

1 **FULLY DISCRETE SCHEME FOR BEAN'S CRITICAL-STATE**
2 **MODEL WITH TEMPERATURE EFFECTS IN**
3 **SUPERCONDUCTIVITY ***

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5 **Abstract.** This paper considers a hyperbolic Maxwell variational inequality with temperature
6 effects arising from Bean's critical-state model in type-II (high-temperature) superconductivity. Here,
7 temperature dependence is included in the critical current density due to its main importance for
8 the realization of superconducting effects, as confirmed through physical measurements. We propose
9 a fully discrete scheme based on the implicit Euler in time and a mixed FEM in space consisting
10 of Nédélec's edge elements for the electric field and piecewise constant elements for the magnetic
11 induction. Furthermore, the initial approximation is specified by a compatibility system given by an
12 elliptic curl-curl variational inequality. This specific setting enables us to derive the well-posedness
13 of the discrete solution with a certain magnetic induction regularity. Our main result is the uniform
14 convergence of the proposed fully discrete method. To prove this result, first of all, we establish
15 stability estimates for the zero-order and first-order terms of the fully discrete solution. These
16 stability estimates along with the underlying nonlinear structure allow us to derive a weak-star
17 convergence result, which in particular yields the well-posedness of the governing Maxwell variational
18 inequality with temperature effects. Finally, through the use of the solution operator for a discrete
19 mixed variational problem in combination with the involved magnetic induction regularity and the
20 weak-star convergence result, we are able to complete the proof of the uniform convergence. The last
21 part of the paper is devoted to the a priori error analysis under a low Sobolev regularity assumption
22 on the electric field. We close this paper by presenting some 3D numerical results, which especially
23 confirm the physical Meissner-Ochsenfeld effect in superconductivity.

24 **Key words.** Maxwell variational inequality, Bean's critical-state model with temperature effects,
25 superconductivity, fully discrete scheme, convergence analysis, error estimates.

26 **AMS subject classifications.** 35Q61, 35L87, 78M10

27 **1. Introduction.** The physical phenomenon of superconductivity is character-
28 ized by zero electrical resistance and repulsion of magnetic fields (Meissner-Ochsenfeld
29 effect) under the condition that the temperature is below some critical level. It was
30 first discovered in 1911 by H. Kamerlingh-Onnes and has gained tremendous theoret-
31 ical and practical attentions ever since. Nowadays, modern magnetic levitation trains,
32 distributed superconducting magnetic energy storage (D-SMES), magnetic resonance
33 imaging (MRI) and magnetic confinement fusion cannot be realized without the use
34 of superconductors, just to mention a few key technologies. A critical-state model de-
35 scribing the magnetization process of penetration and exit of magnetic flux in type-II
36 (high-temperature) superconductors was proposed by Bean [5, 6]. More precisely, his
37 model describes a nonlinear and non-smooth constitutive relation between the (total)
38 current density and the electric field as follows:

39 (B1) The current density strength $|\mathbf{J}|$ cannot exceed the critical current j_c ;

40 (B2) if $|\mathbf{J}|$ is strictly less than j_c , then the electric field \mathbf{E} vanishes;

41 (B3) the electric field \mathbf{E} is parallel to \mathbf{J} .

42 We underline that the (unknown) superconductive region is determined by points
43 $(x, t) \in \Omega \times (0, T)$, for which the strict inequality $|\mathbf{J}(x, t)| < j_c$ is satisfied. Thus, in

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44 this region, there is no electrical resistance so that the electric field must vanish.

45 We refer to Bossavit [7–9] for early contributions towards extended Bean’s law
 46 and the corresponding finite element method. The Bean critical-state model (B1)-(B3)
 47 governed by the eddy current equations with magnetic field dependence $j_c = j_c(\mathbf{H})$
 48 leads to a parabolic quasi-variational inequality (QVI) of obstacle type. Prigozhin [31,
 49 32] was the first, who introduced and analyzed this formulation. Barrett and Prigozhin
 50 [3] analyzed it in a scalar two-dimensional (2D) setting and its dual formulation. The
 51 finite element analysis for the associated parabolic variational inequality in a 2D
 52 setting was investigated in [16] (see also [17] for a similar 2D model using an $\mathbf{E}\text{-}\mathbf{J}$ -
 53 formulation). Furthermore, the numerical analysis for the three-dimensional (3D)
 54 setting was investigated in [15]. Recent results on the numerical analysis for the
 55 parabolic QVI in a 2D setting were obtained in [4]. All the previously mentioned
 56 contributions were devoted to the numerical analysis for the eddy current case. In the
 57 full 3D Maxwell case (cf. [22]), the Bean’s critical state model (B1)-(B3) with $j_c =$
 58 $j_c(x)$ leads to a hyperbolic Maxwell variational inequality of the second kind [35] (see
 59 [33] for the mathematical analysis in a more general setting). The numerical analysis
 60 for this variational inequality is still in its earlier state. We are only aware of the
 61 recent work [35] for the analysis of the semi-discrete spatial Galerkin approximations.

62 This paper is devoted to the fully discrete analysis of the Bean critical-state model
 63 (B1)-(B3) governed by the full 3D Maxwell equations with temperature effects. Let
 64 us underline that in all previously mentioned contributions, temperature dependence
 65 was neglected. However, by the nature of superconductivity, temperature effects play
 66 a major role, since superconducting effects strongly depend on the temperature itself
 67 and can only be reached, if the temperature is underneath some critical level. We
 68 refer to [2, 14] concerning experimental measurements showing the strong temperature
 69 dependence in the critical current $j_c = j_c(x, \theta(x, t))$ and its physical properties. Let
 70 us now formulate the variational inequality we focused on in this paper:

$$71 \quad (VI) \quad \left\{ \begin{array}{l} \int_{\Omega} \epsilon \partial_t \mathbf{E}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) + \mu^{-1} \partial_t \mathbf{B}(t) \cdot (\mathbf{w} - \mathbf{B}(t)) \, dx \\ \quad + \int_{\Omega} \mu^{-1} \mathbf{curl} \, \mathbf{E}(t) \cdot \mathbf{w} - \mu^{-1} \mathbf{B}(t) \cdot \mathbf{curl} \, \mathbf{v} \, dx \\ \quad + \varphi(\theta(t), \mathbf{v}) - \varphi(\theta(t), \mathbf{E}(t)) \geq \int_{\Omega} \mathbf{f}(t) \cdot (\mathbf{v} - \mathbf{E}(t)) \, dx \\ \text{for a.e. } t \in (0, T) \text{ and every } (\mathbf{v}, \mathbf{w}) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{L}^2(\Omega), \\ \quad (\mathbf{E}(0), \mathbf{B}(0)) = (\mathbf{E}_0, \mathbf{B}_0), \end{array} \right.$$

72
 73 with a nonsmooth L^1 -type functional

$$74 \quad \varphi: L^\infty(\Omega) \times \mathbf{L}^1(\Omega) \rightarrow \mathbb{R}, \quad (y, \mathbf{v}) \mapsto \int_{\Omega} j_c(x, y(x)) |\mathbf{v}(x)| \, dx.$$

76 In this setting, $\Omega \subset \mathbb{R}^3$ is a bounded polyhedral domain with a connected Lipschitz-
 77 boundary $\partial\Omega$. The assumption of the connected boundary guarantees that $\{\mathbf{v} \in$
 78 $\mathbf{H}_0(\mathbf{curl}) \cap \mathbf{H}(\text{div}) \mid \mathbf{curl} \, \mathbf{v} = 0, \text{div} \, \mathbf{v} = 0\} = \{0\}$ (cf. [1, Proposition 3.18.]), which is
 79 required for our analysis in connection with the application of the (discrete) Poincaré-
 80 Friedrichs-type inequality [20, Theorem 4.7]. Furthermore, $\mathbf{E} : \Omega \times (0, T) \rightarrow \mathbb{R}^3$
 81 denotes the electric field, $\mathbf{B} : \Omega \times (0, T) \rightarrow \mathbb{R}^3$ the magnetic induction, $\mathbf{f} : \Omega \times (0, T) \rightarrow$
 82 \mathbb{R}^3 the applied current source and $\theta : \Omega \times (0, T) \rightarrow \mathbb{R}^3$ the temperature distribution.
 83 Note that in (VI) and all what follows, we use the abbreviation $\mathbf{E}(t) = \mathbf{E}(\cdot, t)$ (the

84 same notation is also used for other quantities). The precise assumptions for the data
 85 involved in (VI) will be given in Section 2.

86 In $(\mathbf{VI}_{N,h})$, we propose a fully discrete scheme for (VI) based on the implicit
 87 Euler in time and a mixed FEM in space consisting of Nédélec's edge elements [30]
 88 for \mathbf{E} and piecewise constant elements for \mathbf{B} . Furthermore, we consider finite element
 89 approximations for the initial data $(\mathbf{E}_0, \mathbf{B}_0)$ by solving an elliptic **curl-curl** variational
 90 inequality (3.2). This specific setting enables us to prove the well-posedness of $(\mathbf{VI}_{N,h})$
 91 with a magnetic induction regularity in $\mathbf{curl} \mathbf{V}_h$ (see Theorem 3.4), where \mathbf{V}_h denotes
 92 the Nédélec edge element space.

93 Our main goal is the uniform convergence of $(\mathbf{VI}_{N,h})$ towards (VI) (Theorem 3.10),
 94 which in particular yields the global well-posedness for (VI). The proof follows the
 95 following consecutive steps: First of all, by the compatibility system (3.2) and ex-
 96 ploiting the regularity properties of the critical current density and the given data,
 97 we derive stability estimates for the zero-order and first-order terms of the fully dis-
 98 crete solution (Lemmas 3.6 and 3.7). These a priori estimates together with the
 99 mathematical properties of φ allow us to extract weakly-* converging subsequences
 100 whose limits turn out to solve the original variational inequality (Theorem 3.8). In
 101 particular, this implies the well-posedness of (VI). Hereafter, we consider the solution
 102 operator $\Phi_h: \mathbf{H}_0(\mathbf{curl}) \rightarrow \mathbf{V}_h$ associated with a discrete mixed variational problem
 103 (Definition 3.2) and use its properties in combination with the magnetic induction
 104 regularity in $\mathbf{curl} \mathbf{V}_h$ and the weak-star convergence result to complete the proof of
 105 the uniform convergence. The final part of the paper is devoted to the a priori error
 106 analysis for the proposed fully discrete scheme $(\mathbf{VI}_{N,h})$. Under a low Sobolev reg-
 107 ularity assumption on the electric field \mathbf{E} of (VI), we derive a priori estimates for
 108 the error between the fully discrete solution and the continuous one (Theorem 4.4).
 109 The proof is based on the use of the operator $\Phi_h: \mathbf{H}_0(\mathbf{curl}) \rightarrow \mathbf{V}_h$ and the recent
 110 sharp quasi-interpolation results [13, 18, 19]. Last but not least, we refer the reader
 111 to some existing works [11, 12, 24, 25, 28] concerning fully discrete approximations for
 112 time-dependent Maxwell's equations.

113 **2. Preliminaries.** For a given Banach space X , we denote its norm by $\|\cdot\|_X$ and
 114 the duality pairing with the corresponding dual space X^* by $\langle \cdot, \cdot \rangle$. If X is a Hilbert
 115 space, then $(\cdot, \cdot)_X$ stands for its scalar product and $\|\cdot\|_X$ for the induced norm. In
 116 the case of $X = \mathbb{R}^n$, we renounce the subscript in the (Euclidean) norm and write
 117 $|\cdot|$. The Euclidean scalar product is denoted by a dot. Unless otherwise stated, we
 118 identify the dual space X^* with the Hilbert space X itself. The embedding between
 119 two Banach spaces X, Y is denoted by $X \hookrightarrow Y$. Now, we introduce some important
 120 Hilbert spaces throughout this paper:

$$121 \quad \mathbf{H}(\mathbf{curl}) := \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \mathbf{curl} \mathbf{v} \in \mathbf{L}^2(\Omega)\} \quad \text{and} \quad \mathbf{H}(\mathbf{div}) := \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \mathbf{div} \mathbf{v} \in L^2(\Omega)\},$$

123 where **curl** and **div** are understood in the distributional sense. Also, note that we use
 124 bold letters for vector-valued functions and the respective spaces. As usual, $\mathbf{C}_0^\infty(\Omega)$
 125 denotes the space of all infinitely differentiable functions with compact support in
 126 Ω . The spaces $\mathbf{H}_0(\mathbf{curl})$ and $\mathbf{H}_0(\mathbf{div})$ stand for the closure of $\mathbf{C}_0^\infty(\Omega)$ with respect
 127 to the $\mathbf{H}(\mathbf{curl})$ -norm and the $\mathbf{H}(\mathbf{div})$ -norm, respectively. Furthermore, the spaces of
 128 divergence-free vector functions are

$$129 \quad \mathbf{H}(\mathbf{div}=0) := \{\mathbf{v} \in \mathbf{L}^2(\Omega) : (\mathbf{v}, \nabla \phi)_{\mathbf{L}^2(\Omega)} = 0 \quad \forall \phi \in H_0^1(\Omega)\},$$

$$130 \quad \mathbf{H}_0(\mathbf{div}=0) := \{\mathbf{v} \in \mathbf{L}^2(\Omega) : (\mathbf{v}, \nabla \phi)_{\mathbf{L}^2(\Omega)} = 0 \quad \forall \phi \in H^1(\Omega)\},$$

132 which are endowed with the $\mathbf{L}^2(\Omega)$ -norm. Material parameters will occur on the prob-
 133 lem statement, and thus, for a given positive function $\alpha \in L^\infty(\Omega)$, we denote by $\mathbf{L}_\alpha^2(\Omega)$
 134 the weighted $\mathbf{L}^2(\Omega)$ -space with the weighted scalar product $(\alpha \cdot, \cdot)_{\mathbf{L}^2(\Omega)}$. Moreover, we
 135 denote by $C > 0$ a generic constant, that can change during an estimation. Let us
 136 close this section by presenting all the mathematical assumptions for (VI).

