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OPTIMAL CONTROL OF QUASILINEAR $\boldsymbol{H}(\mathbf{curl})$-ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS IN MAGNETOSTATIC FIELD PROBLEMS*

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Abstract. This paper examines the mathematical and numerical analysis for optimal control problems governed by quasilinear $\boldsymbol{H}(\mathbf{curl})$-elliptic partial differential equations. We consider a mathematical model involving isotropic materials with magnetic permeability depending strongly on the magnetic field. Due to the physical and mathematical nature of the problem, it is necessary to include divergence-free constraints on the state and the control. The divergence-free control constraint is treated as an explicit variational equality constraint, whereas a Lagrange multiplier is included in the state equation to deal with the divergence-free state constraint. We investigate the sensitivity analysis of the control-to-state operator and establish the associated optimality conditions. Here, the key tool for proving the KKT theory is the Helmholtz decomposition. An important consequence of the optimality system is a higher regularity result for the optimal control, which we prove under the assumption of a nonmagnetic control region. The second part of the paper deals with the finite element analysis based on the edge elements of Nédélec's first family for the control and state discretization, and the continuous piecewise linear ansatz for the Lagrange multiplier discretization. The discrete Helmholtz decomposition and the discrete compactness property of the Nédélec edge elements are the main tools for the finite element analysis. Our final results include the convergence and a priori error estimates for the finite element approximation. Numerical results illustrating the theoretical findings are presented.

Key words. optimal control, quasilinear curl-curl magnetostatic field problems, Maxwell's equations, divergence-free constraint, Helmholtz decomposition, optimality conditions, regularity, Nédélec's edge elements, convergence analysis, error estimates

AMS subject classifications. 49K20, 35Q60, 65N30

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1. Introduction.

Strong material parameter dependence on electromagnetic fields is a well-known physical phenomenon. In the context of magnetism, for instance, there is a wide variety of ferromagnetic and diamagnetic materials whose physical properties can be significantly influenced by external magnetic fields. Such materials play an essential role in many applications and modern technologies. The governing partial differential equations (PDEs) for this electromagnetic phenomenon feature a quasilinear curl-curl structure. Let us recall the magnetostatic field problem emerging from a special case of Maxwell's equations:

$$\begin{cases} \mathbf{curl}\,\boldsymbol{H} = \boldsymbol{J} & \text{in } \Omega, \\ \boldsymbol{H} = \nu(x, |\boldsymbol{B}|)\boldsymbol{B} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{B} = 0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here, the three-dimensional vector fields $\boldsymbol{H}$, $\boldsymbol{B}$, and $\boldsymbol{J}$ denote, respectively, the magnetic field, the magnetic induction, and the current density. The domain $\Omega \subset \mathbb{R}^3$ is assumed to be bounded with a connected Lipschitz boundary Γ. Furthermore, it

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contains isotropic materials whose physical properties depend strongly on the magnetic field. Therefore, the corresponding magnetic reluctivity is given by a scalar nonlinear function $\nu = \nu(x, |\boldsymbol{B}|)$. We note that the magnetic reluctivity ν is nothing but the inverse of the magnetic permeability μ, i.e., $\nu = \mu^{-1}$. In view of Gauss' law for magnetism, the third equation of (1.1), there is a magnetic vector potential $\boldsymbol{y}$ satisfying

$$\mathbf{curl}\, \boldsymbol{y} = \boldsymbol{B} \text{ in } \Omega, \quad \operatorname{div} \boldsymbol{y} = 0 \text{ in } \Omega. \tag{1.2}$$

Putting (1.1)–(1.2) together and considering a perfectly conducting electric boundary condition, we arrive at a nonlinear elliptic boundary value problem of the following type:

$$\begin{cases} \mathbf{curl}\,(\nu(x, |\mathbf{curl}\, \boldsymbol{y}|)\mathbf{curl}\, \boldsymbol{y}) = \boldsymbol{J} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{y} = 0 & \text{in } \Omega, \\ \boldsymbol{y} \times \boldsymbol{n} = 0 & \text{on } \Gamma, \end{cases} \tag{1.3}$$

where $\boldsymbol{n}$ denotes the unit outward normal to Γ.

This paper investigates the mathematical and numerical analysis for optimal control problems governed by (1.3). For practical applications, we consider a control region given by a Lipschitz domain Ω_c satisfying $\overline{\Omega_c} \subset \Omega$. Typically, Ω_c contains electromagnetic coils, in which the applied current density is acting. Then, we formulate the corresponding optimal control problem as

$$\min J(\boldsymbol{y}, \boldsymbol{u}) := \frac{1}{2} \int_\Omega |\boldsymbol{y} - \boldsymbol{y}_d|^2 \, dx + \frac{\kappa}{2} \int_{\Omega_c} |\boldsymbol{u}|^2 \, dx, \tag{P}$$

subject to

$$\begin{cases} \mathbf{curl}\,(\nu(x, |\mathbf{curl}\, \boldsymbol{y}|)\mathbf{curl}\, \boldsymbol{y}) = \chi_{\Omega_c} \boldsymbol{u} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{y} = 0 & \text{in } \Omega, \\ \boldsymbol{y} \times \boldsymbol{n} = 0 & \text{on } \Gamma, \end{cases} \tag{1.4}$$

and the following divergence-free constraint on the applied electric current flow $\boldsymbol{u}$ with a vanishing normal trace condition:

$$\begin{cases} \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega_c, \\ \boldsymbol{u} \cdot \boldsymbol{n}_c = 0 & \text{on } \Gamma_c. \end{cases} \tag{1.5}$$

In the setting of (P), $\kappa \in \mathbb{R}^+$ is the control cost parameter, $\boldsymbol{y}_d \in \boldsymbol{L}^2(\Omega)$ the desired field, χ_{Ω_c} the characteristic function of Ω_c, and $\boldsymbol{n}_c$ the unit outward normal to $\Gamma_c := \partial\Omega_c$. Note that, in view of the charge conservation law, we need to include the divergence-free control constraint (1.5), as $\boldsymbol{u}$ represents current. This constraint is indeed necessary by the mathematical structure of the state equation itself. More precisely, by the distributional definition of the **curl**-operator, it follows from the first equation in (1.4) that

$$(\boldsymbol{u}, \nabla\psi)_{\boldsymbol{L}^2(\Omega_c)} = 0 \quad \forall \psi \in H^1(\Omega_c). \tag{1.6}$$

This variational equality then implies (1.5).

We employ the $\boldsymbol{L}^2(\Omega_c)$-space for the admissible control space of (P) and consider (1.6) as an explicit variational equality constraint of (P) in place of (1.5). Furthermore, the divergence-free constraint on the state is treated through the standard use of a Lagrange multiplier, which leads to a mixed variational formulation of the state equation (1.4) with a quasilinear saddle point structure. Under standard assumptions on the magnetic reluctivity, we study the sensitivity analysis of the control-to-state mapping $\boldsymbol{u} \mapsto \boldsymbol{y}$ and show its differentiability property. Then, we prove the Karush–Kuhn–Tucker (KKT)-type necessary optimality conditions for (P), where the key point for the proof is the Helmholtz decomposition. Based on the structural property of the optimality system, we investigate the regularity property of the optimal control. Our analysis relies mainly on a material assumption (see Assumption 4.1) stating that the control region Ω_c contains only nonmagnetic materials such as copper, silver, or aluminum. In fact, this assumption is reasonable in practice, as copper is often used for electromagnetic coils. Under this material assumption, we are able to prove a regularity result for the optimal control.

The second part of the paper is devoted to the finite element analysis of (P) in the case of Lipschitz-polyhedral computational domains. Here, we discretize both the state and the control using the lowest order edge elements of Nédélec's first family [25], whereas the Lagrange multipliers for the divergence-free constraints are discretized by piecewise linear elements (see (P_h) in section 5.1). This choice of finite element spaces is indeed suitable for handling (P) and its explicit variational equality constraint (1.6). In particular, the discretization allows us to apply the discrete Helmholtz decomposition and the discrete compactness property of the Nédélec edge elements (see Kikuchi [20]) for the feasible control set of (P_h). By the use of these tools, we derive the discrete KKT optimality system and prove a convergence result for (P_h) as $h \to 0$. Finally, invoking all the theoretical findings, we are able to establish an a priori error estimate for the finite element discretization error in the $\boldsymbol{L}^2(\Omega_c)$-norm:

$$\|\overline{\boldsymbol{u}} - \overline{\boldsymbol{u}}_h\|_{\boldsymbol{L}^2(\Omega_c)} \leq c h^{\frac{1+2\epsilon}{4}},$$

where $\overline{\boldsymbol{u}}$ and $\overline{\boldsymbol{u}}_h$ denote, respectively, the optimal controls for the continuous problem (P) and its finite element approximation (P_h). Here, the exponent $\epsilon > 0$ only depends on the geometry and is equal to 0.5 in the case where both Ω and Ω_c are convex-polyhedral.

We refer the reader to the articles by Langer et al. [4, 5, 21] for the mathematical and numerical analysis of multiharmonic eddy current equations. We believe that the result of this paper can be extended to the multiharmonic problems. Also, our result remains true for the case involving a permanent magnet $\boldsymbol{M}$, i.e., to the case where an additional term $\mathbf{curl}\, \boldsymbol{M}$ is included in the right-hand side of the state equation (1.4). This causes only minor and obvious modification. As far as the theory and the numerical analysis of Maxwell's equations are concerned, we refer to the monographs [2, 7, 17, 22, 24], the articles [3, 6, 12, 13, 24, 25, 26], and all the references therein.

To the best of the author's knowledge, this paper is the first contribution towards the mathematical and numerical analysis for optimal control problems governed by *quasilinear* $\boldsymbol{H}(\mathbf{curl})$-elliptic PDEs arising from electromagnetic phenomena. The novelty of the paper also includes the treatment of the variational equality constraint (1.6) through the Helmholtz decomposition. Note that, in the existing literature, we are only aware of the results for optimal control problems of *linear* eddy current equations (see [21, 30, 32, 33, 34]) and magnetohydrodynamics (MHD) [15, 16]. However, the quasilinear case is not a straightforward extension of the linear one and calls for a more careful investigation. For results on optimal control problems of quasilinear $H^1(\Omega)$-elliptic PDEs, we refer to Casas et al. [8, 9, 10]. The rest of this paper is organized as follows. In next section, we introduce our notation and summarize the main assumptions on the magnetic reluctivity ν. Section 3 is devoted to the first-order analysis of (P), including the sensitivity analysis and the derivation of the optimality conditions. Section 4 is concerned with the finite element approximation of (P) and its rigorous analysis. We conclude this paper by providing some numerical results.

2. Preliminaries. As usual, we denote by c a generic positive constant that can take different values on different occasions. For a given Hilbert space V, we use the notation $\|\cdot\|_V$ and $(\cdot,\cdot)_V$ for a standard norm and a standard inner product in V. The Euclidean norm in $\mathbb{R}^3$ is denoted by $|\cdot|$. Furthermore, if V is continuously embedded in another normed function space Y, we write $V \hookrightarrow Y$. We use a bold typeface to indicate a three-dimensional vector function or a Hilbert space of three-dimensional vector functions. In our analysis, we mainly work with the following well-known Hilbert spaces:

$$
\begin{aligned}
\boldsymbol{H}_0(\mathbf{curl}) &= \Big\{\boldsymbol{q} \in \boldsymbol{L}^2(\Omega) \;\Big|\; \mathbf{curl}\,\boldsymbol{q} \in \boldsymbol{L}^2(\Omega), \quad \boldsymbol{q}\times\boldsymbol{n} = 0 \text{ on } \Gamma\Big\},\\
\boldsymbol{H}(\mathrm{div}) &= \Big\{\boldsymbol{q} \in \boldsymbol{L}^2(\Omega) \;\Big|\; \mathrm{div}\,\boldsymbol{q} \in L^2(\Omega)\Big\},\\
\boldsymbol{H}(\mathbf{curl};\Omega_c) &= \Big\{\boldsymbol{q} \in \boldsymbol{L}^2(\Omega_c) \;\Big|\; \mathbf{curl}\,\boldsymbol{q} \in \boldsymbol{L}^2(\Omega_c)\Big\},\\
\boldsymbol{H}_0(\mathrm{div};\Omega_c) &= \Big\{\boldsymbol{q} \in \boldsymbol{L}^2(\Omega_c) \;\Big|\; \mathrm{div}\,\boldsymbol{q} \in L^2(\Omega_c), \quad \boldsymbol{q}\cdot\boldsymbol{n}_c = 0 \text{ on } \Gamma_c\Big\},
\end{aligned}
$$

where the **curl** and div operators as well as the tangential and normal traces are understood in the sense of distributions (see [14, section 2] or [24, section 3.5]).

Let $\mathbb{R}_0^+$ denote the set of all nonnegative real numbers. The magnetic reluctivity $\nu : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}$, $(x,s) \mapsto \nu(x,s)$, is a Carathéodory function; i.e., for every $s \in \mathbb{R}_0^+$ the function $\nu(\cdot,s)$ is measurable, and ν is continuous with respect to s for almost every $x \in \Omega$. The main mathematical assumption for $\nu : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}$ is summarized in the following.

Assumption 2.1 (cf. [4, 5, 19]). Let $\nu_0 > 0$ denote the magnetic reluctivity in a vacuum. There exists a constant $\underline{\nu} \in (0, \nu_0)$ such that

$$\underline{\nu} \le \nu(x,s) \le \nu_0 \quad \text{for almost all } x \in \Omega \text{ and all } s \in \mathbb{R}_0^+, \tag{2.1}$$

$$\lim_{s\to\infty} \nu(x,s) = \nu_0 \quad \text{for almost all } x \in \Omega. \tag{2.2}$$

Furthermore, for almost every $x \in \Omega$, the mapping $s \mapsto \nu(x,s)s$ satisfies

$$(\nu(x,s)s - \nu(x,\hat{s})\hat{s})(s-\hat{s}) \ge \underline{\nu}(s-\hat{s})^2 \quad \forall s,\hat{s} \in \mathbb{R}_0^+, \tag{2.3}$$

$$|\nu(x,s)s - \nu(x,\hat{s})\hat{s}| \le \overline{\nu}|s-\hat{s}| \qquad \forall s,\hat{s} \in \mathbb{R}_0^+ \tag{2.4}$$

with a fixed constant $\overline{\nu} \in [\nu_0, \infty)$.

According to (1.1), the mapping $s \mapsto \nu(x,s)s$ describes the $|\boldsymbol{B}|$-$|\boldsymbol{H}|$-curve. Based on physical measurements, $|\boldsymbol{B}|$-$|\boldsymbol{H}|$-curves for ferromagnetic materials are nonlinear but monotone (see the above mentioned references). This motivates the monotonicity assumption (2.3). On the other hand, the magnetic reluctivity $\nu(x,\cdot)$ is in general not monotone. It is standard to show that the assumptions (2.3)–(2.4) imply the following result.

LEMMA 2.2. *Let Assumption* 2.1 *be satisfied. Then, for almost every $x \in \Omega$, it holds that*

$$(\nu(x,|\boldsymbol{s}|)\boldsymbol{s} - \nu(x,|\widehat{\boldsymbol{s}}|)\widehat{\boldsymbol{s}}) \cdot (\boldsymbol{s} - \widehat{\boldsymbol{s}}) \geq \underline{\nu}|\boldsymbol{s} - \widehat{\boldsymbol{s}}|^2 \quad \forall \boldsymbol{s}, \widehat{\boldsymbol{s}} \in \mathbb{R}^3, \tag{2.5}$$

$$|\nu(x,|\boldsymbol{s}|)\boldsymbol{s} - \nu(x,|\widehat{\boldsymbol{s}}|)\widehat{\boldsymbol{s}}| \leq L|\boldsymbol{s} - \widehat{\boldsymbol{s}}| \quad \forall \boldsymbol{s}, \widehat{\boldsymbol{s}} \in \mathbb{R}^3 \tag{2.6}$$

with $L = 2\nu_0 + \overline{\nu}$.

Proof. We provide the proof in the appendix.

3. Analysis of (P).

This section is devoted to the first-order analysis of (P). We begin by investigating the weak formulation of the state equation (1.4).

