ON GOVERNORS

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A GOVERNOR is a part of a machine by means of which the velocity of the machine is kept nearly uniform, notwithstanding variations in the driving-power or the resistance.

Most governors depend on the centrifugal force of a piece connected with a shaft of the machine. When the velocity increases, this force increases, and either increases the pressure of the piece against a surface or moves the piece, and so acts on a break or a valve.

In one class of regulators of machinery, which we may call *moderators*[1], the resistance is increased by a quantity depending on the velocity. Thus in some pieces of clockwork the moderator consists of a conical pendulum revolving within a circular case. When the velocity increases, the ball of the pendulum presses against the inside of the case, and the friction checks the increase of velocity.

In Watt's governor for steam-engines the arms open outwards, and so contract the aperture of the steam-valve.

In a water-break invented by Professor J. Thomson, when the velocity is increased, water is centrifugally pumped up, and overflows with a great velocity, and the work is spent in lifting and communicating this velocity to the water.

In all these contrivances an increase of driving-power produces an increase of velocity, though a much smaller increase than would be produced without the moderator.

But if the part acted on by centrifugal force, instead of acting directly on the machine, sets in motion a contrivance which continually increases the resistance as long as the velocity is above its normal value, and reverses its action when the velocity is below that value, the governor will bring the velocity to the same normal value whatever variation (within the working limits of the machine) be made in the driving-power or the resistance.

I propose at present, without entering into any details of mechanism to direct the attention of engineers and mathematicians to the dynamical theory of such governors.

It will be seen that the motion of a machine with its governor consists in general of a uniform motion, combined with a disturbance which may be expressed as the sum of several component motions. These components may be of four different kinds :-

(1) The disturbance may continually increase.
(2) It may continually diminish.
(3) It may be an oscillation of continually increasing amplitude.
(4) It may be an oscillation of continually decreasing amplitude.

[1]See Mr C. W. Siemens "On Uniform Rotation," *Phil. Trans.* 1866, p. 657.

The first and third cases are evidently inconsistent with the stability of the motion; and the second and fourth alone are admissible in a good governor. This condition is mathematically equivalent to the condition that all the possible roots, and all the possible parts of the impossible roots, of a certain equation shall be negative.

I have not been able completely to determine these conditions for equations of a higher degree than the third; but I hope that the subject will obtain the attention of mathematicians.

The actual motions corresponding to these impossible roots are not generally taken notice of by the inventors of such machines, who naturally confine their attention to the way in which it is designed to act; and this is generally expressed by the real root of the equation. If, by altering the adjustments of the machine, its governing power is continually increased, there is generally a limit at which the disturbance, instead of subsiding more rapidly, becomes an oscillating and jerking motion, increasing in violence till it reaches the limit of action of the governor. This takes place when the possible part of one of the impossible roots becomes positive. The mathematical investigation of the motion may be rendered practically useful by pointing out the remedy for these disturbances.

This has been actually done in the case of a governor constructed by Mr Fleeming Jenkin, with adjustments, by which the regulating power of the governor could be altered. By altering these adjustments the regulation could be made more and more rapid, till at last a dancing motion of the governor , accompanied with a jerking motion of the main shaft, shewed that an alteration had taken place among the impossible roots of the equation.

I shall consider three kinds of governors, corresponding to the three kinds of moderators already referred to.

In the first kind, the centrifugal piece has a constant distance from the axis of motion, but its pressure on a surface on which it rubs varies when the velocity varies. In the moderator this friction is itself the retarding force. In the governor this surface is made moveable about the axis, and the friction tends to move it; and this motion is made to act on a break to retard the machine. A constant force acts on the moveable wheel in the opposite direction to that of the friction, which takes off the break when the friction is less than a given quantity.

Mr Jenkin's governor is on this principle. It has the advantage that the centrifugal piece does not change its position, and that its pressure is always the same function of the velocity. It has the disadvantage that the normal velocity depends in some degree on the coefficient of sliding friction between two surfaces which cannot be kept always in the same condition.

