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Productivity Analysis: Roots, Foundations, Trends and Perspectives

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Abstract

The goal of this article is to give a brief overview of productivity analysis, starting with general concepts, its importance and a brief historical excursion and then focusing on various productivity indexes. We also start from very simple productivity indexes to more sophisticated, such as Malmquist Productivity Indexes, which are among the most popular in academic literature these days. A special attention is on the contributions to this literature from Rolf Färe and Shawna Grosskopf (and their many co-authors), and some of the related works they have inspired. A brief discussion of likely perspectives for the area is also provided.

Keywords: Productivity Analysis; Efficiency, Malmquist; Indexes; Growth Accounting;
JEL Codes: D24, E22, E23, E24, O47

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1 Introduction

The notion of productivity is central in many respects, whether viewed locally or globally. Indeed, whatever topic we take, one can often arrive at a point related to some sort of production, and whenever there is production (i.e., some inputs get converted into some outputs), there is also the notion of productivity. The notion of productivity also goes well beyond economics. For example, from physics we know that various particles *produce* various types of radiation, e.g., light, which further leads to (and can be used for) *production* of heat, electricity, etc. From biology, we know that living things are persistently trying to *produce* something (at least themselves) to exist and, possibly, even prosper—this includes plants, viruses, bacteria, animals and of course us, humans. The latter (humans) even study production and productivity, trying to understand how to measure it well, how to analyze and explain it, and also writing various books and articles about it, such as this one. Some (including the author of these lines) also come to the conclusion that productivity is among the most important aspects in this world. Not surprisingly, the Nobel Laureate, Paul Krugman said about it very eloquently and succinctly: "Productivity isn't everything, but in the long run, it's almost everything..." (Krugman (1997)).

The importance of productivity has also been vividly seen from the Covid-19 pandemic: if the virus' basic reproductive rate (so-called R_0, which can be also viewed as a measure of productivity for a virus) is above 1, the virus prospers, while its potential hosts may try to subdue it with various measures to drive it below 1 in the hopes it will eventually degenerate to non-existence or at least to not being a big issue. In a similar sense, one may expect that for humanity to prosper in certain aspects, the productivity for those aspects should also be sustained above certain levels. In turn, such importance of productivity implies the importance of accurate measurements and adequate analysis of productivity, as well as its relative version, usually referred to as *production efficiency*.

Given such importance, it would not be surprising if the origins of the notion of productivity is as old as this world and so can hardly be traced to a person who was the first to study it. Perhaps our colleagues from our very diversely rich international community of researchers on productivity and efficiency can find some such traces in the works of their famous ancient scholars (Imhotep, Baudhayana, Pythagoras, Euclid, Confucius, Liu Xin, Ptolemy, Al-Kindi, Al-Khwarizmi, to mention a few). What I accidentally discovered recently is that the notion of productivity, and the desire to conduct productivity analysis in particular, appears to be at the very origin of the word *econometrics*. Indeed, Ragnar Frisch, who is usually regarded as the 'father of econometrics' and, sometimes, as the one who coined the word econometrics, in his *Econometrica* note (1936, Volume 4, Issue 1) acknowledged that this word was proposed by Pawel Ciompa (1867-1913) in his book (in German) "published in Lwow (Lernberg) in 1910".[1] What is even more interesting for our productivity analysis community is the context where Pawel Ciompa coined this word—according to Bauer (2014):

> "... in 1910, a Ukrainian bank comptroller named Pawel Ciompa coined the term econometrics to describe his new process of bookkeeping data to project trends in worker *productivity*" (Bauer (2014, p.24), *emphasis* added.)

I believe this is an interesting fact for the productivity community, who of course know that the fields of productivity and of econometrics are very closely related and enrich each other and that, interestingly, the former was apparently at (and the motivation for) the conception of the latter.

Since then (and likely before that), many econometricians and other scholars across the world have tried to use or develop new tools to analyze the productivity of many industries. Among them are many legendary scholars: Leontief, Douglas, von Neumann, Hicks, Moorsteen, Tinbergen, Solow, Shephard, Kendrick, Debreu, Kantorovich, Friedman, Koopmans, Farrell, Cooper, Griliches, Jorgenson, to mention just a few who, sadly, have already passed. Their discoveries, knowledge and insightful insights were then passed on and developed further, and quite substantially in different streams by many other scholars. This includes, of course, our community of productivity and efficiency analysis (recently united in ISEAPA) that embraces many of the bright scholars that have produced impactful research output, both theoretical and applied.

The resulting current literature produced by these scholars is truly overwhelming, featuring many articles, chapters, books and working papers. 'Overwhelming' is exactly what we felt when together with Robin Sickles, we tried to summarize this literature with some basic details. This attempt spilled into 15 years of an exciting book project—because the book kept growing as more and more papers, old and new, were brought to our attention. While I will mention many key works in this article, describing them in any fair detail would take another book and so my goal here is much more modest: I will focus more on *some* of the many contributions to the literature from Rolf Färe and Shawna Grosskopf, and some of the related works they have inspired.

There are many reasons for my choice of the focus here and I will mention just a few. First, this was the topic of the JPA Symposium panel at EWEPA in honor of Rolf Färe and Shawna Grosskopf, where I had the great honor to be invited to present an early version of this article. Second, I had the great honor of being their student and so rightfully feel a duty

[1] Lwow is a Polish spelling of Lviv, a western city of Ukraine, which at that time was part of the Austria-Hungarian empire and was called Lemberg.

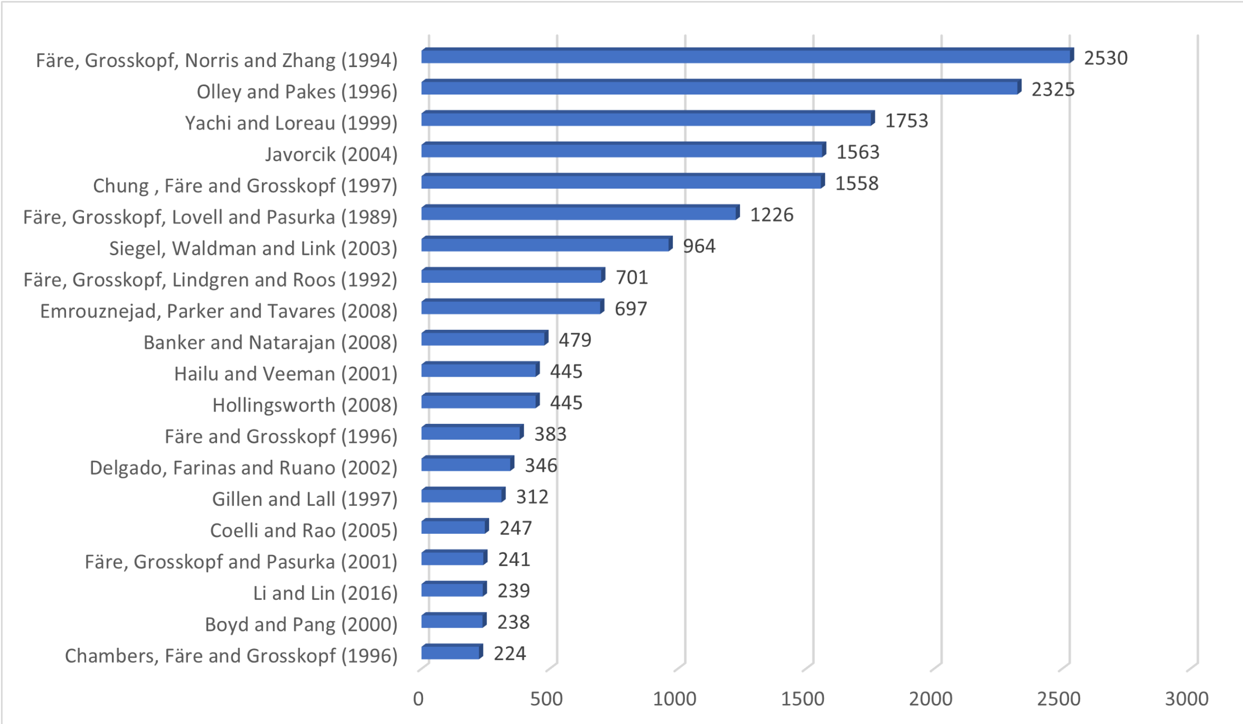

Figure 1: The most cited papers in Scopus for the 'productivity' topic

to write such an article. Last, yet foremost, their contribution to the literature is indeed exceptionally outstanding. The experts in our field know it well, yet some concrete evidence would be useful to support this statement more precisely. To do so, some interesting insights can be captured by various machine learning tools for 'biblioanalytics' on a systematically collected pool of papers (Choi and Oh, 2019; Wang and Zelenyuk, 2021), and summarized, e.g., in Figures 1-3 below.[2]

Figure 1 gives a list of 20 top-cited papers (as of 24 June 2022) and, although the number of citations is not the only (and not without caveats) indicator of an impact of research, it provides an insight about some of the most influential works in the literature.

