Asymptotic estimates of Stirling numbers and related asymptotic problems

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[summary by Helmut Prodinger]

Consider the two families of polynomials x^n and $(x)_n := x(x-1)\cdots(x-n+1)$. They are connected via the formulæ

$$x^n = \sum_{m=0}^{n} S(n,m)(x)_m$$

and

$$(x)_n = \sum_{m=0}^{n} s(n,m)x^m.$$

The coefficients $S(n,m)$ are called Stirling numbers of the second kind, and the coefficients $s(n,m)$ are called Stirling numbers of the first kind. This is the notation of Comtet [2]; other authors use different notations.

These numbers have some combinatorial meanings, e.g. $S(n,m)$ is the numbers of ways to partition the set $\{1,\ldots,n\}$ into exactly m nonempty subsets.

Example.

$$\begin{aligned}&1234\\&1|234; 2|134; 3|124; 4|123; 12|34; 13|24; 14|23\\&1|2|34; 1|3|24; 1|4|23; 2|3|14; 2|4|13; 3|4|12\\&1|2|3|4,\end{aligned}$$

whence $S(4,1)=1$, $S(4,2)=7$, $S(4,3)=6$, $S(4,4)=1$.

There is the handy recursion $S(n,m) = mS(n-1,m) + S(n-1,m-1)$ to compute them.

The (signless) Stirling numbers of the first kind $|s(n,k)|$ enumerate the number of permutations of the set $\{1,\ldots,n\}$ with exactly m cycles.

Example.

$$\begin{aligned}&(1)(2)(3)\\&(12)(3), (13)(2), (23)(1)\\&(123), (132),\end{aligned}$$

whence $|s(3,1)|=2$, $|s(3,2)|=3$, $|s(3,3)|=1$.

There is also an explicit formula

$$S(n,m) = \frac{1}{m!}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}k^n.$$

The asymptotic evaluation of the Stirling numbers has obtained some attention by several authors, but, since there are 2 parameters n and m involved, the range of validity is somehow limited. [1, 3, 4, 5, 6]

The paper [7] (on which the present talk is based) gives an expansion that is uniform in m, as $n \to \infty$.

The approach is based on the saddle point method, starting by expressing the Stirling numbers as Cauchy integrals by means of appropriate generating functions: (m fixed)

$$\sum_n s(n,m)\frac{x^n}{n!} = \frac{\big(\log(1+x)\big)^m}{m!},$$

$$\sum_n S(n,m)\frac{x^n}{n!} = \frac{\big(e^x-1\big)^m}{m!}.$$

Hence

$$S(n,m) = \frac{n!}{m!}\cdot\frac{1}{2\pi i}\oint \frac{\big(e^x-1\big)^m}{x^{n+1}}dx.$$

Rewrite it as

$$S(n,m) = \frac{n!}{m!}\cdot\frac{1}{2\pi i}\oint e^{\phi(x)}\frac{dx}{x},$$

with

$$\phi(x) = -m\log x + m\log(e^x-1).$$

The trick is to introduce a new complex variable t, via

$$\phi(x) = mt + (m-n)\log t + A,$$

where A is not depending on t. It is a linear combination of n and m.

THEOREM 1.

$$S(n,m) \sim e^A m^{n-m}\sqrt{\frac{t_0}{(1+t_0)(x_0-t_0)}}\binom{n}{m}.$$

Here x_0 is the saddle point and t_0 the corresponding t–value. For example, as $m \sim n$, the square root expression may be replaced asymptotically by 1.

The approach for the Stirling numbers of the first kind is similar. The function $\phi(x)$ is now $\phi(x) = n\log(1+t) - m\log t + B$.

THEOREM 2.

$$s(n+1,m+1) \sim (-1)^{n-m}e^B\frac{1}{x_0}\sqrt{\frac{m(n-m)}{n\phi''(x_0)}}\binom{n}{m}$$

Again, if m goes to infinity within a certain ratio of n, the quantities B and x_0 may be replaced by simpler expressions.

Higher order approximations and related topics were also discussed.

Bibliography

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