137 *Assumption 2.1* (Regularity assumptions on the material parameters).

138 (A1) The material parameters $\epsilon, \mu \in L^\infty(\Omega)$ are strictly positive, i.e., there exist
 139 positive constants $\underline{\epsilon}, \bar{\epsilon}, \underline{\mu}, \bar{\mu} \in \mathbb{R}_{>0}$ such that

$$140 \quad \underline{\epsilon} \leq \epsilon(x) \leq \bar{\epsilon} \quad \text{and} \quad \underline{\mu} \leq \mu(x) \leq \bar{\mu} \quad \text{for a.e. } x \in \Omega.$$

142 (A2) For every $y \in \mathbb{R}$, $j_c(\cdot, y): \Omega \rightarrow \mathbb{R}$ is Lebesgue-measurable and nonnegative.

143 (A3) For every $M > 0$, there exists a constant $C(M) > 0$ such that

$$144 \quad 0 \leq j_c(x, y) \leq C(M)$$

146 for a.e. $x \in \Omega$ and every $y \in [-M, M]$.

147 (A4) For every $M > 0$, there exists a constant $L(M) > 0$ such that

$$148 \quad |j_c(x, y) - j_c(x, z)| \leq L(M)|y - z|$$

150 for a.e. $x \in \Omega$ and every $y, z \in [-M, M]$.

151 Let us remark that the local Lipschitz property (A4) and the local boundedness
 152 property (A3) for the temperature dependence in the critical current are justified
 153 by experimental measurements reported in [2, 14]. Note that assumptions (A2)–(A4)
 154 seem to be sharp for our mathematical analysis. In contrast to (A2)–(A4), from the
 155 mathematical point of view, (A1) is not sharp, as our results can be extended to
 156 matrix-valued material parameters ϵ and μ . However, this case leads to a physical
 157 model of an anisotropic material, for which Bean's law (B1)–(B3) is not suitable.
 158 Indeed, (B3) is only reasonable for a scalar-valued resistivity, i.e., not for anisotropic
 159 materials (see [3, 4, 8, 15, 31, 32]). Therefore, due to this physical reason, we only
 160 consider scalar-valued material parameters.

161 *Assumption 2.2* (Regularity assumptions on the given data).

162 (A5) Suppose that

$$163 \quad \mathbf{f} \in \mathcal{C}^{0,1}([0, T], \mathbf{L}^2(\Omega)) \quad \text{and} \quad \theta \in \mathcal{C}^{0,1}([0, T], L^2(\Omega)) \cap \mathcal{C}([0, T], L^\infty(\Omega)).$$

165 (A6) The initial data $(\mathbf{E}_0, \mathbf{B}_0) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}_0(\text{div}=0)$ satisfies the compatibility
 166 system

$$167 \quad (2.1) \quad \left\{ \begin{array}{l} \int_{\Omega} \epsilon \mathbf{E}_0 \cdot (\mathbf{v} - \mathbf{E}_0) + \mu^{-1} \mathbf{B}_0 \cdot (\mathbf{w} - \mathbf{B}_0) dx \\ \quad + \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{E}_0 \cdot \mathbf{w} - \mu^{-1} \mathbf{B}_0 \cdot \mathbf{curl} \mathbf{v} dx \\ \quad + \varphi(\theta(0), \mathbf{v}) - \varphi(\theta(0), \mathbf{E}_0) \geq \int_{\Omega} \mathbf{f}(0) \cdot (\mathbf{v} - \mathbf{E}_0) dx \\ \quad \text{for all } (\mathbf{v}, \mathbf{w}) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{L}^2(\Omega). \end{array} \right.$$

169 **3. Fully discrete scheme.** As pointed out in the introduction, we focus on the
 170 implicit Euler scheme for the time discretization in (VI). To this aim, let us fix $N \in \mathbb{N}$

171 and define an equidistant partition of $[0, T]$ in the following way:

$$172 \quad \tau := \frac{T}{N}, \quad 0 = t_0 < t_1 < \dots < t_N = T \quad \text{with} \quad t_n := n\tau$$

174 for all $n \in \{0, \dots, N\}$. Furthermore, we define

$$175 \quad \mathbf{f}^n := \mathbf{f}(t_n) \in \mathbf{L}^2(\Omega), \quad \varphi^n(\mathbf{v}) := \int_{\Omega} j_c(x, \theta(x, t_n)) |\mathbf{v}(x)| dx \quad \forall n \in \{0, \dots, N\}.$$

177 We choose a family of quasi-uniform triangulations $\{\mathcal{T}_h\}_{h>0}$, i.e.,

$$178 \quad \bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} T \quad \forall h > 0,$$

180 and, for h_T denoting the diameter of T and ρ_T denoting the diameter of the largest
181 ball contained in T , there exist constants $\rho > 0$ and $\nu > 0$ such that

$$182 \quad \frac{h_T}{\rho_T} \leq \rho \quad \text{and} \quad \frac{h}{h_T} \leq \nu \quad \forall T \in \mathcal{T}_h, \quad \forall h > 0.$$

184 The subscript h denotes the maximum of h_T for $T \in \mathcal{T}_h$. The finite element space of
185 Nédélec's first family of edge elements is defined by

$$186 \quad \mathbf{V}_h := \{\mathbf{v}_h \in \mathbf{H}_0(\mathbf{curl}) : \mathbf{v}_h|_T = \mathbf{a}_T + \mathbf{b}_T \times x \text{ with } \mathbf{a}_T, \mathbf{b}_T \in \mathbb{R}^3, \forall T \in \mathcal{T}_h\},$$

188 and the finite element space of piecewise constant functions is denoted by

$$189 \quad \mathbf{W}_h := \{\mathbf{w}_h \in \mathbf{L}^2(\Omega) : \mathbf{w}_h|_T = \mathbf{a}_T \text{ with } \mathbf{a}_T \in \mathbb{R}^3, \forall T \in \mathcal{T}_h\},$$

191 which satisfy $\mathbf{curl} \mathbf{V}_h \subset \mathbf{W}_h$. In addition to these spaces, we introduce the space of
192 continuous piecewise linear elements with vanishing traces by

$$193 \quad \Theta_h := \{\phi_h \in H_0^1(\Omega) : \phi_h|_T = \mathbf{a}_T \cdot x + b_T \text{ with } \mathbf{a}_T \in \mathbb{R}^3, b_T \in \mathbb{R} \quad \forall T \in \mathcal{T}_h\}.$$

195 Moreover, the family $\{\mathcal{T}_h\}_{h>0}$ is chosen such that there exists $\bar{h} > 0$ with

$$196 \quad (3.1) \quad \mathbf{V}_{\bar{h}} \subset \mathbf{V}_h \quad \text{and} \quad \mathbf{W}_{\bar{h}} \subset \mathbf{W}_h \quad \forall 0 < h \leq \bar{h} \leq \bar{h}.$$

197 Having introduced all the required finite element spaces, we now propose the following
198 fully discrete scheme to (VI):

$$199 \quad (\text{VI}_{N,h}) \quad \left\{ \begin{array}{l} \int_{\Omega} \epsilon \delta \mathbf{E}_h^n \cdot (\mathbf{v}_h - \mathbf{E}_h^n) + \mu^{-1} \delta \mathbf{B}_h^n \cdot (\mathbf{w}_h - \mathbf{B}_h^n) dx \\ \quad + \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{E}_h^n \cdot \mathbf{w}_h - \mu^{-1} \mathbf{B}_h^n \cdot \mathbf{curl} \mathbf{v}_h dx \\ \quad + \varphi^n(\mathbf{v}_h) - \varphi^n(\mathbf{E}_h^n) \geq \int_{\Omega} \mathbf{f}^n \cdot (\mathbf{v}_h - \mathbf{E}_h^n) dx \\ \text{for every } (\mathbf{v}_h, \mathbf{w}_h) \in \mathbf{V}_h \times \mathbf{W}_h \text{ and } n \in \{1, \dots, N\} \\ (\mathbf{E}_h^0, \mathbf{B}_h^0) = (\mathbf{E}_{0h}, \mathbf{B}_{0h}), \end{array} \right.$$

200 where

$$201 \quad \delta \mathbf{E}_h^n := \frac{\mathbf{E}_h^n - \mathbf{E}_h^{n-1}}{\tau} \quad \text{and} \quad \delta \mathbf{B}_h^n := \frac{\mathbf{B}_h^n - \mathbf{B}_h^{n-1}}{\tau} \quad \forall n \in \{1, \dots, N\},$$

202

203 Moreover, $(\mathbf{E}_{0h}, \mathbf{B}_{0h}) \in \mathbf{V}_h \times \mathbf{W}_h$ denotes the finite element approximation of the
 204 initial data $(\mathbf{E}_0, \mathbf{B}_0)$, which is defined as the solution to the discrete mixed problem

$$(3.2) \quad \begin{cases} \int_{\Omega} \epsilon \mathbf{E}_{0h} \cdot (\mathbf{v}_h - \mathbf{E}_{0h}) + \mu^{-1} \mathbf{B}_{0h} \cdot (\mathbf{w}_h - \mathbf{B}_{0h}) dx \\ + \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{E}_{0h} \cdot \mathbf{w}_h - \mu^{-1} \mathbf{B}_{0h} \cdot \mathbf{curl} \mathbf{v}_h dx \\ + \varphi(\theta(0), \mathbf{v}_h) - \varphi(\theta(0), \mathbf{E}_{0h}) \geq \int_{\Omega} \mathbf{f}(0) \cdot (\mathbf{v}_h - \mathbf{E}_{0h}) dx \\ \text{for all } (\mathbf{v}_h, \mathbf{w}_h) \in \mathbf{V}_h \times \mathbf{W}_h. \end{cases}$$

207 The well-posedness of (3.2) follows from the classical theory of variational inequalities
 208 [26, Theorem 2.2], as (3.2) is equivalent to an elliptic **curl-curl** variational inequality
 209 (cf. the proof of Theorem 3.4). In view of (3.2), it makes sense to set $(\delta \mathbf{E}_h^0, \delta \mathbf{B}_h^0) :=$
 210 $(\mathbf{E}_{0h}, \mathbf{B}_{0h})$. Indeed, if we replace $(\delta \mathbf{E}_h^n, \delta \mathbf{B}_h^n)$ by $(\mathbf{E}_{0h}, \mathbf{B}_{0h})$ in $(\mathbf{VI}_{N,h})$ and set $n = 0$,
 211 then we arrive exactly at (3.2). Note that $(\delta \mathbf{E}_h^0, \delta \mathbf{B}_h^0) = (\mathbf{E}_{0h}, \mathbf{B}_{0h})$ is important for
 212 our stability analysis (see (3.22) in the proof of Lemma 3.7).

213 *Remark 3.1.* All mathematical findings in this paper remain true, if we replace
 214 \mathbf{W}_h by $\mathbf{W}_h \cap \mathbf{H}_0(\text{div})$. Both \mathbf{W}_h and $\mathbf{W}_h \cap \mathbf{H}_0(\text{div})$ are dense in $\mathbf{L}^2(\Omega)$ and con-
 215 tain $\mathbf{curl} \mathbf{V}_h$. The condition of $\mathbf{curl} \mathbf{V}_h$ being a subspace is necessary to prove the
 216 regularity properties $\mathbf{B}_{0h}, \mathbf{B}_h^n \in \mathbf{curl} \mathbf{V}_h$ for the solutions to (3.2) and $(\mathbf{VI}_{N,h})$; see
 217 Lemma 3.3 and Theorem 3.4. Furthermore, the density property is required for the
 218 derivation of the weak-* convergence result (Theorem 3.8). We note that the choice
 219 $\mathbf{W}_h = \mathbf{curl} \mathbf{V}_h$ is not suitable for our analysis, as $\mathbf{curl} \mathbf{V}_h$ is not dense in $\mathbf{L}^2(\Omega)$.

220 **DEFINITION 3.2.** For every $h > 0$ and $\mathbf{y} \in \mathbf{H}_0(\mathbf{curl})$, we denote the solution
 221 operator of the discrete variational mixed problem

$$(3.3) \quad \begin{cases} (\mathbf{curl} \mathbf{y}_h, \mathbf{curl} \mathbf{v}_h)_{\mathbf{L}^2_{1/\mu}(\Omega)} = (\mathbf{curl} \mathbf{y}, \mathbf{curl} \mathbf{v}_h)_{\mathbf{L}^2_{1/\mu}(\Omega)} & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\mathbf{y}_h, \nabla \psi_h)_{\mathbf{L}^2(\Omega)} = (\mathbf{y}, \nabla \psi_h)_{\mathbf{L}^2(\Omega)} & \forall \psi_h \in \Theta_h \end{cases}$$

224 by $\Phi_h: \mathbf{H}_0(\mathbf{curl}) \rightarrow \mathbf{V}_h$ with $\Phi_h \mathbf{y} := \mathbf{y}_h$.