Figure 2 visualizes the citation network in the productivity literature. Intuitively, it is the network of papers from the pool citing each other, which vividly highlights the key papers and their interactions with the other papers where 'productivity' is a key topic.[3]

[2]The papers and bibliometric data (e.g., number of citations, affiliation of authors, references, etc.) are collected from Scopus using "productivity" and related terms as the keywords, while papers from the Web of Science using the same keywords are also searched for possible complement. Consequently, 1,222 papers are selected for further analysis (as of 24 June 2022). I thank Zhichao Wang for helping with harnessing the data to produce these figures (as an RA task).

[3]This figure was constructed with the help of VOSviewer (Van Eck and Waltman, 2010), where more details can be found. In a nutshell, note: (i) each dot represents a paper, and the size of the dot (and its label) reflects the frequency of the corresponding paper being cited by the papers in the pool; (ii) smaller dots are omitted in the network (to avoid congestion that spoils visualization), while the more prominent

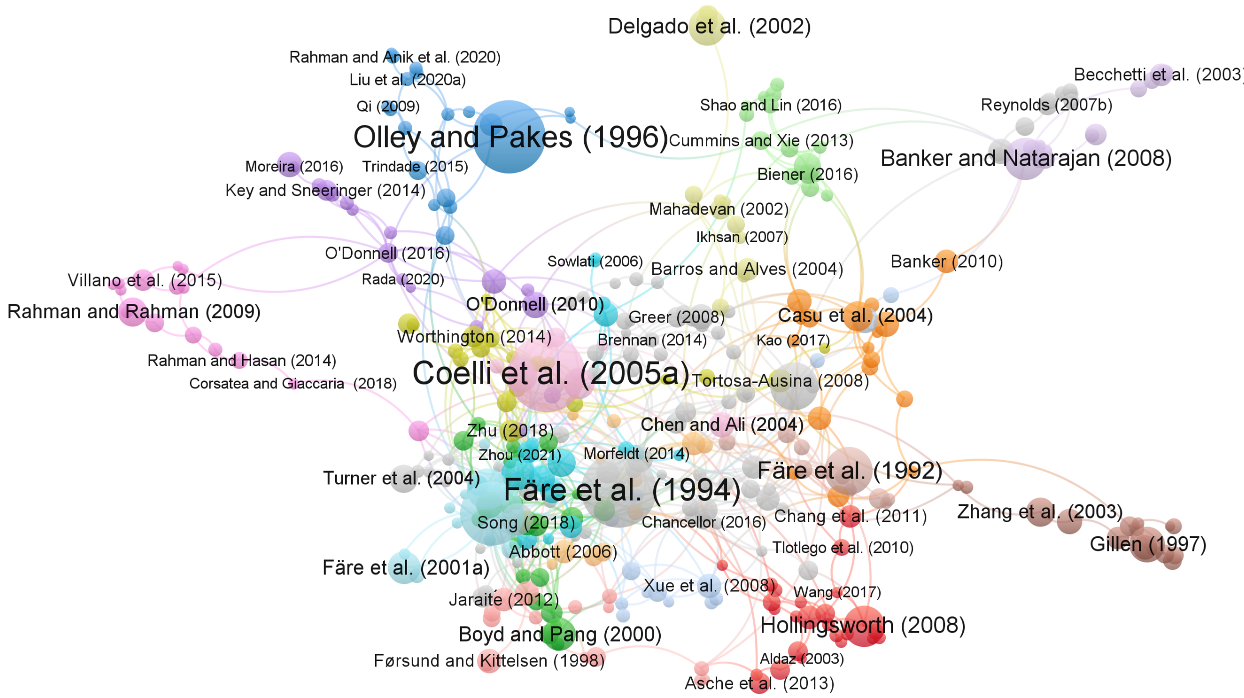

Figure 2: Citation network for the 'productivity' topic.

A closely related picture is seen in Figure 3, which illustrates the co-citation network of the key authors on 'productivity' as a key topic, for papers co-cited by multiple studies from the pool.[4]

While still imperfectly, these three figures quite vividly provide a glimpse of the research literature on productivity in general and the standing and impact of Färe and Grosskopf in particular.[5] More specifically, note that the most cited paper in Figure 1 is the seminal work of Färe and Grosskopf, co-authored with Norris and Zhang, published in the *American Economic Review* (Färe et al. (1994c)), followed by the seminal work of Olley and Pakes (1996) in *Econometrica.* Meanwhile, note also that the 5^{th}, 6^{th}, 8^{th}, 13^{th}, 17^{th} and 20^{th} in the list also have Färe and Grosskopf among co-authors. Notably, most of these works from Färe and Grosskopf are one way or another related to the Malmquist Productivity Index (MPI). The 8^{th} on the list is their paper, co-authored with Lindgren and Roos (Färe et al. (1992c)),

dots with labels float upward; (iii) moreover, the papers are clustered by the vein of the citation relationship, i.e., the papers citing each other following a similar stream of literature tend to be labeled in the same color.

[4]It is also created with a help of VOSviewer (Van Eck and Waltman, 2010) where more details can be found. Briefly, similarly as in Figure 2, the size of the dot (and of the label) reflects the number of times the author is cited; the authors are also grouped in different colors by their frequency of being co-cited, indicating a closer collaboration relationship.

[5]For a more comprehensive and more detailed (albeit a bit less recent) biblioanalytics study, e.g., see Choi and Oh (2019), which focused on the *Journal of Productivity Analysis*, and Wang and Zelenyuk (2021), which focused on performance analysis of hospitals in Australia and its peer. Both of these studies also suggest about the fundamental impact from the research of Färe and Grosskopf, among others.

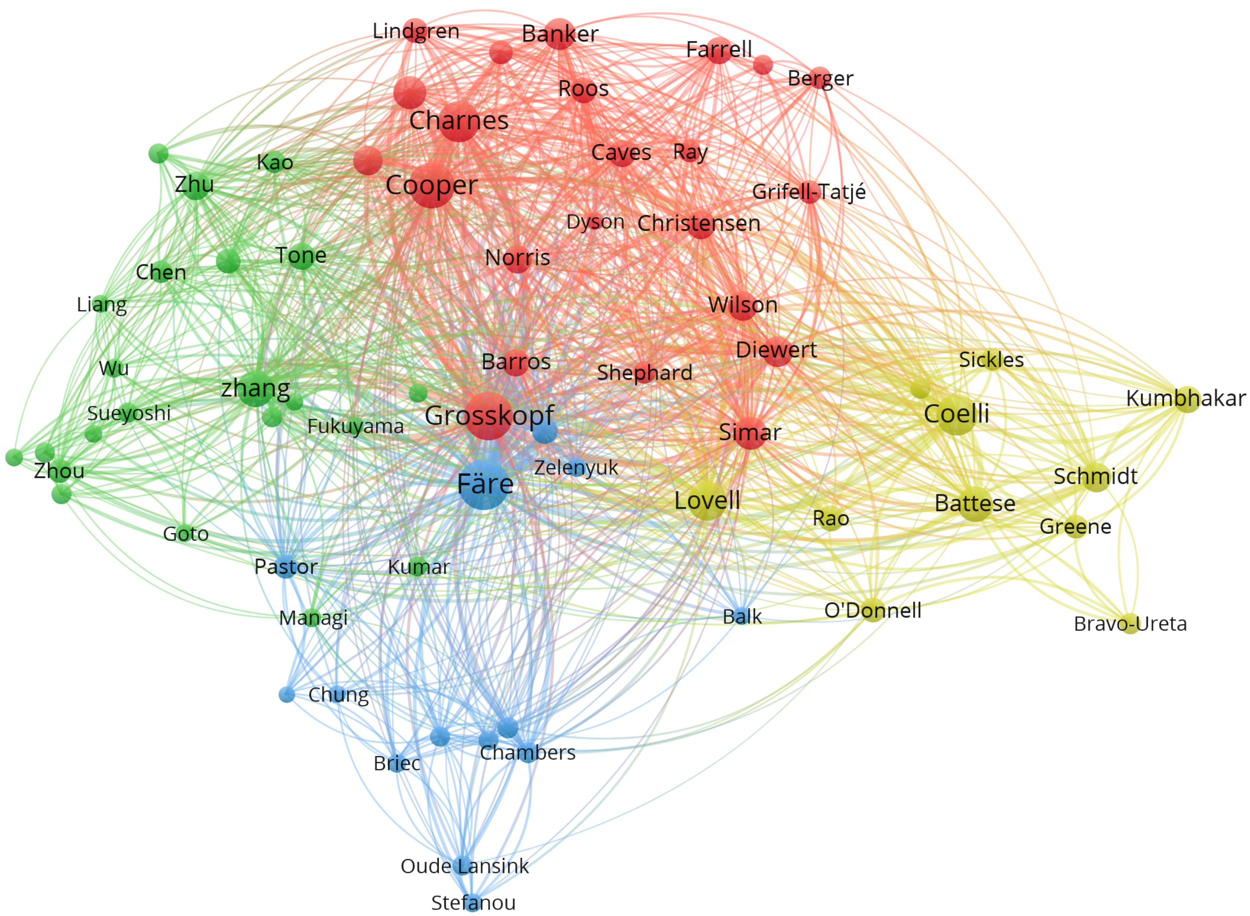

Figure 3: Co-citation network of key authors for the 'productivity' topic.

which is also about MPI for analyzing productivity changes in Swedish pharmacies (from 1980 to 1989). Notably, the latter came before their AER paper and was published in the *Journal of Productivity Analysis (JPA),* the flagship journal for research on productivity and related topics. Among a few others, these JPA and AER articles, are co-responsible for making the MPI (originally proposed in the seminal work of Caves et al. (1982a) in *Econometrica*) the most popular multi-output economic indexes to date. Hence, I will devote more attention to MPIs in this article, yet I will also mention their connection to the simplest productivity measures that are most popular in the general literature.

