Annales Mathematicae et Informaticae
41 (2013) pp. 255–263

Proceedings of the
15th International Conference on Fibonacci Numbers and Their Applications
Institute of Mathematics and Informatics, Eszterházy Károly College
Eger, Hungary, June 25–30, 2012

Bridges between different known integer sequences

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Abstract

In this paper a new method of generating identities for Fibonacci and Lucas numbers is presented. This method is based on some fundamental identities for powers of the golden ratio and its conjugate. These identities give interesting connections between Fibonacci and Lucas numbers and Bernoulli numbers, Catalan numbers, binomial coefficients, δ-Fibonacci numbers, etc.

Keywords: Fibonacci and Lucas numbers, Bernoulli numbers, Bell numbers, Dobinski's formula

MSC: 11B83, 11A07, 39A10

1. Introduction

The authors' fascination with Fibonacci, Lucas and complex numbers has been reflected in the following two nice identities (discovered independently by Rabinowitz [10] and Wituła [7] and, probably, many other, former and future admirers of the Fibonacci and Lucas numbers):

$$(1+\xi+\xi^4)^n = F_{n+1}+F_n(\xi+\xi^4) \quad \text{and} \quad (1+\xi^2+\xi^3)^n = F_{n+1}+F_n(\xi^2+\xi^3), \tag{1.1}$$

where $\xi^5 = 1$, $\xi \in \mathbb{C}$ and $\xi \neq 1$, and F_n denotes the nth Fibonacci number.

2. Basic identities

Let

$$\alpha := 2\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \beta := -2\cos\left(\frac{2}{5}\pi\right) = \frac{1-\sqrt{5}}{2}.$$

Then we have

$$\alpha+\beta=1, \qquad \alpha\beta=-1 \tag{2.1}$$

$$F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}, \qquad n\in\mathbb{Z}, \tag{2.2}$$

$$L_n=\alpha^n+\beta^n, \qquad n=0,1,2,\dots, \tag{2.3}$$

where L_n denotes the nth Lucas number [3, 9].

Then, identities (1.1) can be written in the form

$$F_{n+1}+x^{-1}F_n=x^n, \tag{2.4}$$

for every $x\in\{\alpha,\beta\}$. In other words, we get the divisibility relation of polynomials

$$(x^2-x-1)\,|\,(x^{n+1}-F_{n+1}x-F_n).$$

Similarly (by induction) we can generate the identity

$$L_{n+1}+x^{-1}L_n=(2x-1)\,x^n, \tag{2.5}$$

for every $x\in\{\alpha,\beta\}$. This implies the following divisibility relation of polynomials

$$(x^2-x-1)\,|\,((2x-1)x^{n+1}-L_{n+1}x-L_n).$$

Remark 2.1. If the values F_n and L_n were defined for real subscripts $n\in[0,1)$ (see [15]), then from formulae (2.4) and (2.5) we could easily extend these definitions for any other real subscripts.

In particular, if functions $[0,1]\ni n\mapsto F_n$ and $[0,1]\ni n\mapsto L_n$ are continuous, then from formulae (2.4) and (2.5) we could obtain the continuous extensions of these functions. With this problem also some special problem is connected (see Corollary 2.6 – Dobinski's formula problem).

Immediately from identities (2.4) and (2.5) the next result follows.

Theorem 2.2 (Golden ratio power factorization theorem)**.** *Let $\{k_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Then the following identities hold true*

$$\prod_{n=1}^{N}\Big(F_{k_n+1}+\frac{\sqrt5-1}{2}\,F_{k_n}\Big)=\Big(\frac{1+\sqrt5}{2}\Big)^{\sum\limits_{n=1}^{N}k_n},$$

$$\prod_{n=1}^{N}\Big(F_{k_n+1}-\frac{\sqrt5+1}{2}\,F_{k_n}\Big)=\Big(\frac{1-\sqrt5}{2}\Big)^{\sum\limits_{n=1}^{N}k_n},$$

or in equivalent compact form

$$\prod_{n=1}^{N}\Big(F_{k_n+1}+x^{-1}F_{k_n}\Big)=x^{\sum\limits_{n=1}^{N}k_n},$$

$$\prod_{n=1}^{N} \Big(L_{k_n} + (2x-1)\, F_{k_n} \Big) = 2^N\, x^{\sum\limits_{n=1}^{N} k_n},$$

