu \frac{d}{dt} \mathbf{h}(t) \cdot \mathbf{w} dx + \int_{\Omega} \mathbf{curl} \mathbf{e}(t) \cdot \mathbf{w} - \mathbf{h}(t) \cdot \mathbf{curl} \mathbf{v} dx \geq 0$$

for a.e. $t \in (0, T)$ and all $(\mathbf{v}, \mathbf{w}) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{L}^2(\Omega)$.

Setting $(\mathbf{v}, \mathbf{w}) = -(\mathbf{e}(t), \mathbf{h}(t))$ in (3.2) results in

$$0 \geq \int_{\Omega} \epsilon \frac{d}{dt} \mathbf{e}(t) \cdot \mathbf{e}(t) + \mu \frac{d}{dt} \mathbf{h}(t) \cdot \mathbf{h}(t) dx = \frac{1}{2} \frac{d}{dt} \|\mathbf{e}(t)\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} \|\mathbf{h}(t)\|_{\mathbf{L}_\mu^2(\Omega)}^2 \quad \text{for a.e. } t \in (0, T).$$

In view of this inequality and $\mathbf{e}(0) = \mathbf{h}(0) = 0$, it follows that (3.1) has at most only one solution. *Existence:* Lemma 2.1 implies the existence of a unique $(\mathbf{E}, \mathbf{H}) \in L^\infty((0, T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}(\mathbf{curl})) \cap W^{1,\infty}((0, T), \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_\mu^2(\Omega))$ and a unique $\mathbf{J} \in L^\infty((0, T), \mathbf{L}^\infty(\Omega))$ satisfying (1.1b) and (2.1). Multiplying the first equality in (2.1) by $\mathbf{v} - \mathbf{E}(t)$, with $\mathbf{v} \in \mathbf{H}_0(\mathbf{curl})$, and the second equality in (2.1) by $\mathbf{w} - \mathbf{H}(t)$, with $\mathbf{w} \in \mathbf{L}^2(\Omega)$, and then integrating the resulting equalities over Ω , we obtain

$$(3.3) \quad \int_{\Omega} \epsilon \frac{d}{dt} \mathbf{E}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) dx - \int_{\Omega} \mathbf{curl} \mathbf{H}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) dx$$

$$+ \int_{\Omega} \mathbf{J}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) dx = \int_{\Omega} \mathbf{f}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) dx,$$

$$\int_{\Omega} \mu \frac{d}{dt} \mathbf{H}(t) \cdot (\mathbf{w} - \mathbf{H}(t)) dx + \int_{\Omega} \mathbf{curl} \mathbf{E}(t) \cdot (\mathbf{w} - \mathbf{H}(t)) dx = 0,$$

for a.e. $t \in (0, T)$. Then, applying

$$(3.4) \quad \int_{\Omega} \mathbf{curl} \mathbf{z} \cdot \mathbf{v} \, dx = \int_{\Omega} \mathbf{z} \cdot \mathbf{curl} \mathbf{v} \, dx, \quad \forall (\mathbf{v}, \mathbf{z}) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}(\mathbf{curl}),$$

to the first equality in (3.3), and adding the resulting equality to the second equality in (3.3), it follows that

$$(3.5) \quad \begin{aligned} & \int_{\Omega} \epsilon \frac{d}{dt} \mathbf{E}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) + \mu \frac{d}{dt} \mathbf{H}(t) \cdot (\mathbf{w} - \mathbf{H}(t)) \, dx + \int_{\Omega} \mathbf{curl} \mathbf{E}(t) \cdot (\mathbf{w} - \mathbf{H}(t)) \\ & - \mathbf{H}(t) \cdot \mathbf{curl} (\mathbf{v} - \mathbf{E}(t)) \, dx + \int_{\Omega} \mathbf{J}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) \, dx = \int_{\Omega} \mathbf{f}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) \, dx \\ & \text{for a.e. } t \in (0, T) \text{ and all } (\mathbf{v}, \mathbf{w}) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{L}^2(\Omega). \end{aligned}$$

On the other hand, in view of (1.1b), we have that

$$\begin{aligned} \int_{\Omega} \mathbf{J}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) \, dx &= \int_{\Omega} \mathbf{J}(t) \cdot \mathbf{v} \, dx - \int_{\Omega} g|\mathbf{E}(t)| \, dx \\ &\leq \int_{\Omega} g|\mathbf{v}| \, dx - \int_{\Omega} g|\mathbf{E}(t)| \, dx = \varphi(\mathbf{v}) - \varphi(\mathbf{E}(t)). \end{aligned}$$

Applying this inequality to (3.5), we conclude that $(\mathbf{E}, \mathbf{H}) \in L^\infty((0, T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}(\mathbf{curl})) \cap W^{1,\infty}((0, T), \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_\mu^2(\Omega))$ satisfies (3.1). This completes the proof. \square

COROLLARY 3.2. *Let $\mathbf{f} \in W^{1,\infty}((0, T), \mathbf{L}^2(\Omega))$ and $(\mathbf{E}_0, \mathbf{H}_0) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}(\mathbf{curl})$ satisfying $\mu \mathbf{H}_0 \in \mathbf{H}_0(\text{div}=0)$. Then, the unique solution $(\mathbf{E}, \mathbf{H}) \in L^\infty((0, T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}(\mathbf{curl})) \cap W^{1,\infty}((0, T), \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_\mu^2(\Omega))$ of the variational inequality (3.1) satisfies*

$$\mu \frac{d}{dt} \mathbf{H}(t) \in \mathbf{H}_0(\text{div}=0) \quad \text{for a.e. } t \in (0, T) \quad \text{and} \quad \mu \mathbf{H}(t) \in \mathbf{H}_0(\text{div}=0) \quad \text{for all } t \in [0, T].$$

Proof. Setting $(\mathbf{v}, \mathbf{w}) = (\mathbf{E}(t), \mathbf{H}(t) \pm \nabla \phi)$ in (3.1) with $\phi \in H^1(\Omega)$ yields that

$$(3.6) \quad \int_{\Omega} \mu \frac{d}{dt} \mathbf{H}(t) \cdot \nabla \phi \, dx + \underbrace{\int_{\Omega} \mathbf{curl} \mathbf{E}(t) \cdot \nabla \phi \, dx}_{=0} = 0 \quad \text{for a.e. } t \in (0, T) \text{ and all } \phi \in H^1(\Omega),$$

where we have also used (3.4) and $\mathbf{curl} \nabla \equiv 0$ to deduce that $\int_{\Omega} \mathbf{curl} \mathbf{E}(t) \cdot \nabla \phi \, dx = 0$ for a.e. $t \in (0, T)$. Consequently, $\mu \frac{d}{dt} \mathbf{H}(t) \in \mathbf{H}_0(\text{div}=0)$ holds for a.e. $t \in (0, T)$. Furthermore, integrating (3.6) over the time interval $[0, \tau]$, with $\tau \in [0, T]$, implies that

$$0 = \int_{\Omega} \mu \mathbf{H}(\tau) \cdot \nabla \phi \, dx - \int_{\Omega} \mu \mathbf{H}_0 \cdot \nabla \phi \, dx = \int_{\Omega} \mu \mathbf{H}(\tau) \cdot \nabla \phi \, dx, \quad \forall \phi \in H^1(\Omega), \quad \forall \tau \in [0, T],$$

since $\mu \mathbf{H}_0 \in \mathbf{H}_0(\text{div}=0)$. In conclusion, $\mu \mathbf{H}(\tau) \in \mathbf{H}_0(\text{div}=0)$ holds for all $\tau \in [0, T]$. This completes the proof. \square

We close this section by presenting a variational inequality for (1.1) in the magnetic induction formulation, which is the key basis for our numerical analysis. For the upcoming result, we shall make use of the space

$$\mathbf{R}(\Omega) := \mu \mathbf{H}(\mathbf{curl}) \cap \mathbf{H}_0(\text{div}=0).$$

THEOREM 3.3. *Let $\mathbf{f} \in W^{1,\infty}((0,T), \mathbf{L}^2(\Omega))$ and $(\mathbf{E}_0, \mathbf{B}_0) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{R}(\Omega)$. Then, the hyperbolic mixed variational inequality*

$$(VI) \quad \left\{ \begin{array}{l} \int_{\Omega} \epsilon \frac{d}{dt} \mathbf{E}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) + \mu^{-1} \frac{d}{dt} \mathbf{B}(t) \cdot (\mathbf{w} - \mathbf{B}(t)) \, dx \\ + \int_{\Omega} \mu^{-1} \left(\mathbf{curl} \mathbf{E}(t) \cdot (\mathbf{w} - \mathbf{B}(t)) - \mathbf{B}(t) \cdot \mathbf{curl} (\mathbf{v} - \mathbf{E}(t)) \right) \, dx \\ + \varphi(\mathbf{v}) - \varphi(\mathbf{E}(t)) \geq \int_{\Omega} \mathbf{f}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) \, dx \\ \text{for a.e. } t \in (0, T) \text{ and all } (\mathbf{v}, \mathbf{w}) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{L}^2(\Omega), \\ (\mathbf{E}, \mathbf{B})(0) = (\mathbf{E}_0, \mathbf{B}_0) \end{array} \right.$$

admits a unique solution $(\mathbf{E}, \mathbf{B}) \in L^\infty((0, T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{R}(\Omega)) \cap W^{1,\infty}((0, T), \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{H}_0(\text{div}=0))$. The unique solution \mathbf{E} of (VI) with $\mathbf{H} := \mu^{-1} \mathbf{B}$ is exactly the unique solution of (3.1).