3.1. Mixed variational formulation.

We focus on the mixed variational formulation of (1.4),

$$\begin{cases} \langle A(\boldsymbol{y}), \boldsymbol{v}\rangle + b(\boldsymbol{v}, \phi) = (\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega_c)} & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}), \\ b(\boldsymbol{y}, \psi) = 0 & \forall \psi \in H_0^1(\Omega), \end{cases} \tag{3.1}$$

where the operator $A : \boldsymbol{H}_0(\mathbf{curl}) \to \boldsymbol{H}_0(\mathbf{curl})^*$ and the bilinear form $b : \boldsymbol{H}_0(\mathbf{curl}) \times H_0^1(\Omega) \to \mathbb{R}$ are defined as follows:

$$\langle A(\boldsymbol{y}), \boldsymbol{v}\rangle := \int_\Omega \nu(x, |\mathbf{curl}\,\boldsymbol{y}|)\mathbf{curl}\,\boldsymbol{y} \cdot \mathbf{curl}\,\boldsymbol{v}\,dx \quad \forall \boldsymbol{y}, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}), \tag{3.2}$$

$$b(\boldsymbol{y}, \psi) := \int_\Omega \boldsymbol{y} \cdot \nabla\psi\,dx \qquad \forall \boldsymbol{y} \in \boldsymbol{H}_0(\mathbf{curl}), \forall \psi \in H_0^1(\Omega). \tag{3.3}$$

We note that the mixed variational formulation (3.1) features a nonlinear saddle point structure. If

$$(\boldsymbol{u}, \nabla\psi)_{\boldsymbol{L}^2(\Omega_c)} = 0 \quad \forall \psi \in H^1(\Omega_c), \tag{3.4}$$

then the Lagrange multiplier ϕ vanishes. Therefore, for every control $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c)$ satisfying (3.4), the mixed variational formulation (3.1) is indeed equivalent to the variational formulation of the state equation (1.4). The existence and uniqueness result for (3.1) follows from Scheurer [28]. First, in view of Lemma 2.2, the operator A satisfies

$$\langle A(\boldsymbol{y}) - A(\widehat{\boldsymbol{y}}), \boldsymbol{y} - \widehat{\boldsymbol{y}}\rangle \geq \underline{\nu}\|\mathbf{curl}\,\boldsymbol{y} - \mathbf{curl}\,\widehat{\boldsymbol{y}}\|^2_{\boldsymbol{L}^2(\Omega)} \qquad \forall \boldsymbol{y}, \widehat{\boldsymbol{y}} \in \boldsymbol{H}_0(\mathbf{curl}), \tag{3.5}$$

$$|\langle A(\boldsymbol{y}) - A(\widehat{\boldsymbol{y}}), \boldsymbol{v}\rangle| \leq L\|\boldsymbol{y} - \widehat{\boldsymbol{y}}\|_{\boldsymbol{H}(\mathbf{curl})}\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})} \quad \forall \boldsymbol{y}, \widehat{\boldsymbol{y}}, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \tag{3.6}$$

with $L = 2\nu_0 + \overline{\nu}$. Let us further introduce the subspace

$$\begin{aligned} \boldsymbol{Z} &:= \left\{ \boldsymbol{y} \in \boldsymbol{H}_0(\mathbf{curl}) \;\middle|\; (\boldsymbol{y}, \nabla\psi)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \psi \in H_0^1(\Omega) \right\} \\ &= \left\{ \boldsymbol{y} \in \boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div}) \;\middle|\; \mathrm{div}\,\boldsymbol{y} = 0 \text{ in } \Omega \right\}, \end{aligned}$$

which is equipped with the norm of $\boldsymbol{H}(\mathbf{curl})$. It is well known that the embedding

$$\boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div}) \hookrightarrow \boldsymbol{L}^2(\Omega) \tag{3.7}$$

is compact (see Weck [31]). Also, since the boundary Γ is connected, we have that
(3.8)
$$\boldsymbol{DF}(\Omega) := \Big\{ \boldsymbol{y} \in \boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div}) \;\Big|\; \mathbf{curl}\, \boldsymbol{y} = 0 \text{ in } \Omega,\ \mathrm{div}\, \boldsymbol{y} = 0 \text{ in } \Omega \Big\} = \Big\{ 0 \Big\}.$$

For more details, we refer the reader to [3, 27]. A well-known consequence of the compact embedding (3.7) and (3.8) is the following Poincaré–Friedrichs-type inequality:

$$\|\boldsymbol{y}\|_{\boldsymbol{L}^2(\Omega)} \leq c \|\mathbf{curl}\, \boldsymbol{y}\|_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{y} \in \boldsymbol{Z} \tag{3.9}$$

with a constant $c > 0$ depending only on Ω. Then, using (3.9) in (3.5), we find a constant $\widehat{c} > 0$ depending only on $\underline{\nu}$ and Ω, such that

$$\langle A(\boldsymbol{y}) - A(\widehat{\boldsymbol{y}}), \boldsymbol{y} - \widehat{\boldsymbol{y}} \rangle \geq \widehat{c} \|\boldsymbol{y} - \widehat{\boldsymbol{y}}\|^2_{\boldsymbol{H}(\mathbf{curl})} \quad \forall \boldsymbol{y}, \widehat{\boldsymbol{y}} \in \boldsymbol{Z}. \tag{3.10}$$

Furthermore, the bilinear form $b : \boldsymbol{H}_0(\mathbf{curl}) \times H^1_0(\Omega) \to \mathbb{R}$ fulfills the Ladyzhenskaya–Babuška–Brezzi (LBB) condition:
(3.11)
$$\sup_{\boldsymbol{y} \in \boldsymbol{H}_0(\mathbf{curl})} \frac{|b(\boldsymbol{y}, \psi)|}{\|\boldsymbol{y}\|_{\boldsymbol{H}(\mathbf{curl})}} \geq \frac{|b(\nabla \psi, \psi)|}{\|\nabla \psi\|_{\boldsymbol{H}(\mathbf{curl})}} \underbrace{=}_{(3.3)} \|\nabla \psi\|_{\boldsymbol{L}^2(\Omega)} \geq c \|\psi\|_{H^1_0(\Omega)} \quad \forall \psi \in H^1_0(\Omega)$$

with $c > 0$ depending only on Ω. Note that, since $\psi \in H^1_0(\Omega)$, we have that $\nabla \psi \in \boldsymbol{H}_0(\mathbf{curl})$, and hence we may insert $\boldsymbol{y} = \nabla \psi$ in (3.11) to get the LBB condition. Here, we also used the fact that $\|\nabla \psi\|_{\boldsymbol{H}(\mathbf{curl})} = \|\nabla \psi\|_{\boldsymbol{L}^2(\Omega)}$ since $\mathbf{curl}\, \nabla \equiv 0$. In consequence of (3.6), (3.10), and (3.11), [28, Proposition 2.3] yields the following existence and uniqueness result.

LEMMA 3.1. *Let Assumption* 2.1 *be satisfied. Then, for every* $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c)$, *the mixed variational problem* (3.1) *admits a unique solution* $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{u}) \in \boldsymbol{Z}$ *with a unique Lagrange multiplier* $\phi = \phi(\boldsymbol{u}) \in H^1_0(\Omega)$. *If* $\boldsymbol{u}$ *satisfies* (3.4), *then* $\phi(\boldsymbol{u}) \equiv 0$. *Furthermore, there exists a constant* $c > 0$ *depending only on* $\underline{\nu}$ *and* Ω, *such that*

$$\|\boldsymbol{y}(\boldsymbol{u})\|_{\boldsymbol{H}(\mathbf{curl})} + \|\phi(\boldsymbol{u})\|_{H^1_0(\Omega)} \leq c \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega_c)}$$

for all $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c)$.

We denote the control-to-state operator by

$$\boldsymbol{G} : \boldsymbol{L}^2(\Omega_c) \to \boldsymbol{Z}, \quad \boldsymbol{u} \mapsto \boldsymbol{y},$$

that assigns to every control $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c)$ the unique solution $\boldsymbol{y} \in \boldsymbol{Z}$ of the nonlinear saddle point problem (3.1).

PROPOSITION 3.2. *Let Assumption* 2.1 *be satisfied. If* $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}$ *weakly in* $\boldsymbol{L}^2(\Omega_c)$ *as* $k \to \infty$, *then* $\boldsymbol{G}(\boldsymbol{u}_k) \to \boldsymbol{G}(\boldsymbol{u})$ *strongly in* $\boldsymbol{Z}$ *as* $k \to \infty$.

Proof. We consider a sequence $\{\boldsymbol{u}_k\}_{k=1}^{\infty} \subset \boldsymbol{L}^2(\Omega_c)$ satisfying

$$\boldsymbol{u}_k \rightharpoonup \boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c) \quad \text{weakly in } \boldsymbol{L}^2(\Omega_c) \quad \text{as } k \to \infty. \tag{3.12}$$

For every $k \in \mathbb{N}$, we set $\boldsymbol{y}_k = \boldsymbol{G}(\boldsymbol{u}_k) \in \boldsymbol{Z}$ and $\phi_k = \phi(\boldsymbol{u}_k) \in H^1_0(\Omega)$. In other words, the pair $(\boldsymbol{y}_k, \phi_k)$ satisfies

$$\begin{cases} \langle A(\boldsymbol{y}_k), \boldsymbol{v} \rangle + b(\boldsymbol{v}, \phi_k) = (\boldsymbol{u}_k, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega_c)} & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}), \\ b(\boldsymbol{y}_k, \psi) = 0 & \forall \psi \in H^1_0(\Omega). \end{cases} \tag{3.13}$$

In view of Lemma 3.1, the sequences $\{\boldsymbol{y}_k\}_{k=1}^\infty$ and $\{\phi_k\}_{k=1}^\infty$ are bounded, respectively, in $\boldsymbol{Z}$ and $H_0^1(\Omega)$. For this reason, by possibly extracting subsequences, we find a pair $(\boldsymbol{y}, \phi) \in \boldsymbol{Z} \times H_0^1(\Omega)$ such that

$$\boldsymbol{y}_k \rightharpoonup \boldsymbol{y} \quad \text{weakly in } \boldsymbol{Z} \quad \text{as } k \to \infty, \tag{3.14}$$
$$\phi_k \rightharpoonup \phi \quad \text{weakly in } H_0^1(\Omega) \quad \text{as } k \to \infty. \tag{3.15}$$

By the compactness of the embedding $\boldsymbol{Z} \hookrightarrow \boldsymbol{L}^2(\Omega)$, it follows that

$$\boldsymbol{y}_k \to \boldsymbol{y} \quad \text{strongly in } \boldsymbol{L}^2(\Omega) \quad \text{as } k \to \infty. \tag{3.16}$$

Setting $\boldsymbol{v} = \boldsymbol{y}_k - \boldsymbol{y}$ in (3.13) yields

$$\begin{aligned}\langle A(\boldsymbol{y}_k) - A(\boldsymbol{y}), \boldsymbol{y}_k - \boldsymbol{y}\rangle &= \langle A(\boldsymbol{y}_k), \boldsymbol{y}_k - \boldsymbol{y}\rangle - \langle A(\boldsymbol{y}), \boldsymbol{y}_k - \boldsymbol{y}\rangle \\ &= (\boldsymbol{u}_k, \boldsymbol{y}_k - \boldsymbol{y})_{\boldsymbol{L}^2(\Omega_c)} - \langle A(\boldsymbol{y}), \boldsymbol{y}_k - \boldsymbol{y}\rangle,\end{aligned}$$

where we have also used $b(\boldsymbol{y}_k - \boldsymbol{y}, \phi_k) = 0$. Thus, by (3.10),

$$\widehat{c}\|\boldsymbol{y}_k - \boldsymbol{y}\|^2_{\boldsymbol{H}(\mathbf{curl})} \le (\boldsymbol{u}_k, \boldsymbol{y}_k - \boldsymbol{y})_{\boldsymbol{L}^2(\Omega_c)} - \langle A(\boldsymbol{y}), \boldsymbol{y}_k - \boldsymbol{y}\rangle.$$

Then, passing to the limit $k \to \infty$, the weak convergences (3.12) and (3.14) together with the strong convergence (3.16) imply that

$$\boldsymbol{y}_k \to \boldsymbol{y} \quad \text{strongly in } \boldsymbol{H}_0(\mathbf{curl}) \quad \text{as } k \to \infty. \tag{3.17}$$

It remains to prove that $\boldsymbol{y} = \boldsymbol{G}(\boldsymbol{u})$ and $\phi = \phi(\boldsymbol{u})$. In view of (3.17), there exists a subsequence of $\{\boldsymbol{y}_k\}_{k=1}^\infty$, denoted again by $\{\boldsymbol{y}_k\}_{k=1}^\infty$, such that

$$|\mathbf{curl}\, \boldsymbol{y}_k(x)| \to |\mathbf{curl}\, \boldsymbol{y}(x)| \quad \text{a.e. in } \Omega \quad \text{as } k \to \infty$$

and consequently

$$\nu(x, |\mathbf{curl}\, \boldsymbol{y}_k(x)|) \to \nu(x, |\mathbf{curl}\, \boldsymbol{y}(x)|) \quad \text{a.e. in } \Omega \quad \text{as } k \to \infty.$$

Hence, taking (2.1) into account, Lebesgue's dominated convergence theorem implies that

$$\nu(\cdot, |\mathbf{curl}\, \boldsymbol{y}_k|)\mathbf{curl}\, \boldsymbol{v} \to \nu(\cdot, |\mathbf{curl}\, \boldsymbol{y}|)\mathbf{curl}\, \boldsymbol{v} \quad \text{strongly in } \boldsymbol{L}^2(\Omega) \quad \text{as } k \to \infty$$

for every $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl})$. From the above convergence, (3.15), and (3.17), it follows that

$$(\boldsymbol{u}_k, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega_c)} = \langle A(\boldsymbol{y}_k), \boldsymbol{v}\rangle + b(\boldsymbol{v}, \phi_k) \to \langle A(\boldsymbol{y}), \boldsymbol{v}\rangle + b(\boldsymbol{v}, \phi) \quad \text{as } k \to \infty.$$

On the other hand, the left-hand side in the above equality converges to $(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega_c)}$. In conclusion, the pair $(\boldsymbol{y}, \phi) \in \boldsymbol{Z} \times H_0^1(\Omega)$ satisfies

$$\begin{cases} \langle A(\boldsymbol{y}), \boldsymbol{v}\rangle + b(\boldsymbol{v}, \phi) = (\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega_c)} & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}), \\ b(\boldsymbol{y}, \psi) = 0 & \forall \psi \in H_0^1(\Omega), \end{cases}$$

from which we deduce that $\boldsymbol{y} = \boldsymbol{G}(\boldsymbol{u})$ and $\phi = \phi(\boldsymbol{u})$. ∎

We consider next the control-to-state operator $\boldsymbol{G}$ with range in $\boldsymbol{L}^2(\Omega)$ and denote the corresponding operator by

$$\boldsymbol{\mathcal{S}} : \boldsymbol{L}^2(\Omega_c) \to \boldsymbol{L}^2(\Omega), \quad \boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{I}}\boldsymbol{G},$$

where $\boldsymbol{\mathcal{I}}$ denotes the injection $\boldsymbol{Z} \hookrightarrow \boldsymbol{L}^2(\Omega)$. Employing the operator $\boldsymbol{\mathcal{S}}$, we reformulate the optimal control problem (P) as an optimization problem in Banach spaces:

$$\text{(P)} \qquad \begin{cases} \displaystyle\min_{\boldsymbol{u}\in \boldsymbol{L}^2(\Omega_c)} & f(\boldsymbol{u}) := J(\boldsymbol{\mathcal{S}}(\boldsymbol{u}), \boldsymbol{u}) \\ \text{s.t.} & (\boldsymbol{u}, \nabla\psi)_{\boldsymbol{L}^2(\Omega_c)} = 0 \quad \forall \psi \in H^1(\Omega_c). \end{cases}$$

We introduce the feasible set associated with (P) by

$$\boldsymbol{U}^{feas} := \Big\{ \boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c) \;\Big|\; (\boldsymbol{u}, \nabla\psi)_{\boldsymbol{L}^2(\Omega_c)} = 0 \; \forall \psi \in H^1(\Omega_c) \Big\},$$

and so (P) can be equivalently written as

$$\min_{\boldsymbol{u}\in \boldsymbol{U}^{feas}} f(\boldsymbol{u}).$$

Note that $\boldsymbol{U}^{feas}$ is a convex and closed subset of $\boldsymbol{L}^2(\Omega_c)$. In what follows, a feasible control $\overline{\boldsymbol{u}} \in \boldsymbol{U}^{feas}$ is said to be optimal if and only if it fulfills $f(\overline{\boldsymbol{u}}) \le f(\boldsymbol{u})$ for all $\boldsymbol{u} \in \boldsymbol{U}^{feas}$.