for every $x \in \{\alpha, \beta\}$, *and*

$$\prod_{n=1}^{N} \Big(L_{k_n+1} + \frac{\sqrt{5}-1}{2}\, L_{k_n} \Big) = (\sqrt{5})^N \Big(\frac{1+\sqrt{5}}{2} \Big)^{\sum\limits_{n=1}^{N} k_n},$$

$$\prod_{n=1}^{N} \Big(L_{k_n+1} - \frac{\sqrt{5}+1}{2}\, L_{k_n} \Big) = (-\sqrt{5})^N \Big(\frac{1-\sqrt{5}}{2} \Big)^{\sum\limits_{n=1}^{N} k_n},$$

or in equivalent compact form

$$\prod_{n=1}^{N} \Big(L_{k_n+1} + x^{-1} L_{k_n} \Big) = (2x-1)^N x^{\sum\limits_{n=1}^{N} k_n},$$

for every $x \in \{\alpha, \beta\}$. *The above identities are called "Golden Gate" relations.*

We note that these identities act as links between Fibonacci and Lucas sequences and many other special sequences of numbers, especially many known linear recurrence sequences. Now we will present the collection of such relations.

First let us consider the Bernoulli numbers B_r defined by the following recursion formula [6, 11]:

$$B_0 = 1, \quad \binom{n}{n-1} B_{n-1} + \binom{n}{n-2} B_{n-2} + \ldots + \binom{n}{0} B_0 = 0, \qquad n = 2, 3, \ldots$$

(we note that $B_{2k+1} = 0$, $k = 1, 2, \ldots$). Moreover, $B_k(y)$ denotes here the k-th Bernoulli polynomial defined by

$$B_k(y) = \sum_{l=0}^{k} \binom{k}{l} B_l\, y^{k-l}.$$

Corollary 2.3 (A bridge between Fibonacci, Lucas and Bernoulli numbers)**.** *We have*

$$\prod_{n=1}^{N-1} \Big(F_{n^k+1} + x^{-1} F_{n^k} \Big) = x^{\int_0^N B_k(y)\, dy},$$

$$\prod_{n=1}^{N-1} \Big(L_{n^k} + (2x-1)\, F_{n^k} \Big) = 2^{N-1}\, x^{\int_0^N B_k(y)\, dy}$$

and

$$\prod_{n=1}^{N-1} \Big(L_{n^k+1} + x^{-1} L_{n^k} \Big) = (2x-1)^{N-1} x^{\int_0^N B_k(y)\, dy},$$

for every $x \in \{\alpha, \beta\}$.

Proof. The identities result from the following known relation [6, 11]:

$$\sum_{n=1}^{N-1} n^k = \int_0^N B_k(y)\,dy = \sum_{r=0}^{k} \binom{k}{r} B_r \,\frac{N^{k-r+1}}{k-r+1}. \qquad \square$$

Corollary 2.4 (A bridge between Fibonacci numbers, Lucas numbers and binomial coefficients)**.** *We have*

$$\prod_{k=1}^{\lfloor (n+1)/2 \rfloor} \Big(F_{\binom{n-k}{k-1}+1} + x^{-1} F_{\binom{n-k}{k-1}} \Big) = x^{F_n},$$
$$\prod_{k=1}^{\lfloor (n+1)/2 \rfloor} \Big(L_{\binom{n-k}{k-1}} \pm \sqrt{5}\, F_{\binom{n-k}{k-1}} \Big) = 2^{\lfloor (n+1)/2 \rfloor} \Big(\frac{1 \pm \sqrt{5}}{2} \Big)^{F_n},$$
$$\prod_{k=1}^{\lfloor (n+1)/2 \rfloor} \Big(L_{\binom{n-k}{k-1}+1} + x^{-1} L_{\binom{n-k}{k-1}} \Big) = (2x-1)^{\lfloor (n+1)/2 \rfloor}\, x^{F_n},$$

for every $x \in \{\alpha, \beta\}$.

Proof. All the above identities follow from relation (see [9]):

$$F_n = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \binom{n-k}{k-1}. \qquad \square$$

Note that similar and simultaneously more general relations could be obtained for the incomplete Fibonacci and Lucas $p-$numbers (see [12, 13]).