In the second kind of governor, the centrifugal piece is free to move further from the axis, but is restrained by a force the intensity of which varies with the position of the centrifugal piece in such a way that, if the velocity of rotation has the normal value, the centrifugal piece will be in equilibrium in every position. If the velocity is greater or less than the normal velocity. the centrifugal piece will fly out or fall in without any limit except the limits of motion of the piece. But a break is arranged so that it is made more or less powerful according to the distance of the centrifugal piece from the axis, and thus the oscillations of the centrifugal piece are restrained within narrow limits.

Governors have been constructed on this principle by Sir W. Thomson and by M. Foucault. In the first, the force restraining the centrifugal piece is that of a spring acting between a point of the centrifugal piece and a fixed point at a considerable distance, and the break is a friction-break worked by the reaction of the spring on the fixed point.

In M. Foucault's arrangement, the force acting on the centrifugal piece is the weight of the balls acting downward, and an upward force produced by weights acting on a combination of levers and tending to raise the balls. The resultant vertical force on the balls is proportional to their depth below the centre of motion, which ensures a constant normal velocity. The break is :- in the first place, the variable friction between the combination of levers and the ring on the shaft on which the force is made to act; and, in the second place, a centrifugal air-fan through. which more or less air is allowed to pass, according to the, position of the levers. Both these causes tend to regulate the velocity according to the same law.

The governors designed by the Astronomer-Royal on Mr Siemens's principle for the chronograph and equatorial of Greenwich Observatory depend on nearly similar conditions. The centrifugal piece is here a long conical pendulum, not far removed from the vertical, and it is prevented from deviating much from a fixed angle by the driving-force being rendered nearly constant by means of a differential system. The break of the pendulum consists of a fan which dips into a liquid more or less, according to the angle of the pendulum with the vertical. The break of the principal shaft is worked by the differential apparatus; and the smoothness of motion of the principal shaft is ensured by connecting it with a fly-wheel.

In the third kind of governor a liquid is pumped up and thrown out over the sides of a revolving cup. In the governor on this principle, described by Mr C. W. Siemens, the cup is connected with its axis by a screw and a spring, in such a way that if the axis gets ahead of the cup the cup is lowered and more liquid is pumped up; If this adjustment can be made perfect, the normal velocity of the cup will remain the same through a considerable range of driving-power.

It appears from the investigations that the oscillations in the motion must be checked by some force resisting the motion of oscillation. This may be done in some cases by connecting the oscillating body with a body hanging in a viscous liquid, so that the oscillations cause the body to rise and fall in the liquid.

To check the variations of motion in a revolving shaft, a vessel filled wit 1h viscous liquid may be attached to the shaft. It will have no effect on uniform rotation, but will check periodic alterations of speed.

Similar effects are produced by the viscosity of the lubricating matter in the sliding parts of the machine, and by other unavoidable resistances; so that it is not always necessary to introduce special contrivances to check oscillations.

I shall call all such resistances, if approximately proportional to the velocity, by the name of "viscosity", whatever be their true origin.

In several contrivances a differential system of wheel-work is introduced between the machine and the governor, so that the driving-power acting on the governor is nearly constant.

I have pointed out that, under certain conditions, the sudden disturbances of the machine do not act through the differential system on the governor, or vice versa. When these conditions are fulfilled, the equations of motion are not only simple, but the motion itself is not liable to disturbances depending on the mutual action of the machine and the governor.

Distinction between Moderators and Governors.

In regulators of the first kind, let P be the driving-power and R the resistance, both estimated as if applied to a given axis of the machine. Let V be the normal velocity, estimated for the same axis, and dx/dt the actual velocity, and let M be the moment of inertia of the whole machine reduced to the given axis.