225 The theory of mixed problems (cf. [29, Theorem 2.45]) in combination with the discrete
 226 Poincaré-Friedrichs-type inequality [20, Theorem 4.7] and the discrete LBB condition
 227 (cf. [37, pp. 2802-2803]) implies that for every $h > 0$ and $\mathbf{y} \in \mathbf{H}_0(\mathbf{curl})$, (3.3) admits
 228 a unique solution $\mathbf{y}_h = \Phi_h \mathbf{y} \in \mathbf{V}_h$ satisfying

$$(3.4) \quad \|\Phi_h \mathbf{y} - \mathbf{y}\|_{\mathbf{H}(\mathbf{curl})} \leq C \left(\inf_{\chi_h \in \mathbf{V}_h} \|\mathbf{y} - \chi_h\|_{\mathbf{H}(\mathbf{curl})} \right) \quad \forall \mathbf{y} \in \mathbf{H}_0(\mathbf{curl}).$$

231 with a constant $C > 0$, independent of h and \mathbf{y} . In particular, (3.4) yields

$$(3.5) \quad \|\Phi_h \mathbf{y}\|_{\mathbf{H}(\mathbf{curl})} \leq (C + 1) \|\mathbf{y}\|_{\mathbf{H}(\mathbf{curl})} \quad \forall h > 0, \quad \forall \mathbf{y} \in \mathbf{H}_0(\mathbf{curl}).$$

234 Moreover, (3.4) along with the density property of \mathbf{V}_h :

$$\forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}) \quad \forall \delta > 0 \quad \exists \tilde{h} > 0 \quad \forall h \in (0, \tilde{h}) \quad \exists \mathbf{v}_h \in \mathbf{V}_h : \quad \|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{H}(\mathbf{curl})} \leq \delta$$

236 implies that

$$(3.6) \quad \lim_{h \rightarrow 0} \|\Phi_h \mathbf{y} - \mathbf{y}\|_{\mathbf{H}(\mathbf{curl})} = 0 \quad \forall \mathbf{y} \in \mathbf{H}_0(\mathbf{curl}).$$

239 This operator enables us to show the strong convergence of the discrete initial values
 240 towards $(\mathbf{E}_0, \mathbf{B}_0)$.

241 LEMMA 3.3. Under *Assumptions 2.1 and 2.2*, the discrete approximation of the
 242 initial value $(\mathbf{E}_{0h}, \mathbf{B}_{0h}) \in \mathbf{V}_h \times \mathbf{W}_h$ satisfies $\mathbf{B}_{0h} \in \mathbf{curl} \mathbf{V}_h$ for all $h > 0$ and

$$243 \quad \lim_{h \rightarrow 0} \|\mathbf{E}_{0h} - \mathbf{E}_0\|_{\mathbf{L}_\epsilon^2(\Omega)} = \lim_{h \rightarrow 0} \|\mathbf{B}_{0h} - \mathbf{B}_0\|_{\mathbf{L}_{1/\mu}^2(\Omega)} = 0.$$

244
 245 *Proof.* Let $h > 0$. Inserting $\mathbf{v}_h = \mathbf{E}_{0h}$ in (3.2), we obtain that

$$246 \quad (3.7) \quad \int_{\Omega} \mu^{-1}(\mathbf{B}_{0h} + \mathbf{curl} \mathbf{E}_{0h}) \cdot (\mathbf{w}_h - \mathbf{B}_{0h}) dx = 0 \quad \forall \mathbf{w}_h \in \mathbf{W}_h.$$

247
 248 Since $\mathbf{curl} \mathbf{V}_h \subset \mathbf{W}_h$, we may set $\mathbf{w}_h := 2\mathbf{B}_{0h} + \mathbf{curl} \mathbf{E}_{0h}$ in (3.7), which implies

$$249 \quad (3.8) \quad \mathbf{B}_{0h} = -\mathbf{curl} \mathbf{E}_{0h}.$$

251 Thus, $\mathbf{B}_{0h} \in \mathbf{curl} \mathbf{V}_h \subset \mathbf{H}_0(\text{div}=0)$ follows. Moreover, testing (3.2) with $(\mathbf{v}_h, \mathbf{w}_h) =$
 252 $(2\mathbf{E}_{0h}, 2\mathbf{B}_{0h})$ as well as $(\mathbf{v}_h, \mathbf{w}_h) = (0, 0)$ yields

$$253 \quad \|\mathbf{E}_{0h}\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 + \|\mathbf{B}_{0h}\|_{\mathbf{L}_{1/\mu}^2(\Omega)}^2 + \int_{\Omega} j_c(x, \theta(x, 0)) |\mathbf{E}_{0h}(x)| dx = \int_{\Omega} \mathbf{f}(0) \cdot \mathbf{E}_{0h} dx$$

$$254 \quad (3.9) \quad \Rightarrow \|\mathbf{E}_{0h}\|_{\mathbf{L}_\epsilon^2(\Omega)} \leq \|\epsilon^{-1/2} \mathbf{f}(0)\|_{\mathbf{L}^2(\Omega)}.$$

255 Next, we insert $(\mathbf{v}, \mathbf{w}) = (\mathbf{E}_{0h}, \mathbf{B}_{0h})$ into (2.1) and $(\mathbf{v}_h, \mathbf{w}_h) = (\Phi_h \mathbf{E}_0, 0)$ into (3.2)
 256 and obtain after adding the resulting inequalities together that

$$257 \quad (3.10) \quad \|\mathbf{E}_{0h} - \mathbf{E}_0\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 + \|\mathbf{B}_{0h} - \mathbf{B}_0\|_{\mathbf{L}_{1/\mu}^2(\Omega)}^2$$

$$258 \quad \leq \int_{\Omega} \mathbf{f}(0) \cdot (\mathbf{E}_0 - \Phi_h \mathbf{E}_0) dx + \int_{\Omega} \epsilon \mathbf{E}_{0h} \cdot (\Phi_h \mathbf{E}_0 - \mathbf{E}_0) dx$$

$$259 \quad - \int_{\Omega} \mu^{-1}(\mathbf{B}_{0h} + \mathbf{curl} \mathbf{E}_{0h}) \cdot \mathbf{B}_0 dx + \int_{\Omega} \mu^{-1} \mathbf{B}_{0h} \cdot \mathbf{curl}(\mathbf{E}_0 - \Phi_h \mathbf{E}_0) dx$$

$$260 \quad + \int_{\Omega} j_c(x, \theta(x, 0)) (|\Phi_h \mathbf{E}_0| - |\mathbf{E}_0|) dx.$$

261
 262 Due to (3.8), the third term on the right-hand side of (3.10) vanishes. Moreover,

$$263 \quad \int_{\Omega} \mu^{-1} \mathbf{B}_{0h} \cdot \mathbf{curl}(\mathbf{E}_0 - \Phi_h \mathbf{E}_0) dx \underbrace{=} \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{E}_{0h} \cdot \mathbf{curl}(\mathbf{E}_0 - \Phi_h \mathbf{E}_0) dx \underbrace{=} 0.$$

264
 265 Thus, applying Hölder's inequality together with (A3), (A5) and (3.9) to (3.10) yields

$$266 \quad (3.11) \quad \|\mathbf{E}_{0h} - \mathbf{E}_0\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 + \|\mathbf{B}_{0h} - \mathbf{B}_0\|_{\mathbf{L}_{1/\mu}^2(\Omega)}^2 \leq C \|\Phi_h \mathbf{E}_0 - \mathbf{E}_0\|_{\mathbf{L}_\epsilon^2(\Omega)}$$

267
 268 with a constant $C > 0$ only depending on $\epsilon, \mathbf{f}, j_c$ and θ . Finally, passing to the limit
 269 $h \rightarrow 0$ in (3.11), (3.6) yields the assertion. \square

270 The following theorem proves the well-posedness of $(\mathbf{VI}_{N,h})$ and gives an important
 271 regularity property for the discrete magnetic induction.

272 THEOREM 3.4. Let *Assumptions 2.1 and 2.2* hold. Then, for every $h > 0$ and $N \in$
 273 \mathbb{N} , the system of discrete variational inequalities $(\mathbf{VI}_{N,h})$ admits a unique solution
 274 $\{(\mathbf{E}_h^n, \mathbf{B}_h^n)\}_{n=1}^N \subset \mathbf{V}_h \times \mathbf{curl} \mathbf{V}_h$.

275 *Proof.* Let $N \in \mathbb{N}$, $h > 0$ and $n \in \{1, \dots, N\}$. Furthermore, assume that
 276 $(\mathbf{E}_h^{n-1}, \mathbf{B}_h^{n-1})$ is already known. Using the same arguments as in (3.8), we may de-
 277 couple the variational inequality into two parts by testing $(\mathbf{VI}_{N,h})$ with $\mathbf{v}_h = \mathbf{E}_h^n$ to
 278 obtain that

$$279 \quad (3.12) \quad \delta \mathbf{B}_h^n = -\mathbf{curl} \mathbf{E}_h^n.$$

281 By definition, (3.12) yields the following explicit formula for \mathbf{B}_h^n :

$$282 \quad (3.13) \quad \mathbf{B}_h^n = \mathbf{B}_h^{n-1} - \tau \mathbf{curl} \mathbf{E}_h^n.$$

284 Next, we insert $\mathbf{w}_h = \mathbf{B}_h^n$ in $(\mathbf{VI}_{N,h})$ and employ (3.13) to obtain the variational
 285 inequality

$$287 \quad (3.14) \quad \int_{\Omega} \epsilon \delta \mathbf{E}_h^n \cdot (\mathbf{v}_h - \mathbf{E}_h^n) dx + \int_{\Omega} \tau \mu^{-1} \mathbf{curl} \mathbf{E}_h^n \cdot \mathbf{curl} (\mathbf{v}_h - \mathbf{E}_h^n) dx + \varphi^n(\mathbf{v}_h) \\ 288 \quad - \varphi^n(\mathbf{E}_h^n) \geq \int_{\Omega} \mathbf{f}^n \cdot (\mathbf{v}_h - \mathbf{E}_h^n) + \mu^{-1} \mathbf{B}_h^{n-1} \cdot \mathbf{curl} (\mathbf{v}_h - \mathbf{E}_h^n) dx \quad \forall \mathbf{v}_h \in \mathbf{V}_h.$$

290 The well-posedness of (3.14) is covered by a classical result in [26, Theorem 2.2],
 291 because it is equivalent to an elliptic **curl-curl** variational inequality of the form

$$292 \quad (3.15) \quad a(\mathbf{E}_h^n, \mathbf{v}_h - \mathbf{E}_h^n) + \varphi^n(\mathbf{v}_h) - \varphi^n(\mathbf{E}_h^n) \geq \langle \tilde{\mathbf{f}}^n, \mathbf{v}_h - \mathbf{E}_h^n \rangle \quad \forall \mathbf{v}_h \in \mathbf{V}_h,$$

294 with the continuous and coercive bilinear form $a: \mathbf{V}_h \times \mathbf{V}_h \rightarrow \mathbb{R}$ defined by

$$295 \quad a(\mathbf{u}_h, \mathbf{v}_h) := \int_{\Omega} \tau^{-1} \epsilon \mathbf{u}_h \cdot \mathbf{v}_h dx + \int_{\Omega} \tau \mu^{-1} \mathbf{curl} \mathbf{u}_h \cdot \mathbf{curl} \mathbf{v}_h dx \quad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h$$

297 and the right-hand side $\tilde{\mathbf{f}}^n \in \mathbf{H}_0(\mathbf{curl})^*$ by

$$298 \quad \langle \tilde{\mathbf{f}}^n, \mathbf{v} \rangle := \int_{\Omega} (\mathbf{f}^n + \tau^{-1} \epsilon \mathbf{E}_h^{n-1}) \cdot \mathbf{v} dx + \int_{\Omega} \mu^{-1} \mathbf{B}_h^{n-1} \cdot \mathbf{curl} \mathbf{v} dx \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}).$$

300 Inserting the solution \mathbf{E}_h^n of (3.14) into (3.13), we finally obtain a unique solution

$$301 \quad \{(\mathbf{E}_h^n, \mathbf{B}_h^n)\}_{n=1}^N \subset \mathbf{V}_h \times \mathbf{W}_h$$

303 of $(\mathbf{VI}_{N,h})$. Finally, (3.13) and Lemma 3.3 give $\mathbf{B}_h^n \in \mathbf{curl} \mathbf{V}_h$ by inductive reasoning. \square

304 *Remark 3.5.* The formulas (3.13) and (3.14) will be the foundation for the com-
 305 putation of the numerical solution.