Proof. By assumption, we have that $\mathbf{H}_0 := \mu^{-1} \mathbf{B}_0 \in \mathbf{H}(\mathbf{curl})$, and so $(\mathbf{E}_0, \mathbf{H}_0) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}(\mathbf{curl})$ such that, according to Theorem 3.1, the variational inequality (3.1) admits a unique solution $(\mathbf{E}, \mathbf{H}) \in L^\infty((0, T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}(\mathbf{curl})) \cap W^{1,\infty}((0, T), \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_\mu^2(\Omega))$. We set $\mathbf{B} := \mu \mathbf{H}$. Since $\mu \mathbf{H}_0 = \mathbf{B}_0 \in \mathbf{H}_0(\text{div}=0)$, Corollary 3.2 implies that

$$\mathbf{B} = \mu \mathbf{H} \in L^\infty((0, T), \mathbf{R}(\Omega)) \cap W^{1,\infty}((0, T), \mathbf{H}_0(\text{div}=0)).$$

On the other hand, by virtue of (3.1), $(\mathbf{E}, \mathbf{B}) = (\mathbf{E}, \mu \mathbf{H})$ satisfies

$$(3.7) \quad \begin{aligned} & \int_{\Omega} \epsilon \frac{d}{dt} \mathbf{E}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) + \frac{d}{dt} \mathbf{B}(t) \cdot (\hat{\mathbf{w}} - \mu^{-1} \mathbf{B}(t)) \, dx + \int_{\Omega} \mathbf{curl} \mathbf{E}(t) \cdot (\hat{\mathbf{w}} - \mu^{-1} \mathbf{B}(t)) \\ & - \mu^{-1} \mathbf{B}(t) \cdot \mathbf{curl} (\mathbf{v} - \mathbf{E}(t)) \, dx + \varphi(\mathbf{v}) - \varphi(\mathbf{E}(t)) \geq \int_{\Omega} \mathbf{f}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) \, dx, \end{aligned}$$

for a.e. $t \in (0, T)$ and all $(\mathbf{v}, \hat{\mathbf{w}}) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{L}^2(\Omega)$. Therefore, setting $\hat{\mathbf{w}} = \mu^{-1} \mathbf{w}$, with $\mathbf{w} \in \mathbf{L}^2(\Omega)$, in (3.7) implies that (\mathbf{E}, \mathbf{B}) is a solution of (VI). On the other hand, by a similar transformation, every solution of (VI) satisfies (3.1) with $\mathbf{H} = \mu \mathbf{B}$. Therefore, the assertion follows. \square

4. Semi-Discrete Ritz-Galerkin Approximation. Let $\mathbf{V}_h \subset \mathbf{H}_0(\mathbf{curl})$ and $\mathbf{W}_h \subset \mathbf{L}^2(\Omega)$ be two families of finite-dimensional subspaces, depending on parameters $h > 0$. They are assumed to satisfy the following two conditions:

(A1) The family $\mathbf{V}_h \subset \mathbf{H}_0(\mathbf{curl})$ is dense:

$$\forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}) \, \forall \delta > 0 \, \exists \bar{h} > 0 \, \forall h \in (0, \bar{h}) \, \exists \mathbf{v}_h \in \mathbf{V}_h : \|\mathbf{v}_h - \mathbf{v}\|_{\mathbf{H}(\mathbf{curl})} \leq \delta.$$

(A2) The inclusion $\mathbf{curl} \mathbf{V}_h \subset \mathbf{W}_h$ is satisfied for all $h > 0$.

Let us underline that, in contrast to (A1), the family $\mathbf{W}_h \subset \mathbf{L}^2(\Omega)$ is not necessarily dense. From now on, let $(\mathbf{E}_0, \mathbf{B}_0) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{R}(\Omega)$ and $\mathbf{f} \in W^{1,\infty}((0, T), \mathbf{L}^2(\Omega))$. Then, we formulate

the semi-discrete Ritz-Galerkin approximation for (VI) as follows: Given a proper approximation $(\mathbf{E}_{0h}, \mathbf{B}_{0h}) \in \mathbf{V}_h \times \mathbf{W}_h$ of $(\mathbf{E}_0, \mathbf{B}_0)$, find $(\mathbf{E}_h, \mathbf{B}_h) \in H^1((0, T), \mathbf{V}_h \times \mathbf{W}_h)$ such that

$$(VI_h) \quad \left\{ \begin{array}{l} \int_{\Omega} \epsilon \frac{d}{dt} \mathbf{E}_h(t) \cdot (\mathbf{v}_h - \mathbf{E}_h(t)) + \mu^{-1} \frac{d}{dt} \mathbf{B}_h(t) \cdot (\mathbf{w}_h - \mathbf{B}_h(t)) \, dx \\ + \int_{\Omega} \mu^{-1} \left(\mathbf{curl} \mathbf{E}_h(t) \cdot (\mathbf{w}_h - \mathbf{B}_h(t)) - \mathbf{B}_h(t) \cdot \mathbf{curl} (\mathbf{v}_h - \mathbf{E}_h(t)) \right) \, dx \\ + \varphi(\mathbf{v}_h) - \varphi(\mathbf{E}_h(t)) \geq \int_{\Omega} \mathbf{f}(t) \cdot (\mathbf{v}_h - \mathbf{E}_h(t)) \, dx \\ \text{for a.e. } t \in (0, T) \text{ and all } (\mathbf{v}, \mathbf{w}) \in \mathbf{V}_h \times \mathbf{W}_h, \\ (\mathbf{E}, \mathbf{B})(0) = (\mathbf{E}_{0h}, \mathbf{B}_{0h}). \end{array} \right.$$

Let $\{\boldsymbol{\psi}_h^j\}_{j=1}^{N_h} \subset \mathbf{V}_h$ and $\{\boldsymbol{\eta}_h^j\}_{j=1}^{M_h} \subset \mathbf{W}_h$ denote bases for \mathbf{V}_h and \mathbf{W}_h , respectively. Then, for every $(\mathbf{v}_h, \mathbf{w}_h) \in \mathbf{V}_h \times \mathbf{W}_h$, there exists a unique vector $y = \begin{pmatrix} v \\ w \end{pmatrix} \in \mathbb{R}^{N_h+M_h}$ such that

$$(4.1) \quad \mathbf{v}_h = \sum_{j=1}^{N_h} v_j \boldsymbol{\psi}_h^j \quad \text{and} \quad \mathbf{w}_h = \sum_{j=1}^{M_h} w_j \boldsymbol{\eta}_h^j.$$