Let the governor be so arranged as to increase the resistance or diminish the driving-power by a quantity $F(dx/dt - V)$, then the equation of motion will be

$$\frac{d}{dt}\left(M\frac{dx}{dt}\right) = P - R - F\left(\frac{dx}{dt} - V\right) \tag{1}$$

When the machine has obtained its final rate the first term vanishes, and

$$\frac{dx}{dt} = V\frac{P-R}{F} \tag{2}$$

Hence, if P is increased or R diminished, the velocity will be permanently increased. Regulators of this kind, as Mr Siemens [2], has observed, should be called moderators rather than governors.

In the second kind of regulator, the force $F(dx/dt - V)$, instead of being applied directly to the machine, is applied to an independent moving piece, B, which continually increases the resistance, or diminishes the driving-power, by a quantity depending on the whole motion of B.

If y represents the whole motion of B, the equation of motion of B is

$$\frac{d}{dt}\left(B\frac{dy}{dt}\right) = F\left(\frac{dx}{dt} - V\right) \tag{3}$$

and that of M

$$\frac{d}{dt}\left(M\frac{dx}{dt}\right) = P - R - F\left(\frac{dx}{dt} - V\right) + Gy \tag{4}$$

where G is the resistance applied by B when B moves through one unit of space.

We can integrate the first of these equations at once, and we find

$$B\frac{dy}{dt} = F(x - Vt) \tag{5}$$

so that if the governor B has come to rest $x = Vt$, and not only is the velocity of the machine equal to the normal velocity, but the position of the machine is the same as if no disturbance of the driving-power or resistance had taken place.

Jenkin's Governor. In a governor of this kind, invented by Mr Fleeming Jenkin, and used in electrical experiments, a centrifugal piece revolves on the principal axis, and is kept always at a constant angle by an appendage which slides on the edge of a loose wheel, B, which works on the same axis. The pressure on the edge of this wheel would be proportional to the square of the velocity; but a constant portion

[2] "On Uniform Rotation," *Phil. Trans.* 1866, p. 657.

of this pressure is taken off by a spring which acts on the centrifugal piece. The force acting on B to turn it round is therefore

$$F' \left[\frac{dx}{dt}\right]^2 - C';$$

and if we remember that the velocity varies within very narrow limits, we may write the expression

$$F\left(\frac{dx}{dt} - V_1\right);$$

where F is a new constant, and V_1 is the lowest limit of velocity within which the governor will act.

Since this force necessarily acts on B in the positive direction, and since it is necessary that the break should be taken off as well as put on, a weight W is applied to B tending to turn it in the negative direction; and, for a reason to be afterwards explained, this weight is made to hang in a viscous liquid, so as to bring it to rest quickly.

The equation of motion of B may then be written

$$B\frac{d^2y}{dt^2} = F\left(\frac{dx}{dt} - V_1\right) - Y\frac{dy}{dt} - W, \tag{6}$$

where Y is a coefficient depending on the viscosity of the liquid and on other resistances varying with the velocity, and W is the constant weight.

Integrating this equation with respect to t, we find

$$B\frac{dy}{dt} = F\left(x - V_1 t\right) - Yy - Wt \tag{7}$$

If B has come to rest, we have

$$x = \left(V_1 + \frac{W}{F}\right)t + \frac{Y}{F}y, \tag{8}$$

or the position of the machine is affected by that of the governor, but the final velocity is constant, and

$$V_1 + \frac{W}{F} = V, \tag{9}$$

where V_1 is the normal velocity.

The equation of motion of the machine itself is

$$M\frac{d^2x}{dt^2} = P - R - F\left(\frac{dx}{dt} - V_1\right) - Gy \tag{10}$$

This must be combined with equation (7) to determine the motion of the whole apparatus. The solution is of the form

$$x = A_1 e^{n_1 t} + A_2 e^{n_2 t} + A_3 e^{n_3 t} + Vt \tag{11}$$

where n_1, n_2, n_3 are the roots of the cubic equation

$$MBn^3 + \left(MY + FB\right)n^2 + FYn + FG = 0 \tag{12}$$

If n be a pair of roots of this equation of the form $a \pm \sqrt{-1}b$, then the part of x corresponding to these roots will be of the form

$$e^{at}\cos(bt + \beta). \tag{13}$$

If a is a negative quantity, this will indicate an oscillation the amplitude of which continually decreases. If a is zero, the amplitude will remain constant, and if a is positive, the amplitude will continually increase.