306 The following Lemmas prove the zero-order and first-order stability estimates for the
 307 fully discrete solution to $(\mathbf{VI}_{N,h})$:

308 **LEMMA 3.6.** *Let Assumptions 2.1 and 2.2 be satisfied. Then, there exists a con-*
 309 *stant $C > 0$, depending only on \mathbf{f} , \mathbf{E}_0 , \mathbf{B}_0 and T, ϵ, μ , such that for every $N \in \mathbb{N}$ and*
 310 *$h > 0$, the solution $\{(\mathbf{E}_h^n, \mathbf{B}_h^n)\}_{n=1}^N$ of $(\mathbf{VI}_{N,h})$ fulfills the estimate*

$$312 \quad (3.16) \quad \max_{n \in \{1, \dots, N\}} \|\mathbf{E}_h^n\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 + \max_{n \in \{1, \dots, N\}} \|\mathbf{B}_h^n\|_{\mathbf{L}_{1/\mu}^2(\Omega)}^2 \\ 313 \quad + \sum_{n=1}^N \|\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 + \sum_{n=1}^N \|\mathbf{B}_h^n - \mathbf{B}_h^{n-1}\|_{\mathbf{L}_{1/\mu}^2(\Omega)}^2 + 2\tau \sum_{n=1}^N \varphi^n(\mathbf{E}_h^n) \leq C.$$

314

315 *Proof.* Let $N \in \mathbb{N}$ and $h > 0$. We start by testing $(\mathbf{VI}_{N,h})$ with $(\mathbf{v}_h, \mathbf{w}_h) =$
 316 $(2\mathbf{E}_h^n, 2\mathbf{B}_h^n)$ as well as with $(\mathbf{v}_h, \mathbf{w}_h) = (0, 0)$ to obtain

(3.17)

$$317 \int_{\Omega} \epsilon \delta \mathbf{E}_h^n \cdot \mathbf{E}_h^n dx + \int_{\Omega} \mu^{-1} \delta \mathbf{B}_h^n \cdot \mathbf{B}_h^n dx + \varphi^n(\mathbf{E}_h^n) = \int_{\Omega} \mathbf{f}^n \cdot \mathbf{E}_h^n dx \quad \forall n \in \{1, \dots, N\}.$$

319 Now, fix $i_0 \in \{1, \dots, N\}$ and sum (3.17) up over $\{1, \dots, i_0\}$. Then, applying the
 320 binomial formulas along with the Hölder and Young inequalities, we deduce that

$$321 \begin{aligned} (3.18) \quad & \|\mathbf{E}_h^{i_0}\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{B}_h^{i_0}\|_{\mathbf{L}^2_{1/\mu}(\Omega)} + \sum_{n=1}^{i_0} \|\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\|_{\mathbf{L}^2(\Omega)}^2 \\ & + \sum_{n=1}^{i_0} \|\mathbf{B}_h^n - \mathbf{B}_h^{n-1}\|_{\mathbf{L}^2_{1/\mu}(\Omega)}^2 + 2\tau \sum_{n=1}^{i_0} \varphi^n(\mathbf{E}_h^n) \\ 322 & \leq \frac{2T\tau}{\epsilon} \sum_{n=1}^{i_0} \|\mathbf{f}^n\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{E}_{0h}\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{B}_{0h}\|_{\mathbf{L}^2_{1/\mu}(\Omega)}^2 + \frac{\tau}{2T} \sum_{n=1}^{i_0} \|\mathbf{E}_h^n\|_{\mathbf{L}^2(\Omega)}^2. \end{aligned}$$

325 This, combined with (A5) and Lemma 3.3 and the fact that $\tau/T = 1/N \leq 1$ gives us
 326 an estimate of the form

$$327 \|\mathbf{E}_h^{i_0}\|_{\mathbf{L}^2(\Omega)}^2 \leq C + \sum_{n=1}^{i_0-1} \frac{1}{N} \|\mathbf{E}_h^n\|_{\mathbf{L}^2(\Omega)}^2 \quad \Rightarrow \quad \|\mathbf{E}_h^{i_0}\|_{\mathbf{L}^2(\Omega)}^2 \leq C \exp\left(\sum_{n=1}^{i_0-1} \frac{1}{N}\right) \leq C,$$

329 where we have used the discrete Gronwall inequality. Since i_0 was arbitrary, we see
 330 from (3.18) that the proof is finished. \square

331 **LEMMA 3.7.** *Let Assumptions 2.1 and 2.2 hold. Then, there exists a constant*
 332 *$C > 0$, depending only on $\mathbf{f}, \mathbf{E}_0, \mathbf{B}_0$ and $T, \epsilon, \mu, \theta, j_c$, such that for every $N \in \mathbb{N}$ and*
 333 *$h > 0$ the solution $\{(\mathbf{E}_h^n, \mathbf{B}_h^n)\}_{n=1}^N$ of $(\mathbf{VI}_{N,h})$ satisfies*

$$334 \begin{aligned} (3.19) \quad & \max_{n \in \{1, \dots, N\}} \|\delta \mathbf{E}_h^n\|_{\mathbf{L}^2(\Omega)}^2 + \max_{n \in \{1, \dots, N\}} \|\delta \mathbf{B}_h^n\|_{\mathbf{L}^2_{1/\mu}(\Omega)} \\ & + \sum_{n=1}^N \|\delta \mathbf{E}_h^n - \delta \mathbf{E}_h^{n-1}\|_{\mathbf{L}^2(\Omega)}^2 + \sum_{n=1}^N \|\delta \mathbf{B}_h^n - \delta \mathbf{B}_h^{n-1}\|_{\mathbf{L}^2_{1/\mu}(\Omega)}^2 \leq C \end{aligned}$$

338 and

$$339 \max_{n \in \{1, \dots, N\}} \|\mathbf{curl} \mathbf{E}_h^n\|_{\mathbf{L}^2_{1/\mu}(\Omega)}^2 \leq C.$$

341 *Proof.* Let $N \in \mathbb{N}$, $h > 0$ and $n \in \{1, \dots, N\}$. Inserting $(\mathbf{v}_h, \mathbf{w}_h) = (\mathbf{E}_h^{n-1}, \mathbf{B}_h^{n-1})$
 342 in the n -th inequality of $(\mathbf{VI}_{N,h})$ and then adding it with the $(n-1)$ -th inequality of
 343 $(\mathbf{VI}_{N,h})$ tested with $(\mathbf{v}_h, \mathbf{w}_h) = (\mathbf{E}_h^n, \mathbf{B}_h^n)$ lead to

$$344 \begin{aligned} (3.21) \quad & \int_{\Omega} \epsilon (\delta \mathbf{E}_h^n - \delta \mathbf{E}_h^{n-1}) \cdot (\mathbf{E}_h^n - \mathbf{E}_h^{n-1}) + \mu^{-1} (\delta \mathbf{B}_h^n - \delta \mathbf{B}_h^{n-1}) \cdot (\mathbf{B}_h^n - \mathbf{B}_h^{n-1}) dx \\ & \leq \int_{\Omega} (\mathbf{f}^n - \mathbf{f}^{n-1}) \cdot (\mathbf{E}_h^n - \mathbf{E}_h^{n-1}) dx + \int_{\Omega} j_c(x, \theta(x, t_n)) (|\mathbf{E}_h^{n-1}| - |\mathbf{E}_h^n|) dx \\ 345 & + \int_{\Omega} j_c(x, \theta(x, t_{n-1})) (|\mathbf{E}_h^n| - |\mathbf{E}_h^{n-1}|) dx. \end{aligned}$$

348 We sum (3.21) up over $\{1, \dots, i_0\}$ for a fixed $i_0 \in \{1, \dots, N\}$ and divide the resulting
349 inequality by τ to get

$$\begin{aligned}
350 \quad & \sum_{n=1}^{i_0} \left[\int_{\Omega} \epsilon (\delta \mathbf{E}_h^n - \delta \mathbf{E}_h^{n-1}) \cdot \delta \mathbf{E}_h^n \, dx + \int_{\Omega} \mu^{-1} (\delta \mathbf{B}_h^n - \delta \mathbf{B}_h^{n-1}) \cdot \delta \mathbf{B}_h^n \, dx \right] \\
351 \quad & \leq \sum_{n=1}^{i_0} \left[\int_{\Omega} (\mathbf{f}^n - \mathbf{f}^{n-1}) \cdot \delta \mathbf{E}_h^n \, dx + \int_{\Omega} (j_c(x, \theta(x, t_n)) - j_c(x, \theta(x, t_{n-1}))) \right. \\
352 \quad & \quad \left. \left(\frac{|\mathbf{E}_h^{n-1}| - |\mathbf{E}_h^n|}{\tau} \right) \, dx \right]. \\
353
\end{aligned}$$

354 Then, as in the proof of Lemma 3.6, the binomial formulas along with the Hölder and
355 Young inequalities yield

(3.22)

$$\begin{aligned}
356 \quad & \|\delta \mathbf{E}_h^{i_0}\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 + \|\delta \mathbf{B}_h^{i_0}\|_{\mathbf{L}_{1/\mu}^2(\Omega)}^2 + \sum_{n=1}^{i_0} \|\delta \mathbf{E}_h^n - \delta \mathbf{E}_h^{n-1}\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 + \|\delta \mathbf{B}_h^n - \delta \mathbf{B}_h^{n-1}\|_{\mathbf{L}_{1/\mu}^2(\Omega)}^2 \\
357 \quad & \leq \frac{4T\tau}{\epsilon} \sum_{n=1}^{i_0} \left\| \frac{\mathbf{f}^n - \mathbf{f}^{n-1}}{\tau} \right\|_{\mathbf{L}^2(\Omega)}^2 + \frac{4T\tau}{\epsilon} \sum_{n=1}^{i_0} \left\| \frac{j_c(x, \theta(x, t_n)) - j_c(x, \theta(x, t_{n-1}))}{\tau} \right\|_{\mathbf{L}^2(\Omega)}^2 \\
358 \quad & + \|\mathbf{E}_{0h}\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 + \|\mathbf{B}_{0h}\|_{\mathbf{L}_{1/\mu}^2(\Omega)}^2 + \underbrace{\frac{\tau}{2T} \sum_{n=1}^{i_0} \|\delta \mathbf{E}_h^n\|_{\mathbf{L}_\epsilon^2(\Omega)}^2}_{=\frac{1}{2N}}, \\
359
\end{aligned}$$

360 where we have also used $\delta \mathbf{E}_{0h} = \mathbf{E}_{0h}$ and $\delta \mathbf{B}_{0h} = \mathbf{B}_{0h}$. Therefore, (A4), (A5) and
361 Lemma 3.3 applied to (3.22) imply

$$\begin{aligned}
362 \quad & \|\delta \mathbf{E}_h^{i_0}\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 \leq C + \sum_{n=1}^{i_0-1} \frac{1}{N} \|\delta \mathbf{E}_h^n\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 \quad \Rightarrow \quad \|\delta \mathbf{E}_h^{i_0}\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 \leq C \exp\left(\sum_{n=1}^{i_0-1} \frac{1}{N}\right) \leq C, \\
363
\end{aligned}$$

364 by the discrete Gronwall inequality. Since i_0 was chosen arbitrarily, applying the
365 above estimate to (3.22) yields (3.19). Finally, (3.20) follows immediately from (3.12)
366 and (3.19). \square

367 With these stability estimates at hand, we will establish a weak-* convergence
368 result for $(\mathbf{VI}_{N,h})$, which particularly implies the well-posedness of (VI). First, we
369 denote

$$\begin{aligned}
370 \quad (3.23) \quad & \begin{cases} \mathbf{E}_{N,h}(0) := \mathbf{E}_{0h} \\ \mathbf{E}_{N,h}(t) := \mathbf{E}_h^{n-1} + (t - t_{n-1})\delta \mathbf{E}_h^n \end{cases} \quad \text{and} \quad \begin{cases} \bar{\mathbf{E}}_{N,h}(0) := \mathbf{E}_{0h} \\ \bar{\mathbf{E}}_{N,h}(t) := \mathbf{E}_h^n \end{cases} \\
371
\end{aligned}$$

372 for $t \in (t_{n-1}, t_n]$ and $n \in \{1, \dots, N\}$. In the same way, we define $\mathbf{B}_{N,h}$, $\bar{\mathbf{B}}_{N,h}$ and $\bar{\mathbf{f}}_N$.
373 Furthermore, we introduce the function $\varphi_N : [0, T] \times \mathbf{L}^2(\Omega) \rightarrow \mathbb{R}$ by

$$\begin{aligned}
374 \quad (3.24) \quad & \begin{cases} \varphi_N(0, \mathbf{v}) := \varphi(\theta(0), \mathbf{v}) = \int_{\Omega} j_c(x, \theta(x, 0)) |\mathbf{v}(x)| \, dx \\ \varphi_N(t, \mathbf{v}) := \varphi^n(\mathbf{v}) = \int_{\Omega} j_c(x, \theta(x, t_n)) |\mathbf{v}(x)| \, dx \quad \forall t \in (t_{n-1}, t_n], \end{cases} \\
375
\end{aligned}$$

376 for $\mathbf{v} \in \mathbf{L}^2(\Omega)$ and $n \in \{1, \dots, N\}$. Now, we can rewrite $(\mathbf{VI}_{N,h})$ in the following
 377 manner:

$$\begin{aligned}
 378 \quad (3.25) \quad & \left\{ \begin{aligned}
 & \int_{\Omega} \epsilon \partial_t \mathbf{E}_{N,h}(t) \cdot (\mathbf{v}_h - \bar{\mathbf{E}}_{N,h}(t)) + \mu^{-1} \partial_t \mathbf{B}_{N,h}(t) \cdot (\mathbf{w}_h - \bar{\mathbf{B}}_{N,h}(t)) \, dx \\
 & + \int_{\Omega} \mu^{-1} \mathbf{curl} \bar{\mathbf{E}}_{N,h}(t) \cdot \mathbf{w}_h - \mu^{-1} \bar{\mathbf{B}}_{N,h}(t) \cdot \mathbf{curl} \mathbf{v}_h \, dx \\
 & + \varphi_N(t, \mathbf{v}_h) - \varphi_N(t, \bar{\mathbf{E}}_{N,h}(t)) \geq \int_{\Omega} \bar{\mathbf{f}}_N(t) \cdot (\mathbf{v}_h - \bar{\mathbf{E}}_{N,h}(t)) \, dx \\
 & \text{for every } (\mathbf{v}_h, \mathbf{w}_h) \in \mathbf{V}_h \times \mathbf{W}_h \text{ and a.e. } t \in (0, T) \\
 & (\mathbf{E}_{N,h}(0), \mathbf{B}_{N,h}(0)) = (\mathbf{E}_{0h}, \mathbf{B}_{0h}).
 \end{aligned} \right.
 \end{aligned}$$