Next corollary concerns the Catalan numbers defined in the following way

$$C_n := \frac{1}{n+1} \binom{2n}{n}, \quad n = 0, 1, \ldots$$

Corollary 2.5 (A bridge between Fibonacci numbers, Lucas numbers and Catalan numbers)**.** *We have*

$$\prod_{n=0}^{N} \Big(F_{1+C_{N-n}C_n} + x^{-1} F_{C_{N-n}C_n} \Big) = x^{C_{N+1}}, \tag{2.6}$$

$$\prod_{n=0}^{N} \Big(L_{C_{N-n}C_n} + (2x-1)\, F_{C_{N-n}C_n} \Big) = 2^{N+1} x^{C_{N+1}} \tag{2.7}$$

and

$$\prod_{n=0}^{N} \Big(L_{1+C_{N-n}C_n} + x^{-1} L_{C_{N-n}C_n} \Big) = (2x-1)^{N+1} x^{C_{N+1}}, \tag{2.8}$$

for every $x \in \{\alpha, \beta\}$.

Moreover, if p is prime and $p \equiv 3\,(mod\,4)$, then we have

$$\sqrt[p]{x^2 F_{1+C_{\frac{p-1}{2}}} + x\,F_{C_{\frac{p-1}{2}}}} = x^{\frac{2+C_{(p-1)/2}}{p}}, \tag{2.9}$$

$$\sqrt[p^2]{\left(F_{1+\frac{1}{2}C_{\frac{p^2-1}{2}}} + x^{-1}F_{\frac{1}{2}C_{\frac{p^2-1}{2}}}\right)\left(F_{1+\binom{p-1}{\frac{p-1}{2}}} + x^{-1}F_{\binom{p-1}{\frac{p-1}{2}}}\right)} = x^{\frac{\frac{1}{2}C_{(p^2-1)/2}+\binom{p-1}{\frac{p-1}{2}}}{p^2}}, \tag{2.10}$$

for every $x \in \{\alpha, \beta\}$.

Proof. Identities (2.6)-(2.8) can be obtained from the recursive relation for C_n

$$C_{N+1} = \sum_{n=0}^{N} C_{N-n}C_n, \quad N = 0, 1, \ldots$$

Whereas relations (2.9) and (2.10) result from the fact that if p is prime and $p \equiv 3\,(mod\,4)$, then $p|(2 + C_{\frac{p-1}{2}})$ and $p^2|\left(\frac{1}{2}C_{\frac{p^2-1}{2}} + \binom{p-1}{\frac{p-1}{2}}\right)$ (see [1]). □

Next conclusion is connected with the Bell numbers B_n, $n = 0, 1, \ldots$ [6].

Corollary 2.6 (A bridge between Fibonacci numbers, Lucas numbers and Bell numbers)**.** *We have*

$$\prod_{n=0}^{N}\left(F_{\binom{N}{n}\mathbf{B}_n+1} + x^{-1}F_{\binom{N}{n}\mathbf{B}_n}\right) = x^{\mathbf{B}_{N+1}},$$

$$\prod_{n=0}^{N}\left(L_{\binom{N}{n}\mathbf{B}_n} + (2x-1)\,F_{\binom{N}{n}\mathbf{B}_n}\right) = 2^{N+1}x^{\mathbf{B}_{N+1}},$$

$$\prod_{n=0}^{N}\left(L_{\binom{N}{n}\mathbf{B}_n+1} + x^{-1}L_{\binom{N}{n}\mathbf{B}_n}\right) = (2x-1)^{N+1}x^{\mathbf{B}_{N+1}},$$

for every $x \in \{\alpha, \beta\}$.

Proof. All the above identities follow from the well known recursive relation

$$\mathbf{B}_0 := 1,$$

$$\mathbf{B}_{N+1} = \sum_{n=0}^{N}\binom{N}{n}\mathbf{B}_n, \quad N = 0, 1, \ldots$$ □

We note that for the Bell numbers the following interesting relation, called Dobinski's formula [6], holds:

$$\mathbf{B}_N = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^N}{k!}, \quad N = 0, 1, 2, \ldots$$

In connection with the above formula we formulate a certain problem which can be expressed in the following way. Is it possible to generalize the definition of Fibonacci numbers F_n onto real indices (of Lucas numbers L_n, respectively) such that the following equality will be fulfilled:

$$\prod_{k=0}^{\infty}\left(F_{1+\frac{e^{-1}k^N}{k!}}+x^{-1}F_{\frac{e^{-1}k^N}{k!}}\right)=x^{B_N},$$

for every $x\in\{\alpha,\beta\}$ and $N\in\mathbb{N}$, or

$$\prod_{k=0}^{\infty}\frac{L_{1+\frac{e^{-1}k^N}{k!}}+x^{-1}L_{\frac{e^{-1}k^N}{k!}}}{2x-1}=x^{B_N},$$

for every $x\in\{\alpha,\beta\}$ and $N\in\mathbb{N}$, respectively?