One root of the equation (12) is evidently a real negative quantity. The condition that the real part of the other roots should be negative is

$$\left(\frac{F}{M}+\frac{Y}{B}\right)\frac{Y}{B}-\frac{G}{B}=\text{a positive quantity.} \tag{14}$$

This is the condition of stability of the motion. If it is not fulfilled there will be a dancing motion of the governor, which will increase till it is as great as the limits of motion of the governor. To ensure this stability, the value of Y must be made sufficiently great, as compared with G, by placing the weight W in a viscous liquid if the viscosity of the lubricating materials at the axle is not sufficient.

To determine the value of F, put the break out of gear, and fix the moveable wheel; then, if V and V' be the velocities when the driving-power is P and P',

$$F=\frac{P-P'}{V-V'} \tag{15}$$

To determine G, let the governor act, and let y and y' be the positions of the break when the driving-power is P and P' ,then

$$G=\frac{P-P'}{y-y'}. \tag{16}$$

General Theory of Chronometric Centrifugal Pieces.

Sir W. Thomson's and M. Foucault's Governors. Let A be the moment of Inertia of a revolving apparatus, and θ the angle of revolution. The equation of motion is

$$\frac{d}{dt}\left(A\frac{d\theta}{dt}\right)=L \tag{17}$$

where L is the moment of the applied force round the axis. Now, let A be a function of another variable ϕ (the divergence of the centrifugal piece), and let the kinetic energy of the whole be

$$\tfrac{1}{2}A\left[\frac{d\theta}{dt}\right]^2+\tfrac{1}{2}B\left[\frac{d\phi}{dt}\right]^2$$

where B may also be a function of ϕ, if the centrifugal piece is complex.

If we also assume that P, the potential energy of the apparatus is a function of ϕ then the force tending to *diminish* ϕ, arising from the action of gravity, springs, etc., will be $dP/d\phi$.

The whole energy, kinetic and potential, is

$$E=\tfrac{1}{2}A\left[\frac{d\theta}{dt}\right]^2+\tfrac{1}{2}B\left[\frac{d\phi}{dt}\right]^2+P=\int L d\theta \tag{18}$$

Differentiating with respect to t, we find

$$\begin{aligned}\frac{d\phi}{dt}\left(\tfrac{1}{2}\frac{dA}{d\phi}\left[\frac{d\theta}{dt}\right]^2+\tfrac{1}{2}\frac{dB}{d\phi}\left[\frac{d\phi}{dt}\right]^2+\frac{dP}{d\phi}\right)+A\frac{d\theta}{dt}\frac{d^2\theta}{dt^2}+B\frac{d\phi}{dt}\frac{d^2\phi}{dt^2}\\=L\frac{d\theta}{dt}=\frac{d\theta}{dt}\left(\frac{dA}{d\phi}\frac{d\theta}{dt}\frac{d\phi}{dt}+A\frac{d^2\theta}{dt^2}\right)\end{aligned} \tag{19}$$

whence we have, by eliminating L,

$$\frac{d}{dt}\left(\frac{d\phi}{dt}\right) = \tfrac{1}{2}\frac{dA}{d\phi}\left[\frac{d\theta}{dt}\right]^2 + \tfrac{1}{2}\frac{dB}{d\phi}\left[\frac{d\phi}{dt}\right]^2 - \frac{dP}{d\phi} \tag{20}$$

The first two terms of the right-hand side indicate a force tending to *increase* ϕ depending on the squares of the velocities of the main shaft and of the centrifugal piece. The force indicated by these terms may be called the centrifugal force.

