SIAM J. CONTROL OPTIM.
Vol. 59, No. 3, pp. 2247–2272

SHAPE OPTIMIZATION FOR SUPERCONDUCTORS GOVERNED BY H(CURL)-ELLIPTIC VARIATIONAL INEQUALITIES*

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Abstract. This paper is devoted to the theoretical and numerical study of an optimal design problem in high-temperature superconductivity (HTS). The shape optimization problem is to find an optimal superconductor shape minimizes a certain cost functional under a given target on the electric field over a specific domain of interest. For the governing PDE-model, we consider an elliptic curl-curl variational inequality (VI) of the second kind with an L1-type nonlinearity. In particular, the nonsmooth VI character and the involved H(curl)-structure make the corresponding shape sensitivity analysis challenging. To tackle the nonsmoothness, a penalized dual VI formulation is proposed, leading to the Gateaux differentiability of the corresponding dual variable mapping. This property allows us to derive the distributed shape derivative of the cost functional through rigorous shape calculus on the basis of the averaged adjoint method. The developed shape derivative turns out to be uniformly stable with respect to the penalization parameter, and strong convergence of the penalized problem is guaranteed. Based on the achieved theoretical findings, we propose three-dimensional numerical solutions, realized using a level set algorithm and a Newton method with the Nédélec edge element discretization. Numerical results indicate a favorable and efficient performance of the proposed approach for a specific HTS application in superconducting shielding.

Key words. shape optimization, high-temperature superconductivity, Maxwell variational inequality, Bean's critical-state model, superconducting shielding, level set method

AMS subject classifications. 35J86, 35Q93, 35Q60

DOI. 10.1137/19M1294150

1. Introduction.

The physical phenomenon of superconductivity is characterized by the zero electrical resistance and the expulsion of magnetic fields (Meissner effect) occurring up to a certain level of the operating temperature and magnetic field strength. Nowadays, numerous key technologies can be realized through high-temperature superconductivity (HTS), including magnetic resonance imaging, magnetic levitation, powerful superconducting wires, particle accelerators, magnetic energy storage, and many more. In particular, to improve and optimize their efficiency and reliability, advanced shape optimization (design) methods are highly desirable.

For instance, efficiently designed superconducting shields are a practical way to protect certain areas from magnetic fields. Basically, there are only two possible ways for a magnetic field to penetrate an area shielded by a superconductor—through the material itself and through opened parts such as holes or gaps. The former depends solely on the properties of the material, the operating temperature, and the magnetic

*Received by the editors October 18, 2019; accepted for publication (in revised form) December 10, 2020; published electronically June 21, 2021.
https://doi.org/10.1137/19M1294150
Funding: The first author acknowledges the support of FAPESP, process 2016/24776-6, and of the Brazilian National Council for Scientific and Technological Development (Conselho Nacional de Desenvolvimento Cientifico e Tecnologico—CNPq) through the process 408175/2018-4, "Otimização de forma não suave e controle de problemas de fronteira livre," and through the program "Bolsa de Produtividade em Pesquisa—PQ 2018," processes 302493/2015-8 and 304258/2018-0. The work of the second and third authors was supported by the German Research Foundation Priority Program DFG SPP 1962, project YO 159/2-2.
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field strength, whereas the latter is also highly affected by the geometry. In the case of an HTS coil, for instance, physical experiments [22] show that the enclosed area is still shielded even if the opened ends are directly facing the field lines. On the other hand, if the diameter gets too large, field lines start penetrating the inside. Thus, the following question arises: how should we design superconducting shields in order to save material and still keep the electromagnetic field penetration to a minimum?

In the recent past, the Bean critical-state model for HTS has been extensively studied by several authors. In the eddy current case, it leads to a parabolic Maxwell variational inequality (VI) of the first kind (see [4, 38]), while in the full Maxwell case it gives rise to a hyperbolic Maxwell VI of the second kind [51, 53] (see also [54] regarding hyperbolic Maxwell obstacle problems). For both parabolic and hyperbolic Maxwell VIs, efficient finite element methods have been proposed and analyzed in [3, 10, 47, 48].

This paper focuses on the sensitivity analysis and numerical investigation for a shape optimization problem in HTS. Our task is to find an admissible superconductor shape which minimizes a tracking-type objective functional under a given target on the electric field over a specific domain of interest. For the governing PDE-model, we consider the elliptic (time-discrete) counterpart to the Bean critical-state model governed by Maxwell's equations [47, 51, 53], given by an elliptic **curl-curl** VI of the second kind. To be more precise, let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $L > 0$ be a fixed constant, and

$$\mathcal{O} := \{\omega \subset B : \ \omega \text{ is open, Lipschitz, with uniform Lipschitz constant } L\}$$

for some measurable subset $B \subset \Omega$. For every admissible superconductor shape $\omega \in \mathcal{O}$, let $\boldsymbol{E} = \boldsymbol{E}(\omega) \in \boldsymbol{H}_0(\mathbf{curl})$ denote the associated electric field given as the solution of

$$(\mathrm{VI}_\omega) \qquad a(\boldsymbol{E}, \boldsymbol{v} - \boldsymbol{E}) + \varphi_\omega(\boldsymbol{v}) - \varphi_\omega(\boldsymbol{E}) \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{E})\, dx \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}),$$

with the elliptic **curl-curl** bilinear form $a\colon \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}_0(\mathbf{curl}) \to \mathbb{R}$ defined by

$$a(\boldsymbol{v}, \boldsymbol{w}) := \int_\Omega \nu\, \mathbf{curl}\, \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{w}\, dx + \int_\Omega \varepsilon \boldsymbol{v} \cdot \boldsymbol{w}\, dx \tag{1.1}$$

and the nonsmooth L^1-type functional

$$\varphi_\omega\colon \boldsymbol{L}^1(\Omega) \to \mathbb{R}, \quad \boldsymbol{v} \mapsto j_c \int_\omega |\boldsymbol{v}(x)|\, dx. \tag{1.2}$$

Here, $j_c > 0$ denotes the critical current density of the superconductor ω, and $\epsilon, \nu\colon \Omega \to \mathbb{R}^{3\times 3}$ are the electric permittivity and the magnetic reluctivity, respectively. The right-hand side $\boldsymbol{f}\colon \Omega \to \mathbb{R}^3$ stands for the applied current source. Altogether, the optimal HTS design problem we focus on reads as follows:

$$(\mathrm{P}) \qquad \min_{\omega \in \mathcal{O}} J(\omega) := \frac{1}{2} \int_B \kappa |\boldsymbol{E}(\omega) - \boldsymbol{E}_d|^2\, dx + \int_\omega dx$$

for some given target $\boldsymbol{E}_d : B \to \mathbb{R}^3$ and weight coefficient $\kappa : B \to (0, \infty)$. The precise mathematical assumptions for all data involved in (P) are specified in Assumption 2.1.

To the best of the authors' knowledge, this paper is the first theoretical and numerical study of the shape optimization subject to $\boldsymbol{H}(\mathbf{curl})$-elliptic VI of the second kind. Both the involved $\boldsymbol{H}(\mathbf{curl})$-structure and the nonsmooth VI character make the corresponding analysis truly challenging. We refer to [46, 49, 50] for the optimal control of static Maxwell equations. Quite recently, the optimal control of hyperbolic Maxwell variational inequalities arising in HTS was investigated in [52]. While (P) admits an optimal solution (Theorem 2.4), the differentiability of the dual variable mapping associated with (VI_ω) cannot be guaranteed. This property is, however, indispensable for our shape sensitivity analysis. Therefore, we propose to approximate (P) by replacing (VI_ω) through its penalized dual formulation (3.1), for which the corresponding dual variable mapping is Gateaux-differentiable (Lemma 3.1). This allows us to prove our main theoretical result (Theorem 4.6) on the distributed shape derivative of the cost functional through rigorous shape calculus on the basis of the averaged adjoint method. Importantly, the established shape derivative is uniformly stable with respect to the penalization parameter (Theorem 5.1), and strong convergence of the penalized approach can be guaranteed (Theorem 5.3). In addition, the Newton method is applicable to the penalized dual formulation (3.1). Thus, efficient numerical optimal shapes can be realized by means of a level set algorithm along with the developed shape derivative and a symmetrization strategy. All these theoretical and numerical evidences indicate the favorable performance of our approach to deal with shape optimization problems subject to a VI of the second kind.

Theoretical results on optimal design problems were obtained in [2, 8, 9, 11, 14, 28, 34, 43], but there are few early references for VI-constrained numerical shape optimization (see [13, 21, 32, 42]). Recent publications include [16] regarding a solution algorithm in the infinite dimensional setting for shape optimization problems governed by VIs of the first kind and [12, 30] concerning a shape optimization method based on a regularized variant of VI of the first kind.

The concept of shape derivative [7, 15, 43] is the basis for the sensitivity analysis of shape functionals. We use the *averaged adjoint method* introduced in [44], a Lagrangian-type method for the efficient computation of shape derivatives. Lagrangian methods are commonly used in shape optimization and have the advantage of providing the shape derivative without the need to compute the material derivative of the state (see [1, 5, 7, 17, 18, 20, 37]). Compared to these approaches, the averaged adjoint method is fairly general due to minimal required conditions.

2. Preliminaries.

For a given Banach space V, we denote its norm by $\|\cdot\|_V$. If V is a Hilbert space, then $(\cdot,\cdot)_V$ stands for its scalar product and $\|\cdot\|_V$ for the induced norm. In the case of $V=\mathbb{R}^n$, we renounce the subscript in the (Euclidean) norm and write $|\cdot|$. The Euclidean scalar product is denoted by a dot, and $\otimes$ is the standard outer product for vectors in $\mathbb{R}^3$. Hereinafter, a bold typeset indicates vector-valued functions and their respective spaces. The Banach space $\mathcal{C}^1(\Omega,\mathbb{R}^{3\times 3})$ is equipped with the standard norm, and for $\boldsymbol{\mathcal{C}}^{0,1}(\Omega) := \mathcal{C}^{0,1}(\Omega,\mathbb{R}^3)$ we use

$$\|\boldsymbol{\theta}\|_{\boldsymbol{\mathcal{C}}^{0,1}(\Omega)} = \sup_{x\in\Omega}|\boldsymbol{\theta}(x)| + \sup_{x\neq y\in\Omega}\frac{|\boldsymbol{\theta}(x)-\boldsymbol{\theta}(y)|}{|x-y|}.$$

Now, we introduce the central Hilbert space used throughout this paper:

$$\boldsymbol{H}(\mathbf{curl}) := \{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega)\ :\ \mathbf{curl}\,\boldsymbol{v}\in\boldsymbol{L}^2(\Omega)\},$$

where **curl** is understood in the distributional sense. It is equipped with the corresponding graph norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})} := \left(\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}\right)^{\frac{1}{2}} \quad \forall \boldsymbol{v}\in\boldsymbol{H}(\mathbf{curl}).$$

As usual, $\boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ denotes the space of all infinitely differentiable functions with compact support in Ω. The space $\boldsymbol{H}_0(\mathbf{curl})$ stands for the closure of $\boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ with respect to the $\boldsymbol{H}(\mathbf{curl})$-norm.

Next, we present all the necessary assumptions for the material parameters and the given data in (P) and (VI_ω).

Assumption 2.1 (material parameters and given data).

(A1) The subset $B \subset \Omega$ is a Lipschitz domain, $\boldsymbol{E}_d \in \boldsymbol{\mathcal{C}}^1(B)$, and $\kappa \in \mathcal{C}^1(B)$.

(A2) We assume $j_c \in \mathbb{R}^+$, and the material parameters $\epsilon, \nu \colon \Omega \to \mathbb{R}^{3\times 3}$ are assumed to be $L^\infty(\Omega, \mathbb{R}^{3\times3}) \cap \mathcal{C}^1(B, \mathbb{R}^{3\times3})$, symmetric, and uniformly positive definite, i.e., there exist $\underline{\nu}, \underline{\epsilon} > 0$ such that

$$\xi^\mathsf{T} \nu(x) \xi \geq \underline{\nu}|\xi|^2 \text{ and } \xi^\mathsf{T} \epsilon(x) \xi \geq \underline{\epsilon}|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^3. \tag{2.1}$$

(A3) The right-hand side satisfies $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega) \cap \boldsymbol{\mathcal{C}}^1(B)$.

Remark 2.2.

(i) As pointed out earlier, in the context of superconducting shields, one looks for an optimal superconductor shape ω that minimizes both the electromagnetic field penetration and the volume of material. This can be realized by solving (P) with $\boldsymbol{E}_d = 0$, which obviously satisfies (A1).

(ii) The material assumption (A2) holds true, for instance, in the case of homogeneous HTS material. In this case, ϵ, μ are constant in B.

(iii) A choice for the $\boldsymbol{f}$ satisfying (A3) is given by an induction coil away from the superconducting region B. In this case, $\boldsymbol{f} \equiv 0$ in B.

For every fixed $\omega \subset \mathcal{O}$ the existence of a unique solution $\boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl})$ of (VI_ω) is covered by the fundamental well-posedness result by Lions and Stampacchia. According to [27, Theorem 2.2] we have to verify that the bilinear form a is coercive and continuous. Indeed, in view of (1.1), (A2) yields

$$\begin{aligned} a(\boldsymbol{v}, \boldsymbol{v}) &\geq \min\{\underline{\epsilon}, \underline{\nu}\} \|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathbf{curl})}, \\ |a(\boldsymbol{v}, \boldsymbol{w})| &\leq (\|\nu\|_{L^\infty(\Omega,\mathbb{R}^{3\times3})} + \|\epsilon\|_{L^\infty(\Omega,\mathbb{R}^{3\times3})}) \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})} \|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl})} \end{aligned} \tag{2.2}$$

for all $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl})$. Moreover, according to (1.2), the nonlinearity φ_ω is convex, proper, and lower semicontinuous. Therefore, concluding from [27, Theorem 2.2], (VI_ω) admits for every $\omega \subset \mathcal{O}$ a unique solution $\boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl})$. Additionally, it is well-known (cf. [45]) that there exists a unique $\boldsymbol{\lambda} \in \boldsymbol{L}^\infty(\omega)$ such that

$$\begin{cases} a(\boldsymbol{E}, \boldsymbol{v}) + \displaystyle\int_\omega \boldsymbol{\lambda} \cdot \boldsymbol{v}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}), \\ |\boldsymbol{\lambda}(x)| \leq j_c, \quad \boldsymbol{\lambda}(x) \cdot \boldsymbol{E}(x) = j_c |\boldsymbol{E}(x)| \text{ for a.e. } x \in \omega. \end{cases} \tag{2.3}$$

On the other hand, if there exists a pair $(\boldsymbol{E}, \boldsymbol{\lambda}) \in \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^\infty(\Omega)$ satisfying (2.3), we readily obtain that $\boldsymbol{E}$ is the unique solution of (VI_ω) since it holds that

$$\begin{aligned} \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{E})\, dx &= a(\boldsymbol{E}, \boldsymbol{v} - \boldsymbol{E}) + \int_\omega \boldsymbol{\lambda} \cdot \boldsymbol{v}\, dx - \int_\omega \boldsymbol{\lambda} \cdot \boldsymbol{E}\, dx \\ &\overbrace{\leq}^{(2.3)} a(\boldsymbol{E}, \boldsymbol{v} - \boldsymbol{E}) + \int_\omega j_c |\boldsymbol{v}|\, dx - \int_\omega j_c |\boldsymbol{E}|\, dx \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}). \end{aligned}$$

As a conclusion, the dual formulation (2.3) admits for every $\omega \subset \mathcal{O}$ a unique solution $(\boldsymbol{E}, \boldsymbol{\lambda}) \in \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^\infty(\Omega)$, and its primal solution $\boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl})$ coincides with the solution to (VI_ω). Throughout this paper the following compactness result for the set of domains $\mathcal{O}$ is pivotal to our analysis [15, Theorem 2.4.10].

THEOREM 2.3. *Let Assumption* 2.1 *hold and* $\{\omega_n\}_{n\in\mathbb{N}} \subset \mathcal{O}$. *Then, there exist* $\omega \in \mathcal{O}$ *and a subsequence* $\{\omega_{n_k}\}_{k\in\mathbb{N}}$ *which converges to* ω *in the sense of Hausdorff, and in the sense of characteristic functions. Moreover,* $\overline{\omega}_{n_k}$ *and* $\partial\omega_{n_k}$ *converge in the sense of Hausdorff toward* $\overline{\omega}$ *and* $\partial\omega$*, respectively.*

With Theorem 2.3 at hand, it is possible to prove the existence of an optimal shape for (P) directly. However, as the same result is obtained as a byproduct of Theorem 5.3, we do not give a proof at this point.

THEOREM 2.4. *Under Assumption* 2.1 *the shape optimization problem* (P) *has an optimal solution* $\omega_\star \in \mathcal{O}$.

