On the asymptotic analysis of a class of linear recurrences

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Problems in Combinatorial Enumeration

Examples of recursively definable structures:

- Number of partitions of a set into subsets
  - Bell Numbers
- Partition lattice chains (Babai, Lengyel)
  - Lengyel's Constant
- Analysis of a recursive Program (Knuth)
  - $t(x, y, z) =$ **if** $x \leq y$ **then** y **else**
    $$t(t(x-1, y, z), t(y-1, z, x), t(z-1, x, y))$$

Bell Numbers

- Recurrence: $$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} \ , \quad B_0 = 1$$

- Functional equation for OGF:

$$B(z) = \frac{z}{1-z} B\left(\frac{z}{1-z}\right) + 1$$

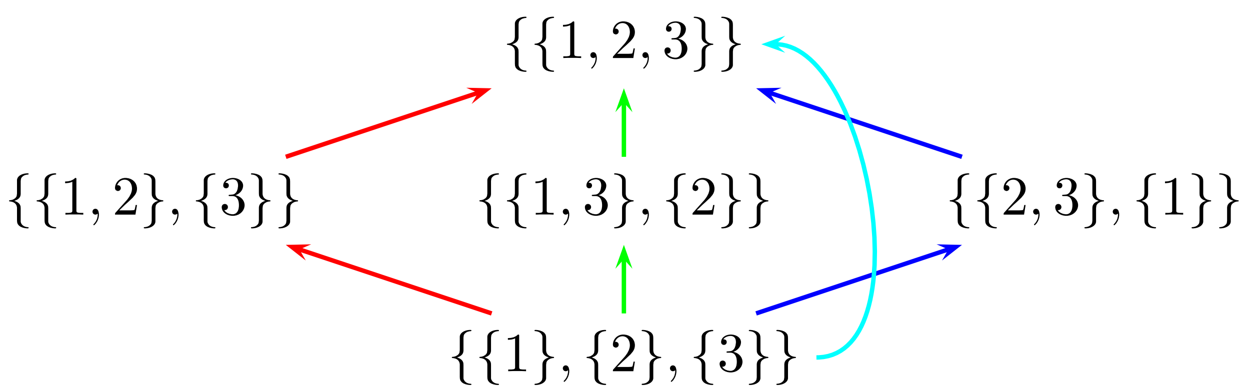
- Asymptotic growth:

$$B_n = \frac{1}{e} \sum_{m=0}^{\infty} \frac{m^n}{m!} \sim \exp\left(e^w(w^2 - w + 1) - \frac{1}{2}\log(w+1) - 1\right)$$

- Scale: $w \exp(w) = n$ Lambert W-function

Partition Lattice Chains

- Poset of partitions of an n-set



- Z_n number of chains from minimal to maximal element

$$Z_1 = 1, \quad Z_2 = 1, \quad , Z_3 = 4, \quad Z_4 = 32, \quad \ldots$$

Partition Lattice Chains (ctd.)

- Recurrence (Lengyel):

$$Z_n = \sum_{k=1}^{n-1} S_{n,k} Z_k \ , \quad S_{n,k} \text{ Stirling numbers 2nd kind}$$

- Functional equation for EGF (Lengyel):

$$Z(z) = \frac{1}{2} Z(e^z - 1) + \frac{z}{2}$$

- Asymptotic growth (Babai, Lengyel):

$$Z_n \sim C_{\text{Lengyel}} (n!)^2 (2 \log 2)^{-n} n^{-1-\frac{1}{3}\log 2}$$

- Lengyel's Constant (Flajolet, Salvy): $C_{\text{Lengyel}} = 1.0986858055\ldots$

Takeuchi Numbers

- Recursive function (Takeuchi):

$$t(x,y,z) = \textbf{if } x \le y \textbf{ then } y \textbf{ else}$$
$$t(t(x-1,y,z), t(y-1,z,x), t(z-1,x,y))$$

- $T(x,y,z)$ number of times the **else** clause is invoked when evaluating $t(x,y,z)$
- $T_n = T(n,0,n+1)$

$$T_1 = 1, \quad T_2 = 4, \quad , T_3 = 14, \quad T_4 = 53, \quad \ldots$$

- Actual value of $t(x,y,z)$ is irrelevant

$$t(x,y,z) = \begin{cases} y & x \le y \\ \begin{cases} z & y \le z \\ x & \text{else} \end{cases} \end{cases}$$

Takeuchi Numbers (ctd.)