379

380 **THEOREM 3.8.** *Let Assumptions 2.1 and 2.2 hold. Then, there exists a pair*

$$381 \quad (\mathbf{E}, \mathbf{B}) \in W^{1,\infty}((0, T), \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{H}_0(\text{div}=0)) \cap L^\infty((0, T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}_0(\text{div}=0))$$

383 *such that for $N = N(h)$ with $N(h) \rightarrow \infty$ as $h \rightarrow 0$ it holds that*

$$\begin{aligned}
 384 \quad & \mathbf{E}_{N,h} \rightharpoonup^* \mathbf{E} \text{ and } \bar{\mathbf{E}}_{N,h} \rightharpoonup^* \bar{\mathbf{E}} \quad \text{weakly-}^* \text{ in } L^\infty((0, T), \mathbf{H}_0(\mathbf{curl})), \\
 385 \quad & \mathbf{B}_{N,h} \rightharpoonup^* \mathbf{B} \text{ and } \bar{\mathbf{B}}_{N,h} \rightharpoonup^* \bar{\mathbf{B}} \quad \text{weakly-}^* \text{ in } L^\infty((0, T), \mathbf{H}_0(\text{div}=0)), \\
 386 \quad & \partial_t \mathbf{E}_{N,h} \rightharpoonup^* \partial_t \mathbf{E} \quad \text{weakly-}^* \text{ in } L^\infty((0, T), \mathbf{L}_\epsilon^2(\Omega)), \\
 387 \quad & \partial_t \mathbf{B}_{N,h} \rightharpoonup^* \partial_t \mathbf{B} \quad \text{weakly-}^* \text{ in } L^\infty((0, T), \mathbf{H}_0(\text{div}=0)),
 \end{aligned}$$

389 *and (\mathbf{E}, \mathbf{B}) is the unique solution to (VI).*

390 *Proof.* First of all, we emphasize that $N = N(h)$ denotes a family of natural
 391 numbers with $N(h) \rightarrow \infty$ for $h \rightarrow 0$. As shown in Lemma 3.6 and Lemma 3.7,
 392 $\{\mathbf{E}_{N,h}\}_{h>0}$, $\{\mathbf{B}_{N,h}\}_{h>0}$, $\{\bar{\mathbf{E}}_{N,h}\}_{h>0}$, $\{\bar{\mathbf{B}}_{N,h}\}_{h>0}$, and $\{\partial_t \mathbf{E}_{N,h}\}_{h>0}$, $\{\partial_t \mathbf{B}_{N,h}\}_{h>0}$ are
 393 bounded in their respective spaces. Therefore, we may extract weakly- * converging
 394 subsequences, which will not be denoted in a special way:

$$395 \quad (3.26) \quad \left\{ \begin{aligned}
 & \bar{\mathbf{E}}_{N,h} \rightharpoonup^* \bar{\mathbf{E}} \quad \text{weakly-}^* \text{ in } L^\infty((0, T), \mathbf{H}_0(\mathbf{curl})), \\
 & \bar{\mathbf{B}}_{N,h} \rightharpoonup^* \bar{\mathbf{B}} \quad \text{weakly-}^* \text{ in } L^\infty((0, T), \mathbf{H}_0(\text{div}=0)), \\
 & \mathbf{E}_{N,h} \rightharpoonup^* \mathbf{E} \quad \text{weakly-}^* \text{ in } L^\infty((0, T), \mathbf{H}_0(\mathbf{curl})), \\
 & \mathbf{B}_{N,h} \rightharpoonup^* \mathbf{B} \quad \text{weakly-}^* \text{ in } L^\infty((0, T), \mathbf{H}_0(\text{div}=0)), \\
 & \partial_t \mathbf{E}_{N,h} \rightharpoonup^* \xi \quad \text{weakly-}^* \text{ in } L^\infty((0, T), \mathbf{L}_\epsilon^2(\Omega)), \\
 & \partial_t \mathbf{B}_{N,h} \rightharpoonup^* \chi \quad \text{weakly-}^* \text{ in } L^\infty((0, T), \mathbf{H}_0(\text{div}=0)),
 \end{aligned} \right.$$

397 for some $\mathbf{E}, \mathbf{B}, \bar{\mathbf{E}}, \bar{\mathbf{B}}, \xi, \chi$ as $h \rightarrow 0$. First of all, we verify that $\mathbf{E} = \bar{\mathbf{E}}$ and $\mathbf{B} = \bar{\mathbf{B}}$.

398 However, this is readily seen by the definition (3.23) and Lemma 3.7 since

$$\begin{aligned}
 399 \quad (3.27) \quad & \|\bar{\mathbf{E}}_{N,h} - \mathbf{E}_{N,h}\|_{L^\infty((0, T), \mathbf{L}_\epsilon^2(\Omega))} \leq \tau \max_{n \in \{1, \dots, N\}} \|\delta \mathbf{E}_h^n\|_{\mathbf{L}_\epsilon^2(\Omega)} \leq C\tau, \\
 400 \quad & \|\bar{\mathbf{B}}_{N,h} - \mathbf{B}_{N,h}\|_{L^\infty((0, T), \mathbf{L}_{1/\mu}^2(\Omega))} \leq \tau \max_{n \in \{1, \dots, N\}} \|\delta \mathbf{B}_h^n\|_{\mathbf{L}_{1/\mu}^2(\Omega)} \leq C\tau.
 \end{aligned}$$

401 Next, derivation in the sense of distributions gives

402

$$\begin{aligned}
403 \quad & \int_0^T (\xi(t), \mathbf{v})_{\mathbf{L}_\epsilon^2(\Omega)} \phi(t) dt \stackrel{(3.26)}{\leftarrow} \int_0^T (\partial_t \mathbf{E}_{N,h}(t), \mathbf{v})_{\mathbf{L}_\epsilon^2(\Omega)} \phi(t) dt \\
404 \quad & = - \int_0^T (\mathbf{E}_{N,h}(t), \mathbf{v})_{\mathbf{L}_\epsilon^2(\Omega)} \phi'(t) dt \stackrel{(3.26)}{\rightarrow} - \int_0^T (\mathbf{E}(t), \mathbf{v})_{\mathbf{L}_\epsilon^2(\Omega)} \phi'(t) dt \\
405 \quad &
\end{aligned}$$

406 for every $\mathbf{v} \in \mathbf{L}^2(\Omega)$ and $\phi \in \mathcal{C}_0^\infty(0, T)$, which yields $\xi = \partial_t \mathbf{E}$ and so

407

$$\mathbf{E} \in W^{1,\infty}((0, T), \mathbf{L}_\epsilon^2(\Omega)) \cap L^\infty((0, T), \mathbf{H}_0(\mathbf{curl})).$$

409 Obviously, the same conclusion can be drawn for $\chi = \partial_t \mathbf{B}$, which implies that $\mathbf{B} \in$
410 $W^{1,\infty}((0, T), \mathbf{H}_0(\mathbf{div}=0))$. Note that

411

$$(\mathbf{E}, \mathbf{B}) \in W^{1,\infty}((0, T), \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{H}_0(\mathbf{div}=0)) \hookrightarrow \mathcal{C}([0, T], \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{H}_0(\mathbf{div}=0))$$

413 implies possibly after a modification on a subset of $[0, T]$ with measure zero that
414 $(\mathbf{E}, \mathbf{B}) \in \mathcal{C}([0, T], \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{H}_0(\mathbf{div}=0))$. Next, we prove the pointwise weak convergence

(3.28)

415

$$\mathbf{E}_{N,h}(t) \rightharpoonup \mathbf{E}(t) \text{ weakly in } \mathbf{L}_\epsilon^2(\Omega) \text{ and } \mathbf{B}_{N,h}(t) \rightharpoonup \mathbf{B}(t) \text{ weakly in } \mathbf{H}_0(\mathbf{div}=0)$$

417 for every $t \in [0, T]$. For that purpose, we fix $t \in (0, T]$, $\mathbf{w} \in \mathbf{L}^2(\Omega)$ and $\phi \in \mathcal{C}^1([0, t])$.
418 Then, integration by parts yields

419

$$\begin{aligned}
420 \quad (3.29) \quad & \int_0^t (\partial_t \mathbf{E}(s), \mathbf{w})_{\mathbf{L}_\epsilon^2(\Omega)} \phi(s) ds \leftarrow \int_0^t (\partial_t \mathbf{E}_{N,h}(s), \mathbf{w})_{\mathbf{L}_\epsilon^2(\Omega)} \phi(s) ds \\
421 \quad & = - \int_0^t (\mathbf{E}_{N,h}(s), \mathbf{w})_{\mathbf{L}_\epsilon^2(\Omega)} \phi'(s) ds + (\mathbf{E}_{N,h}(t), \mathbf{w})_{\mathbf{L}_\epsilon^2(\Omega)} \phi(t) - (\mathbf{E}_{N,h}(0), \mathbf{w})_{\mathbf{L}_\epsilon^2(\Omega)} \phi(0). \\
422 \quad &
\end{aligned}$$

423 Choosing $\phi(0) = 0$ as well as $\phi(t) \neq 0$ and applying integration by parts again gives

424
425

$$\lim_{h \rightarrow 0} (\mathbf{E}_{N,h}(t), \mathbf{w})_{\mathbf{L}_\epsilon^2(\Omega)} = (\mathbf{E}(t), \mathbf{w})_{\mathbf{L}_\epsilon^2(\Omega)} \quad \forall \mathbf{w} \in \mathbf{L}^2(\Omega).$$

426 Applying the above convergence to (3.29) and choosing $\phi(0) \neq 0$ leads to

427
428

$$\lim_{h \rightarrow 0} (\mathbf{E}_{N,h}(0), \mathbf{w})_{\mathbf{L}_\epsilon^2(\Omega)} = (\mathbf{E}(0), \mathbf{w})_{\mathbf{L}_\epsilon^2(\Omega)} \quad \forall \mathbf{w} \in \mathbf{L}^2(\Omega).$$

429 The same results hold also for $\mathbf{B}_{N,h}$, and so we conclude that (3.28) is valid. From
430 [Lemma 3.3](#), (3.23) and (3.28) with $t = 0$, it follows that

431

$$(3.30) \quad \mathbf{E}(0) = \mathbf{E}_0 \text{ and } \mathbf{B}(0) = \mathbf{B}_0.$$

433 We continue and recall the classical identity:

434
435

$$(3.31) \quad \int_0^t (\partial_t \mathbf{E}_{N,h}(s), \mathbf{E}_{N,h}(s))_{\mathbf{L}_\epsilon^2(\Omega)} ds = \frac{1}{2} \|\mathbf{E}_{N,h}(t)\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 - \frac{1}{2} \|\mathbf{E}_{0h}\|_{\mathbf{L}_\epsilon^2(\Omega)}^2.$$

436 Combining (3.31) with (3.28) and Lemma 3.3 yields

$$\begin{aligned}
437 \quad (3.32) \quad & \liminf_{h \rightarrow 0} \int_0^t (\partial_t \mathbf{E}_{N,h}(s), \bar{\mathbf{E}}_{N,h}(s))_{\mathbf{L}_\epsilon^2(\Omega)} ds \\
438 \quad & \stackrel{(3.27)}{=} \liminf_{h \rightarrow 0} \int_0^t (\partial_t \mathbf{E}_{N,h}(s), \mathbf{E}_{N,h}(s))_{\mathbf{L}_\epsilon^2(\Omega)} ds \\
439 \quad & \stackrel{(3.31)}{=} \liminf_{h \rightarrow 0} \frac{1}{2} \|\mathbf{E}_{N,h}(t)\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 - \frac{1}{2} \|\mathbf{E}_{0h}\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 \\
440 \quad & \geq \frac{1}{2} \|\mathbf{E}(t)\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 - \frac{1}{2} \|\mathbf{E}(0)\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 = \int_0^t (\partial_t \mathbf{E}(s), \mathbf{E}(s))_{\mathbf{L}_\epsilon^2(\Omega)} ds, \\
441
\end{aligned}$$

442 where the above inequality holds due to the fact that the squared norm is weakly
443 lower semicontinuous. Analogously, we obtain

$$\begin{aligned}
444 \quad (3.33) \quad & \liminf_{h \rightarrow 0} \int_0^t (\partial_t \mathbf{B}_{N,h}(s), \bar{\mathbf{B}}_{N,h}(s))_{\mathbf{L}_{1/\mu}^2(\Omega)} ds \geq \int_0^t (\partial_t \mathbf{B}(s), \mathbf{B}(s))_{\mathbf{L}_{1/\mu}^2(\Omega)} ds. \\
445
\end{aligned}$$

446 Next, we prove

$$\begin{aligned}
447 \quad (3.34) \quad & \liminf_{h \rightarrow 0} \varphi_N(t, \bar{\mathbf{E}}_{N,h}(t)) ds \geq \varphi(\theta(t), \mathbf{E}(t)) \quad \forall t \in [0, T]. \\
448
\end{aligned}$$

449 For $t = 0$, Lemma 3.3 and (3.23), (3.24), and (3.30) grant even the strong convergence

$$\begin{aligned}
450 \quad & \lim_{h \rightarrow 0} \varphi_N(0, \bar{\mathbf{E}}_{N,h}(0)) = \lim_{h \rightarrow 0} \int_{\Omega} j_c(x, \theta(x, 0)) |\mathbf{E}_{0h}| dx = \varphi(\theta(0), \mathbf{E}(0)). \\
451
\end{aligned}$$