3. Penalized shape optimization approach. As pointed out earlier, our shape sensitivity analysis requires the differentiability of the dual variable mapping $\boldsymbol{E} \mapsto \boldsymbol{\lambda}$ in $\boldsymbol{L}^2(\Omega)$, which cannot be guaranteed in general. To cope with this regularity issue, we approximate (P) by

$$(\mathrm{P}_\gamma) \qquad \min_{\omega\in\mathcal{O}} J_\gamma(\omega) := \frac{1}{2}\int_B \kappa|\boldsymbol{E}^\gamma(\omega) - \boldsymbol{E}_d|^2 + \int_\omega dx,$$

where $\boldsymbol{E}^\gamma := \boldsymbol{E}^\gamma(\omega) \in \boldsymbol{H}_0(\mathbf{curl})$ is specified by the penalized dual formulation of (2.3):

$$(3.1) \qquad \begin{cases} a(\boldsymbol{E}^\gamma, \boldsymbol{v}) + \displaystyle\int_\omega \boldsymbol{\lambda}^\gamma \cdot \boldsymbol{v}\, dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\, dx \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \\ \boldsymbol{\lambda}^\gamma(x) = \dfrac{j_c\gamma\boldsymbol{E}^\gamma(x)}{\max_\gamma\{1, \gamma|\boldsymbol{E}^\gamma(x)|\}} \text{ for a.e. } x \in \omega. \end{cases}$$

In this context, $\max_\gamma \colon \mathbb{R}^3 \to \mathbb{R}$ denotes the Moreau–Yosida-type regularization (cf. [6]) of the max-function given by

$$(3.2) \qquad \max_\gamma\{1, x\} := \begin{cases} x & \text{if } x - 1 \geq \frac{1}{2\gamma}, \\ 1 + \frac{\gamma}{2}\left(x - 1 + \frac{1}{2\gamma}\right)^2 & \text{if } |x-1| \leq \frac{1}{2\gamma}, \\ 1 & \text{if } x - 1 \leq -\frac{1}{2\gamma}. \end{cases}$$

The following lemma summarizes the Gateaux-differentiability result for the dual variable mapping associated with (3.1). For the convenience of the reader, we provide a brief sketch of the proof following [6].

LEMMA 3.1. *Let* $\gamma > 0$ *and Assumption* 2.1 *hold. Then,*

$$(3.3) \qquad \boldsymbol{\Lambda}_\gamma \colon \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega), \quad \boldsymbol{\Lambda}_\gamma(\boldsymbol{e}) := \frac{j_c\gamma\boldsymbol{e}}{\max_\gamma\{1, \gamma|\boldsymbol{e}|\}}$$

is Gateaux-differentiable with the Gateaux-derivative

$$(3.4) \qquad \begin{aligned} \boldsymbol{\Lambda}'_\gamma(\boldsymbol{e})\boldsymbol{w} &= \frac{j_c\gamma\boldsymbol{w}}{\max_\gamma\{1, \gamma|\boldsymbol{e}|\}} \\ &\quad - \gamma\left(\mathbb{1}_{\mathcal{A}_\gamma(\boldsymbol{e})} + \gamma\left(\gamma|\boldsymbol{e}| - 1 + \frac{1}{2\gamma}\right)\mathbb{1}_{\mathcal{S}_\gamma(\boldsymbol{e})}\right)\frac{(\boldsymbol{e}\cdot\boldsymbol{w})\boldsymbol{\Lambda}_\gamma(\boldsymbol{e})}{\max_\gamma\{1, \gamma|\boldsymbol{e}|\}|\boldsymbol{e}|} \quad \forall \boldsymbol{e}, \boldsymbol{w} \in \boldsymbol{L}^2(\Omega), \end{aligned}$$

where $\mathbb{1}_{\mathcal{A}_\gamma(\boldsymbol{e})}$ *and* $\mathbb{1}_{\mathcal{S}_\gamma(\boldsymbol{e})}$ *stand for the characteristic functions of the disjoint sets* $\mathcal{A}_\gamma(\boldsymbol{e}) = \{x \in \Omega : \gamma|\boldsymbol{e}(x)| \geq 1 + 1/2\gamma\}$ *and* $\mathcal{S}_\gamma(\boldsymbol{e}) = \{x \in \Omega : |\gamma|\boldsymbol{e}(x)| - 1| < 1/2\gamma\}$, *respectively. Furthermore,* $\boldsymbol{\Lambda}_\gamma$ *is Lipschitz-continuous and monotone, i.e.,*

$$(\boldsymbol{\Lambda}_\gamma(\boldsymbol{w}_1) - \boldsymbol{\Lambda}_\gamma(\boldsymbol{w}_2), \boldsymbol{w}_1 - \boldsymbol{w}_2)_{\boldsymbol{L}^2(\Omega)} \geq 0 \quad \forall \boldsymbol{w}_1, \boldsymbol{w}_2 \in \boldsymbol{L}^2(\Omega). \tag{3.5}$$

Proof. At first, let us note that the function $\boldsymbol{\xi}_\gamma : \mathbb{R}^3 \to \mathbb{R}^3$, defined by

$$\boldsymbol{\xi}_\gamma(x) := \frac{j_c \gamma x}{\max_\gamma\{1, \gamma|x|\}},$$

is continuously differentiable by the construction of $\max_\gamma : \mathbb{R}^3 \to \mathbb{R}$ (see (3.2)). Moreover, by using the mean value theorem as in [6, Lemma 4.1], $\boldsymbol{\xi}_\gamma$ is also globally Lipschitz continuous and monotone in the following sense:

$$\big(\boldsymbol{\xi}_\gamma(x) - \boldsymbol{\xi}_\gamma(y)\big) \cdot \big(x - y\big) \geq 0 \quad \forall x, y \in \mathbb{R}^3.$$

This readily implies that the same properties hold for $\boldsymbol{\Lambda}_\gamma$. Now, applying the differentiability of $\boldsymbol{\xi}_\gamma : \mathbb{R}^3 \to \mathbb{R}^3$ along with Lebesgue's dominated convergence theorem leads to the directional differentiability of $\boldsymbol{\Lambda}_\gamma : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ with the directional derivative given by (3.4). In view of (3.4), for every $\boldsymbol{e} \in \boldsymbol{L}^2(\Omega)$, the mapping $\boldsymbol{w} \mapsto \boldsymbol{\Lambda}'_\gamma(\boldsymbol{e})\boldsymbol{w}$ is linear and bounded in $\boldsymbol{L}^2(\Omega)$, and so the Gateaux-differentiability follows. ∎

In addition to Lemma 3.1, it is easy to see that the following estimate holds by definition of $\mathcal{S}_\gamma(\boldsymbol{e})$ for every $\boldsymbol{e} \in \boldsymbol{L}^2(\Omega)$:

$$\gamma\left(\gamma|\boldsymbol{e}| - 1 + \frac{1}{2\gamma}\right) \leq 1 \qquad \text{a.e. in } \mathcal{S}_\gamma(\boldsymbol{e}). \tag{3.6}$$

For convenience we define the matrix-valued function $\boldsymbol{\psi}^\gamma : \boldsymbol{L}^2(\Omega) \to L^2(\Omega, \mathbb{R}^{3\times3})$ by

$$\boldsymbol{\psi}^\gamma(\boldsymbol{e}) := \frac{j_c \gamma \boldsymbol{I}_3}{\max_\gamma\{1, \gamma|\boldsymbol{e}|\}} - \gamma\left(\mathbb{1}_{\mathcal{A}_\gamma(\boldsymbol{e})} + \gamma\left(\gamma|\boldsymbol{e}| - 1 + \frac{1}{2\gamma}\right)\mathbb{1}_{\mathcal{S}_\gamma(\boldsymbol{e})}\right)\frac{\boldsymbol{e} \otimes \boldsymbol{\Lambda}_\gamma(\boldsymbol{e})}{\max_\gamma\{1, \gamma|\boldsymbol{e}|\}|\boldsymbol{e}|}, \tag{3.7}$$

where $\boldsymbol{I}_3$ denotes the identity matrix in $\mathbb{R}^{3\times3}$. By multiplying (3.4) with $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ and using $(\boldsymbol{e} \cdot \boldsymbol{w})(\boldsymbol{\Lambda}_\gamma(\boldsymbol{e}) \cdot \boldsymbol{v}) = \big[\big(\boldsymbol{e} \otimes \boldsymbol{\Lambda}_\gamma(\boldsymbol{e})\big)\boldsymbol{v}\big] \cdot \boldsymbol{w}$, for all $\boldsymbol{e}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$, we obtain

$$\boldsymbol{\Lambda}'_\gamma(\boldsymbol{e})\boldsymbol{w} \cdot \boldsymbol{v} = \boldsymbol{\psi}^\gamma(\boldsymbol{e})\boldsymbol{v} \cdot \boldsymbol{w} \quad \forall \boldsymbol{e}, \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega). \tag{3.8}$$

Thanks to the Lipschitz continuity and monotonicity of $\boldsymbol{\Lambda}_\gamma$ (Lemma 3.1) along with (2.2), the operator $\boldsymbol{A} : \boldsymbol{H}_0(\mathbf{curl}) \to \boldsymbol{H}_0(\mathbf{curl})^*$ defined by

$$\langle \boldsymbol{A}(\boldsymbol{v}), \boldsymbol{w} \rangle := a(\boldsymbol{v}, \boldsymbol{w}) + (\boldsymbol{\Lambda}_\gamma(\boldsymbol{v}), \boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl})$$

is strictly monotone, coercive, and (radially) continuous. Thus, the well-posedness of (3.1) follows by the theorem of Minty–Browder [41, Theorem 2.18]. Moreover, (3.2) implies for every $\boldsymbol{e} \in \boldsymbol{L}^2(\Omega)$ that

$$\max_\gamma\{1, \gamma|\boldsymbol{e}|\} \geq \gamma|\boldsymbol{e}| \text{ a.e. in } \Omega. \tag{3.9}$$

Applying this estimate to (3.3) yields that

$$\|\boldsymbol{\Lambda}_\gamma(\boldsymbol{e})\|_{\boldsymbol{L}^\infty(\Omega)} \leq j_c \quad \forall \boldsymbol{e} \in \boldsymbol{L}^2(\Omega). \tag{3.10}$$

Obviously, (3.2) yields for every $\boldsymbol{e} \in \boldsymbol{L}^2(\Omega)$ that $\max_\gamma\{1, \gamma|\boldsymbol{e}|\} \geq 1$ almost everywhere in Ω. Hence, we obtain the following estimate for all $\boldsymbol{e}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$:

$$\int_\Omega |\boldsymbol{\psi}^\gamma(\boldsymbol{e})\boldsymbol{v}\cdot\boldsymbol{w}|\,dx \overbrace{\leq}^{(3.6)} \int_\Omega \frac{j_c\gamma|\boldsymbol{v}\cdot\boldsymbol{w}|}{\max_\gamma\{1,\gamma|\boldsymbol{e}|\}}\,dx + \gamma\int_\Omega \frac{\left|(\boldsymbol{e}\otimes\boldsymbol{\Lambda}_\gamma(\boldsymbol{e}))\boldsymbol{v}\cdot\boldsymbol{w}\right|}{\max_\gamma\{1,\gamma|\boldsymbol{e}|\}|\boldsymbol{e}|}\,dx \overbrace{\leq}^{(3.10)} 2j_c\gamma\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}. \tag{3.11}$$

The next result states the existence of an optimal solution to (P_γ).

THEOREM 3.2. *Let Assumption* 2.1 *hold and* $\gamma > 0$ *be fixed. Then,* (P_γ) *admits an optimal shape* $\omega^\gamma_\star \in \mathcal{O}$.

Proof. Let $\{\omega^\gamma_n\}_{n\in\mathbb{N}} \subset \mathcal{O}$ be a minimizing sequence for (P_γ) with the corresponding states $\boldsymbol{E}^\gamma_n \in \boldsymbol{H}_0(\mathbf{curl})$ solving (3.1) for $\omega = \omega^\gamma_n$ and $\boldsymbol{\lambda}^\gamma_n := \boldsymbol{\Lambda}(\boldsymbol{E}^\gamma_n)$. Thanks to Theorem 2.3, there exists a subsequence of $\{\omega^\gamma_n\}_{n\in\mathbb{N}}$ (with a slight abuse of notation we use the same index for the subsequence) and $\omega^\gamma_\star \subset \mathcal{O}$ such that $\omega^\gamma_n \to \omega^\gamma_\star$ as $n \to \infty$ in the sense of characteristic functions.

We denote the solution to (3.1) for $\omega = \omega^\gamma_\star$ by $\boldsymbol{E}^\gamma_\star \in \boldsymbol{H}_0(\mathbf{curl})$ and $\boldsymbol{\lambda}^\gamma_\star := \boldsymbol{\Lambda}_\gamma(\boldsymbol{E}^\gamma_\star)$. Now, substracting (3.1) for $\boldsymbol{E}^\gamma_n$ from (3.1) for $\boldsymbol{E}^\gamma_\star$ and testing the resulting equation with $\boldsymbol{v} = \boldsymbol{E}^\gamma_\star - \boldsymbol{E}^\gamma_n$ yields

$$\begin{aligned} a(\boldsymbol{E}^\gamma_\star - \boldsymbol{E}^\gamma_n, \boldsymbol{E}^\gamma_\star - \boldsymbol{E}^\gamma_n) &= \int_\Omega (\chi_{\omega^\gamma_\star}\boldsymbol{\lambda}^\gamma_\star - \chi_{\omega^\gamma_n}\boldsymbol{\lambda}^\gamma_n)\cdot(\boldsymbol{E}^\gamma_n - \boldsymbol{E}^\gamma_\star)\,dx \\ &= \int_\Omega (\chi_{\omega^\gamma_\star} - \chi_{\omega^\gamma_n})\boldsymbol{\lambda}^\gamma_n\cdot(\boldsymbol{E}^\gamma_n - \boldsymbol{E}^\gamma_\star)\,dx - \underbrace{\int_\Omega \chi_{\omega^\gamma_\star}(\boldsymbol{\lambda}^\gamma_\star - \boldsymbol{\lambda}^\gamma_n)\cdot(\boldsymbol{E}^\gamma_\star - \boldsymbol{E}^\gamma_n)\,dx}_{=(\boldsymbol{\Lambda}_\gamma(\chi_{\omega^\gamma_\star}\boldsymbol{E}^\gamma_n) - \boldsymbol{\Lambda}_\gamma(\chi_{\omega^\gamma_\star}\boldsymbol{E}^\gamma_\star), \chi_{\omega^\gamma_\star}\boldsymbol{E}^\gamma_n - \chi_{\omega^\gamma_\star}\boldsymbol{E}^\gamma_\star)_{\boldsymbol{L}^2(\Omega)}} \\ &\underbrace{\leq}_{(3.5)} \int_\Omega (\chi_{\omega^\gamma_\star} - \chi_{\omega^\gamma_n})\boldsymbol{\lambda}^\gamma_n\cdot(\boldsymbol{E}^\gamma_n - \boldsymbol{E}^\gamma_\star)\,dx. \end{aligned} \tag{3.12}$$

Thus, (3.12) and (A2) of Assumption 2.1 yield

$$\begin{aligned} &\min\{\underline{\nu},\underline{\epsilon}\}\|\boldsymbol{E}^\gamma_\star - \boldsymbol{E}^\gamma_n\|^2_{\boldsymbol{H}(\mathbf{curl})} \leq \|\chi_{\omega^\gamma_\star} - \chi_{\omega^\gamma_n}\|_{L^2(\Omega)}\|\boldsymbol{\lambda}^\gamma_n\|_{\boldsymbol{L}^\infty(\Omega)}\|\boldsymbol{E}^\gamma_\star - \boldsymbol{E}^\gamma_n\|_{\boldsymbol{H}(\mathbf{curl})} \\ &\overbrace{\Rightarrow}^{(3.9)} \quad \|\boldsymbol{E}^\gamma_\star - \boldsymbol{E}^\gamma_n\|_{\boldsymbol{H}(\mathbf{curl})} \leq \frac{j_c}{\min\{\underline{\nu},\underline{\epsilon}\}}\|\chi_{\omega^\gamma_\star} - \chi_{\omega^\gamma_n}\|_{L^2(\Omega)}. \end{aligned} \tag{3.13}$$

This implies $\boldsymbol{E}^\gamma_n \to \boldsymbol{E}^\gamma_\star$ in $\boldsymbol{H}_0(\mathbf{curl})$ since ω^γ_n converges to $\omega^\gamma_\star$ in the sense of characteristic functions as $n \to \infty$. Hence, we obtain

$$J_\gamma(\omega^\gamma_n) = \frac{1}{2}\int_B \kappa|\boldsymbol{E}^\gamma_n - \boldsymbol{E}_d|^2\,dx + \int_{\omega^\gamma_n} dx \to \frac{1}{2}\int_B \kappa|\boldsymbol{E}^\gamma_\star - \boldsymbol{E}_d|^2\,dx + \int_{\omega^\gamma_\star} dx = J_\gamma(\omega^\gamma_\star).$$

Finally, the assertion follows since ω^γ_n is a minimizing sequence for (P_γ). ∎

4. Shape sensitivity analysis. This section is devoted to the sensitivity analysis of the shape functional $J_\gamma(\omega)$ in (P_γ) for $\gamma > 0$ fixed. We compute the shape derivative using the averaged adjoint method (see [25, 44]). Let $\boldsymbol{T}_t : \Omega \to \Omega$ be the flow of a vector field $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^{0,1}_c(\Omega, \mathbb{R}^3)$ with compact support in B, i.e., $\boldsymbol{T}_t(\boldsymbol{\theta})(X) = x(t, X)$ is the solution to the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t,X) = \boldsymbol{\theta}(x(t,X)) \quad \text{for } t \in [0,\tau], \quad x(0,X) = X \in \Omega, \tag{4.1}$$

for some given $\tau > 0$. It is well-known (see [43, p. 50]) that (4.1) admits a unique solution for a sufficiently small $\tau > 0$. In order to keep the notation short, we write $\boldsymbol{T}_t := \boldsymbol{T}_t(\boldsymbol{\theta})$. Note that $\boldsymbol{T}_t(B) = B$ and $\boldsymbol{T}_t(X) = X$ for every $X \in \Omega \backslash B$ since $\boldsymbol{\theta}$ has compact support in B. For $\omega \in \mathcal{O}$, we introduce the parameterized family of domains $\omega_t := \boldsymbol{T}_t(\omega)$ for all $t \in [0, \tau]$. Let us now recall the definition of shape derivative used in this paper.