Before moving on, it is important to clarify that the contributions of Rolf Färe and Shawna Grosskopf go well beyond productivity indexes and include other important areas, both for theory (e.g., duality theory, theory of various efficiency measures, including directional distance function, aggregation theory, modeling bad outputs and of congestion, network DEA, etc.) and for practice (especially performance of healthcare organizations, but also economic growth, agriculture and environmental economics, energy, banking, etc.) Each of those areas deserve a separate review, and so I will leave them for other opportunities, while mentioning them here only briefly.

The structure of the paper is as follows: Section 2 briefly discusses the challenges in measuring productivity, starting from simple to more complex cases, as well as pointing out some apparently forgotten roots. Section 3 expounds on how to measure productivity using the popular 'Malmquist Approach', leveraging on classical works and mentioning some recent developments. Section 4 briefly mentions other fundamental contributions of Färe and Grosskopf, while Section 5 provides concluding remarks, with a brief discussion of likely perspectives for the area of productivity and efficiency analysis.

2 How to Measure Productivity?

The previous section emphasized the importance of productivity in virtually all aspects of life, which in turn implies the importance of managing productivity. And, "you can't manage it if you can't measure it!..." states a famous saying (usually attributed to Peter Drucker), which succinctly emphasizes the importance of why we need to measure productivity. But, how shall one measure productivity? While sounding simple, this question has been challenging scholars for the last few decades (at least) and different disciplines look at it somewhat differently. In economics, it even became a distinct field (e.g., classified by RePEc as the field of Efficiency and Productivity), embracing a myriad of studies already produced and many yet to come. One way or another, they are usually related to either a simple productivity measure like labor productivity or, recognizing the caveats of the latter, they are related to its generalization aimed to address some of those caveats.

2.1 The Simplest Case: 1-input-1-output case

In economics, one usually (although not always) looks at productivity as the amount of outputs per unit of inputs. This is very easily understood in the case of a single output, call it y, produced from a single input, call it x, and so productivity can be formalized simply as the ratio: output over input. Such a measure is sometimes referred to as *single factor productivity* (SFP), which can be applied for any particular time (or state) t simply as,[6]

$$SFP_t \equiv \frac{y_t}{x_t}. \tag{1}$$

The index measuring changes in productivity in such simple environments can then be simply defined as the ratio of SFP in a period (or state) t to itself in a different period (or state) s, i.e.,

$$SFPI_{st} \equiv \frac{SFP_t}{SFP_s} = \frac{y_t/x_t}{y_s/x_s}. \tag{2}$$

Equivalently, it can be re-arranged as the ratio of y_t/y_s, which is a simple output index, to x_t/x_s, which is a simple input index, i.e.,

$$SFPI_{st} = \frac{y_t/y_s}{x_t/x_s} = \frac{\text{Output Index}}{\text{Input Index}}. \tag{3}$$

This latter representation paves a path to many possible generalizations of SFP for multi-output-multi-input contexts, as will be described below.

2.2 A More Realistic Case: Aggregate output relative to one input

The world is, of course, much more complex than 'a single output produced from a single input' scenario, hence requiring some suitable generalization of (3). One natural approach is to generalize (3), by finding some suitable aggregation procedure to replace the single output y_t with some aggregate output and compare it to a selected input, e.g., labor (which may also be an aggregation of various types of inputs, e.g., different types of labor). A very popular example of this is the Labor Productivity (LP) measure. For example, when applied for a country where one uses a measure of gross domestic product (GDP) as an aggregate proxy

[6] Other names for this measure in the literature include 'average productivity', 'partial productivity' and somewhat misleadingly 'marginal productivity'.

of a myriad of outputs and some aggregate measure of labor at time t (call it L_t), we get

$$LP_t \equiv \frac{GDP_t}{L_t}. \tag{4}$$

The related Labor Productivity Index between s and t is then given by,

$$\begin{aligned} LPI_{st} &\equiv \frac{LP_t}{LP_s} = \frac{GDP_t}{L_t} \Big/ \frac{GDP_s}{L_s}. \\ &= \frac{GDP_t}{GDP_s} \Big/ \frac{L_t}{L_s} = \frac{GDP \text{ Index}_{st}}{Labor \text{ Index}_{st}}. \end{aligned} \tag{5}$$

The questions of suitability of aggregations of many outputs into a single output, like GDP, or many very different types (skills, qualities, etc.) of labor into a scalar-valued measure are subjects by themselves. They usually involve index number theory and particular disciplines of interest (e.g., macroeconomics for GDP_t and labor economics for L_t) and often do not have unique ways, rather some standard practices that evolve (with some improvements) over time.

Even if one were to agree on some unique ways for such aggregations, some important questions of measuring productivity remain. Indeed, *even if there is an agreement on an aggregate output measure, the SFPI measure* (3) *(and its example* (5)*) only accounts for productivity relative to a single input.* A natural question arises: *Can we have a multi-factor productivity measure that accounts for other factors?* The answer is, of course: Yes, we can; yet there are many ways to do it and they may lead to different conclusions! Indeed, as concisely concluded in a methodical work by Diewert (1992b, p.163):

> "The results ... appear to be encouraging from the viewpoint of measuring productivity change: in the one input, one output case, productivity change can indeed be accurately measured. However, as soon as we move to the many output, many input case, the situation is no longer clear cut. Different approaches to productivity measurement can give very different numerical answers."

Therefore, as it often happens in virtually any disciplines, especially in social sciences, the main challenge is to agree on which way to choose. And as usual, there is no panacea-type solution to this question, and no 'ideal' or 'proper' way to do it, despite all the catchy nicknames that were given to such approaches over the history of thought on this topic. Indeed, all approaches have pros and cons, with some caveats attached, which may be more or less critical depending on the contexts and aims of measurement.

2.3 On the Forgotten Roots of Measuring Productivity and its Decomposition

Zvi Griliches, besides his fundamental contributions to economic measurement, provided a nice historical account of how the related literature developed and, in particular, how a closely related concept of the Solow's residual came around (Griliches (1996)).[7] He conjectured that the productivity measurement, based on (1) and (2), goes back (at least) to the works of Copeland (1937) and Copeland and Martin (1938). These two works were in the context of national income analysis at the National Bureau of Economic Research (NBER), the major think tank that was at the forefront of many developments in economics. Interestingly, the paper of Copeland (1937) also included interesting discussions, one of which was by Simon Kuznets himself (who later became a Nobel Laureate, in 1971), where a major focus was on productivity, followed by Copeland's response. Meanwhile, the paper of Copeland and Martin (1938) also included some interesting discussions, one of which was by Milton Friedman himself (who was a student of Kuznets, just 26 year old at that time, who later also became a Nobel Laureate, in 1976). A major focus of those discussions was on productivity and efficiency, their changes and the biases that arise in their measurements. Among the many enlightening thoughts expressed in those discussions and replies, Milton Friedman's seems particularly insightful, as it anticipated and perhaps inspired a wave of research for the next few decades:

> "... Obviously, input is valued only for the output it makes possible. Hence the only way by which the volume of input can be measured is in terms of the volume of output. Were the analysis to stop at this point it would seem as if there were but a single problem—the measurement of 'real output'. We can, however, go somewhat farther, and ask the question—to what extent is the change in output over some specified period a result of a change in the quantity of the available resources, and to what extent does it result from a change in the way in which these resources are employed.[8] In order to answer this question it would be necessary to determine the volume of 'real output' that would have been produced had techniques remained unchanged. A comparison of this series with the actual 'real output' then provides a measure of the change in efficiency. ..." (p.127)

That is, back in the 1930s, inspired by Copeland and Martin (1938), Milton Friedman quite nicely (and perhaps was the first who) conceptualized the idea that productivity change can

[7] Also see Hulten (2001) for the related discussions.