452 Let now $t \in (0, T]$. Then, for every $N \in \mathbb{N}$, there exists a unique $n \in \{1, \dots, N\}$ such
453 that $t \in (t_{n-1}, t_n]$. Hence, the sequence $\tilde{t}_{N,h} := t_n$ fulfills $\tilde{t}_{N,h} \rightarrow t$ as $h \rightarrow 0$. Making
454 use of this sequence, we obtain that

$$\begin{aligned}
455 \quad (3.35) \quad & \liminf_{h \rightarrow 0} \varphi_N(t, \bar{\mathbf{E}}_{N,h}(t)) \\
456 \quad & = \liminf_{h \rightarrow 0} \left(\varphi(\theta(t), \bar{\mathbf{E}}_{N,h}(t)) + \int_{\Omega} (j_c(x, \theta(x, \tilde{t}_{N,h})) - j_c(x, \theta(x, t))) |\bar{\mathbf{E}}_{N,h}(t)| dx \right) \\
457 \quad & \geq \varphi(\theta(t), \mathbf{E}(t)) + \liminf_{h \rightarrow \infty} \int_{\Omega} (j_c(x, \theta(x, \tilde{t}_{N,h})) - j_c(x, \theta(x, t))) |\bar{\mathbf{E}}_{N,h}(t)| dx, \\
458
\end{aligned}$$

459 where we have employed (3.28) and the fact that $\varphi(\theta(t), \cdot): \mathbf{L}_\epsilon^2(\Omega) \rightarrow \mathbb{R}$, for every
460 fixed $t \in [0, T]$, is sequentially weakly lower semicontinuous. In order to pass to the
461 limit in the second term in (3.35), we make use of (A4) and (A5) to deduce after
462 selecting a subsequence that

$$\begin{aligned}
463 \quad (3.36) \quad & \lim_{h \rightarrow 0} j_c(x, \theta(x, \tilde{t}_{N,h})) - j_c(x, \theta(x, t)) = 0 \quad \text{for a.e. } x \in \Omega,
\end{aligned}$$

464 and so, thanks to (A3)–(A5) and Lemma 3.6, Lebesgue's dominated convergence
465 theorem yields

$$\begin{aligned}
466 \quad (3.37) \quad & \lim_{h \rightarrow 0} \int_{\Omega} |j_c(x, \theta(\tilde{t}_{N,h}, x)) - j_c(x, \theta(t, x))| |\bar{\mathbf{E}}_{N,h}(t)| dx = 0. \\
467
\end{aligned}$$

468 In conclusion, (3.34) is valid.

469 Now, we show that (\mathbf{E}, \mathbf{B}) is a solution to (VI): Fix $t \in (0, T]$, $\tilde{h} \in (0, \bar{h}]$, integrate
 470 (3.25) for $h < \tilde{h}$ over $[0, t]$ and test it with $(\mathbf{v}_{\tilde{h}}, \mathbf{w}_{\tilde{h}}) \in \mathbf{V}_{\tilde{h}} \times \mathbf{W}_{\tilde{h}} \subset \mathbf{V}_h \times \mathbf{W}_h$ (cf. (3.1)).
 471 Afterwards, we apply the limit superior to the resulting inequality to deduce

$$\begin{aligned}
 472 \quad (3.38) \quad & \int_0^t (\mathbf{f}(s), \mathbf{v}_h - \mathbf{E}(s))_{\mathbf{L}^2(\Omega)} ds = \lim_{h \rightarrow 0} \int_0^t (\bar{\mathbf{f}}_N(s), \mathbf{v}_h - \bar{\mathbf{E}}_{N,h}(s))_{\mathbf{L}^2(\Omega)} ds \\
 473 \quad & \underbrace{\leq}_{(3.25)} \limsup_{h \rightarrow 0} \left[\int_0^t (\partial_t \mathbf{E}_{N,h}(s), \mathbf{v}_h - \bar{\mathbf{E}}_{N,h}(s))_{\mathbf{L}_\epsilon^2(\Omega)} ds \right. \\
 474 \quad & + \int_0^t (\partial_t \mathbf{B}_{N,h}(s), \mathbf{w}_h - \bar{\mathbf{B}}_{N,h}(s))_{\mathbf{L}_{1/\mu}^2(\Omega)} + (\mathbf{curl} \bar{\mathbf{E}}_{N,h}(s), \mathbf{w}_h)_{\mathbf{L}_{1/\mu}^2(\Omega)} ds \\
 475 \quad & \left. - \int_0^t (\bar{\mathbf{B}}_{N,h}(s), \mathbf{curl} \mathbf{v}_h)_{\mathbf{L}_{1/\mu}^2(\Omega)} ds + \int_0^t \varphi_N(s, \mathbf{v}_h) - \varphi_N(s, \bar{\mathbf{E}}_{N,h}(s)) ds \right] \\
 476 \quad & \underbrace{\leq}_{(3.26), (3.32), (3.33)} \int_0^t (\partial_t \mathbf{E}(s), \mathbf{v} - \mathbf{E}(s))_{\mathbf{L}_\epsilon^2(\Omega)} + (\partial_t \mathbf{B}(s), \mathbf{w} - \mathbf{B}(s))_{\mathbf{L}_{1/\mu}^2(\Omega)} ds \\
 477 \quad & + \int_0^t (\mathbf{curl} \mathbf{E}(s), \mathbf{w})_{\mathbf{L}_{1/\mu}^2(\Omega)} - (\mathbf{B}(s), \mathbf{curl} \mathbf{v})_{\mathbf{L}_{1/\mu}^2(\Omega)} ds \\
 478 \quad & + \int_0^t \varphi(\theta(s), \mathbf{v}) - \varphi(\theta(s), \mathbf{E}(s)) ds, \\
 479 \quad &
 \end{aligned}$$

480 where we have also used (3.34) and Fatou's lemma to obtain convergence of the last
 481 time integral. Since there is no restriction to $\tilde{h} > 0$, the density of $\mathbf{V}_h \subset \mathbf{H}_0(\mathbf{curl})$
 482 and $\mathbf{W}_h \subset \mathbf{L}^2(\Omega)$ yields, if we differentiate (3.38) with respect to t , that $(\mathbf{E}, \mathbf{B}) \in$
 483 $W^{1,\infty}((0, T), \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{H}_0(\text{div}=0)) \cap L^\infty((0, T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}_0(\text{div}=0))$ satisfies the
 484 evolutionary variational inequality (VI).

485 The uniqueness of the solution to (VI) follows by an energy argument: Let
 486 $(\tilde{\mathbf{E}}, \tilde{\mathbf{B}}) \in W^{1,\infty}((0, T), \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_{1/\mu}^2(\Omega)) \cap L^\infty((0, T), \mathbf{H}_0(\mathbf{curl}) \times \mathbf{L}_{1/\mu}^2(\Omega))$ be an-
 487 other solution to (VI). Then, inserting $(\mathbf{v}, \mathbf{w}) = (\tilde{\mathbf{E}}(t), \tilde{\mathbf{B}}(t))$ in (VI) associated with
 488 (\mathbf{E}, \mathbf{B}) and $(\mathbf{v}, \mathbf{w}) = (\mathbf{E}(t), \mathbf{B}(t))$ in (VI) associated with $(\tilde{\mathbf{E}}, \tilde{\mathbf{B}})$, and then adding the
 489 resulting inequalities together, we obtain

$$490 \quad \int_\Omega \epsilon (\partial_t \mathbf{E}(t) - \partial_t \tilde{\mathbf{E}}(t)) \cdot (\mathbf{E}(t) - \tilde{\mathbf{E}}(t)) + \mu^{-1} (\partial_t \mathbf{B}(t) - \partial_t \tilde{\mathbf{B}}(t)) \cdot (\mathbf{B}(t) - \tilde{\mathbf{B}}(t)) dx \leq 0, \\
 491 \quad$$

492 which implies that the difference $(\mathbf{e}(t), \mathbf{b}(t)) = (\mathbf{E}(t) - \tilde{\mathbf{E}}(t), \mathbf{B}(t) - \tilde{\mathbf{B}}(t))$ fulfills

$$493 \quad \frac{1}{2} \frac{d}{dt} \|\mathbf{e}(t)\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} \|\mathbf{b}(t)\|_{\mathbf{L}_{1/\mu}^2(\Omega)}^2 \leq 0. \\
 494 \quad$$

495 Since $\mathbf{e}(0) = \mathbf{b}(0) = 0$, the above inequality yields that $\mathbf{e}(t) = \mathbf{b}(t) = 0$ for all
 496 $t \in [0, T]$. Hence, (\mathbf{E}, \mathbf{B}) is the unique solution to (VI). \square

497 *Remark 3.9.* A main consequence of Theorem 3.8 is the global well-posedness for
 498 (VI). We point out that, based on a direct approach, i.e., without discretization tech-
 499 niques, [33] proved existence and uniqueness results for hyperbolic Maxwell variational
 500 inequalities with a general nonlinearity. However, due to the temperature-dependent
 501 critical current density j_c , [33] cannot be applied to deduce the well-posedness of
 502 (VI). Here, the direct approach requires a substantial extension of [33] to the case of
 503 time-dependent nonlinearities.

504 We now prove our main result on the uniform convergence of (3.25) towards (VI).

505 **THEOREM 3.10.** *Let $N = N(h)$ be a family of natural numbers with $N(h) \rightarrow \infty$*
 506 *for $h \rightarrow 0$. Then, under Assumptions 2.1 and 2.2, the solution $(\mathbf{E}_{N,h}, \mathbf{B}_{N,h})$ to (3.25)*
 507 *converges uniformly to the solution (\mathbf{E}, \mathbf{B}) of (VI), i.e.,*

$$508 \quad \lim_{h \rightarrow 0} \|\mathbf{E}_{N,h} - \mathbf{E}\|_{C([0,T], \mathbf{L}_\epsilon^2(\Omega))} = \lim_{h \rightarrow 0} \|\mathbf{B}_{N,h} - \mathbf{B}\|_{C([0,T], \mathbf{L}_{1/\mu}^2(\Omega))} = 0,$$

$$509 \quad \lim_{h \rightarrow 0} \|\bar{\mathbf{E}}_{N,h} - \mathbf{E}\|_{L^\infty((0,T), \mathbf{L}_\epsilon^2(\Omega))} = \lim_{h \rightarrow 0} \|\bar{\mathbf{B}}_{N,h} - \mathbf{B}\|_{L^\infty((0,T), \mathbf{L}_{1/\mu}^2(\Omega))} = 0.$$

511 *Proof.* First of all, we test (VI) with $(\mathbf{v}, \mathbf{w}) = (\bar{\mathbf{E}}_{N,h}(t), \bar{\mathbf{B}}_{N,h}(t))$ to obtain

$$512 \quad (3.39) \quad \int_{\Omega} \epsilon \partial_t \mathbf{E}(t) \cdot (\bar{\mathbf{E}}_{N,h}(t) - \mathbf{E}(t)) + \mu^{-1} \partial_t \mathbf{B}(t) \cdot (\bar{\mathbf{B}}_{N,h}(t) - \mathbf{B}(t)) dx$$

$$513 \quad + \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{E}(t) \cdot \bar{\mathbf{B}}_{N,h}(t) - \mu^{-1} \mathbf{B}(t) \cdot \mathbf{curl} \bar{\mathbf{E}}_{N,h}(t) dx$$

$$514 \quad + \varphi(\theta(t), \bar{\mathbf{E}}_{N,h}(t)) - \varphi(\theta(t), \mathbf{E}(t)) \geq \int_{\Omega} \mathbf{f}(t) \cdot (\bar{\mathbf{E}}_{N,h}(t) - \mathbf{E}(t)) dx$$

516 for a.e. $t \in (0, T)$. Next, inserting $(\mathbf{v}_h, \mathbf{w}_h) = (\Phi_h \mathbf{E}(t), 0) \in \mathbf{V}_h \times \mathbf{W}_h$ in (3.25) leads
 517 to

$$518 \quad (3.40) \quad \int_{\Omega} \epsilon \partial_t \mathbf{E}_{N,h}(t) \cdot (\mathbf{E}(t) - \bar{\mathbf{E}}_{N,h}(t)) + \mu^{-1} \partial_t \mathbf{B}_{N,h}(t) \cdot (\mathbf{B}(t) - \bar{\mathbf{B}}_{N,h}(t)) dx$$

$$519 \quad + \int_{\Omega} \epsilon \partial_t \mathbf{E}_{N,h}(t) \cdot (\Phi_h \mathbf{E}(t) - \mathbf{E}(t)) dx - \int_{\Omega} \mu^{-1} \partial_t \mathbf{B}_{N,h}(t) \cdot \mathbf{B}(t) dx$$

$$520 \quad - \int_{\Omega} \mu^{-1} \bar{\mathbf{B}}_{N,h}(t) \cdot \mathbf{curl} \Phi_h \mathbf{E}(t) dx + \varphi_N(t, \Phi_h \mathbf{E}(t)) - \varphi_N(t, \bar{\mathbf{E}}_{N,h}(t))$$

$$521 \quad \geq \int_{\Omega} \bar{\mathbf{f}}_N(t) \cdot (\Phi_h \mathbf{E}(t) - \bar{\mathbf{E}}_{N,h}(t)) dx$$

523 for a.e. $t \in (0, T)$. Now, by using the fact that $\partial_t \mathbf{B}_{N,h}(t) = -\mathbf{curl} \bar{\mathbf{E}}_{N,h}(t)$ holds for
 524 a.e. $t \in (0, T)$ (see (3.13) and (3.23)), we obtain

$$525 \quad (3.41) \quad \int_{\Omega} \mu^{-1} (\partial_t \mathbf{B}_{N,h}(t) + \mathbf{curl} \bar{\mathbf{E}}_{N,h}(t)) \cdot \mathbf{B}(t) dx = 0 \quad \text{for a.e. } t \in (0, T).$$