Definition 4.1 (shape derivative). *Let $K : \mathcal{O} \to \mathbb{R}$ be a shape functional. The Eulerian semiderivative of K at $\omega \in \mathcal{O}$ in direction $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}_c^{0,1}(\Omega, \mathbb{R}^3)$ is defined as the limit, if it exists,*

$$dK(\omega)(\boldsymbol{\theta}) := \lim_{t \searrow 0} \frac{K(\omega_t) - K(\omega)}{t},$$

where $\omega_t = \boldsymbol{T}_t(\omega)$. Moreover, K is said to be shape differentiable at ω if it has a Eulerian semiderivative at ω for all $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}_c^{0,1}(\Omega, \mathbb{R}^3)$ and the mapping

$$dK(\omega) : \boldsymbol{\mathcal{C}}_c^{0,1}(\Omega, \mathbb{R}^3) \to \mathbb{R}, \quad \boldsymbol{\theta} \mapsto dK(\omega)(\boldsymbol{\theta})$$

is linear and continuous. In this case $dK(\omega)(\boldsymbol{\theta})$ is called the shape derivative at ω.

In the remainder of this section, we consider the perturbed domain ω_t and denote the corresponding solution of (3.1) for $\omega = \omega_t$ by $\boldsymbol{E}_t^\gamma \in \boldsymbol{H}_0(\mathbf{curl})$.

4.1. Averaged adjoint method. We begin by introducing the Lagrangian $\mathcal{L} : \mathcal{O} \times \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}_0(\mathbf{curl}) \to \mathbb{R}$ associated with (P_γ) as follows:

$$\mathcal{L}(\omega, \boldsymbol{e}, \boldsymbol{v}) := \frac{1}{2} \int_B \kappa |\boldsymbol{e} - \boldsymbol{E}_d|^2 \, dx + \int_\omega dx + a(\boldsymbol{e}, \boldsymbol{v}) + \int_\omega \boldsymbol{\Lambda}_\gamma(\boldsymbol{e}) \cdot \boldsymbol{v} \, dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx, \tag{4.2}$$

where $\boldsymbol{\Lambda}_\gamma$ is given as in (3.3). In view of (4.2), we have for $\omega \in \mathcal{O}$ and $t \in [0, \tau]$ that

$$J_\gamma(\omega_t) = \mathcal{L}(\omega_t, \boldsymbol{E}_t^\gamma, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}). \tag{4.3}$$

Moreover, as $\mathcal{L}$ is linear in $\boldsymbol{v}$, the problem of finding $\boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl})$ such that

$$\partial_{\boldsymbol{v}} \mathcal{L}(\omega_t, \boldsymbol{e}, \boldsymbol{v}; \hat{\boldsymbol{v}}) = a(\boldsymbol{e}, \hat{\boldsymbol{v}}) + \int_{\omega_t} \boldsymbol{\Lambda}_\gamma(\boldsymbol{e}) \cdot \hat{\boldsymbol{v}} \, dx - \int_\Omega \boldsymbol{f} \cdot \hat{\boldsymbol{v}} \, dx = 0 \quad \forall \hat{\boldsymbol{v}} \in \boldsymbol{H}_0(\mathbf{curl})$$

is equivalent to (3.1) with $\omega = \omega_t$ and admits the same unique solution $\boldsymbol{E}_t^\gamma \in \boldsymbol{H}_0(\mathbf{curl})$. In order to pull back the integrals over ω_t to the reference domain ω, one uses the change of variables $x \mapsto \boldsymbol{T}_t(x)$. Furthermore, to avoid the appearance of the composed functions $\boldsymbol{e} \circ \boldsymbol{T}_t$ and $\boldsymbol{v} \circ \boldsymbol{T}_t$ due to this change of variables, we reparameterize the Lagrangian using the following covariant transformation, which is known to be a bijection for $\boldsymbol{H}_0(\mathbf{curl})$ (cf. [31, p. 77]):

$$\Psi_t \colon \boldsymbol{H}_0(\mathbf{curl}) \to \boldsymbol{H}_0(\mathbf{curl}), \qquad \Psi_t(\boldsymbol{e}) := (D\boldsymbol{T}_t^{-\mathsf{T}} \boldsymbol{e}) \circ \boldsymbol{T}_t^{-1}. \tag{4.4}$$

Here $D\boldsymbol{T}_t \colon \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ stands for the Jacobian matrix function of $\boldsymbol{T}_t$ and we denote $D\boldsymbol{T}_t^{-\mathsf{T}} := \big(D\boldsymbol{T}_t^{-1}\big)^{\mathsf{T}}$. It satisfies the important identity (see [19, Lemma 11])

$$\big(\mathbf{curl}\, \Psi_t(\boldsymbol{e})\big) \circ \boldsymbol{T}_t = \xi(t)^{-1} D\boldsymbol{T}_t \, \mathbf{curl}\, \boldsymbol{e} \tag{4.5}$$

with $\xi(t) := \det D\boldsymbol{T}_t$. In this paper we always assume $\tau > 0$ small enough such that $\xi(t) > 0$ for every $t \in [0, \tau]$. That is, the transformation $\boldsymbol{T}_t$ preserves orientation. In view of the above discussion, we introduce the *shape-Lagrangian* $G : [0,\tau]\times \boldsymbol{H}_0(\mathbf{curl})\times \boldsymbol{H}_0(\mathbf{curl}) \to \mathbb{R}$ as

$$\begin{aligned}(4.6)\quad G(t,\boldsymbol{e},\boldsymbol{v}) := \mathcal{L}(\omega_t, \Psi_t(\boldsymbol{e}), \Psi_t(\boldsymbol{v})) = \frac{1}{2}\int_B \kappa |\Psi_t(\boldsymbol{e}) - \boldsymbol{E}_d|^2\,dx + \int_{\omega_t} dx \\ + a(\Psi_t(\boldsymbol{e}),\Psi_t(\boldsymbol{v})) + \int_{\omega_t} \boldsymbol{\Lambda}_\gamma(\Psi_t(\boldsymbol{e}))\cdot \Psi_t(\boldsymbol{v})\,dx - \int_\Omega \boldsymbol{f}\cdot \Psi_t(\boldsymbol{v})\,dx.\end{aligned}$$

The change of variables $x \mapsto \boldsymbol{T}_t(x)$ inside the integrals (4.4) and (4.5) yields

$$\begin{aligned}G(t,\boldsymbol{e},\boldsymbol{v}) = \frac{1}{2}\int_B \kappa\circ \boldsymbol{T}_t |D\boldsymbol{T}_t^{-\top}\boldsymbol{e} - \boldsymbol{E}_d\circ \boldsymbol{T}_t|^2 \xi(t)\,dx + \int_\omega \xi(t)\,dx + \int_\Omega \mathbb{M}_1(t)\,\mathbf{curl}\,\boldsymbol{e}\cdot \mathbf{curl}\,\boldsymbol{v}\,dx \\ (4.7)\qquad + \int_\Omega \mathbb{M}_2(t)\boldsymbol{e}\cdot\boldsymbol{v}\,dx + \int_\omega \mathbb{M}_3(t,\boldsymbol{e})\cdot \boldsymbol{v}\,dx - \int_\Omega (\boldsymbol{f}\circ \boldsymbol{T}_t)\cdot (D\boldsymbol{T}_t^{-\top}\boldsymbol{v})\xi(t)\,dx\end{aligned}$$

with the notation $\mathbb{M}_1(t) := \xi(t)^{-1} D\boldsymbol{T}_t^\top (\nu\circ \boldsymbol{T}_t) D\boldsymbol{T}_t$, $\mathbb{M}_2(t) := \xi(t) D\boldsymbol{T}_t^{-1}(\varepsilon\circ\boldsymbol{T}_t)D\boldsymbol{T}_t^{-\top}$ and $\mathbb{M}_3(t,\boldsymbol{e}) := \xi(t) D\boldsymbol{T}_t^{-1}\boldsymbol{\Lambda}_\gamma(D\boldsymbol{T}_t^{-\top}\boldsymbol{e})$. Note that the problem of finding $\boldsymbol{e}_t \in \boldsymbol{H}_0(\mathbf{curl})$ such that $\partial_{\boldsymbol{v}} G(t,\boldsymbol{e}_t,0;\hat{\boldsymbol{v}}) = 0$ for all $\hat{\boldsymbol{v}} \in \boldsymbol{H}_0(\mathbf{curl})$ is equivalent to (3.1) with $\omega = \omega_t$ after applying the change of variables $x\mapsto \boldsymbol{T}_t(x)$. Hence, it has the same unique solution $\boldsymbol{E}_t^\gamma \in \boldsymbol{H}_0(\mathbf{curl})$.

Next, the shape derivative of J_γ is obtained as the partial derivative with respect to t of the shape-Lagrangian G given by (4.7). For the convenience of the reader, we recall the main result of the averaged adjoint method, adapted to our case.

THEOREM 4.2 (averaged adjoint method). *Let $\gamma > 0$. Moreover, we assume that there exists $\tau \in (0,1]$ such that for every $(t,\boldsymbol{v}) \in [0,\tau]\times \boldsymbol{H}_0(\mathbf{curl})$*

- (H1) *the mapping $[0,1]\ni s\mapsto G(t, s\boldsymbol{E}_t^\gamma + (1-s)\boldsymbol{E}_0^\gamma, \boldsymbol{v})$ is absolutely continuous;*
- (H2) *the mapping $[0,1]\ni s\mapsto \partial_{\boldsymbol{e}} G(t, s\boldsymbol{E}_t^\gamma + (1-s)\boldsymbol{E}_0^\gamma, \boldsymbol{v};\hat{\boldsymbol{e}})$ belongs to $L^1(0,1)$ for every $\hat{\boldsymbol{e}} \in \boldsymbol{H}_0(\mathbf{curl})$;*
- (H3) *there exists a unique $\boldsymbol{P}_t^\gamma \in \boldsymbol{H}_0(\mathbf{curl})$ that solves the averaged adjoint equation*

$$(4.8)\qquad \int_0^1 \partial_{\boldsymbol{e}} G(t, s\boldsymbol{E}_t^\gamma + (1-s)\boldsymbol{E}_0^\gamma, \boldsymbol{P}_t^\gamma;\hat{\boldsymbol{e}})\,ds = 0 \quad \forall \hat{\boldsymbol{e}} \in \boldsymbol{H}_0(\mathbf{curl});$$

- (H4) *the family $\{\boldsymbol{P}_t^\gamma\}_{t\in[0,\tau]}$ satisfies*

$$(4.9)\qquad \lim_{t\searrow 0} \frac{G(t,\boldsymbol{E}_0^\gamma,\boldsymbol{P}_t^\gamma) - G(0,\boldsymbol{E}_0^\gamma,\boldsymbol{P}_t^\gamma)}{t} = \partial_t G(0,\boldsymbol{E}_0^\gamma,\boldsymbol{P}_0^\gamma).$$

Then, J_γ is shape-differentiable in the sense of Definition 4.1 *and it holds that*

$$(4.10)\qquad dJ_\gamma(\omega)(\boldsymbol{\theta}) = \frac{d}{dt}J_\gamma(\omega_t)|_{t=0} = \partial_t G(0,\boldsymbol{E}_0^\gamma,\boldsymbol{P}_0^\gamma),$$

where $\boldsymbol{P}_0^\gamma$ is the so-called adjoint state solution of (4.8) *with $t = 0$.*

Remark 4.3. The main idea of the proof of Theorem 4.2 can be formally understood in the following way. For $t = 0$, the averaged adjoint equation (4.8) coincides with the adjoint equation (see (4.38) for its explicit expression):

$$(4.11)\qquad \partial_{\boldsymbol{e}} G(0,\boldsymbol{E}_0^\gamma,\boldsymbol{P}_0^\gamma;\hat{\boldsymbol{e}}) = 0 \quad \forall \hat{\boldsymbol{e}} \in \boldsymbol{H}_0(\mathbf{curl}).$$

Then, assuming that the material derivative $\partial_t \boldsymbol{E}_t^\gamma|_{t=0}$ exists, we have in view of (4.3) and (4.6)

$$\begin{aligned}dJ_\gamma(\omega)(\boldsymbol{\theta}) &= \frac{d}{dt}J_\gamma(\omega_t)|_{t=0} = \frac{d}{dt}G(t,\boldsymbol{E}_t^\gamma,\boldsymbol{v})|_{t=0} \\ &= \partial_t G(0,\boldsymbol{E}_0^\gamma,\boldsymbol{v}) + \partial_{\boldsymbol{e}} G(0,\boldsymbol{E}_0^\gamma,\boldsymbol{v};\partial_t\boldsymbol{E}_t^\gamma|_{t=0}) \text{ for any } \boldsymbol{v}\in \boldsymbol{H}_0(\mathbf{curl}).\end{aligned}$$

Using (4.11) and choosing $\boldsymbol{v} = \boldsymbol{P}_0^\gamma$ we obtain $\partial_{\boldsymbol{e}} G(0, \boldsymbol{E}_0^\gamma, \boldsymbol{v}; \partial_t \boldsymbol{E}_t^\gamma|_{t=0}) = 0$, and the above equation yields (4.10). The proof of Theorem 4.2 proceeds in a similar way, except that the averaged adjoint $\boldsymbol{P}_t^\gamma$ allows us to obtain the same result without having to introduce the material derivative $\partial_t \boldsymbol{E}_t^\gamma|_{t=0}$. We refer to [44] or [25, Theorem 2.1] for a detailed proof.

We verify that (H1)–(H4) are satisfied so that we may apply Theorem 4.2.

LEMMA 4.4. *Let Assumption* 2.1 *be satisfied. Then,* (H1) *and* (H2) *hold for every* $(t, \boldsymbol{v}) \in [0,1] \times \boldsymbol{H}_0(\mathbf{curl})$.