If the apparatus is so arranged that

$$P = \tfrac{1}{2}A\omega^2 + \text{const} \tag{21}$$

where ω is a constant velocity, the equation becomes

$$\frac{d}{dt}\left(B\frac{d\phi}{dt}\right) = \tfrac{1}{2}\frac{dA}{d\phi}\left(\left[\frac{d\theta}{dt}\right]^2 - \omega^2\right) + \tfrac{1}{2}\frac{dB}{d\phi}\left[\frac{d\phi}{dt}\right]^2 \tag{22}$$

In this case the value of ϕ cannot remain constant unless the angular velocity is equal to ω.

A shaft with a centrifugal piece arranged on this principle has only one velocity of rotation without disturbance. If there be a small disturbance, the equations for the disturbance θ and ϕ may be written

$$A\frac{d^2\theta}{dt^2} + \frac{dA}{d\phi}\omega\frac{d\phi}{dt} = L, \tag{23}$$

$$B\frac{d^2\phi}{dt^2} - \frac{dA}{d\phi}\omega\frac{d\theta}{dt} = 0. \tag{24}$$

The period of such small disturbances is $(dA/d\phi)(AB)^{-1/2}$ revolutions of the shaft.

They will neither increase nor diminish if there are no other terms in the equations.

To convert this apparatus into a governor, let us assume viscosities X and Y in the motions of the main shaft and the centrifugal piece, and a resistance $G\phi$ applied to the main shaft. Putting $(dA/d\phi)\omega = K$, the equations become

$$A\frac{d^2\theta}{dt^2} + X\frac{d\theta}{dt} + K\frac{d\phi}{dt} + G\phi = L, \tag{25}$$

$$B\frac{d^2\phi}{dt^2} + Y\frac{d\phi}{dt} - K\frac{d\theta}{dt} = 0. \tag{26}$$

The condition of stability of the motion indicated by these equations is that all the possible roots, or parts of roots, of the cubic equation

$$ABn^3 + (AY + BX)n^2 + (XY + K^2)n + GK = o \tag{27}$$

shall be negative; and this condition is

$$\left(\frac{X}{A} + \frac{Y}{B}\right)\left(XY + K^2\right) > GK. \tag{28}$$

Combination of Governors. If the break of Thomson's governor is applied to a moveable wheel, as in Jenkin's governor, and if this wheel works a steam-valve, or a more powerful break, we have to consider the motion of three pieces. Without entering into the calculation of the general equations of motion of these pieces, we may confine ourselves to the case of small disturbances, and write the equations

$$\begin{aligned} A\frac{d^2\theta}{dt^2}+X\frac{d\theta}{dt}+K\frac{d\phi}{dt}+T\phi+J\psi&=P-R,\\ B\frac{d^2\phi}{dt^2}+Y\frac{d\phi}{dt}-K\frac{d\theta}{dt}&=0,\\ C\frac{d^2\psi}{dt^2}+Z\frac{d\psi}{dt}-T\phi&=0 \end{aligned} \tag{29}$$

where θ, ϕ, χ are the angles of disturbance of the main shaft, the centrifugal arm, and the moveable wheel respectively, A, B, C their moments of inertia, X, Y, Z the viscosity of their connexions, K is what was formerly denoted by $dA/d\phi = \omega$, and T and J are the powers of Thomson's and Jenkin's breaks respectively.