THEOREM 3.3. *Let Assumption* 2.1 *be satisfied. Then, the optimal control problem* (P) *admits at least one globally optimal control.*

Proof. In view of Proposition 3.2, the existence result follows by classical arguments (cf. [29, section 4.4]). ∎

We remark that, as the operator $\boldsymbol{\mathcal{S}}$ is nonlinear, the objective functional $f : \boldsymbol{L}^2(\Omega_c) \to \mathbb{R}$ is nonconvex. Therefore, the optimal control cannot be expected to be unique.

3.2. Gâteaux differentiability. Our goal now is to establish the Gâteaux differentiability of the operator $\boldsymbol{\mathcal{S}} : \boldsymbol{L}^2(\Omega_c) \to \boldsymbol{L}^2(\Omega)$. To this end, we introduce a (vector) function:

$$\boldsymbol{\mathcal{F}} : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3, \quad \boldsymbol{\mathcal{F}}(x, \boldsymbol{s}) = \nu(x, |\boldsymbol{s}|)\boldsymbol{s}.$$

Using this function, the nonlinear operator $A : \boldsymbol{H}_0(\mathbf{curl}) \to \boldsymbol{H}_0(\mathbf{curl})^*$, defined as in (3.2), can be expressed as

$$\langle A(\boldsymbol{y}), \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{\mathcal{F}}(x, \mathbf{curl}\, \boldsymbol{y}) \cdot \mathbf{curl}\, \boldsymbol{v}\, dx \quad \forall \boldsymbol{y}, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}). \tag{3.18}$$

For our analysis, we require the following differentiability assumption.

Assumption 3.4. For almost every $x \in \Omega$, $\nu(x, \cdot) : (0, \infty) \to \mathbb{R}$ and $\boldsymbol{\mathcal{F}}(x, \cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ are continuously differentiable. There exists a constant $C > 0$ such that

$$\left| \frac{\partial \boldsymbol{\mathcal{F}}_i}{\partial s_j}(x, \boldsymbol{s}) \right| \le C \quad \text{for almost all } x \in \Omega \text{ and all } \boldsymbol{s} \in \mathbb{R}^3 \tag{3.19}$$

for all $i, j = 1, 2, 3$.

From now on, we suppose that Assumption 2.1 and Assumption 3.4 are fulfilled. An instance for the function ν satisfying these assumptions is given in the following.

Example 3.5. *Let* $\Omega = \Omega_1 \cup \ldots \cup \Omega_n$ *with* $n \geq 1$ *and pairwise disjoint subdomains* $\Omega_1, \ldots, \Omega_n \subset \Omega$. *We consider*

$$\nu(x,s) = \nu_0 - \sum_{k=1}^{n} \chi_{\Omega_k}(x)\theta_k \exp(-\alpha_k s^2) \quad \forall (x,s) \in \Omega \times \mathbb{R}_0^+$$

with $\alpha_k > 0$ *and* $0 \leq \theta_k < \nu_0$ *for all* $k = 1, \ldots, n$. *By definition, we have*

$$\boldsymbol{\mathcal{F}}(x,\boldsymbol{s}) = \nu(x,|\boldsymbol{s}|)\boldsymbol{s} = \nu_0 \boldsymbol{s} - \sum_{k=1}^{n} \chi_{\Omega_k}(x)\theta_k \exp(-\alpha_k |\boldsymbol{s}|^2)\boldsymbol{s} \quad \forall (x,\boldsymbol{s}) \in \Omega \times \mathbb{R}^3.$$

For every $i, j = 1,2,3$, *it holds that*

$$\frac{\partial \boldsymbol{\mathcal{F}}_i}{\partial s_j}(x,\boldsymbol{s}) = \nu_0 \delta_{ij} - \sum_{k=1}^{n} \chi_{\Omega_k}(x)\theta_k \exp(-\alpha_k |\boldsymbol{s}|^2)\,(\delta_{ij} - 2\alpha_k s_j s_i) \quad \forall \boldsymbol{s} \in \mathbb{R}^3,$$

where δ_{ij} *denotes the Kronecker delta. Therefore, Assumption* 2.1 *and Assumption* 3.4 *are satisfied for this example.*

We introduce next the Jacobian matrix function

$$\nabla_{\boldsymbol{s}} \boldsymbol{\mathcal{F}} : \Omega \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}, \quad \nabla_{\boldsymbol{s}} \boldsymbol{\mathcal{F}}(x,\boldsymbol{s}) = \left(\frac{\partial \boldsymbol{\mathcal{F}}_i}{\partial \boldsymbol{s}_j}(x,\boldsymbol{s}) \right)_{i,j},$$

and consider the following set:

$$\overline{\boldsymbol{Y}} := \Big\{ \overline{\boldsymbol{y}} \in \boldsymbol{H}_0(\mathbf{curl}) \;\Big|\; \exists\, c = c(\overline{\boldsymbol{y}}) > 0 \;:\; (\nabla_{\boldsymbol{s}} \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\overline{\boldsymbol{y}})\boldsymbol{v}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} \geq c \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \Big\}.$$

Having defined the set $\overline{\boldsymbol{Y}}$, we now turn to the following *linear* saddle point problem: given a $\overline{\boldsymbol{y}} \in \overline{\boldsymbol{Y}}$, find a pair $(\boldsymbol{y}, \phi) \in \boldsymbol{H}_0(\mathbf{curl}) \times H_0^1(\Omega)$ such that

$$\text{(3.20)} \qquad \begin{cases} a_{\overline{\boldsymbol{y}}}(\boldsymbol{y},\boldsymbol{v}) + b(\boldsymbol{v},\phi) = (\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{L}^2(\Omega_c)} & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}), \\ b(\boldsymbol{y},\psi) = 0 & \forall \psi \in H_0^1(\Omega), \end{cases}$$

where the bilinear form $a_{\overline{\boldsymbol{y}}} : \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}_0(\mathbf{curl}) \to \mathbb{R}$ is defined by

$$a_{\overline{\boldsymbol{y}}}(\boldsymbol{y},\boldsymbol{v}) := (\nabla_{\boldsymbol{s}} \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\overline{\boldsymbol{y}})\mathbf{curl}\,\boldsymbol{y}, \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{y}, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}).$$

In view of Assumption 3.4, the matrix function $\nabla_{\boldsymbol{s}} \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\overline{\boldsymbol{y}}) : \Omega \to \mathbb{R}^{3\times 3}$ has entries in $L^\infty(\Omega)$, and hence $a_{\overline{\boldsymbol{y}}}(\cdot,\cdot)$ is a bounded bilinear form. Moreover, as $\overline{\boldsymbol{y}} \in \overline{\boldsymbol{Y}}$, the bilinear form $a_{\overline{\boldsymbol{y}}}$ fulfills

$$a_{\overline{\boldsymbol{y}}}(\boldsymbol{y},\boldsymbol{y}) \geq c \|\mathbf{curl}\,\boldsymbol{y}\|^2_{\boldsymbol{L}^2(\Omega)} \underbrace{\geq}_{(3.9)} c \|\boldsymbol{y}\|^2_{\boldsymbol{H}(\mathbf{curl})} \quad \forall \boldsymbol{y} \in \boldsymbol{Z}$$

with a constant $c > 0$ independent of $\boldsymbol{y}$. In conclusion, we obtain the following existence and uniqueness result.

LEMMA 3.6. *Let $\overline{\boldsymbol{y}} \in \overline{\boldsymbol{Y}}$. Then, for every $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c)$, the linear saddle point problem* (3.20) *admits a unique solution $\boldsymbol{y} \in \boldsymbol{Z}$ with a unique Lagrange multiplier $\phi \in H_0^1(\Omega)$. This solution satisfies*

$$\|\boldsymbol{y}\|_{\boldsymbol{H}(\mathbf{curl})} + \|\phi\|_{H_0^1(\Omega)} \leq c\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega_c)} \tag{3.21}$$

with a constant $c > 0$ independent of $\boldsymbol{u}, \boldsymbol{y}$, and ϕ.

We are now in the position to establish the Gâteaux differentiability of the operator $\boldsymbol{\mathcal{S}} : \boldsymbol{L}^2(\Omega_c) \to \boldsymbol{L}^2(\Omega)$.

PROPOSITION 3.7. *It holds that*

$$(\nabla_{\boldsymbol{s}} \boldsymbol{\mathcal{F}}(x, \boldsymbol{s})\boldsymbol{a}, \boldsymbol{a})_{\mathbb{R}^3} \geq \underline{\nu}|\boldsymbol{a}|^2 \quad \text{for almost all } x \in \Omega \text{ and all } \boldsymbol{s}, \boldsymbol{a} \in \mathbb{R}^3, \tag{3.22}$$

which implies $\overline{\boldsymbol{Y}} = \boldsymbol{H}_0(\mathbf{curl})$. Furthermore, the operator $\boldsymbol{\mathcal{S}} : \boldsymbol{L}^2(\Omega_c) \to \boldsymbol{L}^2(\Omega)$ is Gâteaux differentiable. For all $\overline{\boldsymbol{u}}, \boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c)$, $\boldsymbol{\mathcal{S}}'(\overline{\boldsymbol{u}})\boldsymbol{u} = \boldsymbol{y}$ is given by the unique solution of (3.20) *with $\overline{\boldsymbol{y}} = \boldsymbol{G}(\overline{\boldsymbol{u}})$. Finally, the objective functional $f : \boldsymbol{L}^2(\Omega_c) \to \mathbb{R}$ is Gâteaux differentiable and*

$$f'(\overline{\boldsymbol{u}})\boldsymbol{u} = \int_\Omega (\boldsymbol{\mathcal{S}}(\overline{\boldsymbol{u}}) - \boldsymbol{y}_d) \cdot \boldsymbol{\mathcal{S}}'(\overline{\boldsymbol{u}})\boldsymbol{u}\, dx + \kappa \int_{\Omega_c} \overline{\boldsymbol{u}} \cdot \boldsymbol{u}\, dx \quad \forall \overline{\boldsymbol{u}}, \boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c). \tag{3.23}$$

Proof. Let us first show (3.22). For almost every $x \in \Omega$ and every $\boldsymbol{s} \in \mathbb{R}^3 \setminus \{0\}$, it holds that

$$\begin{aligned}
(\nabla_{\boldsymbol{s}} \boldsymbol{\mathcal{F}}(x, \boldsymbol{s})\boldsymbol{a}, \boldsymbol{a})_{\mathbb{R}^3} &= \nu(x, |\boldsymbol{s}|)|\boldsymbol{a}|^2 + \frac{\nu'(x, |\boldsymbol{s}|)}{|\boldsymbol{s}|}(\boldsymbol{s}, \boldsymbol{a})^2_{\mathbb{R}^3} \\
&= |\boldsymbol{s}|^{-2}(\nu(x, |\boldsymbol{s}|)(|\boldsymbol{s}|^2|\boldsymbol{a}|^2 - (\boldsymbol{s}, \boldsymbol{a})^2_{\mathbb{R}^3}) \\
&\quad + (\nu(x, |\boldsymbol{s}|) + \nu'(x, |\boldsymbol{s}|)|\boldsymbol{s}|)(\boldsymbol{s}, \boldsymbol{a})^2_{\mathbb{R}^3}) \\
&\geq |\boldsymbol{s}|^{-2}(\underline{\nu}(|\boldsymbol{s}|^2|\boldsymbol{a}|^2 - (\boldsymbol{s}, \boldsymbol{a})^2_{\mathbb{R}^3}) + \underline{\nu}(\boldsymbol{s}, \boldsymbol{a})^2_{\mathbb{R}^3}) = \underline{\nu}|\boldsymbol{a}|^2 \quad \forall \boldsymbol{a} \in \mathbb{R}^3,
\end{aligned}$$

where we have used the inequalities $\nu(x, |\boldsymbol{s}|) \geq \underline{\nu}$ and $(\nu(x, |\boldsymbol{s}|) + \nu'(x, |\boldsymbol{s}|)|\boldsymbol{s}|) \geq \underline{\nu}$. Note that the latter inequality follows from (2.3). On the other hand, for almost every $x \in \Omega$ and $\boldsymbol{s} = 0$, we have that

$$\begin{aligned}
(\nabla_{\boldsymbol{s}} \boldsymbol{\mathcal{F}}(x, 0)\boldsymbol{a}, \boldsymbol{a})_{\mathbb{R}^3} = \left(\lim_{\tau \to 0} \frac{\boldsymbol{\mathcal{F}}(x, \tau\boldsymbol{a}) - \boldsymbol{\mathcal{F}}(x, 0)}{\tau}, \boldsymbol{a} \right)_{\mathbb{R}^3} &= \lim_{\tau \to 0} \nu(x, |\tau\boldsymbol{a}|)|\boldsymbol{a}|^2 \\
&= \nu(x, 0)|\boldsymbol{a}|^2 \geq \underline{\nu}|\boldsymbol{a}|^2
\end{aligned}$$

for all $\boldsymbol{a} \in \mathbb{R}^3$. In conclusion, (3.22) is true, and hence $\overline{\boldsymbol{Y}} = \boldsymbol{H}_0(\mathbf{curl})$.

Now, we prove that the control-to-state operator $\boldsymbol{\mathcal{S}} : \boldsymbol{L}^2(\Omega_c) \to \boldsymbol{L}^2(\Omega)$ is Gâteaux differentiable. Let $\overline{\boldsymbol{u}}, \boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c)$ and set $\overline{\boldsymbol{y}} = \boldsymbol{G}(\overline{\boldsymbol{u}})$. For every $\tau > 0$, let $(\boldsymbol{y}_\tau, \phi_\tau) \in \boldsymbol{Z} \times H_0^1(\Omega)$ denote the unique solution of

$$\begin{cases} \langle A(\boldsymbol{y}_\tau), \boldsymbol{v} \rangle + b(\boldsymbol{v}, \phi_\tau) = (\overline{\boldsymbol{u}} + \tau\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega_c)} & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}), \\ b(\boldsymbol{y}_\tau, \psi) = 0 & \forall \psi \in H_0^1(\Omega). \end{cases} \tag{3.24}$$

Then, we have that

$$\langle A(\boldsymbol{y}_\tau) - A(\overline{\boldsymbol{y}}), \boldsymbol{v} \rangle + b(\boldsymbol{v}, \phi_\tau - \overline{\phi}) = (\overline{\boldsymbol{u}} + \tau\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega_c)} - (\overline{\boldsymbol{u}}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega_c)} = \tau(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega_c)} \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}). \tag{3.25}$$

Setting $\boldsymbol{v} = \boldsymbol{y}_\tau - \overline{\boldsymbol{y}}$ in (3.25) and then using (3.10), we obtain

$$\widehat{c}\|\boldsymbol{y}_\tau - \overline{\boldsymbol{y}}\|^2_{\boldsymbol{H}(\mathbf{curl})} \leq \langle A(\boldsymbol{y}_\tau) - A(\overline{\boldsymbol{y}}), \boldsymbol{y}_\tau - \overline{\boldsymbol{y}}\rangle = \tau(\boldsymbol{u}, \boldsymbol{y}_\tau - \overline{\boldsymbol{y}})_{\boldsymbol{L}^2(\Omega)},$$

from which it follows that

$$\widehat{c}\left\|\frac{\boldsymbol{y}_\tau - \overline{\boldsymbol{y}}}{\tau}\right\|_{\boldsymbol{H}(\mathbf{curl})} \leq \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega_c)} \quad \forall \tau > 0. \tag{3.26}$$