[8] Here, Friedman also adds a footnote clarification: "This separation is to a considerable extent artificial: technological change affects not only the way in which resources are employed but also the quantity and character of the resource; themselves."

potentially be decomposed into efficiency change and technology change (and noted on the caveats and challenges involved)—the topic of many key papers in our field in the last few decades. It is also quite interesting to observe from today's perspective, how in the big minds of the profession entertained with the concepts of productivity and efficiency. Also as interesting is the conclusion by Griliches (1996):

> "At this point the gauntlet had been thrown: even though it had been named "efficiency," "technical change," or most accurately a "measure of our ignorance," much of observed economic growth remained unexplained. It was now the turn of the explainers." (p. 1329)

In hindsight, it seems the many works of Rolf Färe and Shawna Grosskopf, as well as those of many other researchers studying productivity, are by and large in response to this gauntlet throw.

2.4 Growth Accounting Approach

The first fundamental breakthrough in measurement of productivity dynamics was due to Robert M. Solow (who also became Nobel Laureate, in 1987), due to his seminal Solow (1957) work, which followed shortly after his other fundamental economic growth theory article, Solow (1956). While the approach he started, usually referred to as growth accounting, has been described in many books, at least a brief discussion of this fundamental approach is well-warranted here.[9]

The basic growth accounting approach typically starts by assuming that technology can be characterized via some simple production function, usually with the so-called Hicks-neutral technology change property, e.g., formally defined as:

$$y_\tau = f_\tau(x) \quad = \quad \mathcal{A}(\tau) \times f(x), \quad \forall x \in \Re^N_+. \tag{6}$$

i.e., the production function is separable into a function characterizing the transformation of inputs into outputs (which is not depending on time) and a residual part, $\mathcal{A}(\tau)$, characterizing technology changes over time (which is not depending on inputs).

Such a simple (and quite restrictive) assumption on technology helps simplifying the measurement of productivity and its changes. Indeed, the simple measure (2) then turns into

[9] We follow Sickles and Zelenyuk (2019) and Zelenyuk (2021), where more details and references can be found.

the following productivity index:

$$\begin{aligned} PI_{st} &= \frac{f_t(x_t)/x_{lt}}{f_s(x_s)/x_{ls}} \\ &= \frac{f(x_t)/x_{lt}}{f(x_s)/x_{ls}} \times \frac{\mathcal{A}(t)}{\mathcal{A}(s)}, \end{aligned} \tag{7}$$

which, note, gives the decomposition of this simple productivity index into changes due to the contributions from all inputs (normalized by one of the inputs) and technology change, $\mathcal{A}(t)/\mathcal{A}(s)$.

Meanwhile, assuming differentiability of f, and given (6), one can represent the growth in output y_t as (Solow (1957)):

$$\mathfrak{g}(y_t) \;=\; \sum_{j=1}^{N} e_{jt}\mathfrak{g}(x_{jt}) + \mathfrak{g}(\mathcal{A}(t)), \tag{8}$$

where $\mathfrak{g}(z_t) := (dz_t/dt)/z_t = d\ln(z_t)/dt \approx (z_{t+\triangle t}/z_t) - 1$, i.e., the growth rate of z_t and $e_{jt} \equiv (\partial f(x_t)/\partial x_{jt}) \times (x_{jt}/f(x_t))$ is the *partial scale elasticity* with respect to input j. If, in addition, one also assumes that technology exhibits constant returns to scale, then

$$\mathfrak{g}(y_t/x_{lt}) \;=\; \sum_{j=1, j\neq l}^{N} e_{jt}\mathfrak{g}(x_{jt}/x_{lt}) + \mathfrak{g}(\mathcal{A}(t)). \tag{9}$$

Hence, equation (9) is a logarithmic version of (7), where due to differentiability of f, the first component in (7) is now being decomposed further, into 'partial productivities' corresponding to each input.

Thus, under the simplifying (though quite restrictive) assumptions, the productivity index (7) can be decomposed into contributions from changes in each input (weighted by their shares that reflect their importance in the production process) and changes in technology. Importantly, the latter component is isolated only as a residual, often referred to as 'Solow residual' and therefore it may hide other sources of change that are not related to the technology change. Besides this important caveat, this approach also has other limitations. One of them is the assumption of a scalar-valued output measure (usually implemented via an aggregation). Another one is the requirement to know the partial scale elasticity corresponding to every element of x. Yet another caveat is the assumption of no inefficiency (i.e., everyone is assumed to be fully efficient), which is an overly optimistic belief that may substantially distort the representation of reality and imply conclusions or policy implications that are very different than would be otherwise. Finally, another serious caveat is the assumption of

Hicks-neutral-type technological change. E.g., in the words of Sickles and Zelenyuk (2019, p.101):

> "Geometrically, this assumption requires that the technology shifts the input-isoquant in a "parallel" fashion. Intuitively, it means that from a technological or engineering point of view, the importance of various types of inputs does not change over time. ... In practice, obviously, technological changes are likely to be biased towards some inputs, e.g., towards the physical and human capital, as progress goes on. Indeed, most production processes in the old days were very labor-intensive and technological progress was making it less and less intensive, and differently for different industries. ... In more recent days, technological process is using much more information and communication technologies (ICT) than before, which is another example of technological bias, etc. Thus, it might be very desirable to have a measure that does not restrict technology to be Hicks-neutral and thus allow a researcher to measure the direction and the size of the bias in the technological change as well as to test for its statistical significance." [10]

While very simple in hindsight, this approach was revolutionary at that stage of knowledge and for many years it remained as the main or at least one of the main methods in applied productivity research. In a sense, many other approaches of measuring productivity dynamics can also be viewed as either generalizations (or as special cases) of this approach. For example, the generalization to a multi-output case was proposed in the seminal work of Jorgenson and Griliches (1967).[11] Also, if one allows for inefficiency at a period τ, characterized by some $e_\tau \in (0, 1]$, implying in this framework that $y_\tau = e_\tau \times f_\tau(x)$, then we get

$$PI_{st} \quad = \quad \frac{f(x_t)/x_{lt}}{f(x_s)/x_{ls}} \times \frac{\mathcal{A}(t)}{\mathcal{A}(s)} \times \frac{e_t}{e_s}, \tag{10}$$

where, the last component of this decomposition is an index measuring efficiency change and is, perhaps, what Milton Friedman had in mind when discussing the work of Copeland and Martin (1938), as quoted above. This decomposition was further generalized by Färe et al. (1994c) and Kumar and Russell (2002), to mention a few as will be discussed in Section 3.

Currently, one can roughly distinguish five major approaches for productivity analysis:[12]

[10] For related discussions, see Jorgenson (2002), Greenwood et al. (1997), Acemoglu (2002) and references therein.

[11] Also see Jorgenson (2002) and references therein.

[12] Of course, this classification is subjective and I recognize that there are potentially many different ways to classify this literature, especially because the different approaches are often interrelated (explicitly or implicitly).

(i) the *growth accounting* approach, starting with the basic Solow's approach (Solow (1957)) and including various modifications (e.g., see Jorgenson and Griliches (1967), Jorgenson and Nishimizu (1978), Jorgenson and Fraumeni (1983), Jorgenson et al. (1987), Jorgenson (1996), Jorgenson (2017)).

(ii) the *productive efficiency* approach, substantially impacted by the seminal works of Farrell (1957),Afriat (1972), Charnes et al. (1978) and Aigner et al. (1977), and advanced further in many other works, within economics/econometrics and operations research literatures. This approach includes the wide literatures on such major methods as Data Envelopment Analysis (DEA) and Stochastic Frontier Analysis (SFA), as well as their various alternatives and related methods.[13]

(iii) the *Olley-Pakes* approach, started by the seminal work of Olley and Pakes (1996) and was elaborated on further in many other works, including Levinsohn and Petrin (2003), Ackerberg et al. (2007), Ackerberg et al. (2015) and Melitz and Polanec (2015), to mention a few.

(iv) the *index number* approach—a classic approach where, e.g., Laspeyres, Paasche, Fisher or Tornqvist indexes are used for obtaining output index and input index, and their ratio would then give, respectively, Laspeyres, Paasche, Fisher or Tornqvist indexes of productivity.

(v) the '*Malmquist approach*'—an approach that originated from the seminal work of Caves et al. (1982a) and further elaborated on by many other works that theoretically connected the index number approach with the neoclassical economic theory, hence providing theoretical justification for the former.[14] This approach is also closely connected to and enriched by the productive efficiency approach. Noteworthy, Färe and Grosskopf (with various co-authors) appear to be the first to establish this connection, by showing how DEA can be used for the estimation of such indexes. This approach will be the main focus in this paper.