The resulting equation in n is of the form

$$\begin{vmatrix} An^2+Xn & Kn+T & J \\ -K & Bn+Y & 0 \\ 0 & -T & Cn^2+Zn \end{vmatrix} = 0 \tag{30}$$

or

$$\begin{aligned} n^5+n^4\left(\frac{X}{A}+\frac{Y}{B}+\frac{Z}{C}\right)+n^3\left[\frac{XYZ}{ABC}\left(\frac{X}{A}+\frac{Y}{B}+\frac{Z}{C}\right)+\frac{K^2}{AB}\right]&\\ +n^2\left(\frac{XYZ+KTC+K^2Z}{ABC}\right)+n\frac{KTZ}{ABC}+\frac{KTZJ}{ABC}&=0. \end{aligned} \tag{31}$$

I have not succeeded in determining completely the conditions of stability of the motion from this equation; but I have found two necessary conditions, which are in fact the conditions of stability of the two governors taken separately. If we write the equation

$$n^5+pn^4+qn^3+rn^2+sn+t=0, \tag{32}$$

then, in order that the possible parts of all the roots shall be negative, it is necessary that

$$pq > r \text{ and } ps > t. \tag{33}$$

I am not able to shew that these conditions are sufficient. This compound governor has been constructed and used.

On the Motion of a Liquid in a Tube revolving about a Vertical Axis.

Mr C. W. Siemens's Liquid Governor. Let ρ be the density of the fluid, k the section of the tube at a point whose distance from the origin measured along the tube is s, r, θ, z the co-ordinates of this point referred to axes fixed with respect to the tube, Q the volume of liquid which passes through any section in unit of time. Also let the following integrals, taken over the whole tube, be

$$\int \rho k r^2 ds = A,\ \int \rho r^2 d\theta = B,\ \int \rho\tfrac{1}{\alpha} ds = C, \tag{34}$$

the lower end of the tube being in the axis of motion.

Let ϕ be the angle of position of the tube about the vertical axis, then the moment of momentum of the liquid in the tube is

$$H = A\frac{d\phi}{dt} + BQ. \tag{35}$$

The moment of momentum of the liquid thrown out of the tube in unit of time is

$$\frac{dH'}{dt} = \rho r^2 Q\frac{d\phi}{dt} + \rho\frac{r}{k}Q^2\cos\alpha, \tag{36}$$

where r is the radius at the orifice, k its section, and α the angle between the direction of the tube there and the direction of motion.

The energy of motion of the fluid in the tube is

$$W = \tfrac{1}{2}A\left[\frac{d\phi}{dt}\right]^2 + BQ\frac{d\phi}{dt} + \tfrac{1}{2}CQ^2. \tag{37}$$

The energy of the fluid which escapes in unit of time is

$$\frac{W'}{dt} = \rho gQ(h+z) + \tfrac{1}{2}\rho r^2 Q\left[\frac{d\phi}{dt}\right]^2 + \rho\frac{r}{k}Q^2\cos\alpha\frac{d\phi}{dt} + \tfrac{1}{2}\frac{\rho}{k^2}Q^3. \tag{38}$$

The work done by the prime mover in turning the shaft in unit of time is

$$L\frac{d\phi}{dt} = \frac{d\phi}{dt}\left(\frac{dH}{dt} + \frac{dH'}{dt}\right). \tag{39}$$

The work spent on the liquid in unit of time is

$$\frac{dW}{dt} + \frac{dW'}{dt}. \tag{40}$$

Equating this to the work done, we obtain the equations of motion

$$A\frac{d^2\phi}{dt^2} + B\frac{dQ}{dt} + \rho r^2 Q\frac{d\phi}{dt} + \rho\frac{r}{k}\cos\alpha Q^2 = L \tag{41}$$

$$B\frac{d^2\phi}{dt^2} + C\frac{dQ}{dt} + \tfrac{1}{2}\frac{\rho}{k^2}Q^2 + \rho g(h+z) - \tfrac{1}{2}\rho r^2\left[\frac{d\phi}{dt}\right]^2 = 0 \tag{42}$$

These equations apply to a tube of given section throughout. If the fluid is in open channels, the values of A and C will depend on the depth to which the channels are filled at each point, and that of k will depend on the depth at the overflow.

In the governor described by Mr C. W. Siemens in the paper already referred to, the discharge is practically limited by the depth of the fluid at the brim of the cup.

The resultant force at the brim is $f = \sqrt{g^2 + \omega^4 r^2}$.