Analogously, setting $\boldsymbol{v} = \nabla\phi_\tau - \nabla\overline{\phi}$ in (3.25) results in

$$\left\|\frac{\phi_\tau - \bar{\phi}}{\tau}\right\|_{H^1_0(\Omega)} \leq c\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega_c)} \quad \forall \tau > 0 \tag{3.27}$$

with a constant $c > 0$ depending only on Ω. In view of (3.26)–(3.27), the sequences $\{\frac{\boldsymbol{y}_\tau - \overline{\boldsymbol{y}}}{\tau}\}_{\tau>0}$ and $\{\frac{\phi_\tau - \bar{\phi}}{\tau}\}_{\tau>0}$ are bounded, respectively, in $\boldsymbol{Z}$ and $H^1_0(\Omega)$. Thus, by possibly extracting subsequences, we find a pair $(\boldsymbol{y}, \phi) \in \boldsymbol{Z} \times H^1_0(\Omega)$ such that

$$\begin{aligned} \frac{\boldsymbol{y}_\tau - \overline{\boldsymbol{y}}}{\tau} &\rightharpoonup \boldsymbol{y} \quad \text{weakly in } \boldsymbol{Z} \quad \text{as } \tau \to 0,\\ \frac{\phi_\tau - \overline{\phi}}{\tau} &\rightharpoonup \phi \quad \text{weakly in } H^1_0(\Omega) \quad \text{as } \tau \to 0. \end{aligned} \tag{3.28}$$

Let $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl})$. According to (3.18), (3.25) is equivalent to

$$(\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\boldsymbol{y}_\tau) - \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\overline{\boldsymbol{y}}), \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + b(\boldsymbol{v}, \phi_\tau - \overline{\phi}) = \tau(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega_c)}.$$

Then, Assumption 3.4 allows us to apply the mean value theorem to obtain

$$\left(\begin{bmatrix} \nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}_1(\cdot, \boldsymbol{\xi}^1_\tau)^T \\ \nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}_2(\cdot, \boldsymbol{\xi}^2_\tau)^T \\ \nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}_3(\cdot, \boldsymbol{\xi}^3_\tau)^T \end{bmatrix} \frac{\mathbf{curl}\,\boldsymbol{y}_\tau - \mathbf{curl}\,\overline{\boldsymbol{y}}}{\tau}, \mathbf{curl}\,\boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)} + b\left(\boldsymbol{v}, \frac{\phi_\tau - \overline{\phi}}{\tau}\right) = (\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega_c)} \tag{3.29}$$

with $\boldsymbol{\xi}^j_\tau(x) = \mathbf{curl}\,\overline{\boldsymbol{y}}(x) + t^j_\tau(x)(\mathbf{curl}\,\boldsymbol{y}_\tau(x) - \mathbf{curl}\,\overline{\boldsymbol{y}}(x))$ a.e. in Ω and $0 < t^j_\tau(x) < 1$ a.e. in Ω for all $j = 1, 2, 3$. Note that $\boldsymbol{y}_\tau \to \overline{\boldsymbol{y}}$ strongly in $\boldsymbol{H}_0(\mathbf{curl})$ as $\tau \to 0$, and so $\boldsymbol{\xi}^j_\tau \to \mathbf{curl}\,\overline{\boldsymbol{y}}$ strongly in $\boldsymbol{L}^2(\Omega)$ as $\tau \to 0$ for all $j = 1, 2, 3$. Hence, in view of (3.19) and by possibly extracting a subsequence of $\{\boldsymbol{\xi}^j_\tau\}_{\tau>0}$ for $j = 1, 2, 3$, Lebesgue's dominated convergence theorem implies that

$$\begin{bmatrix} \nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}_1(\cdot, \boldsymbol{\xi}^1_\tau)^T \\ \nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}_2(\cdot, \boldsymbol{\xi}^2_\tau)^T \\ \nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}_3(\cdot, \boldsymbol{\xi}^3_\tau)^T \end{bmatrix}^T \mathbf{curl}\,\boldsymbol{v} \to \nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\overline{\boldsymbol{y}})^T \mathbf{curl}\,\boldsymbol{v} \quad \text{strongly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \tau \to 0. \tag{3.30}$$

Passing to the limit $\tau \to 0$ in (3.29), it follows from (3.28) and (3.30) that $(\boldsymbol{y}, \phi) \in \boldsymbol{Z} \times H^1_0(\Omega)$ satisfies

$$\begin{cases} (\nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\overline{\boldsymbol{y}})\mathbf{curl}\,\boldsymbol{y}, \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + b(\boldsymbol{v}, \phi) = (\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega_c)} & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}), \\ b(\boldsymbol{y}, \psi) = 0 & \forall \psi \in H^1_0(\Omega). \end{cases}$$

Therefore, by Lemma 3.6, the pair $(\boldsymbol{y}, \phi)$ is unique and solves the linear saddle point problem (3.20). Finally, as the injection $\boldsymbol{Z} \hookrightarrow \boldsymbol{L}^2(\Omega)$ is compact, we come to the conclusion that

$$\frac{\boldsymbol{\mathcal{S}}(\overline{\boldsymbol{u}} + \tau\boldsymbol{u}) - \boldsymbol{\mathcal{S}}(\overline{\boldsymbol{u}})}{\tau} \to \boldsymbol{y} \quad \text{strongly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \tau \to 0.$$

By (3.21), the mapping $\boldsymbol{u} \mapsto \boldsymbol{y}$ is continuous from $\boldsymbol{L}^2(\Omega_c)$ to $\boldsymbol{L}^2(\Omega)$. This yields therefore the Gâteaux differentiability of $\boldsymbol{\mathcal{S}} : \boldsymbol{L}^2(\Omega_c) \to \boldsymbol{L}^2(\Omega)$, and $\boldsymbol{\mathcal{S}}'(\overline{\boldsymbol{u}})\boldsymbol{u} = \boldsymbol{y}$ is given by the solution of (3.20). To conclude the Gâteaux differentiability of $f : \boldsymbol{L}^2(\Omega_c) \to \mathbb{R}$ and (3.23), we write

$$f(\boldsymbol{u}) = F(\boldsymbol{\mathcal{S}}(\boldsymbol{u})) + \frac{\kappa}{2}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega_c)}$$

with $F : \boldsymbol{L}^2(\Omega) \to \mathbb{R}$, $F(\boldsymbol{y}) = \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{y}_d\|^2_{\boldsymbol{L}^2(\Omega)}$. Evidently, the functional $F : \boldsymbol{L}^2(\Omega) \to \mathbb{R}$ is Fréchet differentiable. Therefore, as $\boldsymbol{\mathcal{S}} : \boldsymbol{L}^2(\Omega_c) \to \boldsymbol{L}^2(\Omega)$ is Gâteaux differentiable, the composition $F \circ \boldsymbol{\mathcal{S}} : \boldsymbol{L}^2(\Omega_c) \to \mathbb{R}$ is Gâteaux differentiable, and (3.23) follows immediately from the well-known chain rule. This completes the proof. ∎

3.3. Optimality system. Our main idea to establish the KKT theory for (P) is based on the use of the following Helmholtz decomposition:

$$\boldsymbol{H}(\mathbf{curl}\,;\Omega_c) = \boldsymbol{X} \oplus \nabla H^1(\Omega_c), \tag{3.31}$$

where $\boldsymbol{X} \subset \boldsymbol{H}(\mathbf{curl}\,;\Omega_c)$ is given by

$$\begin{aligned} \boldsymbol{X} :=& \Big\{ \boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl}\,;\Omega_c) \;\Big|\; (\boldsymbol{z}, \nabla\psi)_{\boldsymbol{L}^2(\Omega_c)} = 0 \;\forall \psi \in H^1(\Omega_c) \Big\} \\ =& \Big\{ \boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl}\,;\Omega_c) \cap \boldsymbol{H}_0(\mathrm{div};\Omega_c) \;\Big|\; \mathrm{div}\,\boldsymbol{z} = 0 \text{ in } \Omega_c \Big\}. \end{aligned} \tag{3.32}$$

Note that the decomposition (3.31) follows from a classical projection theorem since $\nabla H^1(\Omega_c)$ is a closed subspace of $\boldsymbol{H}(\mathbf{curl};\Omega_c)$. In what follows, let $\overline{\boldsymbol{u}} \in \boldsymbol{U}^{feas}$ be an optimal control of (P) with the associated state $\overline{\boldsymbol{y}} = \boldsymbol{G}(\overline{\boldsymbol{u}})$. From Proposition 3.7, we have that

$$f'(\overline{\boldsymbol{u}})\boldsymbol{h} = \int_\Omega (\overline{\boldsymbol{y}} - \boldsymbol{y}_d) \cdot \boldsymbol{\mathcal{S}}'(\overline{\boldsymbol{u}})\boldsymbol{h}\,dx + \kappa \int_{\Omega_c} \overline{\boldsymbol{u}} \cdot \boldsymbol{h}\,dx \quad \forall \boldsymbol{h} \in \boldsymbol{L}^2(\Omega_c), \tag{3.33}$$

where $\boldsymbol{\mathcal{S}}'(\overline{\boldsymbol{u}})\boldsymbol{h} = \boldsymbol{y}$ is given by the unique solution of

$$\begin{cases} a_{\overline{\boldsymbol{y}}}(\boldsymbol{y}, \boldsymbol{v}) + b(\boldsymbol{v}, \phi) = (\boldsymbol{h}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega_c)} & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}), \\ b(\boldsymbol{y}, \psi) = 0 & \forall \psi \in H^1_0(\Omega). \end{cases} \tag{3.34}$$

We introduce now the adjoint equation associated with (P) defined by the following *linear* saddle point problem:

$$\begin{cases} a^*_{\overline{\boldsymbol{y}}}(\boldsymbol{p}, \boldsymbol{v}) + b(\boldsymbol{v}, \varphi) = (\overline{\boldsymbol{y}} - \boldsymbol{y}_d, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}), \\ b(\boldsymbol{p}, \psi) = 0 & \forall \psi \in H^1_0(\Omega). \end{cases} \tag{3.35}$$

Here, the bilinear form $a^*_{\overline{\boldsymbol{y}}} : \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}_0(\mathbf{curl}) \to \mathbb{R}$ is defined by

$$a^*_{\overline{\boldsymbol{y}}}(\boldsymbol{p}, \boldsymbol{v}) := (\nabla_{\boldsymbol{s}} \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\overline{\boldsymbol{y}})^T \mathbf{curl}\,\boldsymbol{p}, \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{p}, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}).$$

In view of Lemma 3.6 and Proposition 3.7, the adjoint equation (3.35) admits a unique solution $(\boldsymbol{p}, \varphi) \in \boldsymbol{Z} \times H^1_0(\Omega)$. Setting $\boldsymbol{v} = \boldsymbol{\mathcal{S}}'(\overline{\boldsymbol{u}})\boldsymbol{h} = \boldsymbol{y}$ in (3.35) and $\boldsymbol{v} = \boldsymbol{p}$ in (3.34), and taking into account that $b(\boldsymbol{y}, \varphi) = b(\boldsymbol{p}, \phi) = 0$, we obtain

$$\int_\Omega (\overline{\boldsymbol{y}} - \boldsymbol{y}_d) \cdot \boldsymbol{\mathcal{S}}'(\overline{\boldsymbol{u}})\boldsymbol{h}\,dx = a^*_{\overline{\boldsymbol{y}}}(\boldsymbol{p}, \boldsymbol{y}) = a_{\overline{\boldsymbol{y}}}(\boldsymbol{y}, \boldsymbol{p}) = \int_{\Omega_c} \boldsymbol{h} \cdot \boldsymbol{p}\,dx \quad \forall \boldsymbol{h} \in \boldsymbol{L}^2(\Omega_c). \tag{3.36}$$

From (3.33) and (3.36), it follows therefore that

$$f'(\overline{\boldsymbol{u}})\boldsymbol{h} = \int_{\Omega_c} (\boldsymbol{p} + \kappa \overline{\boldsymbol{u}}) \cdot \boldsymbol{h} \, dx \quad \forall \boldsymbol{h} \in \boldsymbol{L}^2(\Omega_c). \tag{3.37}$$

THEOREM 3.8. *Let* $\overline{\boldsymbol{u}} \in \boldsymbol{U}^{feas}$ *be an optimal control of* (P) *with the associated state* $\overline{\boldsymbol{y}} = \boldsymbol{G}(\overline{\boldsymbol{u}})$. *Then, there exists a unique triple* $(\boldsymbol{p}, \varphi, \nabla\eta) \in \boldsymbol{Z} \times H_0^1(\Omega) \times \nabla H^1(\Omega_c)$ *such that*

$$\begin{cases} \langle A(\overline{\boldsymbol{y}}), \boldsymbol{v}\rangle = (\overline{\boldsymbol{u}}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega_c)} & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}), \\ b(\overline{\boldsymbol{y}}, \psi) = 0 & \forall \psi \in H_0^1(\Omega), \end{cases} \tag{3.38a}$$

$$\begin{cases} a^*_{\overline{\boldsymbol{y}}}(\boldsymbol{p}, \boldsymbol{v}) + b(\boldsymbol{v}, \varphi) = (\overline{\boldsymbol{y}} - \boldsymbol{y}_d, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}), \\ b(\boldsymbol{p}, \psi) = 0 & \forall \psi \in H_0^1(\Omega), \end{cases} \tag{3.38b}$$

$$\overline{\boldsymbol{u}} = -(\kappa^{-1}\boldsymbol{p}_{|\Omega_c} + \nabla\eta). \tag{3.38c}$$

Remark 3.9. The term $b(\boldsymbol{v}, \phi)$ does not appear in (3.38a) since $\overline{\boldsymbol{u}} \in \boldsymbol{U}^{feas}$ (see Lemma 3.1). Furthermore, the gradient $\nabla\eta$ denotes the Lagrange multiplier associated with the variational equality constraint (1.6) in (P). Also, note that the strong PDE formulation for (3.38a)–(3.38b) reads as

$$\begin{cases} \mathbf{curl}\,(\nu(x, |\mathbf{curl}\,\overline{\boldsymbol{y}}|)\mathbf{curl}\,\overline{\boldsymbol{y}}) = \chi_{\Omega_c}\overline{\boldsymbol{u}} & \text{in } \Omega, \\ \operatorname{div} \overline{\boldsymbol{y}} = 0 & \text{in } \Omega, \\ \overline{\boldsymbol{y}} \times \boldsymbol{n} = 0 & \text{on } \Gamma, \\ \mathbf{curl}\,(\nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}(x, \mathbf{curl}\,\overline{\boldsymbol{y}})^T \mathbf{curl}\,\boldsymbol{p}) + \nabla\varphi = \overline{\boldsymbol{y}} - \boldsymbol{y}_d & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{p} = 0 & \text{in } \Omega, \\ \boldsymbol{p} \times \boldsymbol{n} = 0 & \text{on } \Gamma. \end{cases} \tag{3.39}$$

Here, the state equation features a nonlinear isotropic structure, whereas we deal with an anisotropic form in the adjoint equation, as it involves a coefficient matrix function given by the transpose of the Jacobian matrix function $\nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\overline{\boldsymbol{y}}) : \Omega \to \mathbb{R}^{3\times 3}$. We recall that $\boldsymbol{\mathcal{F}}(x, \boldsymbol{s}) = \nu(x, |\boldsymbol{s}|)\boldsymbol{s}$.

Proof of Theorem 3.8. Since $f : \boldsymbol{L}^2(\Omega_c) \to \mathbb{R}$ is Gâteaux differentiable and the feasible set $\boldsymbol{U}^{feas} \subset \boldsymbol{L}^2(\Omega_c)$ is convex, the optimal control $\overline{\boldsymbol{u}}$ satisfies the variational inequality

$$f'(\overline{\boldsymbol{u}})(\boldsymbol{u} - \overline{\boldsymbol{u}}) \geq 0 \quad \forall \boldsymbol{u} \in \boldsymbol{U}^{feas}. \tag{3.40}$$

We refer to [29, Lemma 2.21] for this classical result. Let $(\boldsymbol{p}, \varphi) \in \boldsymbol{Z} \times H_0^1(\Omega)$ denote the unique solution of the adjoint equation (3.38b). In view of (3.37), (3.40) is equivalent to

$$\int_{\Omega_c} (\kappa^{-1}\boldsymbol{p} + \overline{\boldsymbol{u}}) \cdot (\boldsymbol{u} - \overline{\boldsymbol{u}}) \, dx \geq 0 \quad \forall \boldsymbol{u} \in \boldsymbol{U}^{feas}. \tag{3.41}$$

The Helmholtz decomposition (3.31) implies that

$$\underbrace{-\kappa^{-1}\boldsymbol{p}_{|\Omega_c}}_{\in \boldsymbol{H}(\mathbf{curl}\,;\Omega_c)} = \boldsymbol{z} + \nabla\eta, \tag{3.42}$$

with a unique pair $(\boldsymbol{z}, \nabla\eta) \in \boldsymbol{X} \times \nabla H^1(\Omega_c)$. Since

$$(\nabla\eta, \boldsymbol{u} - \overline{\boldsymbol{u}})_{\boldsymbol{L}^2(\Omega_c)} = 0 \quad \forall \boldsymbol{u} \in \boldsymbol{U}^{feas},$$

we obtain from (3.41)–(3.42) that

$$\int_{\Omega_c} (-\boldsymbol{z} + \overline{\boldsymbol{u}}) \cdot (\boldsymbol{u} - \overline{\boldsymbol{u}})\, dx \geq 0 \quad \forall \boldsymbol{u} \in \boldsymbol{U}^{feas}. \tag{3.43}$$

Since $\boldsymbol{X} \subset \boldsymbol{U}^{feas}$, we have that $\boldsymbol{z} \in \boldsymbol{U}^{feas}$ so that we may insert $\boldsymbol{u} = \boldsymbol{z}$ in (3.43) to obtain

$$\int_{\Omega_c} |-\boldsymbol{z} + \overline{\boldsymbol{u}}|^2\, dx \leq 0 \quad \Longrightarrow \quad \overline{\boldsymbol{u}} = \boldsymbol{z},$$

from which (3.38c) follows. This completes the proof. ◻

4. Regularity. We now investigate the regularity property of the optimal control. Our result relies mainly on the following material assumption.