2.5 The Complexity of Productivity Measurement: An Intuitive Explanation

The complexity of measuring productivity was well-explained by Moorsteen (1961), illustrating it with multi-output examples. Importantly, a similar complexity remains even in the single-output case (and even under often unrealistic assumptions of full efficiency). This can

[13]E.g., see Chapters 8 through16 in Sickles and Zelenyuk (2019) for extensive discussions and many references.

[14]Also see Caves et al. (1982b); Diewert and Morrison (1986); Diewert and Wales (1987); Diewert (1992a,b), to mention just a few, and an extensive discussion in chapters 4 and 7 of Sickles and Zelenyuk (2019) with many references therein.)

be visually sensed from Figure 4, reproduced from Färe and Zelenyuk (2021), which uses basic microeconomics concepts of isoquants and isocosts.

Observing Figure 4, one can see that even in a single output case, and with only two inputs, there is quite an ambiguity as to how productivity should really be measured. Also note that the problem is actually harder than in the context of consumer theory where a lot of index number approaches were developed (mainly to measure price changes or inflation). The complexity is indeed three-fold: while for the consumer context the changes occur in one product space, in the production context the changes occur in (i) the input space, (ii) the output-space, and (iii) in the function (or more generally a correspondence) describing the interaction between these two spaces for a firm of interest (which may also differ substantially across firms). The change in the latter is what is referred to as the technological change; and, unlike in the consumer theory where the preferences are typically (and quite naively) assumed as fixed over time, assuming no change in technology over a substantial period of time is too contradictory to reality.

Indeed, as can be seen from Figure 4, just focusing on the input-space, there is an ambiguity of choice of measurement. For example, one can choose to measure the change via a distance between input *isoquants*, which can be either along the ray $0B'$ (which goes through the base-period mix of inputs) or along the ray $0B$ (which goes through the current-period mix of inputs). The difference between the two options might be enormous, as can be seen from the figure.

Alternatively, one can choose to measure the change via a distance between *isocosts*, where again the measurement can be either along the ray $0B'$ or along the ray $0B$ and, the difference between the two options might be also enormous. Even if one agrees to, say, the base-period mix perspective or the current-period mix perspective, the difference between measuring the distances between the isoquants can imply very different conclusions than measuring the distances between the isocosts. Figure 4 also hints that to have the equivalence of these alternative approaches to measurement of productivity one would need to have a very peculiar type of technology where the distances from one isoquant to another are invariant to the location on the isoquant—this will be clarified more formally in the next section. Moreover, some restrictions on the allocative efficiencies are needed for the isocost-based measurement to be equivalent with the isoquant-based measurement. In the case of multiple outputs, similar complexity occurs in the output space, where one needs to decide between different output-isoquants and isorevenues (e.g., see Moorsteen (1961)).

In practice, the isocost and isorevenue approaches are often implemented via the so-called 'statistical approach to index numbers', where one deploys some of the well-studied indexes to represent aggregate changes in inputs and aggregate changes in outputs in (3). Specifically,

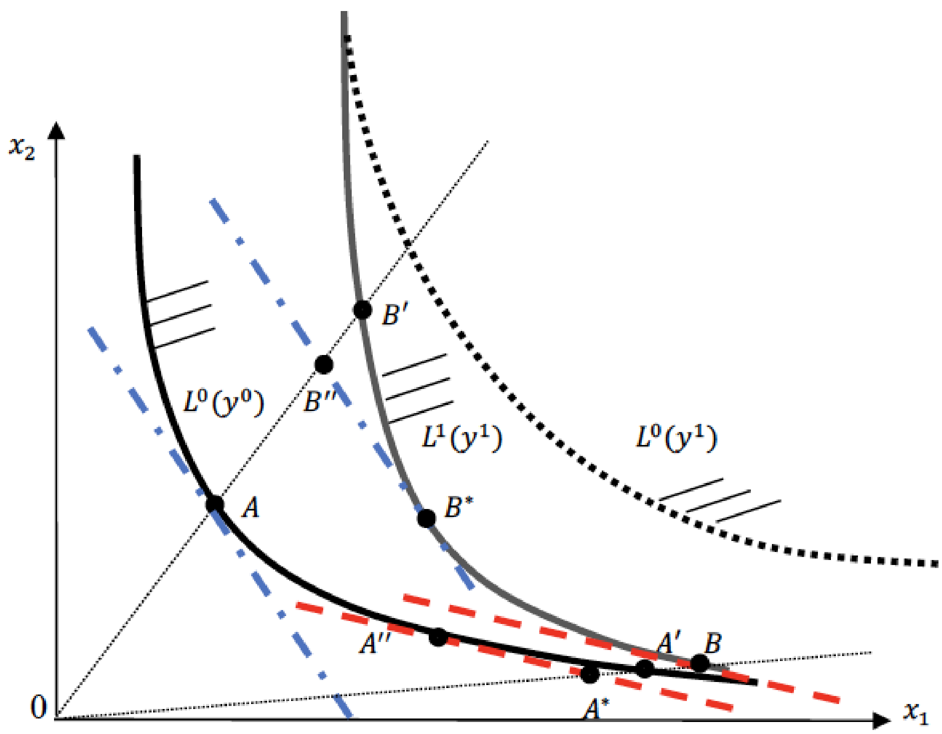

Figure 4: Complexity of Measuring Productivity Changes: An Illustration. (Färe and Zelenyuk (2021, JPA)).

one may use the Laspeyres indexes (i.e., aggregate quantities with base-period price weights) for both, in which case one would get the Laspeyres productivity index. Alternatively, one may use the Paasche indexes (i.e., aggregate quantities with current-period price weights) for both, in which case one would get the Paasche productivity index. Which one to prefer is a matter of taste and is somewhat analogous to the eternal differences between generations: a younger generation tends to look through the lens of current days (as this is what they understand best, perhaps), while an older generation tends to value relative to some 'olden days' (e.g., when that generation was young and joyful).

A way to reconcile the different perspectives is to take some middle ground. One way to do so here is to use what Irving Fisher advocated in the context of price indexes—to take the simple (equally-weighted) geometric average of these Laspeyres and Paasche indexes. This gives the Fisher Productivity Index, e.g., theoretically justified and advocated by Diewert (1992a). Another middle ground is the Törnquist Productivity Index, e.g., theoretically justified and advocated by Caves et al. (1982a). Yet another alternative is the Walsh Productivity Index. [15]

3 The 'Malmquist Approach'

Since the seminal work of Caves et al. (1982a), part of the productivity literature was taken over by what sometimes is broadly referred to as the 'Malmquist Approach.' Roughly, it can be defined as one that embraces the productivity indexes proposed in Caves et al. (1982a) and their variations.[16] We dedicate this section to this important approach, briefly considering some fundamentals and some recent advances.

3.1 Multidimensional Characterization of Technology

To facilitate further discussions, let $x = (x_1, \ldots, x_N)' \in \mathbb{R}^N_+$ represent a vector of inputs for producing a vector of outputs $y = (y_1, \ldots, y_M)' \in \mathbb{R}^M_+$ and suppose the production technology at time t is characterized by the set

$$\Psi^t \;=\; \{(x, y) \,:\, x \text{ can produce } y \text{ at time } t\}. \tag{11}$$

[15] More recently, Mizobuchi and Zelenyuk (2021), inspired by works of Diewert, Färe and Grosskopf among others, developed a generalized framework, based on the *quadratic mean of order-r*, which embraces many other indexes as special cases, and justifies them on some theoretical grounds.

[16] Interestingly, this approach is named in honor of Sten Malmquist, a Swedish economist who considered related ideas about indexes for consumer theory contexts (Malmquist (1953)). In hindsight, considering the contributions to this approach at the very start and over the past four decades, a more appropriate name could be the 'Diewert Approach' or, since it involves Shephard's distance functions, the 'Diewert-Shephard Approach'.

Note that, in principle, N and M can be very large, i.e., allowing for the so-called big-wide data. It is important, however, to assume that Ψ^t satisfies certain regularity conditions to ensure complete characterization of technology via functions, e.g., distance functions that became very popular in productivity and efficiency analysis.[17] To be brief, I will focus on the t-period Shephard's input distance function, defined with respect to Ψ^t,

$$D_i^t(y,x) \;=\; \sup_{\beta}\{\beta > 0 \,:\, (x/\beta, y) \in \Psi^t\}, \tag{12}$$

and the t-period Shephard's output distance function defined with respect to Ψ^t is given by

$$D_o^t(x,y) \;=\; \inf_{\beta}\{\beta > 0 \,:\, (x, y/\beta) \in \Psi^t\}. \tag{13}$$

3.2 Quantity and Productivity Indexes

Following Färe and Zelenyuk (2021), who in turn followed Caves et al. (1982a) and Diewert (1992a) among others, the t-period Malmquist input quantity index is defined as

$$X(x^0, x^1, y|\Psi^t) \;=\; D_i^t(y, x^1)/D_i^t(y, x^0). \tag{14}$$

Note that the index depends on the reference level of output, y, that is kept fixed in this formulae. The choice of this reference, while it appears simple, is the crux of the whole index number theory, which goes back several centuries, to at least the dilemma between proposals of Laspeyres and Paasche and their reconciliation by Fisher, and various debates after that.