- Recurrence (Knuth):

$$T_{n+1} = \sum_{k=0}^{n} \left[\binom{n+k}{n} - \binom{n+k}{n+1} \right] T_{n-k} + \sum_{k=1}^{n+1} \binom{2k}{k} \frac{1}{k+1}$$

- Functional equation for OGF (Knuth):

$$T(z) = zC(z)\,T(zC(z)) + \frac{C(z)-1}{1-z} \,, \quad C(z) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{z^k}{k+1}$$

- Asymptotic growth (Prellberg):

$$T_n \sim C_{\mathrm{Takeuchi}} B_n \exp \frac{1}{2} W(n)^2 \,, \quad C_{\mathrm{Takeuchi}} = 2.2394331040\ldots$$

Common Features

- Linear recurrences:

$$X_n = \sum_{k=1}^{n} c_{n,k} X_{n-k} + b_n$$

- Functional equations for OGF/EGF:

$$X(z) = a(z) X \circ f(z) + b(z)$$

- Parabolic fixed point:

$$f(z) = z + cz^2 + dz^3 + \dots$$

- Caveat: divergence of GF!

Generalization: Recursive Structures

- View combinatorial structures as formed of “atoms”
- Substitution operation $\mathcal{A} = \mathcal{B} \circ \mathcal{C}$:
  “substitute elements of $\mathcal{C}$ for atoms of $\mathcal{B}$”

$$\mathcal{B} \circ \mathcal{C} = \sum_{k \geq 0} \mathcal{B}_k \times \overbrace{\mathcal{C} \times \mathcal{C} \times \ldots \times \mathcal{C}}^{k}$$

- The associated OGF satisfies $A(z) = B(C(z))$
- A recursively definable structure $\mathcal{X}$ is defined by

$$\mathcal{X} = \mathcal{A} \times \mathcal{X} \circ \mathcal{F} + \mathcal{B}$$

- The associated OGF satisfies $X(z) = A(z) X \circ F(z) + B(z)$

Ingredients for General Theory

- Formal solution of the functional equation (leads to divergent FPS)
- Cauchy formula
- Analytic iteration theory near parabolic fixed points (Milnor, Beardon)
- Saddle point analysis

Formal Power Series Solution

Let the FPS $X(z)$ satisfy

$$X(z) = a(z)X \circ f(z) + b(z)$$

with $a(z)$, $f(z)$, and $b(z)$ analytic near $z = 0$ and

$$f(z) = z + cz^2 + dz^3 + \dots \,, \quad c > 0$$

Then

$$X(z) = \sum_{m=0}^{\infty} \left(\prod_{k=0}^{m-1} a \circ f^k(z) \right) b \circ f^m(z)$$

Inversion via Cauchy Formula

From

$$X(z) = \sum_{m=0}^{\infty} \left(\prod_{k=0}^{m-1} a \circ f^k(z) \right) b \circ f^m(z)$$

we compute

$$X_n = [z^n] X(z) = \sum_{m=0}^{\infty} X_{n,m}$$

with

$$X_{n,m} = \frac{1}{2\pi i} \oint \left(\prod_{k=0}^{m-1} a \circ f^k(z) \right) b \circ f^m(z) \frac{dz}{z^{n+1}}$$

Simplification via Homogeneous Eqn.