Assumption 4.1. There exists a Lipschitz domain $\mathcal{D}$ satisfying $\overline{\Omega}_c \subset \mathcal{D} \subset \Omega$ and

$$\nu(x, s) = \nu_0 \quad \forall (x, s) \in \mathcal{D} \times \mathbb{R}_0^+. \tag{4.1}$$

Remark 4.2. We underline that the above material assumption is the key step to prove higher regularity of the optimal control. This assumption is reasonable in the case where the control region Ω_c consists only of nonmagnetic materials (copper, aluminum, silver, etc.) surrounded by an air region $\mathcal{D} \setminus \overline{\Omega}_c$. In such a case, $\mathcal{D}$ is not affected by magnetic fields, and the magnetic reluctivity of Ω_c is approximately the same as that of air so that we may consider the approximation (4.1). Note that copper is commonly used for electromagnetic coils, and so Assumption 4.1 is indeed reasonable.

For our subsequent analysis, we make use of the following embedding:

$$\boldsymbol{H}(\mathbf{curl}; \Omega_c) \cap \boldsymbol{H}_0(\text{div}\,; \Omega_c) \hookrightarrow \boldsymbol{H}^{\frac{1}{2}+\sigma_c}(\Omega_c), \tag{4.2}$$

where the exponent σ_c satisfies

$$\sigma_c \begin{cases} = & 0 & \text{if } \Omega_c \text{ is only Lipschitz,} \\ > & 0 & \text{if } \Omega_c \text{ is Lipschitz-polyhedral,} \\ = & 1/2 & \text{if } \Omega_c \text{ is convex or of class } \mathcal{C}^{1,1}. \end{cases} \tag{4.3}$$

See [11, Theorem 2] and [3, Proposition 3.7 and Theorem 2.17] for the above embedding result.

THEOREM 4.3. *Let Assumption* 4.1 *be satisfied. Then, every optimal control* $\overline{\boldsymbol{u}} \in \boldsymbol{U}^{feas}$ *of* (P) *enjoys the following higher regularity property:*

$$\overline{\boldsymbol{u}} \in \boldsymbol{H}^{\frac{1}{2}+\sigma_c}(\Omega_c), \tag{4.4}$$

$$\mathbf{curl}\,\overline{\boldsymbol{u}} \in \boldsymbol{H}^1(\Omega_c). \tag{4.5}$$

If Ω_c *is of class* $\mathcal{C}^{2,1}$, *then*

$$\overline{\boldsymbol{u}} \in \boldsymbol{H}^2(\Omega_c). \tag{4.6}$$

Proof. Let $\overline{\boldsymbol{u}} \in \boldsymbol{U}^{feas}$ be an optimal control of (P) with the associated state $\overline{\boldsymbol{y}} = \boldsymbol{G}(\overline{\boldsymbol{u}})$. In view of Theorem 3.8, there exists a unique triple $(\boldsymbol{p}, \varphi, \nabla\eta) \in \boldsymbol{Z} \times H_0^1(\Omega) \times \nabla H^1(\Omega_c)$ satisfying (3.38b) and

$$\overline{\boldsymbol{u}} = -(\kappa^{-1}\boldsymbol{p}_{|\Omega_c} + \nabla\eta). \tag{4.7}$$

By (4.7), the optimal control $\overline{\boldsymbol{u}}$ satisfies

$$\overline{\boldsymbol{u}} \in \boldsymbol{X} \subset \boldsymbol{H}(\mathbf{curl};\Omega_c) \cap \boldsymbol{H}_0(\mathrm{div}\,;\Omega_c),$$

and so the regularity property (4.4) follows immediately from (4.2)–(4.3). Also, (4.7) implies that

$$\mathbf{curl}\,\overline{\boldsymbol{u}} = -\kappa^{-1}\mathbf{curl}\,\boldsymbol{p}_{|\Omega_c}. \tag{4.8}$$

Let us prove (4.5). Due to Assumption 4.1, we have that $\boldsymbol{\mathcal{F}}(x,\boldsymbol{s}) = \nu(x,|\boldsymbol{s}|)\boldsymbol{s} = \nu_0\boldsymbol{s}$ for all $x \in \mathcal{D}$ and all $\boldsymbol{s} \in \mathbb{R}^3$. Therefore,

$$\nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}(x,\boldsymbol{s}) = \nu_0 I \quad \forall (x,\boldsymbol{s}) \in \mathcal{D} \times \mathbb{R}^3, \tag{4.9}$$

where $I \in \mathbb{R}^{3\times 3}$ denotes the identity matrix. Now, inserting the test function

$$\boldsymbol{v}_{|\mathcal{D}} = \boldsymbol{w} \in \mathcal{C}_0^\infty(\mathcal{D})^3, \quad \boldsymbol{v}_{|\Omega\setminus\mathcal{D}} = 0$$

in the adjoint equation (3.38b) results in

$$\begin{aligned}\int_{\mathcal{D}} \nu_0 \mathbf{curl}\,\boldsymbol{p}\cdot\mathbf{curl}\,\boldsymbol{w}\,dx \underbrace{=}_{(4.9)} &\int_{\mathcal{D}} \nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}(x,\mathbf{curl}\,\overline{\boldsymbol{y}})^T\mathbf{curl}\,\boldsymbol{p}\cdot\mathbf{curl}\,\boldsymbol{w}\,dx\\ \underbrace{=}_{(3.38\mathrm{b})} &\int_{\mathcal{D}} (\overline{\boldsymbol{y}} - \boldsymbol{y}_d - \nabla\varphi)\cdot\boldsymbol{w}\,dx\\ &\qquad \forall \boldsymbol{w} \in \mathcal{C}_0^\infty(\mathcal{D})^3.\end{aligned}$$

For this reason, the distributional definition of the **curl**-operator yields that

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{p}_{|\mathcal{D}} = \nu_0^{-1}(\overline{\boldsymbol{y}} - \boldsymbol{y}_d - \nabla\varphi)_{|\mathcal{D}} \in \boldsymbol{L}^2(\mathcal{D}).$$

Taking into account that the adjoint state $\boldsymbol{p}$ is divergence-free, it follows therefore that

$$-\boldsymbol{\Delta}\boldsymbol{p}_{|\mathcal{D}} = \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{p}_{|\mathcal{D}} - \nabla\big(\mathrm{div}\,\boldsymbol{p}_{|\mathcal{D}}\big) = \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{p}_{|\mathcal{D}} \in \boldsymbol{L}^2(\mathcal{D}),$$

where $\boldsymbol{\Delta}$ denotes the vector Laplacian. For this reason, a classical result on interior elliptic regularity implies that

$$\boldsymbol{p} \in \boldsymbol{H}^2(\mathcal{O})$$

holds for every open subset $\mathcal{O}$ satisfying $\overline{\mathcal{O}} \subset \mathcal{D}$. In particular, as $\overline{\Omega}_c \subset \mathcal{D}$, we get

$$\boldsymbol{p} \in \boldsymbol{H}^2(\Omega_c) \quad \Longrightarrow \quad \mathbf{curl}\,\boldsymbol{p} \in \boldsymbol{H}^1(\Omega_c). \tag{4.10}$$

Then, the regularity property (4.5) follows from (4.8) and (4.10).

Suppose now that Ω_c is of class $\mathcal{C}^{2,1}$. Multiplying (4.7) with $\nabla\psi$ for any $\psi \in H^1(\Omega_c)$ and then integrating the resulting equality over Ω_c, we obtain

$$\int_{\Omega_c} \nabla\eta \cdot \nabla\psi \, dx = -\kappa^{-1} \int_{\Omega_c} \boldsymbol{p} \cdot \nabla\psi \, dx \quad \forall\psi \in H^1(\Omega_c),$$

where we have also used the fact that $(\overline{\boldsymbol{u}}, \nabla\psi)_{\boldsymbol{L}^2(\Omega_c)} = 0$ for all $\psi \in H^1(\Omega_c)$. As $\boldsymbol{p}$ is divergence-free, Green's formula implies that

$$\int_{\Omega_c} \nabla\eta \cdot \nabla\psi \, dx = -\kappa^{-1} \int_{\Gamma_c} (\boldsymbol{p} \cdot \boldsymbol{n}_c)\psi \, dx, \quad \forall\psi \in H^1(\Omega_c).$$

In view of the $\boldsymbol{H}^2(\Omega_c)$-regularity of the adjoint state $\boldsymbol{p}$ and the $\mathcal{C}^{2,1}$-regularity of the domain Ω_c, we have

$$\boldsymbol{p} \in \boldsymbol{H}^{\frac{3}{2}}(\Gamma_c) \quad \underbrace{\Longrightarrow}_{\boldsymbol{n}_c \in \mathcal{C}^{1,1}(\Gamma_c)^3} \quad \boldsymbol{p} \cdot \boldsymbol{n}_c \in H^{\frac{3}{2}}(\Gamma_c).$$

For this reason, by a well-known result on elliptic regularity (see, e.g., [14, Theorem 1.10]), η enjoys the following regularity:

$$\eta \in H^3(\Omega_c) \quad \Longrightarrow \quad \nabla\eta \in \boldsymbol{H}^2(\Omega_c). \tag{4.11}$$

In conclusion, (4.6) follows from (4.7), (4.10), and (4.11). ∎

5. Finite element analysis. This section is devoted to the finite element analysis of the optimal control problem (P). We focus on a finite element approximation based on the lowest order edge elements of Nédélec's first family [25] for the state and control discretization and the continuous piecewise linear elements for the Lagrange multiplier discretization. From now on, Ω and Ω_c are additionally assumed to be Lipschitz-polyhedral. We consider a family $\{\mathcal{T}_h\}_{h>0}$ of simplicial triangulations $\mathcal{T}_h = \{T\}$ consisting of tetrahedra T such that

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T,$$

and there exists a subset $\mathcal{T}_h^c \subset \mathcal{T}_h$ such that

$$\overline{\Omega}_c = \bigcup_{T \in \mathcal{T}_h^c} T.$$

For every element $T \in \mathcal{T}_h$, h_T denotes the diameter of T, and ρ_T stands for the diameter of the largest ball contained in T. The maximal diameter of all elements is denoted by h, i.e., $h := \max\{h_T \mid T \in \mathcal{T}_h\}$. Finally, we suppose that there exist two positive constants ϱ and ϑ such that

$$\frac{h_T}{\rho_T} \leq \varrho \quad \text{and} \quad \frac{h}{h_T} \leq \vartheta$$

hold for all elements $T \in \mathcal{T}_h$ and all $h > 0$.

Let us denote the space of lowest order edge elements of Nédélec's first family [25] with vanishing tangential traces and the space of piecewise linear elements with vanishing traces by

$$\boldsymbol{V}_h := \left\{ \boldsymbol{y}_h \in \boldsymbol{H}_0(\mathbf{curl}) \mid \boldsymbol{y}_{h|T} = \boldsymbol{a}_T + \boldsymbol{b}_T \times x \quad \text{with } \boldsymbol{a}_T, \boldsymbol{b}_T \in \mathbb{R}^3 \quad \forall T \in \mathcal{T}_h \right\},$$

$$\Theta_h := \left\{ \phi_h \in H_0^1(\Omega) \mid \phi_{h|T} = \boldsymbol{a}_T \cdot x + b_T \quad \text{with } \boldsymbol{a}_T \in \mathbb{R}^3,\, b_T \in \mathbb{R} \quad \forall T \in \mathcal{T}_h \right\}.$$

Note that, by the well-known discrete de Rham diagram (cf. [24, p. 150]), it holds that

$$\nabla \Theta_h \subset \boldsymbol{V}_h. \tag{5.1}$$

Now, the finite element approximation of the mixed variational formulation (3.1) reads as follows: find a pair $(\boldsymbol{y}_h, \phi_h) \in \boldsymbol{V}_h \times \Theta_h$ such that

$$\begin{cases} \langle A(\boldsymbol{y}_h), \boldsymbol{v}_h \rangle + b(\boldsymbol{v}_h, \phi_h) = (\boldsymbol{u}, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega_c)} & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b(\boldsymbol{y}_h, \psi_h) = 0 & \forall \psi_h \in \Theta_h. \end{cases} \tag{5.2}$$

For $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c)$ satisfying

$$(\boldsymbol{u}, \nabla \psi_h)_{\boldsymbol{L}^2(\Omega_c)} = 0 \quad \forall \psi_h \in \Theta_h, \tag{5.3}$$

the corresponding discrete Lagrange multiplier $\phi_h = \phi_h(\boldsymbol{u})$ vanishes. This is an immediate consequence of (5.1). Existence of a unique solution for (5.2) can be justified analogously as in the continuous case. We introduce the space of discrete divergence-free vector functions in $\boldsymbol{V}_h$ by

$$\boldsymbol{Z}_h := \left\{ \boldsymbol{y}_h \in \boldsymbol{V}_h \mid (\boldsymbol{y}_h, \nabla \psi_h)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \psi_h \in \Theta_h \right\}.$$

Notice that $\boldsymbol{Z}_h$ is not a subset of $\boldsymbol{Z}$. However, the discrete counterpart to the Poincaré–Friedrichs-type inequality (3.9) holds and reads as follows:

$$\|\boldsymbol{y}_h\|_{\boldsymbol{L}^2(\Omega)} \leq \beta \|\mathbf{curl}\, \boldsymbol{y}_h\|_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{y}_h \in \boldsymbol{Z}_h \tag{5.4}$$

with a constant $\beta > 0$ independent of h. For the proof of (5.4) in a more general case, we refer to Hiptmair [17, Theorem 4.7]. Therefore, by (3.5) and (5.4), there exists a constant $\widehat{\beta} > 0$ independent of h, such that

$$\langle A(\boldsymbol{y}_h) - A(\widehat{\boldsymbol{y}}_h), \boldsymbol{y}_h - \widehat{\boldsymbol{y}}_h \rangle \geq \widehat{\beta} \|\boldsymbol{y}_h - \widehat{\boldsymbol{y}}_h\|^2_{\boldsymbol{H}(\mathbf{curl})} \quad \forall \boldsymbol{y}_h, \widehat{\boldsymbol{y}}_h \in \boldsymbol{Z}_h. \tag{5.5}$$

Also, in view of (5.1), the bilinear form b satisfies the discrete LBB condition

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{|b(\boldsymbol{v}_h, \psi_h)|}{\|\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl})}} \geq \frac{|b(\nabla \psi_h, \psi_h)|}{\|\nabla \psi_h\|_{\boldsymbol{H}(\mathbf{curl})}} = \|\nabla \psi_h\|_{\boldsymbol{L}^2(\Omega)} \geq c \|\psi_h\|_{H_0^1(\Omega)} \quad \forall \psi_h \in \Theta_h \tag{5.6}$$

with a constant $c > 0$ depending only on Ω. From (3.6), (5.5), and (5.6), [28, Proposition 2.3] yields again the following existence and uniqueness result.