527 Moreover, we know from Theorem 3.4 that $\bar{\mathbf{B}}_{N,h}(t) \in \mathbf{curl} \mathbf{V}_h$, which implies by (3.3)
 528 that

$$529 \quad (3.42) \quad \int_{\Omega} \mu^{-1} \bar{\mathbf{B}}_{N,h}(t) \cdot \mathbf{curl} (\mathbf{E}(t) - \Phi_h \mathbf{E}(t)) dx = 0 \quad \text{for a.e. } t \in (0, T).$$

531 In view of (3.41)–(3.42), adding (3.39) and (3.40) together and then integrating the

532 resulting inequality over the time interval $[0, \sigma]$ with $\sigma \in (0, T]$ yield that

$$\begin{aligned}
533 \quad (3.43) \quad & \int_0^\sigma \int_\Omega \epsilon (\partial_t \mathbf{E}_{N,h}(t) - \partial_t \mathbf{E}(t)) \cdot (\bar{\mathbf{E}}_{N,h}(t) - \mathbf{E}(t)) \, dx dt \\
534 \quad & + \int_0^\sigma \int_\Omega \mu^{-1} (\partial_t \mathbf{B}_{N,h}(t) - \partial_t \mathbf{B}(t)) \cdot (\bar{\mathbf{B}}_{N,h}(t) - \mathbf{B}(t)) \, dx dt \\
535 \quad & \leq \int_0^\sigma \left[\int_\Omega \bar{\mathbf{f}}_N(t) \cdot (\mathbf{E}(t) - \Phi_h \mathbf{E}(t)) + (\mathbf{f}(t) - \bar{\mathbf{f}}_N(t)) \cdot (\mathbf{E}(t) - \bar{\mathbf{E}}_{N,h}(t)) \, dx \right. \\
536 \quad & + \int_\Omega \epsilon \partial_t \mathbf{E}_{N,h}(t) \cdot (\Phi_h \mathbf{E}(t) - \mathbf{E}(t)) \, dx + (\varphi_N(t, \Phi_h \mathbf{E}(t)) - \varphi(\theta(t), \mathbf{E}(t))) \\
537 \quad & \left. + (\varphi(\theta(t), \bar{\mathbf{E}}_{N,h}(t)) - \varphi_N(t, \bar{\mathbf{E}}_{N,h}(t))) \right] dt =: \sum_{i=1}^5 C_i. \\
538
\end{aligned}$$

539 We proceed by showing the convergence of C_i , $i \in \{1, \dots, 5\}$, towards 0 as $h \rightarrow 0$.
540 This obviously exploits the convergence property of Φ_h . Therefore, we use (3.5) and
541 (3.6) to deduce by Lebesgue's dominated convergence theorem that

$$\begin{aligned}
542 \quad (3.44) \quad & \lim_{h \rightarrow 0} \int_0^\sigma \|\Phi_h \mathbf{E}(t) - \mathbf{E}(t)\|_{\mathbf{L}^2_\epsilon(\Omega)} \, dt = 0 \quad \forall \sigma \in [0, T]. \\
543
\end{aligned}$$

544 Now, (A5), Lemma 3.7 and (3.44) imply for $i \in \{1, 3\}$ that $|C_i| \rightarrow 0$ as $h \rightarrow 0$. Also,
545 the Lipschitz continuity of \mathbf{f} (A5) together with Theorem 3.8 implies that $|C_2| \rightarrow 0$
546 as $h \rightarrow 0$. Next, the convergence for C_4 is shown: We begin with

$$\begin{aligned}
547 \quad (3.45) \quad & \left| \int_0^\sigma \varphi_N(t, \Phi_h \mathbf{E}(t)) - \varphi(\theta(t), \mathbf{E}(t)) \, dt \right| \\
548 \quad & \leq \int_0^T |\varphi_N(t, \Phi_h \mathbf{E}(t)) - \varphi_N(t, \mathbf{E}(t))| \, dt + \int_0^T |\varphi_N(t, \mathbf{E}(t)) - \varphi(\theta(t), \mathbf{E}(t))| \, dt. \\
549
\end{aligned}$$

550 Because of (A3) and (A5), the first term on the right-hand side of (3.45) satisfies

$$\begin{aligned}
551 \quad (3.46) \quad & \int_0^T |\varphi_N(t, \Phi_h \mathbf{E}(t)) - \varphi_N(t, \mathbf{E}(t))| \, dt \leq C \int_0^T \|\Phi_h \mathbf{E}(t) - \mathbf{E}(t)\|_{\mathbf{L}^2_\epsilon(\Omega)} \, dt, \\
552
\end{aligned}$$

553 with a constant $C > 0$, independent of h . On the other hand, the second term in
554 (3.45) is estimated by using (A4) and (A5):

$$\begin{aligned}
555 \quad (3.47) \quad & \int_0^T |\varphi_N(t, \mathbf{E}(t)) - \varphi(\theta(t), \mathbf{E}(t))| \, dt \\
556 \quad & \stackrel{(3.24)}{=} \underbrace{\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_\Omega |j_c(x, \theta(t_n, x)) - j_c(x, \theta(t, x))| |\mathbf{E}(t)| \, dx dt}_{\leq C \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \tau \|\mathbf{E}(t)\|_{\mathbf{L}^2_\epsilon(\Omega)} \, dt} \\
557 \quad & \leq C \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \tau \|\mathbf{E}(t)\|_{\mathbf{L}^2_\epsilon(\Omega)} \, dt = C\tau \|\mathbf{E}\|_{L^1((0,T), \mathbf{L}^2_\epsilon(\Omega))}. \\
558
\end{aligned}$$

559 Thus, combining (3.45)–(3.47) gives

$$\begin{aligned}
560 \quad (3.48) \quad & |C_4| \leq C \left(\int_0^T \|\Phi_h \mathbf{E}(t) - \mathbf{E}(t)\|_{\mathbf{L}^2_\epsilon(\Omega)} \, dt + \tau \|\mathbf{E}\|_{L^1((0,T), \mathbf{L}^2_\epsilon(\Omega))} \right) \stackrel{(3.44)}{\xrightarrow{}} 0 \quad \text{as } h \rightarrow 0. \\
561
\end{aligned}$$

562 We reuse the arguments from (3.47) in combination with Lemma 3.6 to obtain the
 563 convergence $|C_5| \rightarrow 0$ as $h \rightarrow 0$. Finally, we extract the desired norms on the left
 564 hand side of (3.43) as follows:

(3.49)

$$565 \quad \int_0^\sigma \int_\Omega \epsilon(\partial_t \mathbf{E}_{N,h}(t) - \partial_t \mathbf{E}(t)) \cdot (\overline{\mathbf{E}}_{N,h}(t) - \mathbf{E}(t)) \, dxdt = \frac{1}{2} \|\mathbf{E}_{N,h}(\sigma) - \mathbf{E}(\sigma)\|_{\mathbf{L}_\epsilon^2(\Omega)}^2$$

$$566 \quad - \frac{1}{2} \|\mathbf{E}_{0h} - \mathbf{E}_0\|_{\mathbf{L}_\epsilon^2(\Omega)}^2 + \int_0^\sigma \int_\Omega \epsilon(\partial_t \mathbf{E}_{N,h}(t) - \partial_t \mathbf{E}(t)) \cdot (\overline{\mathbf{E}}_{N,h}(t) - \mathbf{E}_{N,h}(t)) \, dxdt$$

568 and

$$569 \quad (3.50) \quad \int_0^\sigma \int_\Omega \mu^{-1}(\partial_t \mathbf{B}_{N,h}(t) - \partial_t \mathbf{B}(t)) \cdot (\overline{\mathbf{B}}_{N,h}(t) - \mathbf{B}(t)) \, dxdt$$

$$570 \quad = \frac{1}{2} \|\mathbf{B}_{N,h}(\sigma) - \mathbf{B}(\sigma)\|_{\mathbf{L}_{1/\mu}^2(\Omega)}^2 - \frac{1}{2} \|\mathbf{B}_{0h} - \mathbf{B}_0\|_{\mathbf{L}_{1/\mu}^2(\Omega)}^2$$

$$571 \quad + \int_0^\sigma \int_\Omega \mu^{-1}(\partial_t \mathbf{B}_{N,h}(t) - \partial_t \mathbf{B}(t)) \cdot (\overline{\mathbf{B}}_{N,h}(t) - \mathbf{B}_{N,h}(t)) \, dxdt.$$

573 In view of (3.27) and Lemma 3.7, we have

$$574 \quad (3.51) \quad \left| \int_0^\sigma \int_\Omega \epsilon(\partial_t \mathbf{E}_{N,h}(t) - \partial_t \mathbf{E}(t)) \cdot (\overline{\mathbf{E}}_{N,h}(t) - \mathbf{E}_{N,h}(t)) \, dxdt \right|$$

$$575 \quad \leq \|\partial_t \mathbf{E}_{N,h} - \partial_t \mathbf{E}\|_{L^1((0,T), \mathbf{L}_\epsilon^2(\Omega))} \|\overline{\mathbf{E}}_{N,h} - \mathbf{E}_{N,h}\|_{L^\infty((0,T), \mathbf{L}_\epsilon^2(\Omega))}$$

$$576 \quad \leq C\tau \|\partial_t \mathbf{E}_{N,h} - \partial_t \mathbf{E}\|_{L^1((0,T), \mathbf{L}_\epsilon^2(\Omega))} \leq C\tau (\|\partial_t \mathbf{E}\|_{L^1((0,T), \mathbf{L}_\epsilon^2(\Omega))} + 1),$$

578 and analogously

$$579 \quad (3.52) \quad \left| \int_0^\sigma \int_\Omega \mu^{-1}(\partial_t \mathbf{B}_{N,h}(t) - \partial_t \mathbf{B}(t)) \cdot (\overline{\mathbf{B}}_{N,h}(t) - \mathbf{B}_{N,h}(t)) \, dxdt \right|$$

$$580 \quad \leq C\tau (\|\partial_t \mathbf{B}\|_{L^1((0,T), \mathbf{L}_{1/\mu}^2(\Omega))} + 1).$$

582 From (3.43) and (3.49)–(3.52) combined with the previously proved convergence for
 583 C_i for all $i \in \{1, \dots, 5\}$ and Lemma 3.3, we obtain

$$584 \quad (3.53) \quad \lim_{h \rightarrow 0} \|\mathbf{E}_{N,h}(t) - \mathbf{E}(t)\|_{\mathbf{L}_\epsilon^2(\Omega)} = \lim_{h \rightarrow 0} \|\mathbf{B}_{N,h}(t) - \mathbf{B}(t)\|_{\mathbf{L}_{1/\mu}^2(\Omega)} = 0 \quad \forall t \in [0, T].$$

586 On the other hand, Lemmas 3.6 and 3.7 imply the existence of a positive constant
 587 $C > 0$, independent of N and h , such that

$$588 \quad (3.54) \quad \|(\mathbf{E}_{N,h}, \mathbf{B}_{N,h})\|_{W^{1,\infty}((0,T), \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_{1/\mu}^2(\Omega))} \leq C \quad \forall h > 0,$$

590 which yields the uniform boundedness and the equicontinuity of $\{(\mathbf{E}_{N,h}, \mathbf{B}_{N,h})\}_{h>0} \subset$
 591 $\mathcal{C}([0, T], \mathbf{L}_\epsilon^2(\Omega) \times \mathbf{L}_{1/\mu}^2(\Omega))$. Therefore, by (3.53) and (3.54), the Arzelá-Ascoli theorem
 592 for Banach space-valued functions (cf. [23, Theorem 3.1]) implies the existence of a
 593 subsequence of $\{(\mathbf{E}_{N,h}, \mathbf{B}_{N,h})\}_{h>0}$ converging uniformly towards (\mathbf{E}, \mathbf{B}) . As (\mathbf{E}, \mathbf{B}) is
 594 the unique solution of (VI), independent of the choice of the converging subsequence,
 595 a standard argument implies that the whole sequence converges uniformly, i.e.,

$$596 \quad \lim_{h \rightarrow 0} \|\mathbf{E}_{N,h} - \mathbf{E}\|_{\mathcal{C}([0,T], \mathbf{L}_\epsilon^2(\Omega))} = 0 \quad \underset{(3.27)}{\Rightarrow} \quad \lim_{h \rightarrow 0} \|\overline{\mathbf{E}}_{N,h} - \mathbf{E}\|_{L^\infty((0,T), \mathbf{L}_\epsilon^2(\Omega))} = 0.$$

$$597 \quad \lim_{h \rightarrow 0} \|\mathbf{B}_{N,h} - \mathbf{B}\|_{\mathcal{C}([0,T], \mathbf{L}_{1/\mu}^2(\Omega))} = 0 \quad \underset{(3.27)}{\Rightarrow} \quad \lim_{h \rightarrow 0} \|\overline{\mathbf{B}}_{N,h} - \mathbf{B}\|_{L^\infty((0,T), \mathbf{L}_{1/\mu}^2(\Omega))} = 0.$$

599 This completes the proof. \square

600 **4. A priori error analysis.** We start by providing an error estimate result with
 601 low regularity fields for $\Phi_h: \mathbf{H}_0(\mathbf{curl}) \rightarrow \mathbf{V}_h$ introduced in [Definition 3.2](#).