If the brim is perfectly horizontal, the overflow will be proportional to $x^{3/2}$ (where x is the depth at the brim), and the mean square of the velocity relative to the brim will be proportional to x, or to $Q^{2/3}$.

If the breadth of overflow at the surface is proportional to x^n, where x is the height above the lowest point of overflow, then Q will vary as $x^{n+3/2}$, and the mean square of the velocity of overflow relative to the cup as x or as $1/Q^{n+3/2}$.

If $n = -1/2$, then the overflow and the mean square of the velocity are both proportional to x.

From the second equation we find for the mean square of velocity

$$\frac{Q^2}{k^2} = -\frac{2}{\rho}\left(B\frac{d^2\phi}{dt^2} + C\frac{dQ}{dt}\right) + r^2\left[\frac{d\phi}{dt}\right]^2 - 2g(h+z) \tag{43}$$

If the velocity of rotation and of overflow is constant, this becomes

$$\frac{Q^2}{k^2} = r^2 \left[\frac{d\phi}{dt}\right]^2 - 2g(h+z) \tag{44}$$

From the first equation, supposing, as in Mr Siemens's construction, that $\cos\alpha = 0$ and $B = 0$, we find

$$L = \rho r^2 \frac{d\phi}{dt} \tag{45}$$

In Mr Siemens's governor there is an arrangement by which a fixed relation is established between L and z,

$$L = -Sz \tag{46}$$

whence

$$\frac{Q^2}{k^2} = r^2 \left[\frac{d\phi}{dt}\right]^2 - 2gh + 2\frac{g\rho}{S} r^2 Q \frac{d\phi}{dt} \tag{47}$$

If the conditions of overflow can be so arranged that the mean square of the velocity, represented by Q^2/k^2, is proportional to Q, and if the strength of the spring which determines S is also arranged so that

$$\frac{Q^2}{k^2} = 2\frac{g\rho}{S} r^2 \omega Q \tag{48}$$

the equation will become, if $2gh = \omega^2 r^2$,

$$0 = r^2 \left(\left[\frac{d\phi}{dt}\right]^2 - \omega^2 \right) + 2\frac{g\rho}{S} r^2 Q \left(\frac{d\phi}{dt} - \omega \right), \tag{49}$$

which shews that the velocity of rotation and of overflow cannot be constant unless the velocity of rotation is ω.

The condition about the overflow is probably difficult to obtain accurately in practice; but very good results have been obtained within a considerable range of driving-power by a proper adjustment of the spring. If the rim is uniform, there will be a *maximum* velocity for a certain driving-power. This seems to be verified by the results given at p. 667 of Mr Siemens's paper.

If the flow of the fluid were limited by a hole, there would be a *minimum* velocity instead of a maximum.

The differential equation which determines the nature of small disturbances is in general of the fourth order, but may be reduced to the third by a proper choice of the value of the mean overflow.

Theory of Differential Gearing.

In some contrivances the main shaft is connected with the governor by a wheel or system of wheels which are capable of rotation round an axis, which is itself also capable of rotation about the axis of the main shaft. These two axes may be at right angles, as in the ordinary system of differential bevel wheels; or they may be parallel, as in several contrivances adapted to clockwork.

Let ξ and η represent the angular position about each of these axes respectively, θ that of the main shaft, and ϕ that of the governor; then θ and ϕ are linear functions of ξ and η, and the motion of any point of the system can be expressed in terms either of ξ and η or of θ and ϕ.

Let the velocity of a particle whose mass is m resolved in the direction of x be

$$\frac{dx}{dt} = p_1 \frac{d\xi}{dt} + q_1 \frac{d\eta}{dt} \tag{50}$$

with similar expressions for the other co-ordinate directions, putting suffixes 2 and 3 to denote the values of p and q for these directions. Then Lagrange's equation of motion becomes

$$\Xi\delta\xi + H\delta\eta = \Sigma m \left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z \right) = 0 \tag{51}$$

where Ξ and H are the forces tending to increase ξ and η respectively, no force being supposed to be applied at any other point.