- Let $Y(z)$ be a solution of the *homogeneous* equation

$$Y(z) = a(z)\, Y \circ f(z)$$

Then $X_{n,m}$ simplifies to

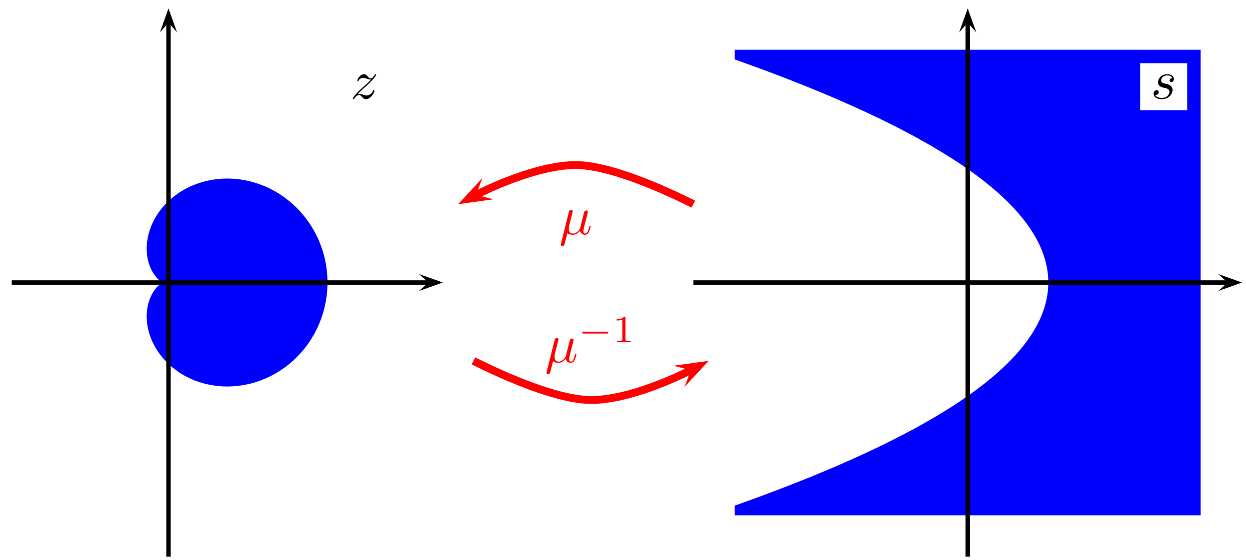
$$X_{n,m} = \frac{1}{2\pi i} \oint \frac{b \circ f^m(z)}{Y \circ f^m(z)} Y(z) \frac{dz}{z^{n+1}}$$

- Needed: existence of $Y(z)$ and analyticity properties
  - Analytic iteration theory (Milnor, Beardon)

Analytic Iteration Theory

"Parabolic Linearization Theorem" $\Rightarrow$ conjugacy of $f(z)$ to a shift

- $f^{-1}(z)$ exists in cardioid domain and maps contractively into it



- $f^{-1}(z) = f^{-1} \circ \mu(s) = \mu(s+1)$ for $s \in \mathcal{D}(\mu)$
- $f^k(z) = \mu\left(\mu^{-1}(z) - k\right)$ for z sufficiently small

Analytic Iteration Theory (ctd.)

- $\mu(s)$ admits a complete asymptotic expansion for $\Re(s) \to \infty$:

$$\mu(s) \sim \frac{1}{cs}\left(1 + \left(1 - \frac{d}{c^2}\right)\frac{\log s}{s} + \sum_{k=2}^{\infty}\sum_{j=0}^{k}\mu_{j,k}\frac{(\log s)^j}{s^k}\right)$$

- $f^{-m} \circ \mu(s) = \mu(s+m)$ admits a complete asymptotic expansion for $m \to \infty$:

$$\mu(s+m) \sim \frac{1}{cm}\left(1 + \left(1 - \frac{d}{c^2} - s\right)\frac{\log m}{m} + \sum_{k=2}^{\infty}\sum_{j=0}^{k}\nu_{j,k}(s)\frac{(\log m)^j}{m^k}\right)$$

Solution of the Homogeneous Eqn.