602 **LEMMA 4.1.** *Let $s \in (0, 1]$. There exists a constant $C > 0$, independent of h and*
 603 *\mathbf{y} , such that*

$$604 \quad \|\mathbf{y} - \Phi_h \mathbf{y}\|_{\mathbf{H}(\mathbf{curl})} \leq Ch^s \|\mathbf{y}\|_{\mathbf{H}_0^s(\mathbf{curl})} \quad \forall \mathbf{y} \in \mathbf{H}_0^s(\mathbf{curl})$$

606 *for all $h > 0$. Here, $\mathbf{H}_0^s(\mathbf{curl}) := \{\mathbf{y} \in \mathbf{H}^s(\Omega) \cap \mathbf{H}_0(\mathbf{curl}) : \mathbf{curl} \mathbf{y} \in \mathbf{H}^s(\Omega)\}$.*

607 The proof is completely analogous to the one of [[19](#), [Theorem 3.3](#)], which follows
 608 from [\(3.4\)](#) in combination with the stable commuting quasi-interpolation operator [[19](#),
 609 [Theorem 2.2](#)] (cf. [[10](#)]) and the sharp approximation result [[18](#), [Corollary 6.5](#)] (cf. [[13](#)]).
 610

611 *Assumption 4.2* (Additional assumptions on the initial data and the solution).
 612 (A7) There exists $s \in (0, 1]$ such that $\mathbf{E}_0 \in \mathbf{H}_0^s(\mathbf{curl})$ and the solution of [\(VI\)](#)
 613 satisfies $\mathbf{E} \in L^1((0, T), \mathbf{H}_0^s(\mathbf{curl}))$.

614 [Assumption 4.2](#) yields the following error estimate for the initial value, which follows
 615 readily from [\(3.11\)](#) by using [\(A7\)](#) and [Lemma 4.1](#).

616 **LEMMA 4.3.** *Let [Assumptions 2.1](#), [2.2](#), and [4.2](#) hold. Then there exists a constant*
 617 *$C > 0$, independent of $h > 0$, such that*

$$618 \quad (4.1) \quad \|\mathbf{E}_{0h} - \mathbf{E}_0\|_{\mathbf{L}_e^2(\Omega)}^2 + \|\mathbf{B}_{0h} - \mathbf{B}_0\|_{\mathbf{L}_{1/\mu}^2(\Omega)}^2 \leq Ch^s \quad \forall h > 0.$$

620 **THEOREM 4.4.** *Let [Assumptions 2.1](#), [2.2](#), and [4.2](#) hold. Then, there exists a*
 621 *constant $C > 0$, independent of N and h , such that*

$$622 \quad \|\mathbf{E}_{N,h} - \mathbf{E}\|_{\mathcal{C}([0,T], \mathbf{L}_e^2(\Omega))}^2 + \|\mathbf{B}_{N,h} - \mathbf{B}\|_{\mathcal{C}([0,T], \mathbf{L}_{1/\mu}^2(\Omega))}^2$$

$$623 \quad \leq C(h^s + \tau)(\|\mathbf{E}\|_{L^1((0,T), \mathbf{H}_0^s(\mathbf{curl}))} + \|\partial_t \mathbf{E}\|_{L^1((0,T), \mathbf{L}_e^2(\Omega))} + \|\partial_t \mathbf{B}\|_{L^1((0,T), \mathbf{L}_{1/\mu}^2(\Omega))} + 1)$$

624 *holds for every $h > 0$ and every $N \in \mathbb{N}$.*

627 *Proof.* The lines of the proof are similar to the proof of [Theorem 3.10](#), but, due to
 628 the regularity assumption on \mathbf{E} ([Assumption 4.2](#)), we may use [Lemma 4.1](#) in place of
 629 [\(3.6\)](#). Thus, we consider again [\(3.43\)](#) and give estimates for C_i , $i \in \{1, \dots, 5\}$, instead
 630 of simply proving their convergence towards 0. The stability results in [Lemma 3.6](#)
 631 and [Lemma 3.7](#) combined with the regularity of \mathbf{E} (see [\(A7\)](#)) as well as the error
 632 estimates for Φ_h in [Lemma 4.1](#) lead to

$$633 \quad (4.2) \quad |C_i| \leq Ch^s \|\mathbf{E}\|_{L^1((0,T), \mathbf{H}_0^s(\mathbf{curl}))} \quad \forall i \in \{1, 3\},$$

635 with a constant C , independent of the time variable, N , and h . To estimate C_2 , we
 636 use the Lipschitz continuity of \mathbf{f} (see [\(A5\)](#)) and [Theorem 3.10](#):

$$637 \quad (4.3) \quad |C_2| \leq C\tau \int_0^\sigma \|\mathbf{E}(t) - \bar{\mathbf{E}}_{N,h}(t)\|_{\mathbf{L}_e^2(\Omega)} dt \leq C\tau.$$

639 Next, C_4 is estimated by applying [Lemma 4.1](#) to [\(3.48\)](#)

$$640 \quad (4.4) \quad |C_4| \leq C(h^s + \tau) \|\mathbf{E}\|_{L^1((0,T), \mathbf{H}_0^s(\mathbf{curl}))}.$$

642 Last but not least, the arguments from [\(3.47\)](#) in combination with [Lemma 3.6](#) imply
 643 $|C_5| \leq C\tau$. The combination of [\(3.43\)](#) and [\(3.49\)](#)-[\(3.50\)](#) with [\(3.51\)](#)-[\(3.52\)](#) as well as
 644 the previously proved estimation for C_i , $i \in \{1, \dots, 5\}$ and [Lemma 4.3](#) finally yields
 645 the desired error estimate. \square

646 *Remark 4.5.* The results by Ern and Guermond [18,19] are also valid for higher-
 647 order finite elements. Therefore, [18,19] together with the higher-order FEM for linear
 648 Maxwell's equations [28] would serve as an important basis for the extension of our
 649 approach to the higher-order case.

650 **5. Numerical Results.** We close this paper by presenting numerical results for
 651 some particular examples for (VI). When it comes to computing the solution (\mathbf{E}, \mathbf{B})
 652 to (VI), Euler's implicit method provides an iterative algorithm, which also enables
 653 us to split the mixed problem into two associated problems as we did in Theorem 3.4.
 654 We recall (3.13), which gives an explicit formula for \mathbf{B}_h^n :

655 (5.1)
$$\mathbf{B}_h^n = \mathbf{B}_h^{n-1} - \tau \operatorname{curl} \mathbf{E}_h^n$$

657 provided that \mathbf{E}_h^n is already computed. In view of (3.15), \mathbf{E}_h^n solves an elliptic **curl-curl**
 658 variational inequality of the form

659 (5.2)
$$a(\mathbf{E}_h^n, \mathbf{v}_h - \mathbf{E}_h^n) + \varphi^n(\mathbf{v}_h) - \varphi^n(\mathbf{E}_h^n) \geq \langle \tilde{\mathbf{f}}^n, \mathbf{v}_h - \mathbf{E}_h^n \rangle \quad \forall \mathbf{v}_h \in \mathbf{V}_h.$$

661 We solve this variational inequality using the semi-smooth Newton method (cf. [21]).
 662 Our computational domain is the cube $\Omega = (-1, 1)^3$ and we apply a circular
 663 current $\mathbf{f}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

664
$$\mathbf{f}(x, y, z) = \begin{cases} 1/R \left(0, -z/(y^2 + z^2)^{1/2}, y/(y^2 + z^2)^{1/2} \right) & \text{for } (x, y, z) \in \Omega_p \\ 0 & \text{for } (x, y, z) \notin \Omega_p \end{cases}$$

665

666 to a cylindrical pipe coil $\Omega_p := \{(x, y, z) \in \mathbb{R}^3 : |x| \leq 0.5, \sqrt{y^2 + z^2} \in [0.3, 0.5]\}$.
 667 The constant $R > 0$ denotes the electrical resistance of the pipe (here: $R = 1$).
 668 All implementations were done with the open-source finite-element computational
 669 platform FENICS [27] and as a visualization tool PARAVIEW was used. For this
 670 study, the uniform tetrahedral mesh was refined around the coil. If we do not include
 671 a superconductor in this setup, the applied current induces an orthogonal magnetic
 672 field, which admits its greatest field strength in the center of the coil.

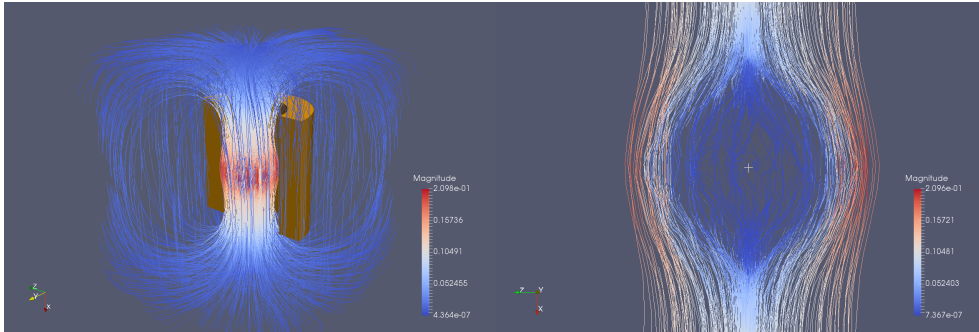


FIG. 1. *First numerical example. Left: Magnetic field lines and the clipped pipe coil. Right: 2D-slice of the magnetic field along the x-z axis.*

673 In the first example, we place a type-II superconducting ball Ω_{sc} with radius 0.2
 674 in the center of the pipe, set $j_c = 80\chi_{\Omega_{sc}}$, $\epsilon = \mu = 1$ and solve the compatibility
 675 system (3.2) for the discrete initial value $(\mathbf{E}_{0h}, \mathbf{B}_{0h})$. In our computation, the mesh

n	0	1	2	3	4	5	6
$j_c(\cdot, \theta(t_n))$	$80\chi_{\Omega_{sc}}$	$50\chi_{\Omega_{sc}}$	$35\chi_{\Omega_{sc}}$	$20\chi_{\Omega_{sc}}$	$10\chi_{\Omega_{sc}}$	$5\chi_{\Omega_{sc}}$	$0,5\chi_{\Omega_{sc}}$
$\theta(t_n)$	$60,0K$	$65,0K$	$67,5K$	$70,0K$	$72,5K$	$75,0K$	$80,0K$
t_n	0	1/6	1/3	1/2	2/3	5/6	1

TABLE 1

Critical current j_c and temperature θ of the superconductor at each the time step.

676 was refined around the superconductor such that we end up with roughly 240.000
677 cells and 1.020.000 degrees of freedom (DOFs) for the mixed finite element space.
678 The resulting solution $(\mathbf{E}_{0h}, \mathbf{B}_{0h})$ of (3.2) exhibits the physical phenomenon of the
679 Meissner-Ochsenfeld effect. In Figure 1, we see how the magnetic field lines get re-
680 pelled by the superconductor and since they are squashed between the superconductor
681 and the coil, one observes the highest magnetic field strength in this area (see Fig-
682 ure 1).

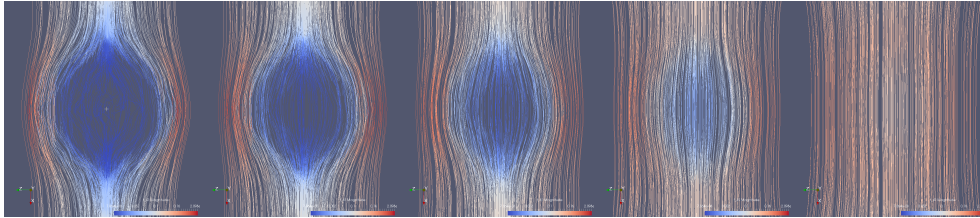


FIG. 2. Evolution of the magnetic field around the superconductor in the time-steps t_n for $n \in \{0, 1, 2, 3, 4\}$.

683 Keeping the observations of the first example in mind, we continue and compute
684 a time-dependent problem, where the solution of the first example serves as the dis-
685 crete initial electromagnetic field, since it satisfies the discrete compatibility system
686 (3.2). We consider the temperature dependence in the critical current density j_c for
687 a superconductor with the nominal composition $Y_{1.2}Ba_{0.8}Cu_2O_x$ as it was suggested
688 in [2]. Moreover, we set $T = 1$ as well as $\tau = 1/6$ and use the same amount of DOFs
689 and cells as in the first example. We place the cooled down superconductor inside
690 the coil in the same way it was done in the first example, but now the temperature θ
691 increases over time (see Table 1), whereas the applied current source \mathbf{f} stays constant.
692 The evolution of the magnetic field over time is shown in Figure 2. One observes that
693 the magnetic field lines in the squashed area start penetrating the superconductor as
694 the temperature becomes larger and larger. As soon as the temperature θ exceeds the
695 threshold $75K$, the magnetical field completely penetrates the superconductor and we
696 can no longer observe the Meissner-Ochsenfeld effect.

697 **Further Research.** As pointed out in the introduction, the Bean critical-state
698 model is a free boundary problem, as it involves unknown superconductive and nor-
699 mal regions, which may change their locations in course of time, depending on the
700 temperature distribution θ and the applied current source \mathbf{f} . Thus, an adaptive mesh
701 refinement strategy based on rigorous a posteriori error estimators will be useful, not
702 only for increasing numerical accuracy, but also for capturing the unknown interfaces
703 between the superconductive and normal regions. We also point out that Theorem 3.8
704 opens a way to study the temperature control in the magnetization process of type-II

705 superconductivity. This leads to a state-constrained optimal control problem governed
 706 by a fully coupled system consisting of (VI) and non-smooth heat equations. This
 707 problem requires a substantial extension of the recently developed optimal control
 708 techniques for electromagnetic problems [34, 36, 37].

709

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