Proof. First, (H1) is a direct consequence of (4.7) and Lemma 3.1. Before we proceed to prove (H2), let us introduce the notation $\mathcal{E}(s) := s\boldsymbol{E}_t^\gamma + (1-s)\boldsymbol{E}_0^\gamma$. Now, fix $\tau \in (0,1]$ and $(t, \boldsymbol{v}) \in [0,\tau] \times \boldsymbol{H}_0(\mathbf{curl})$. Thanks to the Gateaux-differentiability of $\boldsymbol{\Lambda}_\gamma$ (Lemma 3.1), and using (4.7), we may compute

$$\begin{aligned}(4.12)\qquad \partial_{\boldsymbol{e}} G(t, \mathcal{E}(s), \boldsymbol{v}; \hat{\boldsymbol{e}}) &= \int_B \kappa \circ \boldsymbol{T}_t \big(D\boldsymbol{T}_t^{-\mathsf{T}} \hat{\boldsymbol{e}} \cdot (D\boldsymbol{T}_t^{-\mathsf{T}} \mathcal{E}(s) - \boldsymbol{E}_d \circ \boldsymbol{T}_t)\big)\xi(t)\, dx \\ &\quad + \int_\Omega \mathbb{M}_1(t)\, \mathbf{curl}\, \hat{\boldsymbol{e}} \cdot \mathbf{curl}\, \boldsymbol{v} + \mathbb{M}_2(t) \hat{\boldsymbol{e}} \cdot \boldsymbol{v}\, dx + \int_\omega \partial_{\boldsymbol{e}} \mathbb{M}_3\big(t, \mathcal{E}(s)\big)\hat{\boldsymbol{e}} \cdot \boldsymbol{v}\, dx\end{aligned}$$

for every $\hat{\boldsymbol{e}} \in \boldsymbol{H}_0(\mathbf{curl})$, where

$$\begin{aligned}(4.13)\quad \int_\omega \partial_{\boldsymbol{e}} \mathbb{M}_3\big(t, \mathcal{E}(s)\big)\hat{\boldsymbol{e}} \cdot \boldsymbol{v}\, dx \ &= \ \int_\omega \xi(t) D\boldsymbol{T}_t^{-1} \boldsymbol{\Lambda}_\gamma'\big(D\boldsymbol{T}_t^{-\mathsf{T}} \mathcal{E}(s)\big)(D\boldsymbol{T}_t^{-\mathsf{T}} \hat{\boldsymbol{e}}) \cdot \boldsymbol{v}\, dx \\ &\overbrace{=}^{(3.7)\,\&\,(3.8)} \int_\omega \xi(t) D\boldsymbol{T}_t^{-2} \boldsymbol{\psi}^\gamma\big(D\boldsymbol{T}_t^{-\mathsf{T}} \mathcal{E}(s)\big)\boldsymbol{v} \cdot \hat{\boldsymbol{e}}\, dx.\end{aligned}$$

Moreover, the following asymptotic expansions hold (see [43, Lemma 2.31]):

$$(4.14)\quad \xi(t) = 1 + t\,\mathrm{div}(\boldsymbol{\theta}) + o(t),\ D\boldsymbol{T}_t = \boldsymbol{I}_3 + tD\boldsymbol{\theta} + o(t),\ D\boldsymbol{T}_t^{-1} = \boldsymbol{I}_3 - tD\boldsymbol{\theta} + o(t)$$

such that $o(t)/t \to 0$ as $t \to 0$ with respect to $\|\cdot\|_{\mathcal{C}(\Omega)}$ and $\|\cdot\|_{\mathcal{C}(\Omega,\mathbb{R}^{3\times3})}$, respectively. Hence, (4.14) implies that there exists a constant $C > 0$ only dependent on $\boldsymbol{\theta}$ such that

$$(4.15)\qquad \|\xi(t)\|_{L^\infty(\Omega)} + \|D\boldsymbol{T}_t\|_{L^\infty(\Omega,\mathbb{R}^{3\times3})} + \|D\boldsymbol{T}_t^{-1}\|_{L^\infty(\Omega,\mathbb{R}^{3\times3})} \le 1 + C\tau.$$

Applying (4.15) in (4.13) leads to

$$\begin{aligned}(4.16)\quad \left|\int_\omega \partial_{\boldsymbol{e}} \mathbb{M}_3(t, \mathcal{E}(s))\hat{\boldsymbol{e}} \cdot \boldsymbol{v}\, dx\right| \ &\le \ (1+C\tau)^3 \int_\omega \big|\boldsymbol{\psi}^\gamma\big(D\boldsymbol{T}_t^{-\mathsf{T}} \mathcal{E}(s)\big)\boldsymbol{v} \cdot \hat{\boldsymbol{e}}\big|\, dx \\ &\overbrace{\le}^{(3.11)} 2j_c\gamma(1+C\tau)^3 \|\hat{\boldsymbol{e}}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \quad \forall s \in (0,1).\end{aligned}$$

Thus, the mapping $s \mapsto \int_\omega \partial_{\boldsymbol{e}} \mathbb{M}_3(t, \mathcal{E}(s))\hat{\boldsymbol{e}} \cdot \boldsymbol{v}\, dx$ belongs to $L^\infty(0,1) \subset L^1(0,1)$. In a similar way, since $t \in [0,\tau]$ and $\gamma > 0$ are fixed, (4.15) and (A1) of Assumption 2.1 yield

$$\begin{aligned}(4.17)\quad &\int_B \big|\kappa \circ \boldsymbol{T}_t \big(D\boldsymbol{T}_t^{-\mathsf{T}} \hat{\boldsymbol{e}} \cdot D\boldsymbol{T}_t^{-\mathsf{T}} \mathcal{E}(s)\big)\xi(t)\big|\, dx \\ &\le (1+C\tau)^3 \|\kappa\|_{\mathcal{C}(\Omega)} \|\hat{\boldsymbol{e}}\|_{\boldsymbol{L}^2(\Omega)} \|\mathcal{E}(s)\|_{\boldsymbol{L}^2(\Omega)} \\ &\le (1+C\tau)^3 \|\kappa\|_{\mathcal{C}(\Omega)} \|\hat{\boldsymbol{e}}\|_{\boldsymbol{L}^2(\Omega)} \big(\|\boldsymbol{E}_0^\gamma\|_{\boldsymbol{L}^2(\Omega)} + s\|\boldsymbol{E}_t^\gamma - \boldsymbol{E}_0^\gamma\|_{\boldsymbol{L}^2(\Omega)}\big) \\ &\le (1+s)(1+C\tau)^3 \|\kappa\|_{\mathcal{C}(\Omega)} \|\hat{\boldsymbol{e}}\|_{\boldsymbol{L}^2(\Omega)} \big(\|\boldsymbol{E}_t^\gamma - \boldsymbol{E}_0^\gamma\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{E}_0^\gamma\|_{\boldsymbol{L}^2(\Omega)}\big).\end{aligned}$$

As the remaining terms in (4.12) are independent of s, (4.16) and (4.17) imply that the mapping $s \mapsto \partial_{\boldsymbol{e}} G(t, \mathcal{E}(s), \boldsymbol{v}; \hat{\boldsymbol{e}})$ belongs to $L^1(0,1)$ for all $\hat{\boldsymbol{e}} \in \boldsymbol{H}_0(\mathbf{curl})$ and $(t, \boldsymbol{v}) \in [0, \tau] \times \boldsymbol{H}_0(\mathbf{curl})$. Thus, the proof is complete. ∎

LEMMA 4.5. *Let Assumption* 2.1 *hold. Then, there exists* $\tau \in (0,1]$ *such that* (H3) *is satisfied for every* $t \in [0, \tau]$. *Moreover,* (H4) *holds as well.*

Proof. Fix some arbitrary $\tau > 0$ and denote $\mathcal{E}(s) \coloneqq s\boldsymbol{E}_t^\gamma + (1-s)\boldsymbol{E}_0^\gamma$ for $s \in (0,1)$. Let $\tau \in (0,1]$ be arbitrarily fixed. In the following, if necessary, we shall reduce $\tau \in (0,1]$ step by step to prove our result. Let $t \in [0,\tau]$ and $\hat{\boldsymbol{e}} \in \boldsymbol{H}_0(\mathbf{curl})$. Thanks to Lemma 4.4, the left-hand side of (4.8) is well-defined, and our goal is to prove the existence of a unique $\boldsymbol{P}_t^\gamma \in \boldsymbol{H}_0(\mathbf{curl})$ satisfying (4.8). In view of (4.12), we note that (4.8) can be written as

$$B_t(\boldsymbol{P}_t^\gamma, \hat{\boldsymbol{e}}) = F_t(\hat{\boldsymbol{e}}) \quad \forall \hat{\boldsymbol{e}} \in \boldsymbol{H}_0(\mathbf{curl}) \tag{4.18}$$

with $B_t\colon \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}_0(\mathbf{curl}) \to \mathbb{R}$ and $F_t\colon \boldsymbol{H}_0(\mathbf{curl}) \to \mathbb{R}$ defined by

$$\begin{aligned} B_t(\boldsymbol{v}, \hat{\boldsymbol{e}}) &\coloneqq \int_\Omega \mathbb{M}_1(t)\mathbf{curl}\,\hat{\boldsymbol{e}} \cdot \mathbf{curl}\,\boldsymbol{v} + \mathbb{M}_2(t)\hat{\boldsymbol{e}} \cdot \boldsymbol{v}\,dx + \int_0^1\int_\omega \partial_{\boldsymbol{e}}\mathbb{M}_3(t, \mathcal{E}(s))\hat{\boldsymbol{e}} \cdot \boldsymbol{v}\,dx\,ds, \\ F_t(\hat{\boldsymbol{e}}) &\coloneqq -\int_0^1 \int_B \kappa \circ \boldsymbol{T}_t\big(D\boldsymbol{T}_t^{-\mathsf{T}}\hat{\boldsymbol{e}} \cdot \big(D\boldsymbol{T}_t^{-\mathsf{T}}\mathcal{E}(s) - \boldsymbol{E}_d \circ \boldsymbol{T}_t\big)\big)\xi(t)\,dx\,ds. \end{aligned}$$

Thanks to (A2) and (4.15) and (4.16), B_t is a bounded bilinear form. In order to apply the Lax–Milgram lemma, we have to prove the coercivity of B_t. The asymptotic expansions (4.14) show that $\mathbb{M}_1(t)$ and $\mathbb{M}_2(t)$ are small perturbations of ν and ϵ, respectively. Thus, if necessary, we may reduce the number $\tau \in (0,1]$ such that, in view of (2.1), $\mathbb{M}_1(t)$ and $\mathbb{M}_2(t)$ are uniformly positive definite for all $t \in [0,\tau]$ with

$$\int_\Omega \mathbb{M}_1(t)\,\mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{v} + \mathbb{M}_2(t)\boldsymbol{v} \cdot \boldsymbol{v}\,dx \geq C_1\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathbf{curl})} \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \tag{4.19}$$

for some constant $C_1 > 0$ depending only on $\boldsymbol{\theta}, \epsilon$, and ν. In order to keep the notation short, let us define $\mathcal{K}(s) \coloneqq D\boldsymbol{T}_t^{-\mathsf{T}}\mathcal{E}(s) \in \boldsymbol{H}_0(\mathbf{curl})$ as well as the sets $\mathcal{A}_\gamma(s) \coloneqq \mathcal{A}_\gamma(\mathcal{K}(s)) \subset \Omega$ and $\mathcal{S}_\gamma(s) \coloneqq \mathcal{S}_\gamma(\mathcal{K}(s)) \subset \Omega$ for $s \in (0,1)$ (cf. Lemma 3.1). We estimate the third term in B_t which, in view of (3.7) and (4.13), corresponds to

$$\begin{aligned} \int_0^1\int_\omega \partial_e\mathbb{M}_3(t, \mathcal{E}(s))\boldsymbol{v} \cdot \boldsymbol{v}\,dx\,ds &= \int_0^1\int_\omega \xi(t)D\boldsymbol{T}_t^{-2}\Bigg[\frac{j_c\gamma\boldsymbol{I}_3}{\max_\gamma\{1, \gamma|\mathcal{K}(s)|\}} \\ &- \gamma\bigg(\mathbb{1}_{\mathcal{A}_\gamma(s)} + \gamma\Big(\gamma|\mathcal{K}(s)| - 1 + \frac{1}{2\gamma}\Big)\mathbb{1}_{\mathcal{S}_\gamma(s)}\bigg)\frac{\mathcal{K}(s) \otimes \boldsymbol{\Lambda}_\gamma(\mathcal{K}(s))}{\max_\gamma\{1, \gamma|\mathcal{K}(s)|\}|\mathcal{K}(s)|}\Bigg]\boldsymbol{v} \cdot \boldsymbol{v}\,dx\,ds. \end{aligned} \tag{4.20}$$

Therefore, we fix $s \in (0,1)$ and estimate the three summands in (4.20) separately. We begin with the first term and note that (4.14) implies, possibly after reducing $\tau > 0$, that there exists a constant $C > 0$, depending only on $\boldsymbol{\theta}$, such that $\xi(t) \geq 1 - C\tau > 0$, and $D\boldsymbol{T}_t^{-2}\boldsymbol{\eta} \cdot \boldsymbol{\eta} \geq (1 - C\tau)^2|\boldsymbol{\eta}|^2$ for all $\boldsymbol{\eta} \in \mathbb{R}^3$ and almost everywhere in Ω. Hence,

$$\int_\omega j_c\gamma\xi(t)\frac{D\boldsymbol{T}_t^{-2}\boldsymbol{v} \cdot \boldsymbol{v}}{\max_\gamma\{1, \gamma|\mathcal{K}(s)|\}}\,dx \geq (1 - C\tau)^3\int_\omega \frac{j_c\gamma|\boldsymbol{v}|^2}{\max_\gamma\{1, \gamma|\mathcal{K}(s)|\}}\,dx. \tag{4.21}$$

Now, we proceed to estimate the integrals over the disjoint sets $\omega \cap \mathcal{A}_\gamma(s)$ and $\omega \cap \mathcal{S}_\gamma(s)$

appearing in the last two summands in (4.20). We obtain

$$\left|\int_{\omega\cap\mathcal{A}_\gamma(s)} \gamma\xi(t)D\boldsymbol{T}_t^{-2}\frac{\mathcal{K}(s)\otimes\boldsymbol{\Lambda}_\gamma(\mathcal{K}(s))\boldsymbol{v}\cdot\boldsymbol{v}}{\max_\gamma\{1,\gamma|\mathcal{K}(s)|\}|\mathcal{K}(s)|}\,dx\right| \tag{4.22}$$
$$\underbrace{\leq}^{(3.3)\,\&\,(3.10)} \|\xi(t)\|_{L^\infty(\Omega)}\|D\boldsymbol{T}_t^{-1}\|^2_{L^\infty(\Omega,\mathbb{R}^{3\times3})}\int_{\omega\cap\mathcal{A}_\gamma(s)}\frac{j_c\gamma|\boldsymbol{v}|^2}{\max_\gamma\{1,\gamma|\mathcal{K}(s)|\}}\,dx$$
$$\underbrace{\leq}^{(4.15)} (1+C\tau)^3\int_{\omega\cap\mathcal{A}_\gamma(s)}\frac{j_c\gamma|\boldsymbol{v}|^2}{\max_\gamma\{1,\gamma|\mathcal{K}(s)|\}}\,dx.$$

For the last summand, we use the same arguments and also (3.6) to deduce

$$\left|\int_{\omega\cap\mathcal{S}_\gamma(s)}\gamma^2\left(\gamma|\mathcal{K}(s)|-1+\frac{1}{2\gamma}\right)\xi(t)D\boldsymbol{T}_t^{-2}\frac{\mathcal{K}(s)\otimes\boldsymbol{\Lambda}_\gamma(\mathcal{K}(s))\boldsymbol{v}\cdot\boldsymbol{v}}{\max_\gamma\{1,\gamma|\mathcal{K}(s)|\}\,|\mathcal{K}(s)|}\,dx\right| \tag{4.23}$$
$$\leq (1+C\tau)^3\int_{\omega\cap\mathcal{S}_\gamma(s)}\frac{j_c\gamma|\boldsymbol{v}|^2}{\max_\gamma\{1,\gamma|\mathcal{K}(s)|\}}\,dx.$$

Note that the constant $C>0$ in (4.21)–(4.23) is the same in the three inequalities. Thus, we sum up (4.22) and (4.23) and substract the result from (4.21) to obtain

$$\int_\omega \partial_e\mathbb{M}_3(t,\mathcal{E}(s))\boldsymbol{v}\cdot\boldsymbol{v}\,dx \geq \left(1+3(C\tau)^2\right)\int_{\omega\setminus(\mathcal{A}_\gamma(s)\cup\mathcal{S}_\gamma(s))}\frac{j_c\gamma|\boldsymbol{v}|^2}{\max_\gamma\{1,\gamma|\mathcal{K}(s)|\}}\,dx$$
$$-\left(6C\tau+2(C\tau)^3\right)\int_\omega\frac{j_c\gamma|\boldsymbol{v}|^2}{\max_\gamma\{1,\gamma|\mathcal{K}(s)|\}}\,dx.$$

As the first term is nonnegative and $\max_\gamma\{1,\gamma|\mathcal{K}(s)|\}\geq 1$, we conclude for (4.20) that

$$\int_0^1\int_\omega \partial_e\mathbb{M}_3(t,\mathcal{E}(s))\boldsymbol{v}\cdot\boldsymbol{v}\,dx\,ds \geq -(6C\tau+2(C\tau)^3)j_c\gamma\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\omega)}. \tag{4.24}$$

The coercivity of B_t follows, as (4.19) in combination with (4.24) implies that

$$B_t(\boldsymbol{v},\boldsymbol{v}) \geq \underbrace{\left(C_1-6C\tau-2(C\tau)^3\right)}_{=:C_2}\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathbf{curl})}\quad\forall\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl}). \tag{4.25}$$

If necessary, we further reduce $\tau\in(0,1]$ such that $C_2>0$ holds true. In turn, for all $t\in[0,\tau]$, B_t is coercive with the coercivity constant $C_2>0$, independent of t. Ultimately, the Lax–Milgram lemma yields the existence of a unique solution $\boldsymbol{P}_t^\gamma\in\boldsymbol{H}_0(\mathbf{curl})$ of the averaged adjoint equation (4.8). Thus, (H3) holds.