While in principle any y can be chosen for this index, it is important that, intuitively speaking, this reference is 'as relevant as possible' to the environment that the DMU in question faced when making the decision to use x^0 and x^1, while facing technology Ψ^t ($t \in \{0,1\}$). Thus, quite naturally (albeit not without caveats), the usual references for comparing x^0 and x^1 are y^0 and y^1, which give Laspeyres and Paasche versions of the index, respectively,

$$X_L(x^0, x^1, y^0|\Psi^0) \;=\; D_i^0(y^0, x^1)/D_i^0(y^0, x^0), \tag{15}$$

and

$$X_P(x^0, x^1, y^1|\Psi^1) \;=\; D_i^1(y^1, x^1)/D_i^1(y^1, x^0). \tag{16}$$

[17] See Shephard (1953), Färe and Primont (1995), and Sickles and Zelenyuk (2019).

Similarly, t-period Malmquist output quantity index can be defined as

$$Y(y^0, y^1, x|\Psi^t) \quad = \quad D_o^t(x, y^1)/D_o^t(x, y^0) \tag{17}$$

i.e., this index depends on the reference level of input, x, that is kept fixed in these formula. And, again, while in principle any x can be chosen for this index, it is imperative that this reference is 'as relevant as possible' to the environment that the DMU in question faced when making the decisions to produce y^0 and y^1, while facing technology Ψ^0 or Ψ^1. Again, quite naturally (yet not without caveats), the usual references for comparing y^0 and y^1 are x^0 and x^1, which give Laspeyres and Paasche versions of output index, respectively,

$$Y_L(y^0, y^1, x^0|\Psi^0) \quad = \quad D_o^0(x^0, y^1)/D_o^0(x^0, y^0), \tag{18}$$

and

$$Y_P(y^0, y^1, x^1|\Psi^1) \quad = \quad D_o^1(x^1, y^1)/D_o^1(x^1, y^0). \tag{19}$$

In turn, the quantity indexes defined above can be used for constructing a multi-factor productivity index, where, again, we get the Laspeyres version

$$DBPI(x^0, x^1, y^0, y^1|\Psi^0) \equiv \frac{Y_L(y^0, y^1, x^0, \Psi^0)}{X_L(x^0, x^1, y^0, \Psi^0)} = \frac{D_o^0(x^0, y^1)/D_o^0(x^0, y^0)}{D_i^0(y^0, x^1)/D_i^0(y^0, x^0)}, \tag{20}$$

and the Paasche version,

$$DBPI(x^0, x^1, y^0, y^1|\Psi^1) \equiv \frac{Y_P(y^0, y^1, x^1, \Psi^1)}{X_P(x^0, x^1, y^1, \Psi^1)} = \frac{D_o^1(x^1, y^1)/D_o^1(x^1, y^0)}{D_i^1(y^1, x^1)/D_i^1(y^1, x^0)}. \tag{21}$$

And, to reconcile, the geometric mean of the Laspeyres and Paasche perspectives is usually taken, i.e.,

$$\begin{aligned} &DBPI(x^0, x^1, y^0, y^1|\Psi^0, \Psi^1) \equiv \\ &\quad \left[DBPI(x^0, x^1, y^0, y^1|\Psi^0) \times DBPI(x^0, x^1, y^0, y^1|\Psi^1)\right]^{\frac{1}{2}}. \end{aligned} \tag{22}$$

Common names for (22) appear to be 'Hicks–Moorsteen Productivity index' or HMPI (due to Diewert (1992a)) and 'Malmquist TFP index' (due to Bjurek (1996)). However, as pointed out by Färe and Zelenyuk (2021), this index "emerged under different names at different waves of the literature and, perhaps, the name 'Malmquist–Shephard–Hicks–Moorsteen–Diewert–Bjurek productivity index' would probably be the fairest". To make it simpler, Färe and Zelenyuk (2021) referred to it as the Diewert–Bjurek productivity index (DBPI), which was "to credit the two authors who appear to have contributed the most to the origin of this interesting index.[18]

In a nutshell, the HMPI/DBPI approach compares changes in outputs (inputs), somewhat separately from changes in inputs (outputs), in the sense of fixing the latter (and the technology) at either 0 or 1, to obtain a 'separate' output and input quantity indexes and then use them in a ratio form, as in (3). This seems natural, if one thinks 'inside the box' of the formula (3), which is a somewhat narrow way to look at productivity. Indeed, while very natural for a single-input-single-output case, this 'output over input' approach implicitly imposes a type of '*separability*' or '*de-coupling*' of inputs from outputs, which may or may not be justified in practice.[19] Indeed, note that x and y come into realization together, in the sense that x^0 (and not x^1 or any other x) was chosen to produce y^0, while x^1 (and not x^0 or any other x) was chosen to produce y^1 and so it is desirable that (x^0, y^0) is considered as a whole and compared to its analogue, (x^1, y^1) as a whole, in the other period (or state). This way of thinking seems natural for many now, yet it was revolutionary (in the sense of going beyond the narrow definition of 'output over input' for productivity) at the time it was proposed by Caves et al. (1982a). Specifically, in the now seminal *Econometrica* paper, Caves et al. (1982a) introduced a new type of productivity measure that they called the Malmquist Productivity Index (MPI), although perhaps a more appropriate name would be the 'CCD-approach' (crediting the authors), as it is sometimes referred to. Specifically, this approach encompasses several indexes: in the output oriented context, we have

$$MPI_o(x^0, x^1, y^0, y^1|\Psi^0) \equiv \frac{D_o^0(x^1, y^1)}{D_o^0(x^0, y^0)}, \tag{23}$$

and

$$MPI_o(x^0, x^1, y^0, y^1|\Psi^1) \equiv \frac{D_o^1(x^1, y^1)}{D_o^1(x^0, y^0)} \tag{24}$$

[18]There are also other variants of this index in the literature. E.g., one of them is the so-called Färe-Primont Productivity Index (O'Donnell (2014); O'Donnell (2018)), which is essentially a restricted version of HMPI that fixes the base of the measurement to satisfy some restrictive properties. (Also, see Färe and Zelenyuk (2021) for related discussions about fixing the base of measurement.)

[19]As pointed out by one of the anonymous referees, this point of view deserves more discussion, e.g., on whether this '*separability*' or '*de-coupling*' phenomenon has implications in the context of other indexes, e.g., the empirical indexes.

and, to reconcile between these, a simple geometric mean is taken, i.e.,

$$\begin{aligned} MPI_o(x^0,x^1,y^0,y^1|\Psi^0,\Psi^1) &\equiv \left[MPI_o(x^0,x^1,y^0,y^1|\Psi^0) \times MPI_o(x^0,x^1,y^0,y^1|\Psi^1)\right]^{\frac{1}{2}} \\ &\equiv \left[\frac{D_o^0(x^1,y^1)}{D_o^0(x^0,y^0)} \times \frac{D_o^1(x^1,y^1)}{D_o^1(x^0,y^0)}\right]^{\frac{1}{2}}. \end{aligned} \tag{25}$$

Analogously, in the input oriented context, we have

$$MPI_i(x^0,x^1,y^0,y^1|\Psi^0) \equiv \frac{D_i^0(y^1,x^1)}{D_i^0(y^0,x^0)}, \tag{26}$$

and

$$MPI_i(x^0,x^1,y^0,y^1|\Psi^1) \equiv \frac{D_i^1(y^1,x^1)}{D_i^0(y^0,x^0)}, \tag{27}$$

and their geometric mean is then taken to reconcile, i.e.,

$$\begin{aligned} MPI_i(x^0,x^1,y^0,y^1|\Psi^0,\Psi^1) &\equiv \left[MPI_i(x^0,x^1,y^0,y^1|\Psi^0) \times MPI_i(x^0,x^1,y^0,y^1|\Psi^1)\right]^{\frac{1}{2}} \\ &\equiv \left[\frac{D_i^0(y^1,x^1)}{D_i^0(y^0,x^0)} \times \frac{D_i^1(y^1,x^1)}{D_i^1(y^0,x^0)}\right]^{\frac{1}{2}}. \end{aligned} \tag{28}$$

While proposed and theoretically justified by Caves et al. (1982a), it is good to note that the work by Färe and Grosskopf with various co-authors was also fundamentally important, especially for the applied world, as they were the first to show how DEA can be used for the estimation of such indexes. Among others, their work also influenced many applications for various industries or cross-countries studies,[20] as well as further theoretical developments of Malmquist-index-like indexes.[21]

The seminal work of Färe, Grosskopf, Norris and Zhang (1994c) also inspired the stream of literature exploring the productivity dynamics of countries using Penn World Tables (PWT) via DEA, which is briefly described in the next sub-section.[22]

3.3 Decompositions of Productivity Indexes

A particular feature of MPI (and other similar measures) is the possibility of various decompositions. (Recall the paper of Copeland and Martin (1938) and especially its discussion by

[20]E.g., see Färe et al. (1998) and Badunenko et al. (2017) for some reviews, although an update for these seems to be overdue.