Making use of the bases, we introduce the matrices

$$\begin{aligned} \mathbb{M}_h^{(\epsilon)} &:= \left(\int_{\Omega} \epsilon \boldsymbol{\psi}_h^i \cdot \boldsymbol{\psi}_h^j \, dx \right)_{ij} \in \mathbb{R}^{N_h \times N_h}, \quad \mathbb{M}_h^{(\mu)} := \left(\int_{\Omega} \mu^{-1} \boldsymbol{\eta}_h^i \cdot \boldsymbol{\eta}_h^j \, dx \right)_{ij} \in \mathbb{R}^{M_h \times M_h}, \\ \mathbb{K}_h^{(\mu)} &:= \left(\int_{\Omega} \mu^{-1} \boldsymbol{\eta}_h^i \cdot \mathbf{curl} \boldsymbol{\psi}_h^j \, dx \right)_{ij} \in \mathbb{R}^{M_h \times N_h}, \end{aligned}$$

and

$$(4.2) \quad \mathbb{K}_h := \begin{pmatrix} 0 & -\mathbb{K}_h^{(\mu)T} \\ \mathbb{K}_h^{(\mu)} & 0 \end{pmatrix} \in \mathbb{R}^{N_h+M_h \times N_h+M_h}, \quad \mathbb{M}_h := \begin{pmatrix} \mathbb{M}_h^{(\epsilon)} & 0 \\ 0 & \mathbb{M}_h^{(\mu)} \end{pmatrix} \in \mathbb{R}^{N_h+M_h \times N_h+M_h},$$

as well as the following function:

$$(4.3) \quad \varphi_h : \mathbb{R}^{N_h+M_h} \rightarrow \mathbb{R}, \quad \varphi_h(v) = \varphi \left(\sum_{j=1}^{N_h} v_j \boldsymbol{\psi}_h^j \right) = \int_{\Omega} g(x) \left| \sum_{j=1}^{N_h} v_j \boldsymbol{\psi}_h^j(x) \right| \, dx.$$

Now, by the representation through the bases (4.1) and making use of the matrices (4.2) and the function (4.3), we see that (VI_h) is equivalent to the following evolution variational inequality on $\mathbb{R}^{N_h+M_h}$: Find $y \in H^1((0, T), \mathbb{R}^{N_h+M_h})$ such that

$$(4.4) \quad \left\{ \begin{array}{l} \left(v - y(t), \mathbb{M}_h \frac{d}{dt} y(t) + \mathbb{K}_h y(t) \right)_{\mathbb{R}^{N_h+M_h}} + \varphi_h(v) - \varphi_h(y(t)) \\ \geq (v - y(t), z(t))_{\mathbb{R}^{N_h+M_h}}, \\ \text{for all } v \in \mathbb{R}^{N_h+M_h} \text{ and a.e. } t \in (0, T), \\ y(0) = y_0, \end{array} \right.$$

where $y_0 = \begin{pmatrix} E_0 \\ B_0 \end{pmatrix} \in \mathbb{R}^{N_h+M_h}$ is the unique vector satisfying $\mathbf{E}_{0h} = \sum_{j=1}^{N_h} E_{0j} \boldsymbol{\psi}_h^j$ and $\mathbf{B}_{0h} = \sum_{j=1}^{M_h} B_{0j} \boldsymbol{\eta}_h^j$. Moreover, the function $z \in W^{1,\infty}((0, T), \mathbb{R}^{N_h+M_h})$ is defined by

$$z_j(t) := \int_{\Omega} \mathbf{f}(t) \cdot \boldsymbol{\psi}_h^j dx, \quad \forall j \in \{1, \dots, N_h\}, \quad z_j(t) := 0, \quad \forall j \in \{N_h+1, \dots, N_h+M_h\}.$$

By a classical result (see e.g. [3, Theorem 4.1, p. 124]), (4.4) admits a unique solution $y \in W^{1,\infty}((0, T), \mathbb{R}^{N_h+M_h})$. In all what follows, for every $h > 0$, let $(\mathbf{E}_h, \mathbf{B}_h) \in W^{1,\infty}((0, T), \mathbf{V}_h \times \mathbf{W}_h)$ denote the unique solution of the Ritz-Galerkin approximation (VI_h), and let $(\mathbf{E}, \mathbf{B}) \in L^\infty((0, T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{R}(\Omega)) \cap W^{1,\infty}((0, T), \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{H}_0(\text{div}=0))$ denote the unique solution of (VI). Note that, possibly after a modification on a set of $[0, T]$ with measure zero, they also satisfy

$$(\mathbf{E}, \mathbf{B}) \in \mathcal{C}([0, T], \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{H}_0(\text{div}=0)) \quad \text{and} \quad (\mathbf{E}_h, \mathbf{B}_h) \in \mathcal{C}([0, T], \mathbf{V}_h \times \mathbf{W}_h).$$

In the upcoming lemmata, we analyze the structural property of the solution to (VI_h) and its stability. These results will be important for the convergence and error analysis of (VI_h).

LEMMA 4.1. *For a.e. $t \in (0, T)$ and all $h > 0$, it holds that*

$$(4.5) \quad \frac{d}{dt} \mathbf{B}_h(t) = -\mathbf{curl} \mathbf{E}_h(t), \quad \frac{d}{dt} \mathbf{B}(t) = -\mathbf{curl} \mathbf{E}(t).$$

Proof. Let $h > 0$. Setting $(\mathbf{v}_h, \mathbf{w}_h) = 0$ and $(\mathbf{v}_h, \mathbf{w}_h) = (2\mathbf{E}_h(t), 2\mathbf{B}_h(t))$ in (VI_h) implies

$$(4.6) \quad \int_{\Omega} \epsilon \frac{d}{dt} \mathbf{E}_h(t) \cdot \mathbf{E}_h(t) + \mu^{-1} \frac{d}{dt} \mathbf{B}_h(t) \cdot \mathbf{B}_h(t) dx + \varphi(\mathbf{E}_h(t)) = \int_{\Omega} \mathbf{f}(t) \cdot \mathbf{E}_h(t) dx \quad \text{for a.e. } t \in (0, T).$$

Then, applying (4.6) to (VI_h), we see that $(\mathbf{E}_h, \mathbf{B}_h)$ satisfies

$$(4.7) \quad \begin{aligned} & \int_{\Omega} \epsilon \frac{d}{dt} \mathbf{E}_h(t) \cdot \mathbf{v}_h + \mu^{-1} \frac{d}{dt} \mathbf{B}_h(t) \cdot \mathbf{w}_h dx \\ & + \int_{\Omega} \mu^{-1} \left(\mathbf{curl} \mathbf{E}_h(t) \cdot \mathbf{w}_h - \mathbf{B}_h(t) \cdot \mathbf{curl} \mathbf{v}_h \right) dx + \varphi(\mathbf{v}_h) \\ & \geq \int_{\Omega} \mathbf{f}(t) \cdot \mathbf{v}_h dx \quad \text{for a.e. } t \in (0, T) \text{ and all } (\mathbf{v}_h, \mathbf{w}_h) \in \mathbf{V}_h \times \mathbf{W}_h. \end{aligned}$$

As $\mathbf{curl} \mathbf{V}_h \subset \mathbf{W}_h$, we may now set $\mathbf{v}_h = 0$ and $\mathbf{w}_h = -\left(\frac{d}{dt} \mathbf{B}_h(t) + \mathbf{curl} \mathbf{E}_h(t)\right)$ in (4.7) to get

$$\int_{\Omega} \mu^{-1} \left| \frac{d}{dt} \mathbf{B}_h(t) + \mathbf{curl} \mathbf{E}_h(t) \right|^2 dx \leq 0 \quad \implies \quad \frac{d}{dt} \mathbf{B}_h(t) = -\mathbf{curl} \mathbf{E}_h(t) \quad \text{for a.e. } t \in (0, T).$$

Analogously, by setting $(\mathbf{v}, \mathbf{w}) = 0$ and $(\mathbf{v}, \mathbf{w}) = (2\mathbf{E}(t), 2\mathbf{B}(t))$ in (VI), we infer that the solution (\mathbf{E}, \mathbf{B}) of (VI) satisfies

$$(4.8) \quad \begin{aligned} & \int_{\Omega} \epsilon \frac{d}{dt} \mathbf{E}(t) \cdot \mathbf{v} + \mu^{-1} \frac{d}{dt} \mathbf{B}(t) \cdot \mathbf{w} dx + \int_{\Omega} \mu^{-1} \left(\mathbf{curl} \mathbf{E}(t) \cdot \mathbf{w} - \mathbf{B}(t) \cdot \mathbf{curl} \mathbf{v} \right) dx \\ & + \varphi(\mathbf{v}) \geq \int_{\Omega} \mathbf{f}(t) \cdot \mathbf{v} dx \quad \text{for a.e. } t \in (0, T) \text{ and all } (\mathbf{v}, \mathbf{w}) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{L}^2(\Omega). \end{aligned}$$