Now putting

$$\delta x = p_1 d\xi + q_1 d\eta \tag{52}$$

and

$$\frac{d^2x}{dt^2} = p_1 \frac{d^2\xi}{dt^2} + q_1 \frac{d^2\eta}{dt^2} \tag{53}$$

the equation becomes

$$\left(\Xi - \Sigma mp^2 \frac{d^2\xi}{dt^2} - \Sigma mpq \frac{d^2\eta}{dt^2} \right) \delta\xi + \left(H - \Sigma mpq \frac{d^2\xi}{dt^2} - \Sigma mq^2 \frac{d^2\eta}{dt^2} \right) \delta\eta = 0 \tag{54}$$

and since $\delta\xi$ and $\delta\eta$ are independent, the coefficient of each must be zero.

If we now put

$$\Sigma \left(mp^2 \right) = L,\ \Sigma \left(mpq \right) = M,\ \Sigma \left(mqp^2 \right) = N \tag{55}$$

where $p^2 = p_1^2 + p_2^2 + p_3^2$, $pq = p_l q_l + p_2 q_2 + p_3 q_3$, and $q^2 = q_1^2 + q_2^2 + q_3^2$, the equations of motion will be

$$\Xi = L \frac{d^2\xi}{dt^2} + M \frac{d^2\eta}{dt^2} \tag{56}$$

$$H = M \frac{d^2\xi}{dt^2} + N \frac{d^2\eta}{dt^2} \tag{57}$$

If the apparatus is so arranged that $M = 0$, then the two motions will be independent of each other; and the motions indicated by ξ and η will be about conjugate axes — that is, about axes such that the rotation round one of them does not tend to produce a force about the other.

Now let Θ be the driving-power of tile shaft on the differential system, and Φ that of the differential system on the governor; then the equation of motion becomes

$$\Theta\delta\theta + \Phi\delta\phi + \left(\Xi - L \frac{d^2\xi}{dt^2} + M \frac{d^2\eta}{dt^2} \right) \delta\xi + \left(H - M \frac{d^2\xi}{dt^2} + N \frac{d^2\eta}{dt^2} \right) \delta\eta = 0 \tag{58}$$

and if

$$\begin{aligned} \delta\xi &= P\delta\theta + Q\delta\phi \\ \delta\eta &= R\delta\theta + S\delta\phi \end{aligned} \tag{59}$$

and if we put

$$\begin{aligned} L' &= LP^2 + 2MPR + NR^2 \\ M' &= LPQ + M(PS + QR) + NRS \\ N' &= LQ^2 + 2MQS + NS^2 \end{aligned} \tag{60}$$

the equations of motion in θ and ϕ will be

$$\begin{aligned} \Theta + P\Xi + QH &= L'\frac{d^2\theta}{dt^2} + M'\frac{d^2\phi}{dt^2} \\ \Phi + R\Xi + SH &= M'\frac{d^2\theta}{dt^2} + N'\frac{d^2\phi}{dt^2} \end{aligned} \tag{61}$$

If $M' = 0$, then the motions in θ and ϕ will be independent of each other. If M is also 0, then we have the relation

$$LPQ + MRS = 0 \tag{62}$$

and if this is fulfilled, the disturbances of the motion in θ will have no effect on the motion in ϕ. The teeth of the differential system in gear with the main shaft and the governor respectively will then correspond to the centres of percussion and rotation of a simple body, and this relation will be mutual.

In such differential systems a constant force, H, sufficient to keep the governor in a proper state of efficiency, is applied to the axis η, and the motion of this axis is made to work a valve or a break on the main shaft of the machine. Ξ in this case is merely the friction about the axis of ξ. If the moments of inertia of the different parts of the system are so arranged that $M' = 0$, then the disturbance produced by a blow or a jerk on the machine will act instantaneously on the valve, but will not communicate any impulse to the governor.