Next corollary concerns the connection with the δ-Fibonacci numbers defined by relations (see [14]):

$$a_n(\delta)=\sum_{k=0}^{n}\binom{n}{k}F_{k-1}\,(-\delta)^k \tag{2.11}$$

and

$$b_n(\delta)=\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\,F_k\,\delta^k, \tag{2.12}$$

for $\delta\in\mathbb{C}$.

Corollary 2.7 (A bridge between Fibonacci, Lucas and δ-Fibonacci numbers)**.** *For positive integers δ and n we get*

$$\prod_{k=0}^{n}\left(F_{1+\binom{n}{k}F_{k-1}\delta^k}+x^{-1}F_{\binom{n}{k}F_{k-1}\delta^k}\right)=x^{a_n(-\delta)},$$

$$\prod_{k=1}^{n}\left(F_{1+\binom{n}{k}F_k\delta^k}+x^{-1}F_{\binom{n}{k}F_k\delta^k}\right)=x^{-b_n(-\delta)},$$

$$\prod_{k=0}^{n}\left(L_{\binom{n}{k}F_{k-1}\delta^k}+(2x-1)\,F_{\binom{n}{k}F_{k-1}\delta^k}\right)=2^{n+1}\,x^{a_n(-\delta)},$$

$$\prod_{k=1}^{n}\left(L_{\binom{n}{k}F_k\delta^k}\pm\sqrt{5}\,F_{\binom{n}{k}F_k\delta^k}\right)=2^n\,x^{-b_n(-\delta)},$$

$$\prod_{k=0}^{n}\left(L_{1+\binom{n}{k}F_{k-1}\delta^k}+x^{-1}L_{\binom{n}{k}F_{k-1}\delta^k}\right)=(2x-1)^{n+1}x^{a_n(-\delta)},$$

$$\prod_{k=1}^{n}\left(L_{1+\binom{n}{k}F_k\delta^k}+x^{-1}L_{\binom{n}{k}F_k\delta^k}\right)=(2x-1)^{n}x^{-b_n(-\delta)},$$

etc., for every $x\in\{\alpha,\beta\}$. Moreover, we define here $F_{n+1}=F_n+F_{n-1}$, $n\in\mathbb{Z}$.

Let us note that similar relations we have for the incomplete $\delta-$Fibonacci numbers $a_{n,r}(\delta)$ and $b_{n,s}(\delta)$ where

$$a_{n,r}(\delta) := \sum_{k=0}^{r} \binom{n}{k} F_{k-1}(-\delta)^k, \qquad 0 \le r \le n,$$

$$b_{n,s}(\delta) := \sum_{k=1}^{s} \binom{n}{k} (-1)^{k-1} F_k \delta^k, \quad 1 \le s \le n.$$

Now we consider the $r-$generalized Fibonacci sequence $\{G_n\}$ defined as follows

$$G_n = \begin{cases} 0, & \text{if} \quad 0 \le n < r-1, \\ 1, & \text{if} \quad n = r-1, \\ G_{n-1} + G_{n-2} + \ldots + G_{n-r}, & \text{if} \quad n \ge r. \end{cases}$$

Corollary 2.8 (A bridge between Fibonacci, Lucas and classic r-Fibonacci numbers)**.** *Let $r \in \mathbb{N}$, $r \ge 2$. Then the following identities hold true [8]:*

$$\left(F_{1+2^{r-1}G_{n-r}} + x^{-1} F_{2^{r-1}G_{n-r}}\right) \prod_{k=1}^{r-1} \left(F_{1+(\sum\limits_{i=k}^{r-1} 2^{i-1})G_{n-r-k}} + x^{-1} F_{(\sum\limits_{i=k}^{r-1} 2^{i-1})G_{n-r-k}}\right) = x^{G_n},$$

for every $n \ge 2\,r-1$, and

$$\left[\prod_{k=0}^{n} \left(F_{1+G_k^2} + x^{-1} F_{G_k^2}\right)\right] \times \left[\prod_{i=2}^{r-1} \prod_{k=0}^{n-i} \left(F_{1+G_k G_{k+i}} + x^{-1} F_{G_k G_{k+i}}\right)\right] = x^{G_n G_{n+1}},$$

the special case of which is the following Lucas identity

$$\prod_{k=1}^{n} \left(F_{1+F_k^2} + x^{-1} F_{F_k^2}\right) = x^{F_n F_{n+1}},$$

for every $x \in \{\alpha, \beta\}$.