Setting $\mathbf{v} = 0$ and $\mathbf{w} = -\left(\frac{d}{dt}\mathbf{B}(t) + \mathbf{curl}\mathbf{E}(t)\right)$ in (4.8), we get

$$\frac{d}{dt}\mathbf{B}(t) = -\mathbf{curl}\mathbf{E}(t) \quad \text{for a.e. } t \in (0, T).$$

In conclusion, the assertion is valid. \square

LEMMA 4.2. *For every $h > 0$, the solution $(\mathbf{E}_h, \mathbf{B}_h)$ of (VI_h) satisfies the estimate*

$$\begin{aligned} \|(\mathbf{E}_h, \mathbf{B}_h)\|_{\mathcal{C}([0, T], \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_{1/\mu}^2(\Omega))} &\leq \|(\mathbf{E}_{0h}, \mathbf{B}_{0h})\|_{\mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_{1/\mu}^2(\Omega)} \\ &\quad + 2\epsilon^{-1/2}(\|\mathbf{f}\|_{L^1((0, T), \mathbf{L}^2(\Omega))} + T\|g\|_{L^2(\Omega)}). \end{aligned}$$

In particular, if $\{(\mathbf{E}_{0h}, \mathbf{B}_{0h})\}_{h>0} \subset \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_{1/\mu}^2(\Omega)$ is bounded, then $\{(\mathbf{E}_h, \mathbf{B}_h)\}_{h>0}$ is bounded in $\mathcal{C}([0, T], \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_{1/\mu}^2(\Omega))$.

Proof. Let $h > 0$. Integrating (4.6) over the time interval $[0, \tau]$ with $\tau \in [0, T]$ yields

$$\begin{aligned} \frac{1}{2}\|(\mathbf{E}_h, \mathbf{B}_h)(\tau)\|_{\mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_{1/\mu}^2(\Omega)}^2 &= \frac{1}{2}\|(\mathbf{E}_h, \mathbf{B}_h)(0)\|_{\mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_{1/\mu}^2(\Omega)}^2 + \iint_0^\tau \mathbf{f}(t) \cdot \mathbf{E}_h(t) \, dx \, dt + \int_0^\tau \int_\Omega g|\mathbf{E}_h(t)| \, dt \\ &\leq \frac{1}{2}\|(\mathbf{E}_{0h}, \mathbf{B}_{0h})\|_{\mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_{1/\mu}^2(\Omega)}^2 + \|\mathbf{E}_h\|_{\mathcal{C}([0, T], \mathbf{L}^2(\Omega))}(\|\mathbf{f}\|_{L^1((0, T), \mathbf{L}^2(\Omega))} + T\|g\|_{L^2(\Omega)}) \\ &\leq \|(\mathbf{E}_h, \mathbf{B}_h)\|_{\mathcal{C}([0, T], \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_{1/\mu}^2(\Omega))} \left(\frac{1}{2}\|(\mathbf{E}_{0h}, \mathbf{B}_{0h})\|_{\mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_{1/\mu}^2(\Omega)} + \epsilon^{-1/2}(\|\mathbf{f}\|_{L^1((0, T), \mathbf{L}^2(\Omega))} + T\|g\|_{L^2(\Omega)}) \right). \end{aligned}$$

Since the above inequality holds for all $\tau \in [0, T]$, we come to the conclusion that

$$\begin{aligned} \frac{1}{2}\|(\mathbf{E}_h, \mathbf{B}_h)\|_{\mathcal{C}([0, T], \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_{1/\mu}^2(\Omega))} &\leq \frac{1}{2}\|(\mathbf{E}_{0h}, \mathbf{B}_{0h})\|_{\mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_{1/\mu}^2(\Omega)} \\ &\quad + \epsilon^{-1/2}(\|\mathbf{f}\|_{L^1((0, T), \mathbf{L}^2(\Omega))} + T\|g\|_{L^2(\Omega)}). \end{aligned}$$

This completes the proof. \square

4.1. Convergence Analysis. Our goal now is to prove the convergence of the solution of (VI_h) towards the one of (VI) as $h \rightarrow 0$. In all what follows, we endow the Hilbert space $\mathbf{H}_0(\text{div}=0)$ with the weighted scalar product $(\mu^{-1}\cdot, \cdot)_{\mathbf{L}^2(\Omega)}$, and we define the following subspace:

$$(\mathbf{curl}\mathbf{V}_h)^\perp := \{\hat{\mathbf{y}} \in \mathbf{H}_0(\text{div}=0) \mid (\mu^{-1}\hat{\mathbf{y}}, \mathbf{curl}\mathbf{v}_h)_{\mathbf{L}^2(\Omega)} = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h\}.$$

As $\mathbf{curl}\mathbf{V}_h$ is a closed subspace of $\mathbf{H}_0(\text{div}=0)$, the Hilbert projection theorem implies that

$$(4.9) \quad \mathbf{H}_0(\text{div}=0) = \mathbf{curl}\mathbf{V}_h \oplus (\mathbf{curl}\mathbf{V}_h)^\perp.$$

In other words, for every $\mathbf{y} \in \mathbf{H}_0(\text{div}=0)$, there exists a unique pair $(\mathbf{v}_h, \hat{\mathbf{y}}) \in \mathbf{V}_h \times (\mathbf{curl}\mathbf{V}_h)^\perp$ such that $\mathbf{y} = \mathbf{curl}\mathbf{v}_h + \hat{\mathbf{y}}$. We denote the Hilbert projection operator associated with (4.9) by

$$\mathbf{\Pi}_h : \mathbf{H}_0(\text{div}=0) \rightarrow \mathbf{curl}\mathbf{V}_h.$$

Accordingly, the Hilbert projection operator $\mathbf{\Pi}_h : \mathbf{H}_0(\text{div}=0) \rightarrow \mathbf{curl}\mathbf{V}_h$ satisfies

$$(4.10) \quad (\mu^{-1}(\mathbf{\Pi}_h\mathbf{y} - \mathbf{y}), \mathbf{curl}\mathbf{v}_h)_{\mathbf{L}^2(\Omega)} = 0, \quad \forall \mathbf{y} \in \mathbf{H}_0(\text{div}=0), \quad \forall \mathbf{v}_h \in \mathbf{V}_h.$$

LEMMA 4.3. *Assume that the domain Ω is additionally simply connected. Then, the Hilbert projection operator $\mathbf{\Pi}_h : \mathbf{H}_0(\text{div}=0) \rightarrow \mathbf{curl} \mathbf{V}_h$ satisfies*

$$\lim_{h \rightarrow 0} \|\mathbf{\Pi}_h \mathbf{y} - \mathbf{y}\|_{\mathbf{L}^2_{1/\mu}(\Omega)} = 0, \quad \forall \mathbf{y} \in \mathbf{H}_0(\text{div}=0).$$

Proof. Let $\mathbf{y} \in \mathbf{H}_0(\text{div}=0)$. According to definition and (4.10), it holds that

$$\|\mathbf{y}\|_{\mathbf{L}^2_{1/\mu}(\Omega)}^2 = \|\mathbf{\Pi}_h \mathbf{y}\|_{\mathbf{L}^2_{1/\mu}(\Omega)}^2 + \|\mathbf{y} - \mathbf{\Pi}_h \mathbf{y}\|_{\mathbf{L}^2_{1/\mu}(\Omega)}^2 \quad \forall h > 0,$$

from which it follows that $\|\mathbf{\Pi}_h \mathbf{y}\|_{\mathbf{L}^2_{1/\mu}(\Omega)} \leq \|\mathbf{y}\|_{\mathbf{L}^2_{1/\mu}(\Omega)}$ for all $h > 0$. Therefore, we can find a $\tilde{\mathbf{y}} \in \mathbf{H}_0(\text{div}=0)$ and a null sequence $\{h_n\}_{n=1}^\infty$ of positive real numbers such that

$$(4.11) \quad \mathbf{\Pi}_{h_n} \mathbf{y} \rightharpoonup \tilde{\mathbf{y}} \quad \text{weakly in } \mathbf{H}_0(\text{div}=0) \quad \text{as } n \rightarrow \infty.$$

Since Ω is simply connected, it holds that

$$\mathbf{H}_0(\text{div}=0) = \mathbf{curl} \left(\mathbf{H}_0(\mathbf{curl}) \cap \mathbf{H}(\text{div}=0) \right).$$