We finish this proof by verifying (H4). To this aim, let $\{t_k\}_{k\in\mathbb{N}}\subset(0,\tau]$ be a null sequence. First, the sequence $\{\boldsymbol{E}_{t_k}^\gamma\}_{k\in\mathbb{N}}\subset\boldsymbol{H}_0(\mathbf{curl})$ of solutions to the perturbed state equations (3.1) with $\omega=\omega_{t_k}$ is bounded. This follows readily by inserting $\boldsymbol{v}=\boldsymbol{E}_{t_k}^\gamma$ into (3.1), which yields

$$\min(\underline{\nu},\underline{\epsilon})\|\boldsymbol{E}_{t_k}^\gamma\|^2_{\boldsymbol{H}(\mathbf{curl})}\leq a(\boldsymbol{E}_{t_k}^\gamma,\boldsymbol{E}_{t_k}^\gamma)\leq(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}+j_c)\|\boldsymbol{E}_{t_k}^\gamma\|_{\boldsymbol{H}(\mathbf{curl})} \tag{4.26}$$
$$\Rightarrow\quad \|\boldsymbol{E}_{t_k}^\gamma\|_{\boldsymbol{H}(\mathbf{curl})}\leq\min(\underline{\nu},\underline{\epsilon})^{-1}(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}+j_c)\quad\forall k\in\mathbb{N}.$$

Hereafter, we deduce a similar estimate for $\{\boldsymbol{P}^\gamma_{t_k}\}_{k\in\mathbb{N}}$ by testing (4.18) with $\hat{\boldsymbol{e}} = \boldsymbol{P}^\gamma_{t_k}$ and using (4.25) along with (4.15):

$$(4.27)\quad C_2\|\boldsymbol{P}^\gamma_{t_k}\|^2_{\boldsymbol{H}_0(\mathbf{curl})} \le B_t(\boldsymbol{P}^\gamma_{t_k}, \boldsymbol{P}^\gamma_{t_k}) = F_t(\boldsymbol{P}^\gamma_{t_k}) \le \|\kappa\|_{\mathcal{C}(\Omega)}(1+C\tau)^3(\|\boldsymbol{E}^\gamma_{t_k}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{E}^\gamma_0\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{E}_d\|_{\boldsymbol{L}^2(\Omega)})\|\boldsymbol{P}^\gamma_{t_k}\|_{\boldsymbol{L}^2(\Omega)} \quad \forall k\in\mathbb{N}.$$

Since the constant C_2 and C are independent of $k\in\mathbb{N}$, the above estimate implies the boundedness of $\{\boldsymbol{P}^\gamma_{t_k}\}_{k\in\mathbb{N}} \subset \boldsymbol{H}_0(\mathbf{curl})$. Hence, there exists a subsequence $\{t_{k_j}\}_{j\in\mathbb{N}} \subset \{t_k\}_{k\in\mathbb{N}}$ converging weakly in $\boldsymbol{H}_0(\mathbf{curl})$ to some $\boldsymbol{P}^\star \in \boldsymbol{H}_0(\mathbf{curl})$. By (4.14) and as the solution of (4.18) is unique, passing to the limit $t = t_{k_j} \to 0$ in (4.18) yields $\boldsymbol{P}^\star = \boldsymbol{P}^\gamma_0$. Since $\boldsymbol{P}^\gamma_0$ is independent of the choice of the subsequence $\{t_{k_j}\}_{j\in\mathbb{N}}$, a standard argument implies the weak convergence of the whole sequence:

$$(4.28)\qquad \boldsymbol{P}^\gamma_{t_k} \rightharpoonup \boldsymbol{P}^\star \quad \text{weakly in } \boldsymbol{H}_0(\mathbf{curl}) \quad \text{as } k\to\infty.$$

Let us now consider the differential quotient

$$(4.29)\qquad \begin{aligned} \frac{G(t_k,\boldsymbol{E}^\gamma_0,\boldsymbol{P}^\gamma_{t_k}) - G(0,\boldsymbol{E}^\gamma_0,\boldsymbol{P}^\gamma_{t_k})}{t_k} &= \int_B \frac{\mathbb{M}_0(t_k)-\mathbb{M}_0(0)}{t_k}\,dx + \int_\omega \frac{\xi(t_k)-\xi(0)}{t_k}\,dx \\ &+ \int_\Omega \frac{\mathbb{M}_1(t_k)-\mathbb{M}_1(0)}{t_k}\,\mathbf{curl}\,\boldsymbol{E}^\gamma_0\cdot\mathbf{curl}\,\boldsymbol{P}^\gamma_{t_k} + \frac{\mathbb{M}_2(t_k)-\mathbb{M}_2(0)}{t_k}\boldsymbol{E}^\gamma_0\cdot\boldsymbol{P}^\gamma_{t_k}\,dx \\ &+ \int_\omega \frac{\mathbb{M}_3(t_k,\boldsymbol{E}^\gamma_0)-\mathbb{M}_3(0,\boldsymbol{E}^\gamma_0)}{t_k}\cdot\boldsymbol{P}^\gamma_{t_k}\,dx - \int_\Omega \frac{\mathbb{M}_4(t_k)-\mathbb{M}_4(0)}{t_k}\cdot\boldsymbol{P}^\gamma_{t_k}\,dx \end{aligned}$$

with $\mathbb{M}_0(t_k) := \frac{1}{2}\kappa\circ\boldsymbol{T}_{t_k}|D\boldsymbol{T}^{-\mathsf{T}}_{t_k}\boldsymbol{E}^\gamma_0 - \boldsymbol{E}_d\circ\boldsymbol{T}_{t_k}|^2\xi(t_k)$ and $\mathbb{M}_4(t_k) := \xi(t_k)D\boldsymbol{T}^{-1}_{t_k}(\boldsymbol{f}\circ\boldsymbol{T}_{t_k})$. First, (4.14) yields the strong convergence

$$(4.30)\qquad \lim_{k\to\infty}\frac{\xi(t_k)-\xi(0)}{t_k} = \operatorname{div}\boldsymbol{\theta} \quad \text{in } \mathcal{C}(\Omega).$$

Moreover, thanks to Assumption 2.1, (4.14), and $\operatorname{supp}\boldsymbol{\theta} \subset\subset B$, we obtain the strong convergence of $\big(\mathbb{M}_i(t_k)-\mathbb{M}_i(0)\big)/t_k$, $i=0,1,2,4$, as $k\to\infty$ in $L^\infty(\Omega)$:

$$(4.31)\qquad \lim_{k\to\infty}\frac{\mathbb{M}_0(t_k)-\mathbb{M}_0(0)}{t_k} = \frac{1}{2}(\widetilde{\nabla\kappa}\cdot\boldsymbol{\theta} + \kappa\operatorname{div}\boldsymbol{\theta})|\boldsymbol{E}^\gamma_0-\boldsymbol{E}_d|^2 - \kappa(\boldsymbol{E}^\gamma_0-\boldsymbol{E}_d)\cdot(D\boldsymbol{\theta}^{\mathsf{T}}\boldsymbol{E}^\gamma_0 - \widetilde{D\boldsymbol{E}_d}\boldsymbol{\theta})$$

$$(4.32)\qquad \lim_{k\to\infty}\frac{\mathbb{M}_1(t_k)-\mathbb{M}_1(0)}{t_k} = -(\operatorname{div}\boldsymbol{\theta})\nu + D\boldsymbol{\theta}^{\mathsf{T}}\nu + \nu D\boldsymbol{\theta} + \widetilde{D\nu}\boldsymbol{\theta},$$

$$(4.33)\qquad \lim_{k\to\infty}\frac{\mathbb{M}_2(t_k)-\mathbb{M}_2(0)}{t_k} = (\operatorname{div}\boldsymbol{\theta})\varepsilon - D\boldsymbol{\theta}\varepsilon - \varepsilon D\boldsymbol{\theta}^{\mathsf{T}} + \widetilde{D\varepsilon}\boldsymbol{\theta},$$

$$(4.34)\qquad \lim_{k\to\infty}\frac{\mathbb{M}_4(t_k)-\mathbb{M}_4(0)}{t_k} = (\operatorname{div}\boldsymbol{\theta})\boldsymbol{f} - D\boldsymbol{\theta}\boldsymbol{f} + \widetilde{D\boldsymbol{f}}\boldsymbol{\theta}.$$

Note that $\widetilde{\nabla\kappa}$ denotes the zero extension of $\nabla\kappa_{|B} \in \boldsymbol{\mathcal{C}}(B)$ to Ω. The same notation is used for $\widetilde{D\boldsymbol{E}_d}, \widetilde{D\varepsilon}, \widetilde{D\nu}, \widetilde{D\boldsymbol{f}}$. Similarly, by the Gateaux-differentiability of $\boldsymbol{\Lambda}_\gamma$ (see Lemma 3.1), (3.8), and (4.28), we deduce that

$$(4.35)\qquad \lim_{k\to\infty}\frac{\mathbb{M}_3(t_k)-\mathbb{M}_3(0)}{t_k}\cdot\boldsymbol{P}^\gamma_{t_k} = \big((\operatorname{div}\boldsymbol{\theta})\boldsymbol{\Lambda}_\gamma(\boldsymbol{E}^\gamma_0) - D\boldsymbol{\theta}\boldsymbol{\Lambda}_\gamma(\boldsymbol{E}^\gamma_0)\big)\cdot\boldsymbol{P}^\gamma_0 - \boldsymbol{\psi}^\gamma(\boldsymbol{E}^\gamma_0)\boldsymbol{P}^\gamma_0\cdot(D\boldsymbol{\theta}^{\mathsf{T}}\boldsymbol{E}^\gamma_0).$$

From (4.30)–(4.35) along with the weak convergence (4.28) and supp $\boldsymbol{\theta} \subset\subset B$, it follows that

$$
\begin{aligned}
(4.36)\quad & \lim_{k\to\infty} \frac{G(t_k, \boldsymbol{E}_0^\gamma, \boldsymbol{P}_{t_k}^\gamma) - G(0, \boldsymbol{E}_0^\gamma, \boldsymbol{P}_{t_k}^\gamma)}{t_k} \\
&= \int_B \frac{1}{2}(\nabla\kappa \cdot \boldsymbol{\theta} + \kappa \operatorname{div} \boldsymbol{\theta})|\boldsymbol{E}_0^\gamma - \boldsymbol{E}_d|^2 - \kappa(\boldsymbol{E}_0^\gamma - \boldsymbol{E}_d) \cdot (D\boldsymbol{\theta}^\mathsf{T} \boldsymbol{E}_0^\gamma + D\boldsymbol{E}_d \boldsymbol{\theta})\, dx \\
&\quad + \int_\omega \operatorname{div} \boldsymbol{\theta}\, dx + \int_B \big(-(\operatorname{div} \boldsymbol{\theta})\nu + D\boldsymbol{\theta}^\mathsf{T} \nu + \nu D\boldsymbol{\theta} + D\nu\boldsymbol{\theta}\big) \operatorname{\mathbf{curl}} \boldsymbol{E}_0^\gamma \cdot \operatorname{\mathbf{curl}} \boldsymbol{P}_0^\gamma\, dx \\
&\quad + \int_B \big((\operatorname{div} \boldsymbol{\theta})\varepsilon - D\boldsymbol{\theta}\varepsilon - \varepsilon D\boldsymbol{\theta}^\mathsf{T} + D\varepsilon\boldsymbol{\theta}\big)\boldsymbol{E}_0^\gamma \cdot \boldsymbol{P}_0^\gamma\, dx \\
&\quad + \int_\omega (\operatorname{div} \boldsymbol{\theta})\boldsymbol{\Lambda}_\gamma(\boldsymbol{E}_0^\gamma) \cdot \boldsymbol{P}_0^\gamma - D\boldsymbol{\theta}\boldsymbol{\Lambda}_\gamma(\boldsymbol{E}_0^\gamma) \cdot \boldsymbol{P}_0^\gamma - \boldsymbol{\psi}^\gamma(\boldsymbol{E}_0^\gamma)\boldsymbol{P}_0^\gamma \cdot (D\boldsymbol{\theta}^\mathsf{T} \boldsymbol{E}_0^\gamma)\, dx \\
&\quad - \int_B (D\boldsymbol{f}\boldsymbol{\theta} + (\operatorname{div} \boldsymbol{\theta})\boldsymbol{f}) \cdot \boldsymbol{P}_0^\gamma - \boldsymbol{f} \cdot D\boldsymbol{\theta}^\mathsf{T} \boldsymbol{P}_0^\gamma\, dx \\
&= \lim_{k\to\infty} \frac{G(t_k, \boldsymbol{E}_0^\gamma, \boldsymbol{P}_0^\gamma) - G(0, \boldsymbol{E}_0^\gamma, \boldsymbol{P}_0^\gamma)}{t_k} = \partial_t G(0, \boldsymbol{E}_0^\gamma, \boldsymbol{P}_0^\gamma).
\end{aligned}
$$

Thus, (H4) is valid. □

In the case $t = 0$, the solution $\boldsymbol{P}_0^\gamma \in \boldsymbol{H}_0(\mathbf{curl})$ of (4.8) also satisfies the equation

$$
(4.37)\qquad \partial_{\boldsymbol{e}} \mathcal{L}(\omega, \boldsymbol{E}_0^\gamma, \boldsymbol{P}_0^\gamma; \hat{\boldsymbol{e}}) = 0 \quad \forall \hat{\boldsymbol{e}} \in \boldsymbol{H}_0(\mathbf{curl}).
$$

By definition (4.2) and by (4.13) we conclude that (4.37) is equivalent to

$$
(4.38)\quad a(\hat{\boldsymbol{e}}, \boldsymbol{P}_0^\gamma) + \int_\omega \boldsymbol{\psi}^\gamma(\boldsymbol{E}_0^\gamma)\boldsymbol{P}_0^\gamma \cdot \hat{\boldsymbol{e}}\, dx = -\int_B \kappa(\boldsymbol{E}_0^\gamma - \boldsymbol{E}_d) \cdot \hat{\boldsymbol{e}}\, dx \quad \forall \hat{\boldsymbol{e}} \in \boldsymbol{H}_0(\mathbf{curl}).
$$

We refer to (4.38) as the *adjoint equation* and we write for simplicity $(\boldsymbol{E}^\gamma, \boldsymbol{P}^\gamma) = (\boldsymbol{E}_0^\gamma, \boldsymbol{P}_0^\gamma)$. We now have all the elements at hand to prove the shape differentiability of J_γ and write the distributed expression of the shape derivative of J_γ.

THEOREM 4.6. *Let Assumption* 2.1 *be satisfied,* $\gamma > 0$, $\omega \in \mathcal{O}$, *and* $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}_c^{0,1}(\Omega)$ *with a compact support in* B. *Furthermore,* $\boldsymbol{E}^\gamma \in \boldsymbol{H}_0(\mathbf{curl})$ *and* $\boldsymbol{P}^\gamma \in \boldsymbol{H}_0(\mathbf{curl})$ *denote the solutions to* (3.1) *and* (4.38), *respectively. Then, the functional* J_γ *in* (P_γ) *is shape differentiable with*

$$
(4.39)\qquad dJ_\gamma(\omega)(\boldsymbol{\theta}) = \partial_t G(0, \boldsymbol{E}^\gamma, \boldsymbol{P}^\gamma) = \int_B S_1^\gamma : D\boldsymbol{\theta} + \boldsymbol{S}_0^\gamma \cdot \boldsymbol{\theta}\, dx,
$$

where $S_1^\gamma \in L^1(B, \mathbb{R}^{3\times 3})$ *and* $\boldsymbol{S}_0^\gamma \in \boldsymbol{L}^1(B)$ *are given by*

$$
\begin{aligned}
S_1^\gamma &= \Big[\frac{\kappa}{2}|\boldsymbol{E}^\gamma - \boldsymbol{E}_d|^2 + \chi_\omega - \nu \operatorname{\mathbf{curl}} \boldsymbol{E}^\gamma \cdot \operatorname{\mathbf{curl}} \boldsymbol{P}^\gamma + \varepsilon \boldsymbol{E}^\gamma \cdot \boldsymbol{P}^\gamma + \chi_\omega \boldsymbol{\Lambda}_\gamma(\boldsymbol{E}^\gamma) \cdot \boldsymbol{P}^\gamma \\
&\qquad - \boldsymbol{f} \cdot \boldsymbol{P}^\gamma\Big]\boldsymbol{I}_3 - \kappa \boldsymbol{E}^\gamma \otimes (\boldsymbol{E}^\gamma - \boldsymbol{E}_d) + \nu \operatorname{\mathbf{curl}} \boldsymbol{E}^\gamma \otimes \operatorname{\mathbf{curl}} \boldsymbol{P}^\gamma \\
&\quad + \nu^\mathsf{T} \operatorname{\mathbf{curl}} \boldsymbol{P}^\gamma \otimes \operatorname{\mathbf{curl}} \boldsymbol{E}^\gamma - \boldsymbol{P}^\gamma \otimes \varepsilon \boldsymbol{E}^\gamma - \boldsymbol{E}^\gamma \otimes \varepsilon^\mathsf{T} \boldsymbol{P}^\gamma + \boldsymbol{P}^\gamma \otimes \boldsymbol{f} \\
&\quad - \chi_\omega \boldsymbol{\Lambda}_\gamma(\boldsymbol{E}^\gamma) \otimes \boldsymbol{P}^\gamma - \boldsymbol{E}^\gamma \otimes \boldsymbol{\psi}^\gamma(\boldsymbol{E}^\gamma)\boldsymbol{P}^\gamma, \\
\boldsymbol{S}_0^\gamma &= \frac{\nabla\kappa}{2}|\boldsymbol{E}^\gamma - \boldsymbol{E}_d|^2 - \kappa D\boldsymbol{E}_d^\mathsf{T}(\boldsymbol{E}^\gamma - \boldsymbol{E}_d) + (D\nu^\mathsf{T} \operatorname{\mathbf{curl}} \boldsymbol{E}^\gamma) \operatorname{\mathbf{curl}} \boldsymbol{P}^\gamma \\
&\qquad + (D\epsilon^\mathsf{T} \boldsymbol{E}^\gamma)\boldsymbol{P}^\gamma - D\boldsymbol{f}^\mathsf{T} \boldsymbol{P}^\gamma.
\end{aligned}
$$

Proof. Thanks to Lemmas 4.4 and 4.5, we may apply the averaged adjoint method (see Theorem 4.2). This yields that J_γ is shape-differentiable in the sense of Definition 4.1 and the shape derivative satisfies (4.10) with $\partial_t G(0, \boldsymbol{E}^\gamma, \boldsymbol{P}^\gamma)$ given by (4.36). Since $D\epsilon$ is a third-order tensor, its transpose $D\epsilon^\mathsf{T}$ satisfies $D\epsilon\boldsymbol{\theta}\boldsymbol{E}^\gamma \cdot \boldsymbol{P}^\gamma = (D\epsilon^\mathsf{T}\boldsymbol{E}^\gamma)\boldsymbol{P}^\gamma \cdot \boldsymbol{\theta}$, and $D\nu^\mathsf{T}$ satisfies a similar property; see [40, Proposition 3.1]. Furthermore, for vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$ we have the relations $D\boldsymbol{\theta} : (\boldsymbol{x} \otimes \boldsymbol{y}) = \boldsymbol{x} \cdot D\boldsymbol{\theta}\boldsymbol{y} = D\boldsymbol{\theta}^\mathsf{T}\boldsymbol{x} \cdot \boldsymbol{y}$. Applying these to (4.36) and combining it with (4.10), the tensor expression (4.39) for the shape derivative follows. The fact that $S_1^\gamma \in L^1(B, \mathbb{R}^{3\times 3})$ and $\boldsymbol{S}_0^\gamma \in \boldsymbol{L}^1(B)$ is a straightforward consequence of the regularity of $\boldsymbol{E}^\gamma, \boldsymbol{P}^\gamma$ and of the other terms involved in $\boldsymbol{S}_0^\gamma$ and S_1^γ. This completes the proof. ∎

5. Stability and convergence analysis.

In this section we analyze the stability of the shape derivative (4.39) with respect to the penalization parameter $\gamma > 0$. Furthermore, the strong convergence of (P$_\gamma$) toward (P) as $\gamma \to \infty$ is studied. The latter also implies the existence of an optimal shape for (P) (see Theorem 2.4).