[21]E.g., see Berg et al. (1992), Shestalova (2003), Pastor and Lovell (2005), Pastor et al. (2011), Afsharian and Ahn (2015).

[22]In turn, this inspired the related literature that used Stochastic Frontier Analysis (SFA) for estimation MPI (see Badunenko et al. (2017) for related references, and for a recent review of SFA, see Kumbhakar et al. (2022a,b)).

Milton Friedman, briefly mentioned in Section 2.1 above.) Although not the first, and not the only, the most popular of such decompositions appears to be the one usually attributed to Färe et al. (1994c), which breaks down the MPI into efficiency change and technical change as follows:

$$\begin{aligned} MPI_o(x^0, x^1, y^0, y^1|\Psi^0, \Psi^1) =& \frac{D_o^1(x^1, y^1)}{D_o^0(x^0, y^0)} \times \left[\frac{D_o^0(x^0, y^0)}{D_o^1(x^0, y^0)} \times \frac{D_o^0(x^1, y^1)}{D_o^1(x^1, y^1)}\right]^{\frac{1}{2}}, \\ \equiv& \quad \mathit{Eff}\Delta_{01} \times \quad \mathit{Tech}\Delta_{01}, \end{aligned} \tag{29}$$

where $\mathit{Eff}\Delta_{01}$ denotes the efficiency change between period 0 and 1, while $\mathit{Tech}\Delta_{01}$ denotes the technical change between period 0 and period 1.[23]

Another very interesting decomposition that connects the simple productivity indexes discussed in Section 2 with MPI was proposed in a follow-up to Färe et al. (1994c) by Kumar and Russell (2002). In particular, they noted that in a scalar-valued–output case, when technology in a period t can be characterized by some production function $f^t(K^t, L^t)$ satisfying constant returns to scale, the labor productivity index can be described as:

$$\begin{aligned} LPI_{01} \equiv \frac{y^1/L^1}{y^0/L^0} =& \frac{D_o^1(K^1, L^1, y^1)}{D_o^0(K^0, L^0, y^0)} \times \left[\frac{f^1(K^0/L^0)}{f^0(K^0/L^0)} \times \frac{f^1(K^1/L^1)}{f^0(K^1/L^1)}\right]^{1/2} \\ & \times \left[\frac{f^0(K^1/L^1)}{f^0(K^0/L^0)} \times \frac{f^1(K^1/L^1)}{f^1(K^0/L^0)}\right]^{1/2} \\ =& \quad \mathit{Eff}\Delta_{01} \times \quad \mathit{Tech}\Delta_{01} \quad \times \quad KLACC\Delta_{01}, \end{aligned} \tag{30}$$

i.e., it can be decomposed into MPI (which can also be decomposed into technical change and efficiency change) and the change in capital accumulation (per unit of labor) between periods 0 and 1, denoted by $KLACC\Delta_{01}$.

Like Färe et al. (1994c), Kumar and Russell (2002) also used PWT and found interesting evidence about the sources of productivity dynamics. This work was further elaborated by Henderson and Russell (2005), who also used PWT but also added a Human Capital component to (30), and Badunenko et al. (2008) who looked at a larger set of countries (including former USSR republics) with more recent PWT, and Badunenko and Romero-Avila (2013) who added a 'financial development' component to (30), among others. Nice

[23] It appears that the first decomposition of MPI goes back to Nishimizu and Page (1982) in a parametric context. As mentioned, many alternative (or further) decompositions of MPI were offered in the literature, e.g, Färe et al. (1997), proposed an interesting decomposition of the technical change component of the MPI that tries to measure the bias of technical change, and its sources (input-biased vs. output biased technological change).

historical accounts of the MPI and its decompositions (along with discussions of theoretical and empirical issues) can be found in, e.g., Färe et al. (1998) and Grosskopf (2003).[24]

It is also worth remembering that for all these and other methods, data plays a key role in what conclusions are reached, e.g., see a recent work of Meng et al. (2022) that showed how some important conclusions from Kumar and Russell (2002) and Henderson and Russell (2005) changed due to the recent update of the data vintage from the PWT.

3.4 Equivalences and Differences

An important question is how are the different indexes related and whether they provide the same or at least approximate information. This important question is at the core of the index number literature and has been studied widely and it would be unfair to miss it here, although we must examine it only briefly.[25]

While there are a lot of technical details, the essence of the matter (as pointed out by Färe and Zelenyuk (2021)) can be seen by noting that we can have:

$$X_L(x^0, x^1, y^0, \Psi^0) \;=\; X_P(x^0, x^1, y^1, \Psi^1) \tag{31}$$

if and only if we have

$$D_i^1(y^1, x^1) \;=\; f^1(y^1)h(x^1) \tag{32}$$

and

$$D_i^0(y^0, x^0) \;=\; f^0(y^0)h(x^0), \tag{33}$$

where $h(.)$, $f^1(.)$ and $f^0(.)$ are some functions whose properties are inherited from $D_i^1(y, x)$ and $D_i^0(y, x)$.

An analogous situation exists for the output orientation. Intuitively, this means that technology must satisfy various types of Hicks-neutrality and of homotheticity, which are very restrictive assumptions, as well as constant returns to scale or its generalizations. Meanwhile, the equivalence of the input and output oriented MPIs (i.e., (25) and (28)) requires constant

[24] Also see Färe et al. (1994b), Grifell-Tatjé and Lovell (1995, 1999, 2015), Färe et al. (1997), Ray and Desli (1997), Simar and Wilson (1998a), Wheelock and Wilson (1999), Balk (2001), Lovell (2003), Zofio (2007), Fried et al. (2008), Diewert and Fox (2017) and a review in Badunenko et al. (2017) and more references therein.

[25] E.g., see Diewert (1992b,a), Chambers and Färe (1994), Färe and Grosskopf (1996a), Balk et al. (2003), Peyrache (2013) and most recently in Färe et al. (2020) and references therein. A textbook discussion of this topic can also be found in Chapter 4 and 7 in Sickles and Zelenyuk (2019).

returns to scale (e.g., see Berg et al. (1992) and Färe and Grosskopf (1996a)).

Overall, Färe and Zelenyuk (2021) concluded:

> "Because of this complexity, it appears infeasible to establish general conclusions about superiority of any of the indexes we discussed here, be they "true" indexes that are based on economic principles or any empirical indexes. Only under fairly restrictive assumptions about technology and how it changes over time or assumptions about the dynamics as prices is it possible to establish some of the results."

While the indexes discussed above (and many other) were originally designed to compare multidimensional points at any two periods or states, there is often a practical need for multilateral comparisons involving many periods or states. The task of generalizing to multilateral comparisons might look like a simple task, yet it turns out to be not so simple, as can be seen from the debates at various points of the index numbers literature. The problem is indeed more complicated, although the essence is very similar and rooted to what was discussed just above. Indeed, and as pointed out by Färe and Zelenyuk (2021), the requirement of equality of the Laspeyres and Paasche versions generalizes in multilateral context to what is often referred to as the 'transitivity' or *circularity* property of indexes, which may seem 'natural', yet can be also extremely restrictive and misleading in resulting conclusions.

For multi-output-multi-input indexes, this circularity property can sometimes be achieved very easily, just by fixing the base of measurement.[26] However, such imposition of a fixed base (or fixed weight) may also lead to an even more serious problem—a lower relevance and possibly even irrelevance of the chosen base (or some of its parts) to the quantities being measured. This problem is less severe (and less evident) for relatively small periods, and can be taken as an approximation error, yet for long spans it could be dramatic. Indeed, if one takes, for example the last 100 years span—when the world experienced dramatic changes in the types of inputs, the types of outputs and in deployed technologies—the fixed base at the beginning of the period could be quite irrelevant for comparing what happened 100 years after it. Apparently, after realizing such a problem, Irving Fisher himself, who was an advocate of circularity for indexes in his famous Fisher (1911) book (in the context of price indexes), corrected himself about a decade later, acknowledging the serious problems of this property, stating:

> "... for the only definite error which I have found among my former conclusions has to do with the so-called "circular test" which I originally, with other writers, accepted as sound, but which, in this book, I reject as theoretically unsound."

[26] E.g., see O'Donnell (2018) for some examples, related discussions and references therein.