LEMMA 5.1. *For every $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c)$, the discrete mixed variational formulation* (5.2) *admits a unique solution $\boldsymbol{y}_h = \boldsymbol{y}_h(\boldsymbol{u}) \in \boldsymbol{Z}_h$ with a unique Lagrange multiplier $\phi_h = \phi_h(\boldsymbol{u}) \in \Theta_h$. If $\boldsymbol{u}$ satisfies* (5.3)*, then $\phi_h(\boldsymbol{u}) \equiv 0$. Furthermore, there exists a constant $c > 0$ independent of h and $\boldsymbol{u}$, such that*

$$\|\boldsymbol{y}_h(\boldsymbol{u})\|_{\boldsymbol{H}(\mathbf{curl})} + \|\phi_h(\boldsymbol{u})\|_{H_0^1(\Omega)} \leq c \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega_c)}$$

for all $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c)$.

We denote the discrete control-to-state operator by

$$\boldsymbol{G}_h : \boldsymbol{L}^2(\Omega_c) \to \boldsymbol{Z}_h, \quad \boldsymbol{u} \mapsto \boldsymbol{y}_h.$$

This operator assigns to every $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c)$ the unique solution $\boldsymbol{y}_h \in \boldsymbol{Z}_h$ of the discrete nonlinear saddle point problem (5.2). In view of (5.5), the operator $\boldsymbol{G}_h$ is Lipschitz-continuous:

$$\|\boldsymbol{G}_h(\boldsymbol{u}) - \boldsymbol{G}_h(\widehat{\boldsymbol{u}})\|_{\boldsymbol{H}(\mathbf{curl})} \leq \frac{1}{\widehat{\beta}}\|\boldsymbol{u} - \widehat{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega_c)} \quad \forall \boldsymbol{u}, \widehat{\boldsymbol{u}} \in \boldsymbol{L}^2(\Omega_c), \tag{5.7}$$

where the positive constant $\widehat{\beta}$, defined as in (5.5), is independent of $\boldsymbol{u}$, $\widehat{\boldsymbol{u}}$ and h. Finally, by the monotonicity properties (3.10) and (5.5) as well as a Galerkin orthogonality argumentation (see [28, Theorem 2.1]), the quasi optimality of the error in the finite element discretization is obtained as follows.

LEMMA 5.2. *There exists a constant $c > 0$ independent of h and $\boldsymbol{u}$, such that*

(5.8)
$$\|\boldsymbol{G}(\boldsymbol{u}) - \boldsymbol{G}_h(\boldsymbol{u})\|_{\boldsymbol{H}(\mathbf{curl})} \leq c\left(\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h}\|\boldsymbol{G}(\boldsymbol{u}) - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl})} + \inf_{\psi_h \in \Theta_h}\|\phi(\boldsymbol{u}) - \psi_h\|_{H_0^1(\Omega)}\right)$$

for all $h > 0$ and all $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c)$.

An immediate consequence of the above lemma is the following convergence result.

COROLLARY 5.3. *If $\boldsymbol{u}_h \to \boldsymbol{u}$ strongly in $\boldsymbol{L}^2(\Omega_c)$ as $h \to 0$, then $\boldsymbol{G}_h(\boldsymbol{u}_h) \to \boldsymbol{G}(\boldsymbol{u})$ strongly in $\boldsymbol{H}_0(\mathbf{curl})$ as $h \to 0$.*

5.1. Finite element discretization of (P). We now turn to the finite element formulation of the optimal control problem (P). The discrete control space is given by

$$\boldsymbol{U}_h := \left\{\boldsymbol{u}_h \in \boldsymbol{H}(\mathbf{curl}\,;\Omega_c) \;\middle|\; \boldsymbol{u}_{h|T} = \boldsymbol{a}_T + \boldsymbol{b}_T \times x \quad \text{with } \boldsymbol{a}_T, \boldsymbol{b}_T \in \mathbb{R}^3 \quad \forall T \in \mathcal{T}_h^c\right\},$$

and the scalar space of continuous piecewise linear functions on $\overline{\Omega}_c$ is denoted by

$$\Theta_{c,h} := \left\{\psi_h \in H^1(\Omega_c) \;\middle|\; \psi_{h|T} = \boldsymbol{a}_T \cdot x + b_T \quad \text{with } \boldsymbol{a}_T \in \mathbb{R}^3,\, b_T \in \mathbb{R} \quad \forall T \in \mathcal{T}_h^c\right\}.$$

By the construction, it holds that

$$\boldsymbol{v}_h \in \boldsymbol{V}_h \quad \Longrightarrow \quad \boldsymbol{v}_{h|\overline{\Omega}_c} \in \boldsymbol{U}_h, \tag{5.9}$$

and, analogously to (5.1),

$$\nabla\Theta_{c,h} \subset \boldsymbol{U}_h. \tag{5.10}$$

We consider the following finite element approximation for the optimal control problem (P):

$$(\mathrm{P}_h) \qquad \begin{cases} \min\limits_{\boldsymbol{u}_h \in \boldsymbol{U}_h} \quad f_h(\boldsymbol{u}_h) := \dfrac{1}{2}\|\boldsymbol{G}_h(\boldsymbol{u}_h) - \boldsymbol{y}_d\|^2_{\boldsymbol{L}^2(\Omega)} + \dfrac{\kappa}{2}\|\boldsymbol{u}_h\|^2_{\boldsymbol{L}^2(\Omega_c)} \\ \text{s.t.} \quad (\boldsymbol{u}_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \psi_h \in \Theta_{c,h}. \end{cases}$$

As in the continuous case, we denote the feasible set corresponding to (P_h) by

$$\boldsymbol{U}_h^{feas} := \left\{ \boldsymbol{u}_h \in \boldsymbol{U}_h \;\Big|\; (\boldsymbol{u}_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega_c)} = 0 \quad \forall \psi_h \in \Theta_{c,h} \right\}.$$

For every $h > 0$, the existence of an optimal control of (P_h) follows from the Weierstraß lemma.

On the discrete level involving the finite element spaces $\boldsymbol{U}_h$, $\Theta_{c,h}$ and $\boldsymbol{U}_h^{feas}$, the discrete counterpart to the Helmholtz decomposition (3.31) reads as follows:

$$\boldsymbol{U}_h = \boldsymbol{U}_h^{feas} \oplus \nabla\Theta_{c,h}. \tag{5.11}$$

Invoking the decomposition (5.11), the KKT-type necessary optimality conditions for (P_h) are obtained in the following form.

THEOREM 5.4. *Let* $\overline{\boldsymbol{u}}_h \in \boldsymbol{U}_h^{feas}$ *be an optimal control of* (P_h) *with the associated discrete state* $\overline{\boldsymbol{y}}_h = \boldsymbol{G}_h(\overline{\boldsymbol{u}}_h)$. *Then, there exists a unique triple* $(\boldsymbol{p}_h, \varphi_h, \nabla\eta_h) \in \boldsymbol{Z}_h \times \Theta_h \times \nabla\Theta_{c,h}$ *such that*

$$\begin{cases} \langle A(\overline{\boldsymbol{y}}_h), \boldsymbol{v}_h\rangle = (\overline{\boldsymbol{u}}_h, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega_c)} & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b(\overline{\boldsymbol{y}}_h, \psi_h) = 0 & \forall \psi_h \in \Theta_h, \end{cases} \tag{5.12a}$$

$$\begin{cases} a^*_{\overline{\boldsymbol{y}}_h}(\boldsymbol{p}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, \varphi_h) = (\overline{\boldsymbol{y}}_h - \boldsymbol{y}_d, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b(\boldsymbol{p}_h, \psi_h) = 0 & \forall \psi_h \in \Theta_h, \end{cases} \tag{5.12b}$$

$$\overline{\boldsymbol{u}}_h = -(\kappa^{-1}\boldsymbol{p}_{h|\overline{\Omega}_c} + \nabla\eta_h). \tag{5.12c}$$

Proof. For every $h > 0$, the discrete control-to-state operator $\boldsymbol{G}_h : \boldsymbol{L}^2(\Omega_c) \to \boldsymbol{Z}_h$ is Gâteaux differentiable. The proof is completely analogous to the continuous case (Proposition 3.7) by using the discrete properties (5.1), (5.5) and the fact that the spaces $\boldsymbol{V}_h$ and Θ_h are finite-dimensional. Let $(\boldsymbol{p}_h, \varphi_h) \in \boldsymbol{Z}_h \times \Theta_h$ denote the unique solution of the discrete adjoint state (5.12b). According to (5.9), we have $\boldsymbol{p}_{h|\overline{\Omega}_c} \in \boldsymbol{U}_h$. Then, the discrete Helmholtz decomposition (5.11) implies that

$$-\kappa^{-1}\boldsymbol{p}_{h|\overline{\Omega}_c} = \boldsymbol{z}_h + \nabla\eta_h$$

with a unique pair $(\boldsymbol{z}_h, \nabla\eta_h) \in \boldsymbol{U}_h^{feas} \times \nabla\Theta_{c,h}$. By similar arguments as in the proof of Theorem 3.8, we obtain that $\overline{\boldsymbol{u}}_h = \boldsymbol{z}_h$, from which (5.12c) follows. ∎

Let us next denote by $\boldsymbol{\pi}_h : \boldsymbol{L}^2(\Omega_c) \to \boldsymbol{U}_h$ the $\boldsymbol{L}^2$-projection onto $\boldsymbol{U}_h$, i.e.,

$$(\boldsymbol{\pi}_h \boldsymbol{u}, \boldsymbol{q}_h)_{\boldsymbol{L}^2(\Omega_c)} = (\boldsymbol{u}, \boldsymbol{q}_h)_{\boldsymbol{L}^2(\Omega_c)} \quad \forall \boldsymbol{q}_h \in \boldsymbol{U}_h. \tag{5.13}$$

By (5.10) and (5.13), it holds that

$$(\boldsymbol{u}, \nabla\psi)_{\boldsymbol{L}^2(\Omega_c)} = 0 \quad \forall \psi \in H^1(\Omega_c) \quad \Longrightarrow \quad (\boldsymbol{\pi}_h \boldsymbol{u}, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega_c)} = 0 \quad \forall \psi_h \in \Theta_{c,h}.$$

In other words, the operator $\boldsymbol{\pi}_h$ satisfies

$$\boldsymbol{u} \in \boldsymbol{U}^{feas} \quad \Longrightarrow \quad \boldsymbol{\pi}_h \boldsymbol{u} \in \boldsymbol{U}_h^{feas}. \tag{5.14}$$

In the upcoming theorem, we prove a convergence result for the finite element approximation (P_h) as $h \to 0$. The main tools for the proof are the discrete KKT optimality conditions (5.12) and the discrete compactness property of the Nédélec edge elements as $h \to 0$ (see Kikuchi [20]).

THEOREM 5.5. *Let* $\{\overline{\boldsymbol{u}}_h\}_{h>0}$ *be a sequence of optimal controls of* (P_h). *Then, there exists a subsequence of* $\{\overline{\boldsymbol{u}}_h\}_{h>0}$ *converging strongly in* $\boldsymbol{L}^2(\Omega_c)$ *towards an optimal solution of* (P) *as* $h \to 0$. *Every* $\boldsymbol{L}^2(\Omega_c)$*-converging subsequence of* $\{\overline{\boldsymbol{u}}_h\}_{h>0}$ *converges strongly in* $\boldsymbol{L}^2(\Omega_c)$ *towards an optimal solution of* (P) *as* $h \to 0$.

Proof. For every $h > 0$, we set $\overline{\boldsymbol{y}}_h = \boldsymbol{G}_h(\overline{\boldsymbol{u}}_h)$. According to Theorem 5.4, for every $h > 0$, there exists a unique triple $(\boldsymbol{p}_h, \varphi_h, \nabla\eta_h) \in \boldsymbol{Z}_h \times \Theta_h \times \nabla\Theta_{c,h}$ satisfying (5.12). Setting $\boldsymbol{v}_h = \boldsymbol{p}_h$ in (5.12b) gives

$$(\nabla_{\boldsymbol{s}}\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\overline{\boldsymbol{y}}_h)^T \mathbf{curl}\,\boldsymbol{p}_h, \mathbf{curl}\,\boldsymbol{p}_h)_{\boldsymbol{L}^2(\Omega)} = (\overline{\boldsymbol{y}}_h - \boldsymbol{y}_d, \boldsymbol{p}_h)_{\boldsymbol{L}^2(\Omega)} \quad \forall h > 0.$$

From (3.22) and (5.4), it follows that

$$\|\mathbf{curl}\,\boldsymbol{p}_h\|_{\boldsymbol{L}^2(\Omega)} \leq \underline{\nu}^{-1}\beta\|\overline{\boldsymbol{y}}_h - \boldsymbol{y}_d\|_{\boldsymbol{L}^2(\Omega)} \quad \forall h > 0. \tag{5.15}$$

Since $\overline{\boldsymbol{u}}_h$ is an optimal control of (P_h) and $0 \in \boldsymbol{U}_h^{feas}$ for all $h > 0$, it holds that

$$\frac{\kappa}{2}\|\overline{\boldsymbol{u}}_h\|^2_{\boldsymbol{L}^2(\Omega_c)} \leq f_h(\overline{\boldsymbol{u}}_h) \leq f_h(0) = \frac{1}{2}\|\boldsymbol{y}_d\|^2_{\boldsymbol{L}^2(\Omega)} \quad \forall h > 0, \tag{5.16}$$

and so the sequence $\{\overline{\boldsymbol{u}}_h\}_{h>0}$ is bounded with respect to the $\boldsymbol{L}^2(\Omega_c)$-topology. Then, Lemma 5.1 implies the boundedness of the sequence $\{\overline{\boldsymbol{y}}_h\}_{h>0}$ in $\boldsymbol{H}_0(\mathbf{curl})$. Therefore, (5.15) yields that

$$\|\mathbf{curl}\,\boldsymbol{p}_h\|_{\boldsymbol{L}^2(\Omega)} \leq c \quad \forall h > 0 \tag{5.17}$$

with a constant $c > 0$ independent of h. In view of (5.9)–(5.10), (5.12c) implies

$$\mathbf{curl}\,\overline{\boldsymbol{u}}_h = -\kappa^{-1}\mathbf{curl}\,\boldsymbol{p}_{h|\Omega_c} \quad \forall h > 0. \tag{5.18}$$

Therefore, from (5.16)–(5.18), it follows that

$$\|\overline{\boldsymbol{u}}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega_c)} \leq c \quad \forall h > 0, \tag{5.19}$$

with a constant $c > 0$ independent of h. On the other hand, $\overline{\boldsymbol{u}}_h$ satisfies

$$(\overline{\boldsymbol{u}}_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \psi_h \in \Theta_{c,h} \quad \forall h > 0. \tag{5.20}$$

By (5.19)–(5.20), the discrete compactness result [20, Theorem 1] implies the existence of a subsequence of $\{\overline{\boldsymbol{u}}_h\}_{h>0}$, denoted again by $\{\overline{\boldsymbol{u}}_h\}_{h>0}$, such that

$$\overline{\boldsymbol{u}}_h \to \overline{\boldsymbol{u}} \quad \text{strongly in } \boldsymbol{L}^2(\Omega_c) \quad \text{as } h \to 0 \tag{5.21}$$

for some $\overline{\boldsymbol{u}} \in \boldsymbol{H}(\mathbf{curl};\Omega_c)$. Let $\mathcal{I}_h : \mathcal{C}(\overline{\Omega}_c) \to \Theta_{c,h}$ denote the nodal interpolation operator corresponding to the finite element space $\Theta_{c,h}$. Inserting $\psi_h = \mathcal{I}_h\psi$ with $\psi \in \mathcal{C}^\infty(\overline{\Omega}_c)$ in (5.20) and then passing to the limit $h \to 0$, we obtain

$$0 = (\overline{\boldsymbol{u}}_h, \nabla\mathcal{I}_h\psi)_{\boldsymbol{L}^2(\Omega)} \to (\overline{\boldsymbol{u}}, \nabla\psi)_{\boldsymbol{L}^2(\Omega)} \quad \text{as } h \to 0.$$

Hence, as $\mathcal{C}^\infty(\overline{\Omega}_c) \subset H^1(\Omega_c)$ is dense, it follows that

$$(\overline{\boldsymbol{u}}, \nabla\psi) = 0 \quad \forall \psi \in H^1(\Omega_c).$$

This implies that $\overline{\boldsymbol{u}} \in \boldsymbol{U}^{feas}$.