5.1. Stability analysis of the shape derivative.

THEOREM 5.1. *Let $\omega \in \mathcal{O}$ and Assumption 2.1 hold. Then, the following stability estimate holds:*

$$|dJ_\gamma(\omega)(\boldsymbol{\theta})| \le C\|\boldsymbol{\theta}\|_{\boldsymbol{\mathcal{C}}^{0,1}(B)} \quad \forall \boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}_c^{0,1}(\Omega),\ \operatorname{supp}\boldsymbol{\theta} \subset\subset B \tag{5.1}$$

with a constant $C = C(j_c, \kappa, \epsilon, \nu, \boldsymbol{f}, \boldsymbol{E}_d, B, \omega)$ independent of γ.

Proof. First, the distributed shape derivative from (4.39) yields the estimate

$$|dJ_\gamma(\omega)(\boldsymbol{\theta})| \le \big(\|S_1^\gamma\|_{L^1(B,\mathbb{R}^{3\times 3})} + \|\boldsymbol{S}_0^\gamma\|_{\boldsymbol{L}^1(B)}\big)\|\boldsymbol{\theta}\|_{\boldsymbol{\mathcal{C}}^{0,1}(B)}. \tag{5.2}$$

In order to derive upper bounds for $\|S_1^\gamma\|_{L^1(B,\mathbb{R}^{3\times 3})}$ and $\|\boldsymbol{S}_0^\gamma\|_{\boldsymbol{L}^1(B)}$, we begin by proving that the families $\{\boldsymbol{E}^\gamma\}_{\gamma>0}$ and $\{\boldsymbol{P}^\gamma\}_{\gamma>0}$ are uniformly bounded in $\boldsymbol{H}_0(\mathbf{curl})$. In view of (4.26), we have

$$\|\boldsymbol{E}^\gamma\|_{\boldsymbol{H}(\mathbf{curl})} \le \min(\underline{\nu}, \underline{\epsilon})^{-1}(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + j_c) =: C_{\boldsymbol{E}}. \tag{5.3}$$

Moreover, we set $t, s = 0$ in (4.13), which yields

$$\int_\omega \partial_{\boldsymbol{e}}\mathbb{M}_3(0, \mathcal{E}(0))(\boldsymbol{P}^\gamma) \cdot \boldsymbol{P}^\gamma\, dx = \int_\omega \boldsymbol{\psi}^\gamma(\boldsymbol{E}^\gamma)\boldsymbol{P}^\gamma \cdot \boldsymbol{P}^\gamma\, dx \ge 0. \tag{5.4}$$

In fact, the nonnegativity of (5.4) follows by similar calculations as (4.20)–(4.24) in the special case $t, s, \tau = 0$. As $\boldsymbol{P}^\gamma$ is the unique solution to (4.38), inserting $\hat{\boldsymbol{e}} = \boldsymbol{P}^\gamma$ implies with (A2)

$$\begin{aligned}\min(\underline{\epsilon}, \underline{\nu})\|\boldsymbol{P}^\gamma\|^2_{\boldsymbol{H}(\mathbf{curl})} &\le a(\boldsymbol{P}^\gamma, \boldsymbol{P}^\gamma) \\ &= -\int_B \kappa(\boldsymbol{E}^\gamma - \boldsymbol{E}_d) \cdot \boldsymbol{P}^\gamma\, dx - \int_\omega \boldsymbol{\psi}^\gamma(\boldsymbol{E}^\gamma)\boldsymbol{P}^\gamma \cdot \boldsymbol{P}^\gamma\, dx.\end{aligned}$$

Hence, we obtain a uniform bound for $\boldsymbol{P}^\gamma$ by means of (5.3) and (5.4), i.e.,

$$\|\boldsymbol{P}^\gamma\|_{\boldsymbol{H}(\mathbf{curl})} \le \|\kappa\|_{\mathcal{C}(\overline{\Omega})} \min(\underline{\epsilon}, \underline{\nu})^{-1}\big(C_{\boldsymbol{E}} + \|\boldsymbol{E}_d\|_{\boldsymbol{L}^2(B)}\big) =: C_{\boldsymbol{P}}. \tag{5.5}$$

With (5.3) and (5.5) at hand, we may now estimate both terms in (5.2) separately. Therefore, let us introduce the notation (see Theorem 4.6)

$$S_1^\gamma =: \sum_{i=1}^{14} \Theta_i, \tag{5.6}$$

where $\Theta_i \in L^1(B,\mathbb{R}^{3\times 3})$ for every $i \in \{1,\ldots,14\}$. Now, Assumption 2.1, (3.10), (5.3) and (5.5) together with Hölder's and Young's inequalities yield

$$
\begin{aligned}
&\sum_{i=1}^{6} \|\Theta_i\|_{L^1(B,\mathbb{R}^{3\times 3})} \\
&\leq \int_B \frac{|\kappa|}{2}|\boldsymbol{E}^\gamma - \boldsymbol{E}_d|^2 + \chi_\omega \, dx + \int_B |\nu \,\mathbf{curl}\, \boldsymbol{E}^\gamma \cdot \mathbf{curl}\, \boldsymbol{P}^\gamma| + |\epsilon \boldsymbol{E}^\gamma \cdot \boldsymbol{P}^\gamma| \, dx \\
&\quad + \int_\omega |\boldsymbol{\Lambda}_\gamma(\boldsymbol{E}^\gamma)\cdot \boldsymbol{P}^\gamma| \, dx + \int_B |\boldsymbol{f}\cdot \boldsymbol{P}^\gamma| \, dx \\
&\overbrace{\leq}^{(5.3)\,(5.5)\,\&\,(3.10)} \|\kappa\|_{\mathcal{C}(B)}\big(C_{\boldsymbol{E}}^2 + \|\boldsymbol{E}_d\|^2_{\boldsymbol{L}^2(B)}\big) + |\omega| + \big(\|\nu\|_{\mathcal{C}(B,\mathbb{R}^{3\times 3})} + \|\epsilon\|_{\mathcal{C}(B,\mathbb{R}^{3\times 3})}\big) C_{\boldsymbol{E}} C_{\boldsymbol{P}} \\
&\quad + (j_c\sqrt{|\omega|} + \|\boldsymbol{f}\|_{\boldsymbol{L}^2(B)}) C_{\boldsymbol{P}}.
\end{aligned} \tag{5.7}
$$

For the remaining terms, we use again Assumption 2.1, (3.10), (5.3), and (5.5) as well as the identity $|\boldsymbol{x}\otimes \boldsymbol{y}| = |\boldsymbol{x}|\cdot|\boldsymbol{y}|$ for all $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3$ to infer

$$
\begin{aligned}
\sum_{i=7}^{13} \|\Theta_i\|_{L^1(B,\mathbb{R}^{3\times 3})} &\leq \frac{1}{2}\|\kappa\|_{\mathcal{C}(B)}\big(3C_{\boldsymbol{E}}^2 + \|\boldsymbol{E}_d\|^2_{\boldsymbol{L}^2(B)}\big) \\
&\quad + 2(\|\nu\|_{\mathcal{C}(B,\mathbb{R}^{3\times 3})} + \|\epsilon\|_{\mathcal{C}(B,\mathbb{R}^{3\times 3})}) C_{\boldsymbol{E}} C_{\boldsymbol{P}} + (\|\boldsymbol{f}\|_{\boldsymbol{L}^2(B)} + j_c\sqrt{|\omega|}) C_{\boldsymbol{P}},
\end{aligned} \tag{5.8}
$$

where we have also used Young's inequality to obtain the first term in (5.8). Moreover, the last summand of S_1^γ can be estimated as follows:

$$
\begin{aligned}
\|\Theta_{14}\|_{L^1(B,\mathbb{R}^{3\times 3})} &= \|\boldsymbol{E}^\gamma \otimes \boldsymbol{\psi}^\gamma(\boldsymbol{E}^\gamma)\boldsymbol{P}^\gamma\|_{L^1(\Omega,\mathbb{R}^{3\times 3})} \leq \int_\omega |\boldsymbol{\psi}^\gamma(\boldsymbol{E}^\gamma)\boldsymbol{P}^\gamma|\cdot|\boldsymbol{E}^\gamma| \, dx \\
&\overbrace{\leq}^{(3.6)\,\&\,(3.7)} \int_\omega \left(\frac{j_c\gamma|\boldsymbol{P}^\gamma|}{\max_\gamma\{1,\gamma|\boldsymbol{E}^\gamma|\}} + \frac{\gamma|\boldsymbol{E}^\gamma\otimes\boldsymbol{\Lambda}_\gamma(\boldsymbol{E}^\gamma)|\cdot|\boldsymbol{P}^\gamma|}{\max_\gamma\{1,\gamma|\boldsymbol{E}^\gamma|\}|\boldsymbol{E}^\gamma|}\right)|\boldsymbol{E}^\gamma| \, dx \\
&\overbrace{\leq}^{(3.9)\,\&\,(3.10)} \int_\omega 2j_c|\boldsymbol{P}^\gamma| \, dx \leq 2j_c\sqrt{|\omega|}C_{\boldsymbol{P}}.
\end{aligned} \tag{5.9}
$$

Gathering (5.7)–(5.9) we deduce the final estimate for S_1^γ:

$$
\begin{aligned}
\|S_1^\gamma\|_{L^1(B,\mathbb{R}^{3\times 3})} &\leq \frac{1}{2}\|\kappa\|_{\mathcal{C}(B)}\big(5C_{\boldsymbol{E}}^2 + 3\|\boldsymbol{E}_d\|^2_{\boldsymbol{L}^2(B)}\big) + |\omega| \\
&\quad + 3\big(\|\nu\|_{\mathcal{C}(B,\mathbb{R}^{3\times 3})} + \|\epsilon\|_{\mathcal{C}(B,\mathbb{R}^{3\times 3})}\big) C_{\boldsymbol{E}} C_{\boldsymbol{P}} + \big(2\|\boldsymbol{f}\|_{\boldsymbol{L}^2(B)} + 4j_c\sqrt{|\omega|}\big) C_{\boldsymbol{P}}.
\end{aligned} \tag{5.10}
$$

Again, (5.3) and (5.5) with Hölder's and Young's inequalities imply for $\boldsymbol{S}_0^\gamma$

$$
\begin{aligned}
\|\boldsymbol{S}_0^\gamma\|_{\boldsymbol{L}^1(B)} &\leq \int_B \frac{1}{2}|\nabla\kappa|\cdot|\boldsymbol{E}^\gamma - \boldsymbol{E}_d|^2 + |\kappa D\boldsymbol{E}_d^\top(\boldsymbol{E}^\gamma - \boldsymbol{E}_d)| \, dx \\
&\quad + \int_B |D\nu^\top \,\mathbf{curl}\, \boldsymbol{E}^\gamma|\cdot|\,\mathbf{curl}\, \boldsymbol{P}^\gamma| + |D\epsilon^\top \boldsymbol{E}^\gamma|\cdot|\boldsymbol{P}^\gamma| + |D\boldsymbol{f}^\top \boldsymbol{P}^\gamma| \, dx \\
&\leq \frac{1}{2}\|\kappa\|_{\mathcal{C}^1(B)}\big(3C_{\boldsymbol{E}}^2 + 5\|\boldsymbol{E}_d\|^2_{\boldsymbol{H}^1(B)}\big) \\
&\quad + \big(\|\nu\|_{\mathcal{C}^1(B,\mathbb{R}^{3\times 3})} + \|\epsilon\|_{\mathcal{C}^1(B,\mathbb{R}^{3\times 3})}\big) C_{\boldsymbol{P}} C_{\boldsymbol{E}} + \|\boldsymbol{f}\|_{\boldsymbol{H}^1(B)} C_{\boldsymbol{P}}.
\end{aligned} \tag{5.11}
$$

Finally, we combine (5.2), (5.10), and (5.11) to conclude

$$|dJ_\gamma(\omega)(\boldsymbol{\theta})| \leq \Big[4\|\kappa\|_{\mathcal{C}^1(B)}\big(C_{\boldsymbol{E}}^2+\|\boldsymbol{E}_d\|^2_{\boldsymbol{H}^1(B)}\big)+4\big(\|\nu\|_{\mathcal{C}^1(B,\mathbb{R}^{3\times 3})}+\|\epsilon\|_{\mathcal{C}^1(B,\mathbb{R}^{3\times 3})}\big)C_{\boldsymbol{E}}C_{\boldsymbol{P}} + |\omega| + \big(3\|\boldsymbol{f}\|_{\boldsymbol{H}^1(B)} + 4j_c\sqrt{|\omega|}\big)C_{\boldsymbol{P}}\Big]\|\boldsymbol{\theta}\|_{\boldsymbol{\mathcal{C}}^{0,1}(B)}.$$

Hence, the proof is finished. ∎

5.2. Convergence of the regularized shape optimization problem. Our aim is to prove the strong convergence of (P_γ) toward (P). For this purpose, we recall a helpful result which states the strong convergence of the solution to (3.1) for a fixed $\omega \in \mathcal{O}$. This result goes back to [6, Corollary 4.3]. Since the argumentation has to be modified and adapted to our case, we include a complete proof below.

LEMMA 5.2. *Let Assumption* 2.1 *be satisfied and* $\omega \in \mathcal{O}$. *Moreover, for every* $\gamma > 0$, *let* $(\boldsymbol{E}^\gamma, \boldsymbol{\lambda}^\gamma) \in \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^\infty(\omega)$ *denote the solution to* (3.1). *Then,*

$$(\boldsymbol{E}^\gamma, \boldsymbol{\lambda}^\gamma) \to (\boldsymbol{E}, \boldsymbol{\lambda}) \quad \text{strongly in } \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}_0(\mathbf{curl})^* \text{ as } \gamma \to \infty, \tag{5.12}$$

where $(\boldsymbol{E}, \boldsymbol{\lambda}) \in \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^\infty(\omega)$ *is the unique solution to* (2.3).