- Substitute $z = \mu(s)$:

$$Y(z) = a(z)Y \circ f(z) \quad \Longrightarrow \quad Y \circ \mu(s) = a \circ \mu(s) Y \circ \mu(s-1)$$

- Solution is given by

$$Y \circ \mu(s) = \lim_{n \to \infty} \frac{a \circ \mu(1) a \circ \mu(2) \ldots a \circ \mu(n) \left(a \circ \mu(n)\right)^s}{a \circ \mu(s+1) a \circ \mu(s+2) \ldots a \circ \mu(s+n)}$$

which defines an analytic function in $\mathcal{D}(\mu)$

- Asymptotics as $n \to \infty$:

$$\frac{Y \circ \mu(s+n)}{Y \circ \mu(n)} \sim \left(a \circ \mu(n)\right)^s$$

Asymptotics of $X_{n,m}$

$$X_{n,m} = \frac{1}{2\pi i}\oint \frac{b\circ f^m(z)}{Y\circ f^m(z)} Y(z)\frac{dz}{z^{n+1}}$$

- Substitute $z = \mu(s+m)$:

$$\sim \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{b\circ\mu(s)}{Y\circ\mu(s)} Y\circ\mu(s+m)\frac{d\mu(s+m)}{\mu(s+m)^{n+1}}$$

- Asymptotics of $Y\circ\mu(s+m)$:

$$\sim \frac{Y\circ\mu(m)}{2\pi i}\int_{\mathcal{C}} \frac{b\circ\mu(s)}{Y\circ\mu(s)}(a\circ\mu(m))^s\frac{d\mu(s+m)}{\mu(s+m)^{n+1}}$$

- Asymptotics of $\mu(s+m)$:

$$\sim (cm)^n m^{-1-(1-\frac{d}{c^2})\frac{n}{m}}\frac{Y\circ\mu(m)}{2\pi i}\int_{\mathcal{C}} \frac{b\circ\mu(s)}{Y\circ\mu(s)}\left(a\circ\mu(m)e^{\frac{n}{m}}\right)^s ds$$

Asymptotics of X_n

$$X_n = \sum_{m=0}^{\infty} X_{n,m} \ , \quad X_{n,m} \sim \ldots \int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} \left(a \circ \mu(m) e^{\frac{n}{m}}\right)^s ds$$

- Saddle point analysis:

Saddle at $a \circ \mu(m) e^{\frac{n}{m}} = 1$

- Sum simplifies to

$$X_n \sim C \sum_m (cm)^n \frac{Y \circ \mu(m)}{m} \left(a \circ \mu(m)\right)^{(1-\frac{d}{c^2}) \log m}$$

with

$$C = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} ds$$

The Saddle Point Condition

- $a(z) = a_k z^k + \ldots$, $\mu(s) \sim (cs)^{-1}$
- Saddle at
$$a_k (cm)^{-k} e^{\frac{n}{m}} = 1$$
- Different behavior according to
  - $k = 0$:
$$m = -\frac{n}{\log a_0} \qquad 0 < a_0 < 1$$
  - $k = 1$:
$$m = \frac{n}{W(cn/a_1)} \qquad a_1 > 0$$
  - $k \geq 1$:
$$m = \frac{n}{kW(cn/ka_k^{1/k})} \qquad a_k > 0$$

Asymptotics of $Y \circ \mu(m)$

- $a(z) = a_k z^k + \ldots,\ \mu(s) \sim (cs)^{-1}(1 + (1 - \frac{d}{c^2})\frac{\log s}{s})$
- Homogeneous equation

$$Y \circ \mu(m) \sim a_k (cm)^{-k} e^{k(1-\frac{d}{c^2})\frac{\log m}{m}} Y \circ \mu(m-1)$$

- Different behavior according to
  - $k = 0$:

$$Y \circ \mu(m) \sim C_0 a_0^m$$

  - $k = 1$:

$$Y \circ \mu(m) \sim C_1 \frac{a_1^m}{c^m m!} e^{(1-\frac{d}{c^2})\frac{1}{2}(\log m)^2}$$

  - $k \geq 1$:

$$Y \circ \mu(m) \sim C_k \frac{a_k^m}{(c^m m!)^k} e^{k(1-\frac{d}{c^2})\frac{1}{2}(\log m)^2}$$

Main Results

THEOREM 1: Let the FPS $X(z) = \sum_{n=0}^{\infty} X_n z^n$ satisfy

$$X(z) = a(z) X \circ f(z) + b(z)$$

with $f(z) = z + cz^2 + dz^3 + \ldots$, $a(z) = a_0 + \ldots$, and $b(z)$ analytic near zero.
If $c > 0$ and $0 < a_0 < 1$ then

$$X_n \sim D n! \, (-c/\log a_0)^n \, n^{(1-\frac{d}{c^2})\log a_0 - 1}$$

as $n \to \infty$, where

$$D = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} ds (-\log a_0)^{-(1-\frac{d}{c^2})\log a_0}$$

Main Results (ctd.)