Let us finally show that $\overline{\boldsymbol{u}}$ is an optimal solution of (P). To this end, let $\hat{\boldsymbol{u}} \in \boldsymbol{U}^{feas}$ be an optimal control of (P). In view of (5.14), $\boldsymbol{\pi}_h \hat{\boldsymbol{u}} \in \boldsymbol{U}_h^{feas}$ for all $h > 0$, and hence

$$f_h(\overline{\boldsymbol{u}}_h) \leq f_h(\boldsymbol{\pi}_h \hat{\boldsymbol{u}}) \quad \forall h > 0,$$

since $\overline{\boldsymbol{u}}_h$ is an optimal control of (P_h) for every $h > 0$. Consequently, by Corollary 5.3 and (5.21), it follows that

$$f(\overline{\boldsymbol{u}}) = \lim_{h \to 0} f_h(\overline{\boldsymbol{u}}_h) \leq \lim_{h \to 0} f_h(\boldsymbol{\pi}_h \hat{\boldsymbol{u}}) = f(\hat{\boldsymbol{u}}), \tag{5.22}$$

where we have also used the fact that $\boldsymbol{\pi}_h \hat{\boldsymbol{u}} \to \hat{\boldsymbol{u}}$ strongly in $\boldsymbol{L}^2(\Omega_c)$ as $h \to 0$. Since $\overline{\boldsymbol{u}} \in \boldsymbol{U}^{feas}$ and $\hat{\boldsymbol{u}}$ is an optimal control of (P), (5.22) concludes that $\overline{\boldsymbol{u}}$ is an optimal control of (P). ☐

5.2. Error estimate. Let $\{\overline{\boldsymbol{u}}_h\}_{h>0}$ be a sequence of optimal controls of (P_h). According to Theorem 5.5, there exists a subsequence of $\{\overline{\boldsymbol{u}}_h\}_{h>0}$, denoted again by $\{\overline{\boldsymbol{u}}_h\}_{h>0}$, converging strongly in $\boldsymbol{L}^2(\Omega_c)$ towards an optimal control $\overline{\boldsymbol{u}} \in \boldsymbol{U}^{feas}$ of (P). In what follows, we consider this converging sequence $\{\overline{\boldsymbol{u}}_h\}_{h>0}$. We set $\overline{\boldsymbol{y}} = \boldsymbol{G}(\overline{\boldsymbol{u}})$ and $\overline{\boldsymbol{y}}_h = \boldsymbol{G}_h(\overline{\boldsymbol{u}}_h)$ for all $h > 0$. Our final goal is to derive a priori error estimation for the $\boldsymbol{L}^2(\Omega_c)$-error between $\overline{\boldsymbol{u}}$ and $\overline{\boldsymbol{u}}_h$.

Assumption 5.6. Assume the Lipschitz polyhedron Ω_c to be simply connected, and let Assumption 4.1 be satisfied. Furthermore, let the optimal control $\overline{\boldsymbol{u}}$ satisfy

$$\mathbf{curl}\, \boldsymbol{G}(\overline{\boldsymbol{u}}) \in \boldsymbol{H}(\mathbf{curl}) \tag{5.23}$$

and the following standard quadratic growth condition: there exist two constants $r, m > 0$ such that

$$f(\overline{\boldsymbol{u}}) + \frac{r}{2} \|\overline{\boldsymbol{u}} - \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega_c)} \leq f(\boldsymbol{u}) \tag{5.24}$$

for all $\boldsymbol{u} \in \boldsymbol{U}^{feas}$ satisfying $\|\overline{\boldsymbol{u}} - \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega_c)} \leq m$.

Remark 5.7. Quadratic growth conditions such as (5.24) are in general obtained by second-order sufficient conditions (SSC). Let us remark that SSC for (P) would require the twice continuous Fréchet differentiability of the control-to-state operator and certainly call for a very careful second-order analysis, which we do not perform in our present paper. We refer to Casas and Tröltzsch [10] for rigorous SSC in the context of quasilinear $H^1(\Omega)$-elliptic PDEs.

We recall again the embedding result [3, Proposition 3.7 and Theorem 2.17] stating that

$$\begin{cases} \boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div}) \hookrightarrow \boldsymbol{H}^{\frac{1}{2}+\sigma}(\Omega) \\ \boldsymbol{H}(\mathbf{curl}) \cap \boldsymbol{H}_0(\mathrm{div}) \hookrightarrow \boldsymbol{H}^{\frac{1}{2}+\sigma}(\Omega) \end{cases} \text{with } \sigma \begin{cases} > \ 0, \\ = \ 1/2 \quad \text{if } \Omega \text{ is convex.} \end{cases} \tag{5.25}$$

For the main theorem below, let $\epsilon \in (0, 1/2]$ be defined as follows:

$$\epsilon := \min\{\sigma_c, \sigma\}$$

with σ_c as in (4.2)–(4.3) and σ as in (5.25). Note that if both Ω_c and Ω are convex, then $\epsilon = 1/2$.

THEOREM 5.8. *Under Assumption* 5.6*, there is a constant* $c > 0$ *independent of* h*, such that*

$$\|\overline{\boldsymbol{u}} - \overline{\boldsymbol{u}}_h\|_{\boldsymbol{L}^2(\Omega_c)} \leq c h^{\frac{1+2\epsilon}{4}}$$

for all sufficiently small h. If both Ω and Ω_c are convex, then

$$\|\overline{\boldsymbol{u}} - \overline{\boldsymbol{u}}_h\|_{\boldsymbol{L}^2(\Omega_c)} \leq ch^{\frac{1}{2}}$$

for all sufficiently small h.

In the following, we present a series of lemmas which we use for proving Theorem 5.8. For the first lemma, let $\boldsymbol{\mathcal{N}_h}$ denote the curl-conforming Nédélec interpolant.

LEMMA 5.9 (see Alonso and Valli [1, Proposition 5.6] or Monk [24, Theorem 5.41 and Remark 5.42]). *Let $\tau \in (0, 1/2]$. There exists a constant $c > 0$ independent of h, such that*

$$\|\boldsymbol{y} - \boldsymbol{\mathcal{N}_h}\boldsymbol{y}\|_{\boldsymbol{H}(\mathbf{curl})} \leq ch^{\frac{1}{2}+\tau}(\|\boldsymbol{y}\|_{\boldsymbol{H}^{\frac{1}{2}+\tau}(\Omega)} + \|\mathbf{curl}\,\boldsymbol{y}\|_{\boldsymbol{H}^{\frac{1}{2}+\tau}(\Omega)}) \tag{5.26}$$

for all $\boldsymbol{y}$ satisfying $\boldsymbol{y}, \mathbf{curl}\,\boldsymbol{y} \in \boldsymbol{H}^{\frac{1}{2}+\tau}(\Omega)$. In view of the Nédélec interpolant property, the operator $\boldsymbol{\pi}_h$ defined as in (5.13) *satisfies*

$$\|\boldsymbol{u} - \boldsymbol{\pi}_h\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega_c)} \leq ch^{\frac{1}{2}+\tau}(\|\boldsymbol{u}\|_{\boldsymbol{H}^{\frac{1}{2}+\tau}(\Omega_c)} + \|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{H}^{\frac{1}{2}+\tau}(\Omega_c)}) \tag{5.27}$$

for all $\boldsymbol{u}$ satisfying $\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{H}^{\frac{1}{2}+\tau}(\Omega_c)$.

LEMMA 5.10. *Let Assumption* 5.6 *be satisfied. Then, there exists a positive constant c independent of h, such that*

$$\|\boldsymbol{G}(\overline{\boldsymbol{u}}) - \boldsymbol{G}_h(\boldsymbol{u})\|_{\boldsymbol{H}(\mathbf{curl})} \leq ch^{\frac{1}{2}+\sigma} + \frac{1}{\widehat{\beta}}\|\overline{\boldsymbol{u}} - \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega_c)} \quad \forall \boldsymbol{u} \in \boldsymbol{L}^2(\Omega_c) \tag{5.28}$$

with $\sigma \in (0, 1/2]$ as in (5.25) *and $\widehat{\beta}$ as in* (5.5).

Proof. Invoking the estimate (5.8) and the Lipschitz continuity (5.7), we obtain

$$\begin{aligned} \|\boldsymbol{G}(\overline{\boldsymbol{u}}) - \boldsymbol{G}_h(\boldsymbol{u})\|_{\boldsymbol{H}(\mathbf{curl})} &\leq \|\boldsymbol{G}(\overline{\boldsymbol{u}}) - \boldsymbol{G}_h(\overline{\boldsymbol{u}})\|_{\boldsymbol{H}(\mathbf{curl})} + \|\boldsymbol{G}_h(\overline{\boldsymbol{u}}) - \boldsymbol{G}_h(\boldsymbol{u})\|_{\boldsymbol{H}(\mathbf{curl})} \\ &\leq c \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\boldsymbol{G}(\overline{\boldsymbol{u}}) - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl})} + \frac{1}{\widehat{\beta}}\|\overline{\boldsymbol{u}} - \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega_c)}, \end{aligned} \tag{5.29}$$

where we have also used the fact that $\phi(\overline{\boldsymbol{u}}) = 0$ since $\overline{\boldsymbol{u}} \in \boldsymbol{U}^{feas}$. The assumption (5.23) implies that

$$\mathbf{curl}\,\boldsymbol{G}(\overline{\boldsymbol{u}}) \in \boldsymbol{X}(\Omega) := \left\{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}) \cap \boldsymbol{H}_0(\mathrm{div}) \;\middle|\; \mathrm{div}\,\boldsymbol{v} = 0 \text{ in } \Omega\right\},$$

and hence, by the embedding result (5.25), it follows that

$$\boldsymbol{G}(\overline{\boldsymbol{u}}), \mathbf{curl}\,\boldsymbol{G}(\overline{\boldsymbol{u}}) \in \boldsymbol{H}^{\frac{1}{2}+\sigma}(\Omega).$$

Then, (5.26) yields

$$\|\boldsymbol{G}(\overline{\boldsymbol{u}}) - \boldsymbol{\mathcal{N}_h}\boldsymbol{G}(\overline{\boldsymbol{u}})\|_{\boldsymbol{H}(\mathbf{curl})} \leq ch^{\frac{1}{2}+\sigma} \tag{5.30}$$

with a constant $c > 0$ independent of h. Therefore, inserting $\boldsymbol{v}_h = \boldsymbol{\mathcal{N}_h}\boldsymbol{G}(\overline{\boldsymbol{u}})$ in (5.29) and using (5.30), the estimate (5.28) follows. ∎

Let us remark that every optimal control $\overline{\boldsymbol{u}}_h$ of (P_h) cannot be expected to be a feasible control of (P), as it is generally not divergence-free. However, we know that every $\overline{\boldsymbol{u}}_h$ is discretely divergence-free and belongs to the Nédélec finite element space $\boldsymbol{U}_h$ so that we can find a continuous divergence-free approximation $\overline{\boldsymbol{u}}^h$ with the following properties:

LEMMA 5.11. *Let Assumption* 5.6 *be satisfied. For every $h > 0$, there is a $\overline{\boldsymbol{u}}^h \in \boldsymbol{H}(\mathbf{curl}\,;\Omega_c)$ satisfying*

$$\begin{cases} \mathbf{curl}\,\overline{\boldsymbol{u}}^h = \mathbf{curl}\,\overline{\boldsymbol{u}}_h, \\ (\overline{\boldsymbol{u}}^h, \nabla\psi)_{\boldsymbol{L}^2(\Omega_c)} = 0 \quad \forall \psi \in H^1(\Omega_c), \\ \|\overline{\boldsymbol{u}}^h - \overline{\boldsymbol{u}}_h\|_{\boldsymbol{L}^2(\Omega_c)} \le c h^{\frac{1}{2}+\sigma_c} \|\mathbf{curl}\,\overline{\boldsymbol{u}}_h\|_{\boldsymbol{L}^2(\Omega_c)} \end{cases} \tag{5.31}$$

with a constant $c > 0$ independent of h, and $\sigma_c \in (0, 1/2]$ as in (4.2)–(4.3)*.*

Lemma 5.11 is a well-known result in literature (see, e.g., [17, Lemma 4.5] or [24, Lemma 7.6]). We note that the proof particularly requires the use of the following estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^{\frac{1}{2}+\sigma_c}(\Omega_c)} \le c\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega_c)} \quad \forall \boldsymbol{u} \in \boldsymbol{X} \tag{5.32}$$

with a constant $c > 0$ depending only on Ω_c. Here, we recall that $\boldsymbol{X}$ contains all divergence-free vector functions in $\boldsymbol{H}(\mathbf{curl};\Omega_c) \cap \boldsymbol{H}_0(\mathrm{div};\Omega_c)$; see (3.32). By our assumption that the Lipschitz polyhedron Ω_c is simply connected, the inequality (5.32) is indeed satisfied by a well-known Poincaré–Friedrichs-type inequality, analogous to (3.9), for vector functions in $\boldsymbol{X}$ together with the embedding result (4.2). We now have all the required components to prove Theorem 5.8.

Proof of Theorem 5.8. The second property in (5.31) yields that

$$\overline{\boldsymbol{u}}^h \in \boldsymbol{U}^{feas} \quad \forall h > 0. \tag{5.33}$$

Furthermore, as justified in (5.19), the sequence $\{\mathbf{curl}\,\overline{\boldsymbol{u}}_h\}_{h>0}$ is bounded in $\boldsymbol{L}^2(\Omega_c)$. For this reason, we get from the third property in (5.31) that

$$\|\overline{\boldsymbol{u}}^h - \overline{\boldsymbol{u}}_h\|_{\boldsymbol{L}^2(\Omega_c)} \le c h^{\frac{1}{2}+\sigma_c} \tag{5.34}$$

with a constant $c > 0$ independent of h. In particular, since $\overline{\boldsymbol{u}}_h$ converges strongly in $\boldsymbol{L}^2(\Omega_c)$ towards $\overline{\boldsymbol{u}}$ as $h \to 0$, we can find an $\overline{h} \in (0, 1)$ such that

$$\|\overline{\boldsymbol{u}}^h - \overline{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega_c)} \le m \quad \forall h \in (0, \overline{h}]. \tag{5.35}$$

By (5.33) and (5.35), we may insert $\boldsymbol{u} = \overline{\boldsymbol{u}}^h$ in the quadratic growth condition (5.24) to obtain

$$\begin{aligned} \frac{r}{2}\|\overline{\boldsymbol{u}} - \overline{\boldsymbol{u}}^h\|^2_{\boldsymbol{L}^2(\Omega_c)} &\le f(\overline{\boldsymbol{u}}^h) - f(\overline{\boldsymbol{u}}) \\ &= f(\overline{\boldsymbol{u}}^h) - f(\overline{\boldsymbol{u}}) - f_h(\boldsymbol{\pi}_h\overline{\boldsymbol{u}}) + f_h(\boldsymbol{\pi}_h\overline{\boldsymbol{u}}) \quad \forall h \in (0, \overline{h}]. \end{aligned}$$

Since $\overline{\boldsymbol{u}}_h$ is an optimal control of (P$_h$) and $\boldsymbol{\pi}_h\overline{\boldsymbol{u}} \in \boldsymbol{U}_h^{feas}$, we have that $f_h(\overline{\boldsymbol{u}}_h) \le f_h(\boldsymbol{\pi}_h\overline{\boldsymbol{u}})$ for all $h > 0$, and so

$$\begin{aligned} \frac{r}{2}\|\overline{\boldsymbol{u}} - \overline{\boldsymbol{u}}^h\|^2_{\boldsymbol{L}^2(\Omega_c)} &\le f(\overline{\boldsymbol{u}}^h) - f(\overline{\boldsymbol{u}}) - f_h(\overline{\boldsymbol{u}}_h) + f_h(\boldsymbol{\pi}_h\overline{\boldsymbol{u}}) \\ &\le |f(\overline{\boldsymbol{u}}) - f_h(\boldsymbol{\pi}_h\overline{\boldsymbol{u}})| + |f(\overline{\boldsymbol{u}}^h) - f_h(\overline{\boldsymbol{u}}_h)| := I + II \quad \forall h \in (0, \overline{h}]. \end{aligned} \tag{5.36}$$

As we have already proven in Theorem 4.3, the optimal control $\overline{\boldsymbol{u}}$ enjoys the regularity

$$\overline{\boldsymbol{u}} \in \boldsymbol{H}^{\frac{1}{2}+\sigma_c}(\Omega_c) \quad \text{and} \quad \mathbf{curl}\, \overline{\boldsymbol{u}} \in \boldsymbol{H}^1(\Omega_c).$$