Proof. At first, we introduce $(\boldsymbol{z}^\gamma, \boldsymbol{\xi}^\gamma) \in \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^\infty(\Omega)$ as the solution to the auxiliary problem

$$\begin{cases} a(\boldsymbol{z}^\gamma, \boldsymbol{v}) + \displaystyle\int_\omega \boldsymbol{\xi}^\gamma \cdot \boldsymbol{v}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}), \\ \boldsymbol{\xi}^\gamma(x) = \Xi^\gamma(\boldsymbol{z}^\gamma) := \dfrac{j_c\gamma \boldsymbol{z}^\gamma(x)}{\max\{1, \gamma|\boldsymbol{z}^\gamma(x)|\}} \quad \text{for a.e. } x \in \omega. \end{cases} \tag{5.13}$$

We note that the mapping $\boldsymbol{\Xi}^\gamma \colon \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ is monotone in the sense of (3.5) being the derivative of the convex functional $\int_\omega j_c \Psi_\gamma(\boldsymbol{v})\, dx$, where $\Psi_\gamma \colon \mathbb{R}^3 \to \mathbb{R}$ denotes the Huber regularization of $|\cdot|$:

$$\Psi_\gamma(x) := \begin{cases} |x| - \dfrac{1}{2\gamma} & \text{for } |x| \geq \dfrac{1}{\gamma}, \\ \dfrac{\gamma}{2}|x|^2 & \text{for } |x| < \dfrac{1}{\gamma}. \end{cases}$$

Thus, the well-posedness of (5.13) follows from the Minty–Browder theorem [41, Theorem 2.18] by completely analoguous arguments used for (3.1). Now, by substracting (5.13) from (3.1), inserting $\boldsymbol{v} = \boldsymbol{E}^\gamma - \boldsymbol{z}^\gamma$, we obtain that

$$a(\boldsymbol{E}^\gamma - \boldsymbol{z}^\gamma, \boldsymbol{E}^\gamma - \boldsymbol{z}^\gamma) + \int_\omega \left(\frac{j_c\gamma\boldsymbol{E}^\gamma}{\max_\gamma\{1, \gamma|\boldsymbol{E}^\gamma|\}} - \frac{j_c\gamma\boldsymbol{z}^\gamma}{\max\{1, \gamma|\boldsymbol{z}^\gamma|\}}\right) \cdot (\boldsymbol{E}^\gamma - \boldsymbol{z}^\gamma)\, dx = 0.$$

This implies

$$\begin{aligned} a(\boldsymbol{E}^\gamma - \boldsymbol{z}^\gamma, \boldsymbol{E}^\gamma - \boldsymbol{z}^\gamma) + \int_\omega \left(\frac{j_c\gamma\boldsymbol{E}^\gamma}{\max\{1, \gamma|\boldsymbol{E}^\gamma|\}} - \frac{j_c\gamma\boldsymbol{z}^\gamma}{\max\{1, \gamma|\boldsymbol{z}^\gamma|\}}\right) \cdot \big(\boldsymbol{E}^\gamma - \boldsymbol{z}^\gamma\big)\, dx \\ = \int_\omega \left(\frac{j_c\gamma\boldsymbol{E}^\gamma}{\max_\gamma\{1, \gamma|\boldsymbol{E}^\gamma|\}} - \frac{j_c\gamma\boldsymbol{E}^\gamma}{\max\{1, \gamma|\boldsymbol{E}^\gamma|\}}\right) \cdot \big(\boldsymbol{z}^\gamma - \boldsymbol{E}^\gamma\big)\, dx. \end{aligned}$$

Thanks to the monotonicity of $\boldsymbol{\Xi}^\gamma$, the second summand on the left-hand side is nonnegative. Hence, the coercivity property (2.2) yields

$$C\|\boldsymbol{E}^\gamma - \boldsymbol{z}^\gamma\|^2_{\boldsymbol{H}(\mathbf{curl})} \le j_c\gamma \int_\omega \left(\frac{\max\{1,\gamma|\boldsymbol{E}^\gamma|\} - \max_\gamma\{1,\gamma|\boldsymbol{E}^\gamma|\}}{\max_\gamma\{1,\gamma|\boldsymbol{E}^\gamma|\}\max\{1,\gamma|\boldsymbol{E}^\gamma|\}} \right) \boldsymbol{E}^\gamma \cdot \left(\boldsymbol{z}^\gamma - \boldsymbol{E}^\gamma\right) dx.$$

Since $0 \le \max_\gamma\{1,x\} - \max\{1,x\} \le \frac{1}{4\gamma}$ holds for every $x \in \mathbb{R}$, it follows that

$$C\|\boldsymbol{E}^\gamma - \boldsymbol{z}^\gamma\|^2_{\boldsymbol{H}(\mathbf{curl})} \le \frac{j_c}{4\gamma} \int_\omega \underbrace{\frac{\gamma|\boldsymbol{E}^\gamma|}{\max\{1,\gamma|\boldsymbol{E}^\gamma|\}^2}}_{\le 1} |\boldsymbol{E}^\gamma - \boldsymbol{z}^\gamma|\, dx. \tag{5.14}$$

Therefore, after applying the Hölder inequality, (5.14) implies

$$\lim_{\gamma\to\infty} \|\boldsymbol{E}^\gamma - \boldsymbol{z}^\gamma\|_{\boldsymbol{H}(\mathbf{curl})} = 0. \tag{5.15}$$

The next step is to verify the strong convergence $\boldsymbol{z}^\gamma \to \boldsymbol{E}$ as $\gamma \to \infty$. We proceed similarly as before by substracting (5.13) from (2.3) and inserting $\boldsymbol{v} = \boldsymbol{E} - \boldsymbol{z}^\gamma$ to deduce that

$$C\|\boldsymbol{E} - \boldsymbol{z}^\gamma\|^2_{\boldsymbol{H}(\mathbf{curl})} \le a(\boldsymbol{E} - \boldsymbol{z}^\gamma, \boldsymbol{E} - \boldsymbol{z}^\gamma) = \int_\omega (\boldsymbol{\xi}^\gamma - \boldsymbol{\lambda}) \cdot (\boldsymbol{E} - \boldsymbol{z}^\gamma)\, dx. \tag{5.16}$$

Next, we exploit the properties of $\boldsymbol{\lambda}$ and $\boldsymbol{\xi}^\gamma$ from (2.3) and (5.13) to estimate the right-hand side of (5.16). For this purpose, we divide ω into the disjoint sets $\mathcal{A}\cap\mathcal{A}_\gamma$, $\mathcal{A}\cap\mathcal{I}_\gamma$, $\mathcal{I}\cap\mathcal{A}_\gamma$, and $\mathcal{I}\cap\mathcal{I}_\gamma$ where

$$\begin{aligned} \mathcal{A} &\coloneqq \{x\in\omega \;:\; |\boldsymbol{E}(x)| > 0\}, & \mathcal{I} &\coloneqq \omega\setminus\mathcal{A},\\ \mathcal{A}_\gamma &\coloneqq \{x\in\omega \;:\; \gamma|\boldsymbol{z}^\gamma(x)| > 1\}, & \mathcal{I}_\gamma &\coloneqq \omega\setminus\mathcal{A}_\gamma. \end{aligned}$$

Now, we establish pointwise estimates for the integrand in (5.16). For $x\in\mathcal{A}\cap\mathcal{A}_\gamma$, (2.3) and (5.13) imply

$$\begin{aligned} &(\boldsymbol{\lambda}(x) - \boldsymbol{\xi}^\gamma(x)) \cdot (\boldsymbol{z}^\gamma(x) - \boldsymbol{E}(x))\\ &= \boldsymbol{\lambda}(x)\cdot\boldsymbol{z}^\gamma(x) - \boldsymbol{\lambda}(x)\cdot\boldsymbol{E}(x) + \boldsymbol{\xi}^\gamma(x)\cdot\boldsymbol{E}(x) - \boldsymbol{\xi}^\gamma(x)\cdot\boldsymbol{z}^\gamma(x)\\ &\le j_c|\boldsymbol{z}^\gamma(x)| - j_c|\boldsymbol{E}(x)| + j_c|\boldsymbol{E}(x)| - j_c|\boldsymbol{z}^\gamma(x)| = 0. \end{aligned}$$

For $x\in\mathcal{A}\cap\mathcal{I}_\gamma$, (2.3) and (5.13) yield $j_c\boldsymbol{z}^\gamma(x) = \gamma^{-1}\boldsymbol{\xi}^\gamma(x)$, $|\boldsymbol{\xi}^\gamma(x)| \le j_c$, $|\boldsymbol{z}^\gamma(x)| \le \gamma^{-1}$, and $|\boldsymbol{\lambda}(x)| = j_c$. Hence, we can derive

$$\begin{aligned} &(\boldsymbol{\lambda}(x) - \boldsymbol{\xi}^\gamma(x)) \cdot (\boldsymbol{z}^\gamma(x) - \boldsymbol{E}(x))\\ &= \boldsymbol{\lambda}(x)\cdot\boldsymbol{z}^\gamma(x) - j_c|\boldsymbol{E}(x)| - \gamma j_c|\boldsymbol{z}^\gamma(x)|^2 + \boldsymbol{\xi}^\gamma(x)\cdot\boldsymbol{E}(x)\\ &\le \frac{1}{\gamma}j_c - j_c|\boldsymbol{E}(x)| + j_c|\boldsymbol{E}(x)| - \gamma j_c|\boldsymbol{z}^\gamma(x)|^2 \le \frac{1}{\gamma}j_c. \end{aligned}$$

For $x\in\mathcal{I}\cap\mathcal{A}_\gamma$, we have $\boldsymbol{E}(x) = 0$ and thus

$$(\boldsymbol{\lambda}(x) - \boldsymbol{\xi}^\gamma(x)) \cdot (\boldsymbol{z}^\gamma(x) - \boldsymbol{E}(x)) = (\boldsymbol{\lambda}(x) - \boldsymbol{\xi}^\gamma(x))\cdot\boldsymbol{z}^\gamma(x) \le 0,$$

where the last inequality follows from (5.13). Finally, for $x\in\mathcal{I}\cap\mathcal{I}_\gamma$ we have $\boldsymbol{E}(x) = 0$, $j_c\boldsymbol{z}^\gamma(x) = \gamma^{-1}\boldsymbol{\xi}^\gamma(x)$ as well as $|\boldsymbol{z}^\gamma(x)| \le \gamma^{-1}$. This implies

$$(\boldsymbol{\lambda}(x) - \boldsymbol{\xi}^\gamma(x)) \cdot (\boldsymbol{z}^\gamma(x) - \boldsymbol{E}(x)) \le \frac{1}{\gamma}j_c - \gamma j_c|\boldsymbol{z}^\gamma|^2 \le \frac{1}{\gamma}j_c.$$

After taking all the pointwise estimates above together in (5.16), it follows that

$$C\|\boldsymbol{E}-\boldsymbol{z}^\gamma\|^2_{\boldsymbol{H}(\mathbf{curl})} \leq j_c\gamma^{-1}|\omega| \quad \Rightarrow \quad \lim_{\gamma\to\infty}\|\boldsymbol{E}-\boldsymbol{z}^\gamma\|_{\boldsymbol{H}(\mathbf{curl})} = 0. \tag{5.17}$$

Therefore, (5.15) and (5.17) imply the strong convergence $\|\boldsymbol{E}^\gamma - \boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl})} \to 0$ as $\gamma \to \infty$. Finally, in view of (2.3) and (5.13), we have that

$$\sup_{\boldsymbol{v}\in \boldsymbol{H}_0(\mathbf{curl})\setminus\{0\}} \frac{\int_\omega (\boldsymbol{\lambda}^\gamma - \boldsymbol{\lambda})\cdot \boldsymbol{v}\,dx}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})}} = \sup_{\boldsymbol{v}\in \boldsymbol{H}_0(\mathbf{curl})\setminus\{0\}} \frac{a(\boldsymbol{E}-\boldsymbol{E}^\gamma, \boldsymbol{v})}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})}} \leq C\|\boldsymbol{E}-\boldsymbol{E}^\gamma\|_{\boldsymbol{H}(\mathbf{curl})},$$

which implies the convergence $\boldsymbol{\lambda}^\gamma \to \boldsymbol{\lambda}$ in $\boldsymbol{H}_0(\mathbf{curl})^*$ as $\gamma \to \infty$. ∎

Let us point out that in (5.12) we extended the Lagrange multipliers $\boldsymbol{\lambda}^\gamma, \boldsymbol{\lambda}$ by zero as functions in $\boldsymbol{L}^2(\Omega)$, i.e., we set $\boldsymbol{\lambda}^\gamma(x) = 0$ and $\boldsymbol{\lambda}(x) = 0$ for all $x \in \Omega\setminus\omega$. This zero extension shall also be used in the following theorem.

THEOREM 5.3. *Let Assumption* 2.1 *hold and* $\{\gamma_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ *be such that* $\gamma_n \to \infty$ *as* $n \to \infty$. *Then, there exists a subsequence of* $\{\gamma_n\}_{n\in\mathbb{N}}$, *still denoted by* $\{\gamma_n\}_{n\in\mathbb{N}}$, *such that the sequence of solutions* $\{\omega^{\gamma_n}\}_{n\in\mathbb{N}}$ *of* (P$_\gamma$) *with* $\gamma = \gamma_n$ *converges toward an optimal solution* $\omega_\star \subset \mathcal{O}$ *of* (P) *in the sense of Hausdorff and in the sense of characteristic functions.*

Moreover, $\{(\boldsymbol{E}^{\gamma_n}(\omega^{\gamma_n}), \boldsymbol{\lambda}^{\gamma_n}(\omega^{\gamma_n}))\}_{n\in\mathbb{N}}$ *and* $(\boldsymbol{E}(\omega_\star), \boldsymbol{\lambda}(\omega_\star))$ *as the solutions of* (3.1) *for* $\omega = \omega^{\gamma_n}$ *and* (2.3) *for* $\omega = \omega_\star$, *respectively, satisfy*

$$\lim_{\gamma\to\infty}\|\boldsymbol{E}^{\gamma_n}(\omega^{\gamma_n}) - \boldsymbol{E}(\omega_\star)\|_{\boldsymbol{H}(\mathbf{curl})} = 0, \tag{5.18}$$

$$\lim_{\gamma\to\infty}\|\boldsymbol{\lambda}^{\gamma_n}(\omega^{\gamma_n}) - \boldsymbol{\lambda}(\omega_\star)\|_{\boldsymbol{H}_0(\mathbf{curl})^*} = 0, \tag{5.19}$$

where $\boldsymbol{\lambda}^{\gamma_n}(\omega^{\gamma_n})$ *(resp.,* $\boldsymbol{\lambda}(\omega_\star)$ *) is extended by zero in* $\Omega\setminus\omega^{\gamma_n}$ *(resp., in* $\Omega\setminus\omega_\star$*).*

Proof. Thanks to Theorem 2.3 and $\gamma_n \to \infty$, there exists $\omega_\star \in \mathcal{O}$ such that, possibly for a subsequence,

$$\omega^{\gamma_n} \to \omega_\star \quad \text{as } n \to \infty \tag{5.20}$$

in the sense of Hausdorff and in the sense of characteristic functions. Furthermore, we have the estimate

$$\begin{aligned}\|\boldsymbol{E}^{\gamma_n}(\omega^{\gamma_n}) - \boldsymbol{E}(\omega_\star)\|_{\boldsymbol{H}(\mathbf{curl})} \leq \|\boldsymbol{E}^{\gamma_n}(\omega^{\gamma_n}) - \boldsymbol{E}^{\gamma_n}(\omega_\star)\|_{\boldsymbol{H}(\mathbf{curl})}& \\ + \|\boldsymbol{E}^{\gamma_n}(\omega_\star) - \boldsymbol{E}(\omega_\star)\|_{\boldsymbol{H}(\mathbf{curl})}&.\end{aligned} \tag{5.21}$$

Now, by virtue of Lemma 5.2, the second term on the right-hand side of (5.21) converges to 0 as $n \to \infty$. For the first term we observe (for every $n \in \mathbb{N}$) that the arguments used to derive (3.13) are applicable. Thus, we substract (3.1) for $\boldsymbol{E}^{\gamma_n}(\omega^{\gamma_n})$ and (3.1) for $\boldsymbol{E}^{\gamma_n}(\omega_\star)$ and test the resulting equation with $\boldsymbol{v} = \boldsymbol{E}^{\gamma_n}(\omega_\star) - \boldsymbol{E}^{\gamma_n}(\omega^{\gamma_n})$. Hereafter, analoguously to (3.12), calculations involving (3.5) yield

$$\|\boldsymbol{E}^{\gamma_n}(\omega^{\gamma_n}) - \boldsymbol{E}^{\gamma_n}(\omega_\star)\|_{\boldsymbol{H}(\mathbf{curl})} \leq \frac{j_c}{\min\{\underline{\nu}, \underline{\epsilon}\}}\|\chi_{\omega_\star} - \chi_{\omega^{\gamma_n}}\|_{L^2(\Omega)} \quad \forall n \in \mathbb{N}. \tag{5.22}$$

Combining Lemma 5.2 and (5.20)–(5.22) together leads to (5.18).