Corollary 2.9. *We have also ($x \in \{\alpha, \beta\}$):*

$$\left(F_{F_{n+1}+1} + x^{-1} F_{F_{n+1}}\right)\left(F_{F_{n-1}+1} + x^{-1} F_{F_{n-1}}\right) = x^{L_n},$$

$$\left(L_{F_{n+1}} \pm \sqrt{5} F_{F_{n+1}}\right)\left(L_{F_{n-1}} \pm \sqrt{5}\, F_{F_{n-1}}\right) = 4\left(\frac{1 \pm \sqrt{5}}{2}\right)^{L_n},$$

$$\left(L_{F_{n+1}+1} + x^{-1} L_{F_{n+1}}\right)\left(L_{F_{n-1}+1} + x^{-1} L_{F_{n-1}}\right) = 5\, x^{L_n},$$

since $F_{n+1} + F_{n-1} = L_n$, $n \in \mathbb{N}$. Furthermore, we have

$$\left(F_{L_{n+1}+1} + x^{-1} F_{L_{n+1}}\right)\left(F_{L_{n-1}+1} + x^{-1} F_{L_{n-1}}\right) = x^{5\,F_n},$$

$$\left(L_{L_{n+1}} \pm \sqrt{5}\,F_{L_{n+1}}\right)\left(L_{L_{n-1}} \pm \sqrt{5}\,F_{L_{n-1}}\right) = 4\left(\frac{1\pm\sqrt{5}}{2}\right)^{5\,F_n},$$
$$\left(L_{L_{n+1}+1} + x^{-1}L_{L_{n+1}}\right)\left(L_{L_{n-1}+1} + x^{-1}L_{L_{n-1}+1}\right) = 5\,x^{5\,F_n},$$

since $L_{n+1} + L_{n-1} = 5\,F_n$, $n \in \mathbb{N}$.

Remark 2.10. Note that Theorem 2.2 is connected, in some way, with the following very important Zeckendorf's theorem [6]:

For every number $n \in \mathbb{N}$ *there exists exactly one increasing sequence* $2 \le k_1 < \ldots < k_r$, *where* $r = r(n) \in \mathbb{N}$, *such that* $k_{i+1} - k_i \ge 2$ *for* $i = 1, 2, \ldots, r-1$, *and*

$$n = F_{k_1} + F_{k_2} + \ldots + F_{k_r}.$$

For example, we have

$$1000 = 987 + 13 = F_{16} + F_7,$$

that is

$$\left(\sqrt{5}\,F_{987} \pm L_{987}\right)\left(\sqrt{5}\,F_{13} \pm L_{13}\right) = 2\,L_{1000} \pm 2\sqrt{5}\,F_{1000} =$$
$$= \left(L_{987} \pm \sqrt{5}\,F_{987}\right)\left(L_{13} \pm \sqrt{5}\,F_{13}\right) = 4\left(\frac{1\pm\sqrt{5}}{2}\right)^{1000}.$$

3. Final remark

Finally, we note that identities (2.4), considered at the beginning of this paper, were discussed by many authors. For example, S. Alikhani and Y. Peng [2] basing on (2.4) have proven that α^n, for every $n \in \mathbb{N}$, cannot be a root of any chromatic polynomial. Furthermore, D. Gerdemann [5] has used the first of identities (2.4) for analyzing the, so called, Golden Ratio Division Algorithm. Consequently, he has discovered a semi-combinatorial proof of the following beautiful theorem.

Theorem 3.1. *For nonconsecutive integers* $a_1, \ldots, a_k$, *the following two statements are equivalent (for every* $m \in \mathbb{N}$*):*

$$m\,F_n = F_{n+a_1} + F_{n+a_2} + \ldots + F_{n+a_k},$$
$$m = \alpha^{a_1} + \alpha^{a_2} + \ldots + \alpha^{a_k}.$$

Acknowledgements. The Authors are grateful to the valuable remarks of the Referee which gave the possibility to improve presentation of the paper.

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