See, e.g., [15, Theorem 36(2), p. 48]. Therefore, there exist $\mathbf{v}, \tilde{\mathbf{v}} \in \mathbf{H}_0(\mathbf{curl}) \cap \mathbf{H}(\text{div}=0)$ such that

$$(4.12) \quad \mathbf{curl} \mathbf{v} = \mathbf{y} \quad \text{and} \quad \mathbf{curl} \tilde{\mathbf{v}} = \tilde{\mathbf{y}}.$$

Since the family $\mathbf{V}_h \subset \mathbf{H}_0(\mathbf{curl})$ is dense (see (A1)), we can find a subsequence of the null sequence $\{h_n\}_{n=1}^\infty$, which we denote again by $\{h_n\}_{n=1}^\infty$, so that there exists $\{\mathbf{v}_{h_n}\}_{n=1}^\infty$ satisfying $\mathbf{v}_{h_n} \in \mathbf{V}_{h_n}$, for all $n \in \mathbb{N}$, and

$$(4.13) \quad \mathbf{v}_{h_n} \rightarrow \tilde{\mathbf{v}} - \mathbf{v} \quad \text{strongly in } \mathbf{H}_0(\mathbf{curl}) \quad \text{as } n \rightarrow \infty.$$

Next, by virtue of (4.10), it holds for all $n \in \mathbb{N}$ that $(\mu^{-1}(\mathbf{\Pi}_{h_n} \mathbf{y} - \mathbf{y}), \mathbf{curl} \mathbf{v}_{h_n})_{\mathbf{L}^2(\Omega)} = 0$. Then, passing to the limit $n \rightarrow \infty$, (4.11) and (4.13) yield

$$(\mu^{-1}(\tilde{\mathbf{y}} - \mathbf{y}), \mathbf{curl} \tilde{\mathbf{v}} - \mathbf{curl} \mathbf{v})_{\mathbf{L}^2(\Omega)} = 0 \quad \underset{(4.12)}{\implies} \quad \tilde{\mathbf{y}} = \mathbf{y}.$$

Since the weak limit $\tilde{\mathbf{y}} = \mathbf{y}$ is independent of the sequence $\{h_n\}_{n=1}^\infty$, a classical argument implies that (4.11) is satisfied for the whole sequence, i.e.,

$$(4.14) \quad \mathbf{\Pi}_h \mathbf{y} \rightharpoonup \mathbf{y} \quad \text{weakly in } \mathbf{H}_0(\text{div}=0) \quad \text{as } h \rightarrow 0.$$

On the other hand, we have

$$(4.15) \quad \|\mathbf{\Pi}_h \mathbf{y}\|_{\mathbf{L}^2_{1/\mu}(\Omega)}^2 = (\mu^{-1} \mathbf{\Pi}_h \mathbf{y}, \mathbf{\Pi}_h \mathbf{y})_{\mathbf{L}^2(\Omega)} \underset{(4.10)}{=} (\mu^{-1} \mathbf{y}, \mathbf{\Pi}_h \mathbf{y})_{\mathbf{L}^2(\Omega)} \rightarrow (\mu^{-1} \mathbf{y}, \mathbf{y})_{\mathbf{L}^2(\Omega)} = \|\mathbf{y}\|_{\mathbf{L}^2_{1/\mu}(\Omega)}^2,$$

as $h \rightarrow 0$. Now, the assertion follows from (4.14)-(4.15). \square

ASSUMPTION 4.4. *For every $h > 0$, there exists a linear bounded operator $\Phi_h : \mathbf{H}_0(\mathbf{curl}) \rightarrow \mathbf{V}_h$ satisfying*

$$(4.16) \quad (\mu^{-1} \mathbf{curl} \Phi_h \mathbf{y}, \mathbf{curl} \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} = (\mu^{-1} \mathbf{curl} \mathbf{y}, \mathbf{curl} \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \quad \forall \mathbf{y} \in \mathbf{H}_0(\mathbf{curl}),$$

and

$$(4.17) \quad \lim_{h \rightarrow 0} \|\Phi_h \mathbf{y} - \mathbf{y}\|_{L^2_\epsilon(\Omega)} = 0 \quad \forall \mathbf{y} \in \mathbf{H}_0(\mathbf{curl}).$$

Furthermore, there exists a constant $\hat{c} > 0$, independent of h and \mathbf{y} , such that

$$(4.18) \quad \|\Phi_h \mathbf{y}\|_{L^2_\epsilon(\Omega)} \leq \hat{c} \|\mathbf{y}\|_{\mathbf{H}(\mathbf{curl})} \quad \forall h > 0, \quad \forall \mathbf{y} \in \mathbf{H}_0(\mathbf{curl}).$$

REMARK 4.5. Assumption 4.4 can be realised through the solution operator of a mixed discrete variational problem, which we will discuss in Section 5.

LEMMA 4.6. Under Assumption 4.4, it holds that

$$(4.19) \quad \Pi_h \frac{d}{dt} \mathbf{B}(t) = -\mathbf{curl}(\Phi_h \mathbf{E}(t))$$

for a.e. $t \in (0, T)$ and all $h > 0$.

Proof. Since the solution of (VI) satisfies $\mathbf{B}(t) \in \mathbf{H}_0(\text{div}=0)$ for all $t \in [0, T]$ and $\frac{d}{dt} \mathbf{B}(t) \in \mathbf{H}_0(\text{div}=0)$ for a.e. $t \in (0, T)$, the property (4.10) implies

$$(4.20) \quad (\mu^{-1}(\Pi_h \mathbf{B}(t) - \mathbf{B}(t)), \mathbf{curl} \mathbf{v}_h)_{L^2(\Omega)} = 0 \quad \text{for all } t \in [0, T] \text{ and all } \mathbf{v}_h \in \mathbf{V}_h,$$

$$(4.21) \quad \left(\mu^{-1} \left(\Pi_h \frac{d}{dt} \mathbf{B}(t) - \frac{d}{dt} \mathbf{B}(t) \right), \mathbf{curl} \mathbf{v}_h \right)_{L^2(\Omega)} = 0 \quad \text{for a.e. } t \in (0, T) \text{ and all } \mathbf{v}_h \in \mathbf{V}_h.$$

On the other hand, Lemma 4.1 implies

$$\begin{aligned} 0 &= (\mu^{-1}(\frac{d}{dt} \mathbf{B}(t) + \mathbf{curl} \mathbf{E}(t)), \mathbf{curl} \mathbf{v}_h)_{L^2(\Omega)} \\ &\stackrel{(4.16) \& (4.21)}{=} (\mu^{-1}(\Pi_h \frac{d}{dt} \mathbf{B}(t) + \mathbf{curl}(\Phi_h \mathbf{E}(t))), \mathbf{curl} \mathbf{v}_h)_{L^2(\Omega)} \quad \text{for a.e. } t \in (0, T) \text{ and all } \mathbf{v}_h \in \mathbf{V}_h. \end{aligned}$$

Consequently, as $\Pi_h \frac{d}{dt} \mathbf{B}(t)$ and $\mathbf{curl}(\Phi_h \mathbf{E}(t))$ belong to $\mathbf{curl} \mathbf{V}_h$ for a.e. $t \in (0, T)$, it follows that $\Pi_h \frac{d}{dt} \mathbf{B}(t) = -\mathbf{curl}(\Phi_h \mathbf{E}(t))$ for a.e. $t \in (0, T)$. \square

THEOREM 4.7. Let Assumption 4.4 be satisfied. Then,

$$\begin{aligned} \|\mathbf{E}(\tau) - \mathbf{E}_h(\tau)\|_{L^2_\epsilon(\Omega)}^2 + \|\mathbf{B}(\tau) - \mathbf{B}_h(\tau)\|_{L^2_{1/\mu}(\Omega)}^2 &\leq \|\mathbf{E}_0 - \mathbf{E}_{0h}\|_{L^2_\epsilon(\Omega)}^2 + \|\mathbf{B}_0 - \mathbf{B}_{0h}\|_{L^2_{1/\mu}(\Omega)}^2 \\ &\quad + 2 \int_0^\tau \int_\Omega \epsilon \frac{d}{dt} \mathbf{E}_h(t) \cdot (\Phi_h \mathbf{E}(t) - \mathbf{E}(t)) + \mu^{-1} \mathbf{B}_h(t) \cdot \left(\Pi_h \frac{d}{dt} \mathbf{B}(t) - \frac{d}{dt} \mathbf{B}(t) \right) \\ &\quad + g |\Phi_h \mathbf{E}(t) - \mathbf{E}(t)| + \mathbf{f}(t) \cdot (\mathbf{E}(t) - \Phi_h \mathbf{E}(t)) \, dx \, dt \end{aligned}$$

holds for all $\tau \in [0, T]$ and all $h > 0$.