Furthermore, substracting (2.3) for $\omega = \omega_\star$ and (3.1) for $\omega = \omega^{\gamma_n}$ implies

$$\sup_{\boldsymbol{v}\in \boldsymbol{H}_0(\mathbf{curl})} \frac{(\boldsymbol{\lambda}^{\gamma_n}(\omega^{\gamma_n}) - \boldsymbol{\lambda}(\omega_\star), \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})}} = \sup_{\boldsymbol{v}\in \boldsymbol{H}_0(\mathbf{curl})} \frac{a(\boldsymbol{E}(\omega_\star) - \boldsymbol{E}^{\gamma_n}(\omega^{\gamma_n}), \boldsymbol{v})}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})}} \tag{5.23}$$
$$\overbrace{\le}^{(A2)} \max\{\|\epsilon\|_{L^\infty(\Omega,\mathbb{R}^{3\times 3})}, \|\nu\|_{L^\infty(\Omega,\mathbb{R}^{3\times 3})}\}\|\boldsymbol{E}(\omega_\star) - \boldsymbol{E}^{\gamma_n}(\omega^{\gamma_n})\|_{\boldsymbol{H}(\mathbf{curl})}.$$

Thus, (5.19) follows from (5.18). It remains to verify that $\omega_\star \in \mathcal{O}$ is in fact a minimizer of (P). First, we note that since ω^{γ_n} is a solution of (P_γ) for $\gamma = \gamma_n$, the following estimate holds:

$$J_{\gamma_n}(\omega^{\gamma_n}) = \min_{\omega\in\mathcal{O}} J_{\gamma_n}(\omega) \le J_{\gamma_n}(\omega) \quad \forall \omega \in \mathcal{O}. \tag{5.24}$$

Finally, gathering all the previous results, we obtain for every $\omega \in \mathcal{O}$ that

$$\begin{aligned} J(\omega_\star) = \quad & \frac{1}{2}\int_B \kappa|\boldsymbol{E}(\omega_\star) - \boldsymbol{E}_d|^2 dx + \int_{\omega_\star} dx \\ \overbrace{=}^{(5.18)\,\&\,(5.20)} \quad & \lim_{n\to\infty} \frac{1}{2}\int_B \kappa|\boldsymbol{E}^{\gamma_n}(\omega^{\gamma_n}) - \boldsymbol{E}_d|^2 dx + \int_{\omega^{\gamma_n}} dx = \lim_{n\to\infty} J_{\gamma_n}(\omega^{\gamma_n}) \\ \overbrace{\le}^{(5.24)} \quad & \lim_{n\to\infty} J_{\gamma_n}(\omega) = \lim_{n\to\infty} \frac{1}{2}\int_B \kappa|\boldsymbol{E}^{\gamma_n}(\omega) - \boldsymbol{E}_d|^2 dx + \int_\omega dx \\ \overbrace{=}^{(5.12)} \quad & \frac{1}{2}\int_B \kappa|\boldsymbol{E}(\omega) - \boldsymbol{E}_d|^2 dx + \int_\omega dx = J(\omega). \end{aligned}$$

This shows $J(\omega_\star) \le J(\omega)$ for every $\omega \in \mathcal{O}$, which yields the assertion. ∎

Remark 5.4. As we have obtained the optimal shape $\omega_\star \in \mathcal{O}$ in (5.20) as the limit of the optimal shapes for (P_γ), Theorem 2.4 follows immediately from Theorem 5.3.

6. Numerical tests. Our algorithm to obtain a numerical approximation for the optimal shape $\omega_\star$ of (P) is based on a variant of the level set method where the distributed shape derivative (Theorem 4.6) is used to obtain a descent direction (see [25]). We consider the proposed approach (P_γ) with $\gamma = 7\cdot 10^4$. The forward problems (3.1) are computed using the Newton method with a finite element discretization based on the first family of Nédélec's edge elements [33] at roughly 2,000,000 DOFs. As announced in the introduction, we apply our algorithm to two problems stemming from HTS, also widely known as type-II superconductivity.

We choose $\Omega = [-2, 3]^3$ and $B = [0, 1]^3$. For simplicity, we take the material parameters $\epsilon = \nu = \boldsymbol{I}_3$ (cf. (A2)). Moreover, $\boldsymbol{f}$ is a circular current

$$\boldsymbol{f}(x, y, z) = \begin{cases} \dfrac{R}{\sqrt{(y-0.5)^2 + (z-0.5)^2}}(0,\ -z + 0.5,\ y - 0.5) & \text{for } (x, y, z) \in \Omega_p, \\ 0 & \text{for } (x, y, z) \notin \Omega_p, \end{cases}$$

applied to a pipe coil $\Omega_p \subset \Omega$ which is defined by

$$\Omega_p := \left\{(x, y, z) \in \Omega \,:\, |z - 0.5| \le 0.5,\ \sqrt{(x-0.5)^2 + (y-0.5)^2} \in [1.2, 1.6]\right\}.$$

The constant $R > 0$ denotes the electrical resistance of Ω_p (here: $R = 10^{-3}$). As $\Omega_p \cap B = \emptyset$, we have $\boldsymbol{f} \equiv 0$ in B and (A3) is satisfied. Without a superconductor in the system, this current would induce an orthogonal magnetic field which admits its highest field strength inside the coil.

At each new iteration of the optimization algorithm, we use the distributed expression (4.39) of the shape derivative to obtain a new descent direction $\mathbf{\Theta}$. More precisely, let $\boldsymbol{V}_h \subset \boldsymbol{H}^1(B) \cap \boldsymbol{\mathcal{C}}^{0,1}(\overline{B})$ be the space of piecewise linear and continuous finite elements on B. Given a positive definite bilinear form $\mathcal{B} : \boldsymbol{V}_h \times \boldsymbol{V}_h \to \mathbb{R}$, the problem is to find $\mathbf{\Theta} \in \boldsymbol{V}_h$ such that

$$\mathcal{B}(\mathbf{\Theta}, \boldsymbol{\xi}) = -dJ_\gamma(\omega)(\boldsymbol{\xi}) \ \forall \boldsymbol{\xi} \in \boldsymbol{V}_h. \tag{6.1}$$

With this choice, the solution $\mathbf{\Theta}$ of (6.1) is defined on B and is a descent direction since $dJ_\gamma(\omega)(\mathbf{\Theta}) = -\mathcal{B}(\mathbf{\Theta}, \mathbf{\Theta}) < 0$ if $\mathbf{\Theta} \neq 0$. In our algorithm we choose

$$\mathcal{B}(\mathbf{\Theta}, \boldsymbol{\xi}) = \int_B \alpha_1 D\mathbf{\Theta} : D\boldsymbol{\xi} + \alpha_2 \mathbf{\Theta} \cdot \boldsymbol{\xi} \, dx + \alpha_3 \int_{\partial B} (\mathbf{\Theta} \cdot \boldsymbol{n})(\boldsymbol{\xi} \cdot \boldsymbol{n}) \, ds \tag{6.2}$$

with $\alpha_1 = 0.5$, $\alpha_2 = 0.5$, and $\alpha_3 = 1.0$. Moreover, the geometry was optimized in the class of shapes with two symmetries with respect to the planes $x = 0.5$ and $y = 0.5$. This is achieved by symmetrizing $\mathbf{\Theta}$ with respect to these axis, and it can be shown that the symmetrized vector field is still a descent direction. The description of the symmetrization strategy can be found in the extended version of this paper (see [26]).

Then, the moving domain $\omega_t = \boldsymbol{T}_t(\mathbf{\Theta})$ corresponding to the descent direction $\mathbf{\Theta}$ is represented implicitly as the zero sublevel set of a Lipschitz continuous function $\phi : B \times [0, \tau] \to \mathbb{R}$, i.e.,

$$\omega_t = \{x \in B \mid \phi(x,t) < 0\} \quad \text{and} \quad \partial\omega_t = \{x \in B \mid \phi(x,t) = 0\},$$

assuming $|\nabla\phi(\cdot, t)| \neq 0$ on $\partial\omega_t$. It can be shown that the evolution of ϕ corresponding to the flow $\boldsymbol{T}_t(\mathbf{\Theta})$ is determined by the following transport equation:

$$\partial_t \phi(x,t) + \mathbf{\Theta}(x) \cdot \nabla_x \phi(x,t) = 0 \text{ in } B \times [0, \tau]. \tag{6.3}$$

A Lax–Friedrichs flux is used for the discretization of (6.3); we refer to [24] for a detailed description of this level set method including its implementation in a two-dimensional (2D) framework. All codes are written in Python with the open-source finite-element computational software FEniCS [29]. We used ParaView to visualize the 3D plots.

6.1. First example. We set $\boldsymbol{E}_d \equiv 0$ in compliance with (A1) to find the optimal shape of a superconductor that minimizes both the electromagnetic field penetration and the volume of material. This example is motivated by the HTS application in the superconducting shielding (cf. [22]). We take $\kappa \equiv 8 \cdot 10^7$, which is a reasonable choice considering that the electric field strength is roughly $|\boldsymbol{E}| \approx 10^{-3}$ due to the weak applied current strength $|\boldsymbol{f}|$. The initial shape consists of material attached to the boundary of B (see Figure 1(a)). In Figures 1(b) to 1(d) we see some snapshots of the evolving shape generated by our algorithm. The algorithm generates two connected components on the top and the bottom of the (lateral) boundary. It is interesting to observe that the magnetic field ($\mathbf{curl}\,\boldsymbol{E}$) hits the boundary of the bounding box B from above and, despite the small amount of material used, the field lines do not penetrate through the inside of the area enclosed by the superconductor (see Figures 2(b) and 2(d)). Moreover, in Figure 2 we can compare the magnetic field

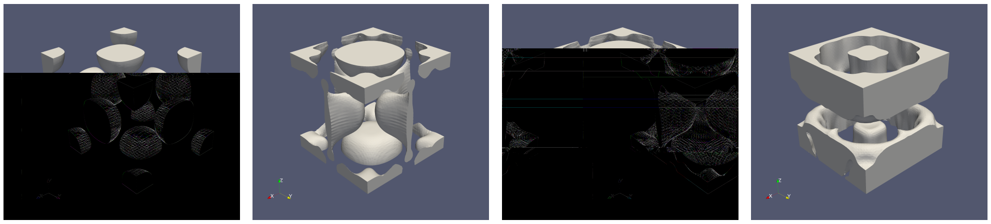

FIG. 1. *Shapes generated by the algorithm at iterations* 0, 42, 45, 143.

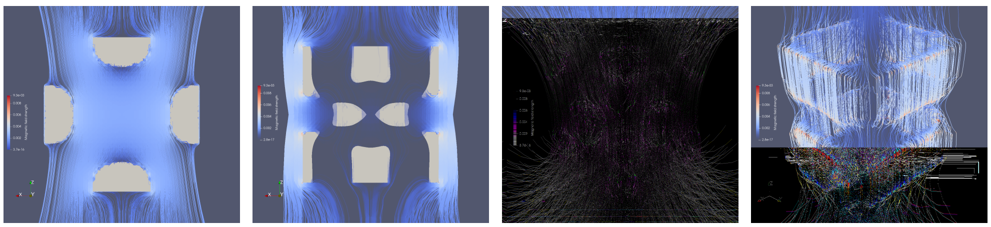

FIG. 2. *Different views on the magnetic field at the initial and the final iteration.* (a)–(b) 2*D slice in the center.* (c)–(d) *Total shot from the same view as Figure* 1.

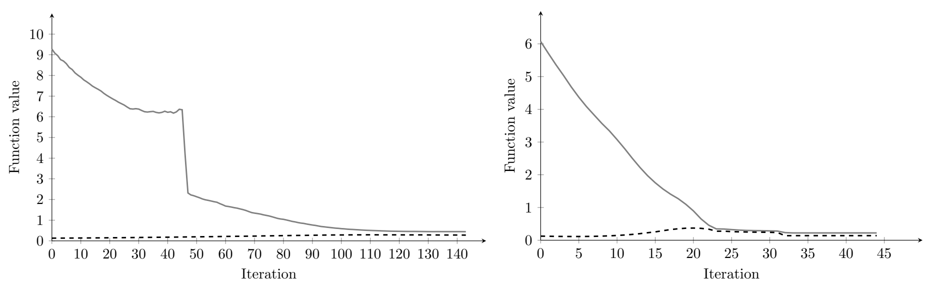

FIG. 3. *Function value (solid) and volume (dashed):* 1. *Example (left),* 2. *Example (right).*

penetration for the initial and the final shape from different camera perspectives. The interior of the initial shape is barely protected from penetration, whereas the final shape redirects the magnetic field lines such that they are condensed on the outside of B.

In the final iteration the functional value is around 0.444 at a volume of roughly 0.278, which is only 27.8% of the volume of B. The E-field fraction in the cost functional amounts roughly to 0.166. This means that there is only a weak magnetic field left in small areas of B. The penetration is mostly between the connected components on the lateral surface of the conducting material. The development of the functional value as well as the volume fraction is documented in Figure 3(a) and the minimal value is reached after roughly 125 iterations. Thereafter, it remains almost constant.

We also observe a slight increase of the cost functional at iterations 43 and 44, due to a topological change in the design. Indeed, at iteration 42 the components on the lateral sides of the cube are disconnected (see Figure 1(b)) and then merge at iteration 45 (see Figure 1(c)). This is a well-known issue with the level set method;

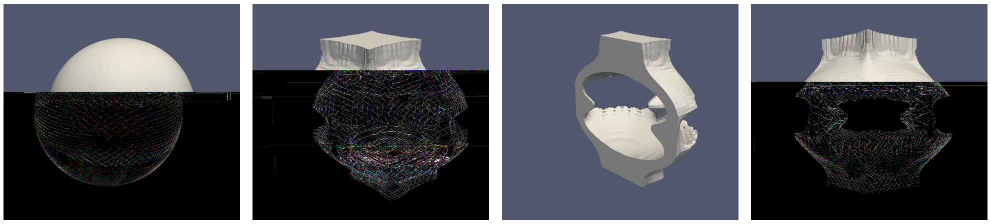

FIG. 4. *The original superconductor and the final shape generated by the algorithm in the second example. The third figure is the final shape clipped along the plane* $x = 0.5$*.*

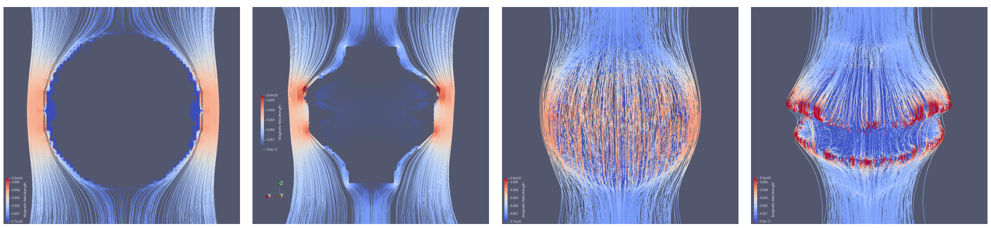

FIG. 5. *Different views on the magnetic field of the original and the final superconductor. Left: 2D slice in the center. Right: Total shot from the same view as Figures* 4(a) *and* 4(b)*.*

see [23] for a recent study on this topic. However, in this example the increase in the functional value is negligible and immediately compensated by a sharp decrease.

6.2. Second example. We place a superconducting ball ω_b with radius $r_b = 0.5$ inside B (see Figure 4(a)) and compute $\boldsymbol{E}_d$ as the corresponding solution of (3.1). The resulting magnetic field is displayed in Figures 5(a) and 5(c). The initial shape and parameters are the same as in the first example (see Figure 1(a)). In the end, we obtain two bell-shaped components connected by small transitions on the boundary. In Figures 4(b) to 4(d) we see this shape from different camera positions. It corresponds to a functional value of 0.223 where the electric field costs get as low as 0.071 at a volume fraction of 0.153. As the original superconductor was a ball with radius 0.5, our algorithm computed an optimal shape with around 70% less material. The development of the functional value and the volume is documented in Figure 3(b). Moreover, the descent in this example is smoother and notably faster than the first example. This could be due to $\boldsymbol{E}_d \not\equiv 0$, which gives more structure than $\boldsymbol{E}_d \equiv 0$, and thus the algorithm has fewer possibilities to design the superconductor and converges faster.

We underline that the optimization problem (P_γ) is highly nonlinear and nonconvex. Therefore, the solution generated by our algorithm can only be expected to be a local solution. In fact, nonconvex optimization problems may admit many different (or even infinitely many) solutions. With regard to this, one may address an open question whether there exists a form ω_{exact} which on the one hand minimizes (P_γ) and at the same time yields the desired target for the corresponding solution to (3.1). This issue is highly related to the question of exact controllability. Previous results in this direction for Maxwell's equations and optimal design can be found in [35, 36, 39].

6.3. Convergence tests with respect to $\boldsymbol{\gamma}$. Let us now report on a numerical test to verify our theoretical convergence result (Theorem 5.3). Since no analytical solution is available for the limit case (P), we consider the computed solution of (P_γ)

for $\gamma = \gamma_{\mathrm{ref}} = 7 \cdot 10^6$ as the reference solution and test the convergence behavior of (P_γ) with respect to γ. More precisely, this is quantitatively clarified by the experimental order of convergence (EOC):

$$\mathrm{EOC}_k := \left| \frac{\log(\mathrm{Error}_k) - \log(\mathrm{Error}_{k-1})}{\log(\gamma_k) - \log(\gamma_{k-1})} \right|,$$

where $\mathrm{Error}_k = \|\chi_{\omega^{\gamma_k}} - \chi_{\omega^{\gamma_{\mathrm{ref}}}}\|_{L^1(\Omega)} + \|\boldsymbol{E}^{\gamma_k} - \boldsymbol{E}^{\gamma_{\mathrm{ref}}}\|_{\boldsymbol{H}(\mathbf{curl})}$. Our numerical results with $\gamma_1 = 10$, $\gamma_2 = 1000$, $\gamma_3 = 7 \cdot 10^4$, and $\gamma_4 = 7 \cdot 10^5$ reveal a first guess for EOC of around 0.45 as confirmed by the following values: $\mathrm{EOC}_2 = 0.468$, $\mathrm{EOC}_3 = 0.516$, $\mathrm{EOC}_4 = 0.425$.

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