THEOREM 2: Let the FPS $X(z)=\sum_{n=0}^{\infty} X_n z^n$ satisfy

$$X(z)=a(z)X\circ f(z)+b(z)$$

with $f(z)=z+cz^2+dz^3+\ldots$, $a(z)=a_1 z+\ldots$, and $b(z)$ analytic near zero.
If $c>0$ and $a_1>0$ then

$$X_n \sim Dc^n e^{-\frac{1}{2}(1-\frac{d}{c^2})W(\frac{c}{a_1}n)^2}\sum_{m=0}^{\infty}\frac{m^n}{m!}\left(\frac{a_1}{c}\right)^m$$

as $n\to\infty$, where

$$D=\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{b\circ\mu(s)}{Y\circ\mu(s)}ds e^{\frac{1}{2}(1-\frac{d}{c^2})(\log\frac{a_1}{c})^2}$$

Application: Bell Numbers

- Functional equation for OGF:

$$B(z) = \frac{z}{1-z} B\left(\frac{z}{1-z}\right) + 1$$

- $a(z) = \frac{z}{1-z}$, $f(z) = \frac{z}{1-z}$, $b(z) = 1$
- $\mu(s) = 1/s$, $Y \circ \mu(s) = 1/\Gamma(s)$

- Asymptotics:

$$B_n \sim D \sum_{m=0}^{\infty} \frac{m^n}{m!}$$

as $n \to \infty$, where

$$D = \frac{1}{2\pi i} \int_{\mathcal{C}} \Gamma(s) ds = \frac{1}{e} \quad \text{(sum of residues)}$$

Application: Partition Lattice Chains

- Functional equation for EGF:

$$Z(z) = \frac{1}{2} Z(e^z - 1) + \frac{z}{2}$$

- $a(z) = \frac{1}{2}$, $f(z) = e^z - 1$, $b(z) = \frac{z}{2}$
- $\mu(s) \sim \frac{2}{s}(1 - \frac{\log s}{3s} + \ldots)$, $Y \circ \mu(s) = 2^s$
- Asymptotics:

$$Z_n \sim D(n!)^2 \, (2\log 2)^{-n} \, n^{-1-\frac{1}{3}\log 2}$$

as $n \to \infty$, where

$$D = \frac{1}{2} \, (\log 2)^{\frac{1}{3}\log 2} \, \frac{1}{2\pi i} \int_{\mathcal{C}} 2^s \mu(s) ds = 1.0986858055\ldots$$

Application: Takeuchi Numbers

- Functional equation for OGF:

$$T(z) = zC(z)\,T(zC(z)) + \frac{C(z)-1}{1-z}\,, \quad C(z) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{z^k}{k+1}$$

- $a(z) = zC(z)$, $f(z) = zC(z)$, $b(z) = \frac{C(z)-1}{1-z}$
- $\mu(s) \sim \frac{1}{s}(1 - \frac{\log s}{s} + \ldots)$, $Y \circ \mu(s) \sim e^{-\frac{1}{2}(\log s)^2}/\Gamma(s)$
- Asymptotics:

$$T_n \sim D \sum_{m=0}^{\infty} \frac{m^n}{m!} e^{\frac{1}{2}W(n)^2} = D' B_n e^{\frac{1}{2}W(n)^2}$$

as $n \to \infty$, where

$$D' = \frac{e}{2\pi i} \int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} ds = 2.2394331040\ldots$$

Summary

- Interesting application of analytic iteration theory and classical complex analysis in the services of asymptotic enumeration
- A large class of linear recurrences corresponding to recursively definable structures can be treated asymptotically
- Further applications?

Other results:

- Numerical evaluation of the constants to about 50 decimal places
- Computation of the next terms in the asymptotic expansion (by a different, non-rigorous method)

To be done:

- Computation of the contour integrals determining the constants