Proof. Setting $(\mathbf{v}_h, \mathbf{w}_h) = (\Phi_h \mathbf{E}(t), 0)$ in (VI_h) implies

$$\begin{aligned} \int_\Omega \epsilon \frac{d}{dt} \mathbf{E}_h(t) \cdot (\Phi_h \mathbf{E}(t) - \mathbf{E}_h(t)) - \mu^{-1} \frac{d}{dt} \mathbf{B}_h(t) \cdot \mathbf{B}_h(t) \, dx - \int_\Omega \mu^{-1} \mathbf{B}_h(t) \cdot \mathbf{curl} \Phi_h \mathbf{E}(t) \, dx \\ + \varphi(\Phi_h \mathbf{E}(t)) - \varphi(\mathbf{E}_h(t)) \geq \int_\Omega \mathbf{f}(t) \cdot (\Phi_h \mathbf{E}(t) - \mathbf{E}_h(t)) \, dx \quad \text{for a.e. } t \in (0, T). \end{aligned}$$

Similarly, inserting $(\mathbf{v}, \mathbf{w}) = (\mathbf{E}_h(t), \mathbf{B}_h(t))$ in (VI) yields

$$\begin{aligned} & \int_{\Omega} \epsilon \frac{d}{dt} \mathbf{E}(t) \cdot (\mathbf{E}_h(t) - \mathbf{E}(t)) + \mu^{-1} \frac{d}{dt} \mathbf{B}(t) \cdot (\mathbf{B}_h(t) - \mathbf{B}(t)) \, dx \\ & + \int_{\Omega} \mu^{-1} \left(\mathbf{curl} \mathbf{E}(t) \cdot \mathbf{B}_h(t) - \mathbf{B}(t) \cdot \mathbf{curl} \mathbf{E}_h(t) \right) \, dx \\ & + \varphi(\mathbf{E}_h(t)) - \varphi(\mathbf{E}(t)) \geq \int_{\Omega} \mathbf{f}(t) \cdot (\mathbf{E}_h(t) - \mathbf{E}(t)) \, dx \quad \text{for a.e. } t \in (0, T). \end{aligned}$$

Then, using the fact that $\frac{d}{dt} \mathbf{B}_h = -\mathbf{curl} \mathbf{E}_h$ and $\frac{d}{dt} \mathbf{B} = -\mathbf{curl} \mathbf{E}$ (Lemma 4.1) as well as $\mathbf{\Pi}_h \frac{d}{dt} \mathbf{B} = -\mathbf{curl}(\mathbf{\Phi}_h \mathbf{E})$ (Lemma 4.6), we obtain by adding the above two inequalities that

$$\begin{aligned} & \int_{\Omega} \epsilon \frac{d}{dt} \mathbf{E}_h(t) \cdot (\mathbf{\Phi}_h \mathbf{E}(t) - \mathbf{E}_h(t)) - \mu^{-1} \frac{d}{dt} \mathbf{B}_h(t) \cdot \mathbf{B}_h(t) + \mu^{-1} \mathbf{B}_h(t) \cdot \mathbf{\Pi}_h \frac{d}{dt} \mathbf{B}(t) \\ & + \epsilon \frac{d}{dt} \mathbf{E}(t) \cdot (\mathbf{E}_h(t) - \mathbf{E}(t)) + \mu^{-1} \frac{d}{dt} \mathbf{B}(t) \cdot (\mathbf{B}_h(t) - \mathbf{B}(t)) - \mu^{-1} \mathbf{B}_h(t) \cdot \frac{d}{dt} \mathbf{B}(t) \\ & + \mu^{-1} \frac{d}{dt} \mathbf{B}_h(t) \cdot \mathbf{B}(t) + g|\mathbf{\Phi}_h \mathbf{E}(t)| - g|\mathbf{E}(t)| \, dx \geq \int_{\Omega} \mathbf{f}(t) \cdot (\mathbf{\Phi}_h \mathbf{E}(t) - \mathbf{E}(t)) \, dx, \end{aligned}$$

for a.e. $t \in (0, T)$, from which it follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{E}(t) - \mathbf{E}_h(t)\|_{\mathbf{L}_{\epsilon}^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} \|\mathbf{B}(t) - \mathbf{B}_h(t)\|_{\mathbf{L}_{1/\mu}^2(\Omega)}^2 \leq \int_{\Omega} \epsilon \frac{d}{dt} \mathbf{E}_h(t) \cdot (\mathbf{\Phi}_h \mathbf{E}(t) - \mathbf{E}(t)) \\ & + \mu^{-1} \mathbf{B}_h(t) \cdot \left(\mathbf{\Pi}_h \frac{d}{dt} \mathbf{B}(t) - \frac{d}{dt} \mathbf{B}(t) \right) + g|\mathbf{\Phi}_h \mathbf{E}(t) - \mathbf{E}(t)| + \mathbf{f}(t) \cdot (\mathbf{E}(t) - \mathbf{\Phi}_h \mathbf{E}(t)) \, dx. \end{aligned}$$

Integrating this inequality over the time interval $[0, \tau]$ with $\tau \in [0, T]$, we conclude that the assertion is valid. \square

With Lemma 4.2, Lemma 4.3, and Theorem 4.7 at hand, we are able to prove a strong convergence result for the Ritz-Galerkin approximation (\mathbf{VI}_h) .

THEOREM 4.8. *Assume that Ω is additionally simply connected and let Assumption 4.4 be satisfied. Furthermore, suppose that*

$$(4.22) \quad \lim_{h \rightarrow 0} \|\mathbf{E}_{0h} - \mathbf{E}_0\|_{\mathbf{L}_{\epsilon}^2(\Omega)} = \lim_{h \rightarrow 0} \|\mathbf{B}_{0h} - \mathbf{B}_0\|_{\mathbf{L}_{1/\mu}^2(\Omega)} = 0.$$

If $\mathbf{E} \in W^{1,1}((0, T), \mathbf{H}_0(\mathbf{curl}))$ or $\left\{ \frac{d}{dt} \mathbf{E}_h \right\}_{h>0}$ is bounded in $L^{1+\theta}((0, T), \mathbf{L}_{\epsilon}^2(\Omega))$ for $\theta > 0$, then

$$\lim_{h \rightarrow 0} \|\mathbf{E}_h(t) - \mathbf{E}(t)\|_{\mathbf{L}_{\epsilon}^2(\Omega)} = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} \|\mathbf{B}_h(t) - \mathbf{B}(t)\|_{\mathbf{L}_{1/\mu}^2(\Omega)} = 0 \quad \forall t \in [0, T].$$

Proof. Thanks to $\mathbf{B} \in W^{1,\infty}((0, T), \mathbf{H}_0(\text{div}=0))$, applying Lemma 4.3 along with Lebesgue's dominated convergence theorem and Lemma 4.2, we obtain that

$$(4.23) \quad \lim_{h \rightarrow 0} \int_{\Omega} \int_0^{\tau} \mu^{-1} \mathbf{B}_h(t) \cdot \left(\mathbf{\Pi}_h \frac{d}{dt} \mathbf{B}(t) - \frac{d}{dt} \mathbf{B}(t) \right) \, dx \, dt = 0.$$

Similarly, by (4.17)–(4.18) and $\mathbf{E} \in L^\infty((0, T), \mathbf{H}_0(\mathbf{curl}))$, Lebesgue's dominated convergence theorem implies

$$(4.24) \quad \lim_{h \rightarrow 0} \|\Phi_h \mathbf{E} - \mathbf{E}\|_{L^p((0, T), \mathbf{L}_\epsilon^2(\Omega))} = 0, \quad \forall p \in [1, \infty).$$