Thus, by (5.27), it follows that

$$\|\overline{\boldsymbol{u}} - \boldsymbol{\pi}_h \overline{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega_c)} \leq c h^{\frac{1}{2}+\sigma_c} \tag{5.37}$$

with a constant $c > 0$ independent of h. Applying Lemma 5.10 and (5.37), we deduce that

(5.38)

$$\begin{aligned} I &= \left| \frac{1}{2}\|\boldsymbol{G}(\overline{\boldsymbol{u}}) - \boldsymbol{y}_d\|^2_{\boldsymbol{L}^2(\Omega)} - \frac{1}{2}\|\boldsymbol{G}_h(\boldsymbol{\pi}_h\overline{\boldsymbol{u}}) - \boldsymbol{y}_d\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\kappa}{2}\|\overline{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega_c)} - \frac{\kappa}{2}\|\boldsymbol{\pi}_h\overline{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega_c)} \right| \\ &\leq \frac{1}{2}\|\boldsymbol{G}(\overline{\boldsymbol{u}}) - \boldsymbol{G}_h(\boldsymbol{\pi}_h\overline{\boldsymbol{u}})\|^2 + \|\boldsymbol{G}(\overline{\boldsymbol{u}}) - \boldsymbol{G}_h(\boldsymbol{\pi}_h\overline{\boldsymbol{u}})\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{G}_h(\boldsymbol{\pi}_h\overline{\boldsymbol{u}}) - \boldsymbol{y}_d\|_{\boldsymbol{L}^2(\Omega)} \\ &\qquad + \frac{\kappa}{2}\|\overline{\boldsymbol{u}} - \boldsymbol{\pi}_h\overline{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega_c)} + \kappa\|\overline{\boldsymbol{u}} - \boldsymbol{\pi}_h\overline{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega_c)}\|\boldsymbol{\pi}_h\overline{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega_c)} \\ &\leq c h^{\frac{1}{2}+\epsilon} \quad \forall h \in (0, \overline{h}] \end{aligned}$$

with a constant $c > 0$ independent of h. Analogously, using Lemma 5.10 and (5.34), we obtain

$$II \leq c h^{\frac{1}{2}+\epsilon} \quad \forall h \in (0, \overline{h}] \tag{5.39}$$

with a constant $c > 0$ independent of h. In conclusion, the assertion follows from the estimates (5.34), (5.36), (5.38), and (5.39). ∎

6. Numerical experiment. We present short numerical results serving as a numerical illustration for Theorem 5.8. Let us consider a fairly simple academic example (see Figure 6.1) with

$$\Omega = (-3/8; 3/8)^3,$$

where the control region is given by

$$\Omega_c = \{(-1/8; 1/8)^2 \times (-1/16; 1/16)\} \setminus (-1/16; 1/16)^3.$$

Furthermore, we define

$$\Omega_s := (-1/4; 1/4) \times (-5/16; -3/16) \times (-1/4; 1/4).$$

In this subdomain, the magnetic reluctivity is chosen to be nonlinear. For the data involved in (P), we consider

$$\begin{aligned} \kappa &= 0.1, \quad \nu_0 = 1, \\ \boldsymbol{y}_d(x) &= 10^3(x_2/(x_1^2 + x_2^2), -x_1/(x_1^2 + x_2^2), 0), \\ \nu(x, s) &= \nu_0 - (\nu_0 - 10^{-3})\chi_{\Omega_s}(x)\exp(-0.1 s^2). \end{aligned}$$

Note that ν satisfies Assumption 2.1 and Assumption 3.4 such that the control-to-state operator $\boldsymbol{\mathcal{S}}$ as well as the objective functional f are Gâteaux differentiable (Proposition 3.7). Furthermore, it fulfills Assumption 4.1, and so the higher regularity result from Theorem 4.3 holds true for this example. We employ the Augmented

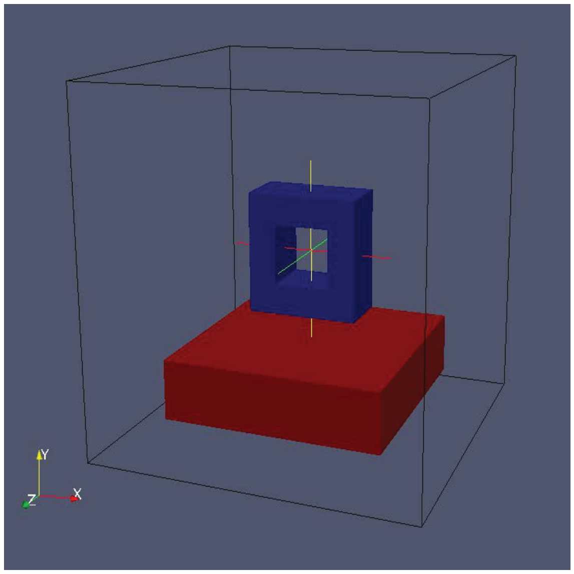

FIG. 6.1. *Computational domain Ω, the control region Ω_c (blue), and the subdomain Ω_s (red).*

Lagrangian-SQP method (see [18, section 6]) for solving (P_h). Based on our numerical observation, this method is suitable for dealing with the equality divergence-free control constraint. Our future goal is to investigate Augmented Lagrangian methods for solving (P), including their efficient numerical realization based on an appropriate preconditioning strategy. We utilized the open source software FEniCS [23] for all computations presented below, and the zero function was used for the initialization of the Augmented Lagrangian method. Also, note that Ω was triangulated with a regular simplicial mesh of mesh size h. The computed optimal controls for different discretization mesh levels are depicted in Figure 6.2. We observe that the computed solution $\overline{\boldsymbol{u}}_h$ at the finest mesh exhibits a reasonable electric current flow structure.

To check the experiment order of convergence, we make use of the quantity

$$\mathrm{EOC} = \frac{\log(\|\overline{\boldsymbol{u}}_{h_1} - \overline{\boldsymbol{u}}_{ref}\|_{\boldsymbol{L}^2(\Omega_c)}) - \log(\|\overline{\boldsymbol{u}}_{h_2} - \overline{\boldsymbol{u}}_{ref}\|_{\boldsymbol{L}^2(\Omega_c)})}{\log h_1 - \log h_2}$$

for two consecutive mesh sizes h_1 and h_2. Here, $\overline{\boldsymbol{u}}_{ref}$ denotes the reference optimal control. As the analytical solution of (P) is unknown, the reference solution $\overline{\boldsymbol{u}}_{ref}$ is set to be the computed one at the finest mesh, i.e., $\overline{\boldsymbol{u}}_{ref} = \overline{\boldsymbol{u}}_h$ with $h = \sqrt{2} \cdot 2^{-7}$. According to Theorem 5.8, since Ω_c is not convex, we can only expect convergence of order $1/4 + \epsilon/2$ with $\epsilon \in (0, 1/2)$. In fact, our numerical result illustrates this theoretical prediction with $\varepsilon \approx 1/4$. This can be observed from Table 6.1, where DOF denotes the number of degrees of freedom in the finite element mesh.

TABLE 6.1
Convergence behavior of $\overline{\boldsymbol{u}}_h$.

h	$\sqrt{2} \cdot 2^{-4}$	$\sqrt{2} \cdot 2^{-5}$	$\sqrt{2} \cdot 2^{-6}$
DOF	15625	117649	912673
$\|\overline{\boldsymbol{u}}_h - \overline{\boldsymbol{u}}_{ref}\|_{\boldsymbol{L}^2(\Omega_c)}$	0.10280693499	0.08348144948	0.06428130530
EOC	-	0.30041003165	0.37705642535

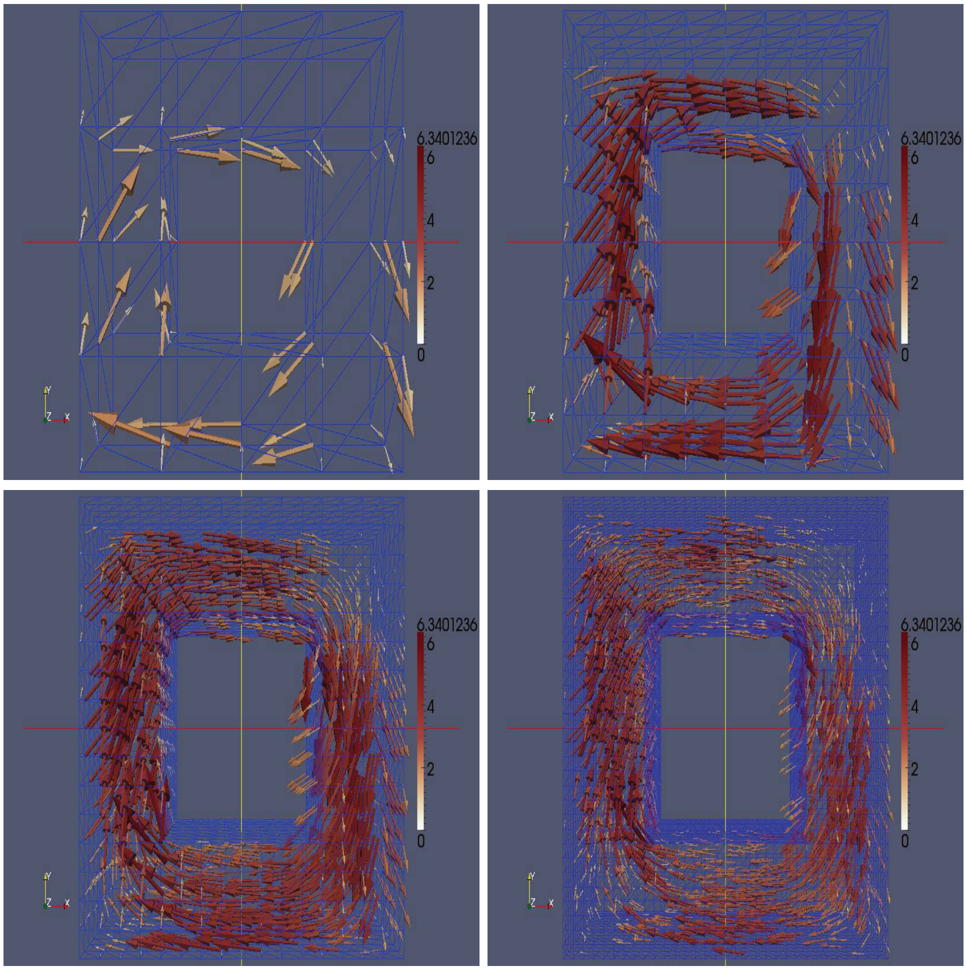

FIG. 6.2. *Computed optimal controls* $\overline{\boldsymbol{u}}_h$ *with* $h = \sqrt{2} \cdot 2^{-4}$ *(upper left),* $h = \sqrt{2} \cdot 2^{-5}$ *(upper right),* $h = \sqrt{2} \cdot 2^{-6}$ *(lower left), and* $h = \sqrt{2} \cdot 2^{-7}$ *(lower right).*

Appendix.

Proof of Lemma 2.2. Let $\boldsymbol{s}, \widehat{\boldsymbol{s}} \in \mathbb{R}^3$. By the assumption (2.3), it holds that

$$\big(\nu(x, |\boldsymbol{s}|)|\boldsymbol{s}| - \nu(x, |\widehat{\boldsymbol{s}}|)|\widehat{\boldsymbol{s}}|\big)\big(|\boldsymbol{s}| - |\widehat{\boldsymbol{s}}|\big) \geq \underline{\nu}\big(|\boldsymbol{s}| - |\widehat{\boldsymbol{s}}|\big)^2,$$

which implies

$$\nu(x, |\boldsymbol{s}|)|\boldsymbol{s}|^2 \geq \underline{\nu}\big(|\boldsymbol{s}| - |\widehat{\boldsymbol{s}}|\big)^2 + \big(\nu(x, |\boldsymbol{s}|) + \nu(x, |\widehat{\boldsymbol{s}}|)\big)|\boldsymbol{s}||\widehat{\boldsymbol{s}}| - \nu(x, |\widehat{\boldsymbol{s}}|)|\widehat{\boldsymbol{s}}|^2. \tag{A.1}$$

From this inequality, we obtain that

$$\begin{aligned}
&(\nu(x, |\boldsymbol{s}|)\boldsymbol{s} - \nu(x, |\widehat{\boldsymbol{s}}|)\widehat{\boldsymbol{s}}) \cdot (\boldsymbol{s} - \widehat{\boldsymbol{s}}) \\
&\quad = \nu(x, |\boldsymbol{s}|)|\boldsymbol{s}|^2 + \nu(x, |\widehat{\boldsymbol{s}}|)|\widehat{\boldsymbol{s}}|^2 - \big(\nu(x, |\boldsymbol{s}|) + \nu(x, |\widehat{\boldsymbol{s}}|)\big)\boldsymbol{s} \cdot \widehat{\boldsymbol{s}} \\
&\quad \underbrace{\geq}_{(A.1)} \underline{\nu}\big(|\boldsymbol{s}| - |\widehat{\boldsymbol{s}}|\big)^2 + \big(\nu(x, |\boldsymbol{s}|) + \nu(x, |\widehat{\boldsymbol{s}}|)\big)\big(|\boldsymbol{s}||\widehat{\boldsymbol{s}}| - \boldsymbol{s} \cdot \widehat{\boldsymbol{s}}\big)
\end{aligned}$$

$$
\underbrace{\geq}_{(2.1)} \underline{\nu}\big(|\boldsymbol{s}| - |\widehat{\boldsymbol{s}}|\big)^2 + 2\underline{\nu}(|\boldsymbol{s}||\widehat{\boldsymbol{s}}| - \boldsymbol{s}\cdot\widehat{\boldsymbol{s}})
= \underline{\nu}\big((|\boldsymbol{s}| - |\widehat{\boldsymbol{s}}|)^2 + 2|\boldsymbol{s}||\widehat{\boldsymbol{s}}| - 2\boldsymbol{s}\cdot\widehat{\boldsymbol{s}}\big)
= \underline{\nu}|\boldsymbol{s} - \widehat{\boldsymbol{s}}|^2.
$$

Thus, (2.5) is valid. Let us now prove (2.6). We have that

$$
\begin{aligned}
&|\nu(x,|\boldsymbol{s}|)\boldsymbol{s} - \nu(x,|\widehat{\boldsymbol{s}}|)\widehat{\boldsymbol{s}}| \\
&\quad = |\nu(x,|\boldsymbol{s}|)(\boldsymbol{s} - \widehat{\boldsymbol{s}}) + (\nu(x,|\boldsymbol{s}|) - \nu(x,|\widehat{\boldsymbol{s}}|))\widehat{\boldsymbol{s}}| \\
&\quad \leq \nu(x,|\boldsymbol{s}|)|\boldsymbol{s} - \widehat{\boldsymbol{s}}| + \big|\,(\nu(x,|\boldsymbol{s}|) - \nu(x,|\widehat{\boldsymbol{s}}|))|\widehat{\boldsymbol{s}}|\,\big| \\
&\quad = \nu(x,|\boldsymbol{s}|)|\boldsymbol{s} - \widehat{\boldsymbol{s}}| + \big|\,\nu(x,|\boldsymbol{s}|)(|\widehat{\boldsymbol{s}}| - |\boldsymbol{s}|) + \nu(x,|\boldsymbol{s}|)|\boldsymbol{s}| - \nu(x,|\widehat{\boldsymbol{s}}|)|\widehat{\boldsymbol{s}}|\,\big| \\
&\quad \leq 2\nu(x,|\boldsymbol{s}|)|\boldsymbol{s} - \widehat{\boldsymbol{s}}| + \big|\,\nu(x,|\boldsymbol{s}|)|\boldsymbol{s}| - \nu(x,|\widehat{\boldsymbol{s}}|)|\widehat{\boldsymbol{s}}|\,\big|.
\end{aligned}
$$

Hence, by the assumption (2.4), it follows that

$$
|\nu(x,|\boldsymbol{s}|)\boldsymbol{s} - \nu(x,|\widehat{\boldsymbol{s}}|)\widehat{\boldsymbol{s}}| \leq (2\nu(x,|\boldsymbol{s}|) + \overline{\nu})|\boldsymbol{s} - \widehat{\boldsymbol{s}}| \underbrace{\leq}_{(2.1)} (2\nu_0 + \overline{\nu})|\boldsymbol{s} - \widehat{\boldsymbol{s}}| = L|\boldsymbol{s} - \widehat{\boldsymbol{s}}|.
$$

This completes the proof. ∎

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