1. Case: Suppose that $\mathbf{E} \in W^{1,1}((0, T), \mathbf{H}_0(\mathbf{curl})) \hookrightarrow \mathcal{C}([0, T], \mathbf{H}_0(\mathbf{curl}))$. Then, for all $\tau \in [0, T]$ and all $h > 0$, integration by parts implies

$$(4.25) \quad \begin{aligned} \iint_{0\Omega}^\tau \epsilon \frac{d}{dt} \mathbf{E}_h(t) \cdot (\Phi_h \mathbf{E}(t) - \mathbf{E}(t)) \, dx \, dt &= - \iint_{0\Omega}^\tau \epsilon \mathbf{E}_h(t) \cdot \left(\Phi_h \frac{d}{dt} \mathbf{E}(t) - \frac{d}{dt} \mathbf{E}(t) \right) \, dx \, dt \\ &+ \int_{\Omega} \epsilon \mathbf{E}_h(\tau) \cdot (\Phi_h \mathbf{E}(\tau) - \mathbf{E}(\tau)) \, dx - \int_{\Omega} \epsilon \mathbf{E}_{0h} \cdot (\Phi_h \mathbf{E}_0 - \mathbf{E}_0) \, dx. \end{aligned}$$

Therefore, employing again (4.17)–(4.18), Lebesgue's dominated convergence theorem and Lemma 4.2, we obtain that

$$(4.26) \quad \lim_{h \rightarrow 0} \iint_{0\Omega}^\tau \epsilon \frac{d}{dt} \mathbf{E}_h(t) \cdot (\Phi_h \mathbf{E}(t) - \mathbf{E}(t)) \, dx \, dt = 0 \quad \forall \tau \in [0, T].$$

In conclusion, applying (4.22)–(4.26) to Theorem 4.7 yields the desired convergence.

2. Case: Suppose that $\{\frac{d}{dt} \mathbf{E}_h\}_{h>0}$ is bounded in $L^{1+\theta}((0, T), \mathbf{L}_\epsilon^2(\Omega))$ for some $\theta > 0$. In this case, applying (4.22)–(4.24) to Theorem 4.7 immediately leads to the desired convergence. \square

5. Mixed Finite Element Method. Throughout this section, Ω is assumed to be a simply connected Lipschitz polyhedral domain with a connected boundary $\partial\Omega$. We consider a family $\{\mathcal{T}_h\}_{h>0}$ of simplicial triangulations $\mathcal{T}_h = \{T\}$ consisting of tetrahedra T such that

$$\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} T.$$

For each element $T \in \mathcal{T}_h$, h_T denotes the diameter of T , and ρ_T stands for the diameter of the largest ball contained in T . The maximal diameter of all elements is denoted by h , i.e., $h := \max\{h_T \mid T \in \mathcal{T}_h\}$. Finally, we suppose that there exist two positive constants ϱ and ϑ such that

$$\frac{h_T}{\rho_T} \leq \varrho \quad \text{and} \quad \frac{h}{h_T} \leq \vartheta$$

hold for all elements $T \in \mathcal{T}_h$ and all $h > 0$. We choose the space of lowest order edge elements of Nédélec's first family [24] for the finite-dimensional subspace $\mathbf{V}_h \subset \mathbf{H}_0(\mathbf{curl})$, i.e., we set

$$(5.1) \quad \mathbf{V}_h = \{ \mathbf{v}_h \in \mathbf{H}_0(\mathbf{curl}) \mid \mathbf{v}_h|_T = \mathbf{a}_T + \mathbf{b}_T \times x \text{ with } \mathbf{a}_T, \mathbf{b}_T \in \mathbb{R}^3, \forall T \in \mathcal{T}_h \}.$$

On the other hand, there are three possibilities for the choice of the subspace \mathbf{W}_h :

$$(5.2a) \quad \mathbf{W}_h = \{ \mathbf{w}_h \in \mathbf{L}^2(\Omega) \mid \mathbf{w}_h|_T = \mathbf{a}_T \text{ with } \mathbf{a}_T \in \mathbb{R}^3, \forall T \in \mathcal{T}_h \},$$

$$(5.2b) \quad \mathbf{W}_h = \{ \mathbf{w}_h \in \mathbf{H}_0(\text{div}) \mid \mathbf{w}_h|_T = \mathbf{a}_T + \mathbf{b}_T \cdot x \text{ with } \mathbf{a}_T, \mathbf{b}_T \in \mathbb{R}^3, \forall T \in \mathcal{T}_h \},$$

$$(5.2c) \quad \mathbf{W}_h = \mathbf{curl} \mathbf{V}_h.$$

We remark that (5.2a) is the space of piecewise constant elements. The second one (5.2b) is the space of Raviart-Thomas face elements (cf. [15]), whereas (5.2c) is the divergence-free Raviart-Thomas finite element space. Note that $\mathbf{curl} \mathbf{V}_h$ contains all piecewise constant divergence-free elements, which is a closed subspace of the Raviart-Thomas finite element space (see [23, p. 150]). For this reason, the choices for \mathbf{W}_h and \mathbf{V}_h satisfy the assumptions (A1)-(A2) (see p. 7). We note that, in the context of linear time-domain Maxwell's equations, the use of the finite element spaces \mathbf{V}_h and (5.2b) for \mathbf{W}_h was proposed in [22]. Let us now construct $\Phi_h : \mathbf{H}_0(\mathbf{curl}) \rightarrow \mathbf{V}_h$ from Assumption 4.4 (see p. 11). To this aim, we introduce the bilinear forms

$$\begin{aligned} a : \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}_0(\mathbf{curl}) &\rightarrow \mathbb{R}, & a(\mathbf{y}, \mathbf{v}) &:= (\mu^{-1} \mathbf{curl} \mathbf{y}, \mathbf{curl} \mathbf{v})_{\mathbf{L}^2(\Omega)}, \\ b : \mathbf{H}_0(\mathbf{curl}) \times H_0^1(\Omega) &\rightarrow \mathbb{R}, & b(\mathbf{y}, \psi) &:= (\mathbf{y}, \nabla \psi)_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

and let Θ_h denote the space of piecewise linear elements with vanishing traces:

$$\Theta_h := \{ \phi_h \in H_0^1(\Omega) \mid \phi_h|_T = \mathbf{a}_T \cdot \mathbf{x} + b_T \text{ with } \mathbf{a}_T \in \mathbb{R}^3, b_T \in \mathbb{R}, \forall T \in \mathcal{T}_h \}.$$

We define the linear and bounded operator $\Phi_h : \mathbf{H}_0(\mathbf{curl}) \rightarrow \mathbf{V}_h$ as follows: For every $\mathbf{y} \in \mathbf{H}_0(\mathbf{curl})$, let $\Phi_h \mathbf{y} := \mathbf{y}_h$ denote the unique solution of the discrete variational mixed problem

$$(5.3) \quad \begin{cases} a(\mathbf{y}_h, \mathbf{v}_h) = a(\mathbf{y}, \mathbf{v}_h), & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{y}_h, \psi_h) = b(\mathbf{y}, \psi_h), & \forall \psi_h \in \Theta_h. \end{cases}$$

Note that the theory of mixed problems (see [23, Theorem 2.45]; cf. [17]) implies that (5.3) admits a unique solution $\mathbf{y}_h = \Phi_h \mathbf{y} \in \mathbf{V}_h$ satisfying

$$(5.4) \quad \|\Phi_h \mathbf{y} - \mathbf{y}\|_{\mathbf{H}(\mathbf{curl})} \leq c \left(\inf_{\mathbf{x}_h \in \mathbf{V}_h} \|\mathbf{y} - \mathbf{x}_h\|_{\mathbf{H}(\mathbf{curl})} \right), \quad \forall h > 0,$$

with a constant $c > 0$, independent of h and \mathbf{y} . In particular, (5.4) yields

$$\begin{aligned} \lim_{h \rightarrow 0} \|\Phi_h \mathbf{y} - \mathbf{y}\|_{\mathbf{H}(\mathbf{curl})} &= 0, \quad \forall \mathbf{y} \in \mathbf{H}_0(\mathbf{curl}), \\ \|\Phi_h \mathbf{y}\|_{\mathbf{H}(\mathbf{curl})} &\leq (c+1) \|\mathbf{y}\|_{\mathbf{H}(\mathbf{curl})}, \quad \forall h > 0, \quad \forall \mathbf{y} \in \mathbf{H}_0(\mathbf{curl}). \end{aligned}$$

In conclusion, Assumption 4.4 is satisfied for $\Phi_h : \mathbf{H}_0(\mathbf{curl}) \rightarrow \mathbf{V}_h$, and so our convergence result (Theorem 4.8) is applicable for (5.1)-(5.2).

5.1. Error Estimates. We close this paper by proving error estimates for the proposed mixed finite element method. First of all, we recall classical error estimates for the curl-conforming Nédélec interpolant \mathcal{N}_h and the divergence-conforming Raviart-Thomas interpolant π_h (see [23, Section 5]).

LEMMA 5.1 (Theorem 5.25, Theorem 5.41, and Remark 5.42 in [23]). *Let $s \in (1/2, 1]$. There exists a constant $c > 0$, independent of h and \mathbf{y} , such that*

$$(5.5) \quad \|\mathbf{y} - \mathcal{N}_h \mathbf{y}\|_{\mathbf{H}(\mathbf{curl})} \leq ch^s \|\mathbf{y}\|_{\mathbf{H}_0^s(\mathbf{curl})}, \quad \forall \mathbf{y} \in \mathbf{H}_0^s(\mathbf{curl}), \quad \forall h > 0,$$

$$(5.6) \quad \|\mathbf{y} - \pi_h \mathbf{y}\|_{\mathbf{L}^2(\Omega)} \leq ch^s \|\mathbf{y}\|_{\mathbf{H}^s(\Omega)}, \quad \forall \mathbf{y} \in \mathbf{H}^s(\Omega), \quad \forall h > 0,$$

where $\mathbf{H}_0^s(\mathbf{curl}) := \{ \mathbf{y} \in \mathbf{H}^s(\Omega) \cap \mathbf{H}_0(\mathbf{curl}) \mid \mathbf{curl} \mathbf{y} \in \mathbf{H}^s(\Omega) \}$.

The following auxiliary result is also well known, which follows from the above lemma and the embedding result $\mathbf{H}_0(\mathbf{curl}) \cap \mathbf{H}(\text{div}=0) \hookrightarrow \mathbf{H}^\xi(\Omega)$ for some $\xi > \frac{1}{2}$ (see [2, Proposition 3.7]).

LEMMA 5.2. *Let $s \in (1/2, 1]$. There exists a constant $c > 0$, independent of h and \mathbf{y} , such that*

$$\|\mathbf{y} - \mathbf{\Pi}_h \mathbf{y}\|_{\mathbf{L}^2_{1/\mu}(\Omega)} \leq ch^s \|\mathbf{y}\|_{\mathbf{H}^s(\Omega)},$$

for all $\mathbf{y} \in \mathbf{H}_0(\text{div}=0) \cap \mathbf{H}^s(\Omega)$ and all $h > 0$.

Now, we have all the ingredients at hand to prove an a priori error estimate for (5.1)-(5.2).

ASSUMPTION 5.3. *There exists an $s \in (1/2, 1]$ such that $\mathbf{E}_0 \in \mathbf{H}_0^s(\mathbf{curl})$ and the solution (\mathbf{E}, \mathbf{B}) of (VI) satisfies the following additional regularity property:*

$$\mathbf{E} \in W^{1,1}((0, T), \mathbf{H}_0^s(\mathbf{curl})).$$

THEOREM 5.4. *Let \mathbf{V}_h be as in (5.1) and \mathbf{W}_h be given either by (5.2a), (5.2b), or (5.2c). Suppose that Assumption 5.3 is satisfied with $s \in (1/2, 1]$. Furthermore, there is a constant $c > 0$, independent of h , such that*

$$(5.7) \quad \|\mathbf{E}_{0h} - \mathbf{E}_0\|_{\mathbf{L}^2_\epsilon(\Omega)}^2 + \|\mathbf{B}_{0h} - \mathbf{B}_0\|_{\mathbf{L}^2_{1/\mu}(\Omega)}^2 \leq ch^s, \quad \forall h > 0.$$

Then, there exists a constant $c > 0$, independent of h and t , such that

$$\|\mathbf{E}_h(t) - \mathbf{E}(t)\|_{\mathbf{L}^2_\epsilon(\Omega)}^2 + \|\mathbf{B}_h(t) - \mathbf{B}(t)\|_{\mathbf{L}^2_{1/\mu}(\Omega)}^2 \leq ch^s,$$

for all $h > 0$ and all $t \in [0, T]$.

Proof. Let $h > 0$ and $t \in [0, T]$. Applying the integration by parts formula (4.25) as well as (5.7) and Lemma 4.2 to Theorem 4.7, we find a constant $c > 0$, independent of h and t , such that

$$\begin{aligned} & \|\mathbf{E}(t) - \mathbf{E}_h(t)\|_{\mathbf{L}^2_\epsilon(\Omega)}^2 + \|\mathbf{B}(t) - \mathbf{B}_h(t)\|_{\mathbf{L}^2_{1/\mu}(\Omega)}^2 \leq c \left(h^s + \int_0^t \|\Phi_h \frac{d}{dt} \mathbf{E}(\tau) - \frac{d}{dt} \mathbf{E}(\tau)\|_{\mathbf{L}^2_\epsilon(\Omega)} \right. \\ & \left. + \left\| \frac{d}{dt} \mathbf{B}(\tau) - \mathbf{\Pi}_h \frac{d}{dt} \mathbf{B}(\tau) \right\|_{\mathbf{L}^2_{1/\mu}(\Omega)} d\tau + \|\Phi_h \mathbf{E}(t) - \mathbf{E}(t)\|_{\mathbf{L}^2_\epsilon(\Omega)} + \|\Phi_h \mathbf{E}_0 - \mathbf{E}_0\|_{\mathbf{L}^2_\epsilon(\Omega)} \right), \\ & \leq ch^s \left(1 + \int_0^t \left\| \frac{d}{dt} \mathbf{E}(\tau) \right\|_{\mathbf{H}_0^s(\mathbf{curl})} + \|\mathbf{curl} \mathbf{E}(\tau)\|_{\mathbf{H}^s(\Omega)} d\tau + \|\mathbf{E}(t)\|_{\mathbf{H}_0^s(\mathbf{curl})} \right), \end{aligned}$$

where we have used $\frac{d}{dt} \mathbf{B} = \mathbf{curl} \mathbf{E}$ and the error estimate results for Φ_h and $\mathbf{\Pi}_h$ (Lemmas 5.1–5.2 and (5.4)) to deduce the last inequality. In conclusion, the assertion is valid. \square

REMARK 5.5. *A fully discrete approximation of (VI) is obtained for instance by employing the implicit Euler method in (VI_h) . More precisely, introducing the time step $\Delta t = \frac{T}{N}$, with $N \in \mathbb{N}$, we consider an equidistant partition of the interval $[0, T]$ as follows $0 = t_1 < t_2 < \dots < t_N = T$, where $t_n = n\Delta t$ for all $n = 0, \dots, N$. Then, applying the implicit backward Euler method to the semi-discrete Ritz-Galerkin approximation (VI_h) , we arrive at the following fully discrete scheme:*

$$(\text{VI}_{h,\Delta t}) \quad \begin{cases} \int_{\Omega} \epsilon \frac{\mathbf{E}_h^n - \mathbf{E}_h^{n-1}}{\Delta t} \cdot (\mathbf{v}_h - \mathbf{E}_h^n) + \mu^{-1} \frac{\mathbf{B}_h^n - \mathbf{B}_h^{n-1}}{\Delta t} \cdot (\mathbf{w}_h - \mathbf{B}_h^n) dx \\ + \int_{\Omega} \mu^{-1} (\mathbf{curl} \mathbf{E}_h^n \cdot (\mathbf{w}_h - \mathbf{B}_h^n) - \mathbf{B}_h^n \cdot \mathbf{curl} (\mathbf{v}_h - \mathbf{E}_h^n)) dx \\ + \varphi(\mathbf{v}_h) - \varphi(\mathbf{E}_h^n) \geq \int_{\Omega} \mathbf{f}(n\Delta t) \cdot (\mathbf{v}_h - \mathbf{E}_h^n) dx \quad \forall (\mathbf{v}, \mathbf{w}) \in \mathbf{V}_h \times \mathbf{W}_h, \\ n = 1, \dots, N, \quad (\mathbf{E}_h^0, \mathbf{B}_h^0) = (\mathbf{E}_{0h}, \mathbf{B}_{0h}). \end{cases}$$

Our future goal is to examine the fully discrete finite element approximations of (VI) such as $(VI_{h,\Delta t})$. While various implicit and explicit fully discrete schemata are available for parabolic $H^1(\Omega)$ -type variational inequalities (see e.g. [16]), we are only aware of the work by Elliott and Kashima [13] on the fully discrete numerical analysis for a parabolic Maxwell variational inequality of the first kind. Their results and [22] serve as an important basis for our future investigation.

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