— Fisher (1922, p.xii-xiii)

and also,

> "... The only formulae which conform perfectly to the circular test are index numbers which have *constant weights*, i.e. weights which are the same for all sides of the "triangle" or segments of the "circle," i.e. for every pair of times or places compared. ... But, clearly, constant weighting is not theoretically correct... We cannot justify using the same weights for comparing the price level of 1913, not only with 1914 and 1915, but with 1860, 1776, 1492, and the times of Diocletian, Rameses II, and the Stone Age!"
>
> — Fisher (1922, p. 274-275, original *emphasis* retained).

In his seminal paper in *Econometrica*, Eichhorn (1976) used the functional equations approach to provide a formal proof for this statement (in the context of price indexes), which was refined further by Funke et al. (1979). More specifically, they showed that the only index satisfying the circularity property (along with other relevant properties) is the Cobb-Douglas (i.e., geometric mean) type index that has *fixed weights*. More discussion on this can be found in Färe and Zelenyuk (2021) who provided further theoretical justifications and illustrated their importance with an empirical example (using data from Kumar and Russell (2002)).[27]

Finally, while there might still be different views on this topic, here it is a good place to conclude this section with the thoughts on it from Färe and Grosskopf:

> Since satisfaction of the circular test is in effect asking that productivity and technical change be path independent, one would expect that this would require imposing a lot of structure on the problem. Requiring that technology be ... Hicks neutral is, we believe, extremely restrictive. As a consequence, we find ourselves in agreement with Fisher (1922). We would rather abandon the circular test and allow for the possibility of nonneutral technical change. We find the march of time to be a natural "path" upon which technical change should be allowed to be dependent. (Färe and Grosskopf (1996a, p. 91).)

3.5 Statistical Aspects

It is worth emphasizing here that the Malmquist-type indexes are theoretical concepts, sometimes called 'true' indexes, in the sense that they are based on (primal) characterizations of

[27] Also see related discussions in Konüs and Byushgens (1926), Frisch (1930, 1936), Samuelson and Swamy (1974), Balk and Althin (1996), Coelli et al. (2005), Diewert and Fox (2017) and Sickles and Zelenyuk (2019, p. 130-137).

the 'true' technology sets (via distance functions). Of course, in practice, the true technology sets are not known and must be estimated with some methods, e.g., DEA or SFA. This raises the question of statistical accuracy of such estimations, which in turn raises the question of statistical properties (consistency, convergence rates and limiting distributions). The early attempts on this matter were well reviewed in Grosskopf (1996), while more recent developments were well reviewed in Simar and Wilson (2013) and Simar and Wilson (2015), although some important results were developed later on and are in progress as this is written.

In a nutshell, the first substantial breakthrough on this topic is due to Simar and Wilson (1999) which explained how to bootstrap MPIs (leveraging on Korostelev et al. (1995); Kneip et al. (1998); Simar and Wilson (1998b) among others). More recently, Simar and Wilson (2019); Kneip et al. (2021) derived several new central limit theorems for the sample mean of MPIs (leveraging on Kneip et al. (2015) among others), while Pham et al. (2023) derived new central limit theorems (CLTs) for the more general aggregates of MPIs, which involve economically justified weights in the aggregation (leveraging on Zelenyuk (2006); Simar and Zelenyuk (2018); Kneip et al. (2021) among others). These new developments were needed because when DEA is used to estimate MPIs (or its components), the estimates are consistent and asymptotically unbiased (under certain conditions), yet except for peculiar cases the bias dominates the variance in terms of asymptotic convergence rates. This dominance remains even for averages of MPIs (or its components), which leads to faulty statistical inference via the standard CLT. The new statistical developments in these above-mentioned papers involve jackknife bias correction and theoretical derivations of the new CLTs for such bias corrected estimates. The resulting CLTs then enable practitioners to construct statistical confidence intervals (or interval-estimates) for MPIs and conduct well-grounded statistical hypotheses tests about them.

4 Other Contributions of Färe and Grosskopf

It is important to clarify that the contributions of Rolf Färe and Shawna Grosskopf go well beyond the topic I focused on here (productivity indexes) and include other important areas both for theory and applications. In fact, there are several distinct areas (with sub-areas) that deserve separate papers on each and I will only mention them briefly here.

The first to mention is the *duality theory in economics*—the area thoroughly developed by Ronald W. Shephard (Shephard (1953, 1970)) and other giants of the profession (e.g., see Diewert (1971, 1974); Jorgenson and Lau (1974); Samuelson and Swamy (1974)). Working with Shephard at the University of California at Berkley, Rolf Färe continued advancing this area with others, especially with his old-time friends Robert Chambers, Shawna Grosskopf and Daniel Primont. (e.g. Färe and Primont (1995); Chambers et al. (1996b); Färe and Primont (2006); Färe and Grosskopf (2000)).

A closely related key area is *theory of various efficiency measures*, including *directional distance function*, and some contributions from Färe or Grosskopf here are Färe and Lovell (1978); Chung (1996); Chambers et al. (1998); Färe et al. (2007, 2019) to mention a few.[28]

Another key area is modeling *bad outputs* and *congestion* and their applications in environmental economics—some key works contributions from Färe or Grosskopf here include the pioneering *Econometrica* article by Färe and Svensson (1980) as well as Färe and Grosskopf (1983); Färe et al. (1989b); Chung et al. (1997b); Färe and Grosskopf (2003, 2004), to mention a few, which in turn influenced many others (e.g., Kuosmanen (2005); Kuosmanen and Podinovski (2009); Podinovski and Kuosmanen (2011); Pham and Zelenyuk (2019), to mention a few).

Färe and Grosskopf were also the first to propose the so-called *network DEA* (Färe and Grosskopf (1996a)), which started a new and growing literature (e.g., see Kao (2014) for a review).

Substantial footprints of Färe and Grosskopf can also be found in applied economics literature, i.e., impacting actual practice, both in academia and outside of it (industries and governments). Besides the area of empirical cross-country studies of economic growth and that were substantially impacted by Färe et al. (1994c), their impact is also vivid for the area of performance analysis of healthcare organizations (e.g., see Färe et al. (1989a, 1992c, 1994a); Grosskopf and Valdmanis (1987)), energy and environment (Färe et al. (1983, 2010, 2013); Ball et al. (2015); Bostian et al. (2016)), to mention just a couple.

Last, yet not least, an important area (closely related to virtually all the areas above) where they contributed with various co-authors is *aggregation theory* for efficiency and productivity indexes. Some important contributions here include the works of Färe et al. (1992a); Färe and Zelenyuk (2003); Färe et al. (2004, 2008); Färe and Zelenyuk (2012); Färe and Karagiannis (2014). Their work also substantially influenced the works of many other researchers on this topic (e.g., Li and Ng (1995); Zelenyuk (2006); Simar and Zelenyuk (2007); Mayer and Zelenyuk (2014); Zelenyuk (2015); Karagiannis (2015); Simar and Zelenyuk (2018); Färe and Zelenyuk (2019) and Pham et al. (2023), to mention a few).[29]

[28] Also see related earlier ideas by Allais (1943); Diewert (1983); Luenberger (1992), among others.

[29] Many of these works were extending the aggregation theorem and ideas of Koopmans (1957) as well as ideas of structural efficiency from Farrell (1957) and Førsund and Hjalmarsson (1979).

5 Concluding Remarks

Productivity—*Quo Vadis*? In other words, where will the field of Productivity be going? While I am not an oracle to answer this intriguing for our community question, a few remarks (perhaps totally incorrect) seem worth making here. So, I will try. In my humble opinion, besides continuing to explore and refine the same important points that our community has been exploring for the last 100+ years, a few important and somewhat novel topics are likely to become trendy in the next decade or so.

First of all, attaining a greater synthesis of the five major approaches for productivity analysis (as per classification in the section 2.4) seems to be a natural (or wishful) path for future research endeavors. This seems especially overdue for the case of synthesizing the *Olley-Pakes* approach with the *productive efficiency* approach, the closely related '*Malmquist approach*' and more generally the *index number* approach.

Second, yet as important, more combinations of our usual models and their estimators (in all the five approaches) with the recent developments in *Causal Inference (CI)* seem well-warranted, if not overdue—to better understand causal effects on productivity. Third, and as important, combinations of our usual models and their estimators with *Machine Learning (ML) and Artificial Intelligence (AI).* The aims here could be to automatize the performance (productivity and efficiency) analysis to enable quick and simple use of many alternative models and many alternative estimators of such models and a selection of the most appropriate ones (if any) that fit or explain the data best. A particular sub-topic here is the handling of *big data* and related challenges. The fourth, equally important and perhaps most challenging direction of research is about the combinations of our usual models and their estimators with both CI and ML/AI methods. The way that I see it will possibly be in the future is that companies and government agencies will have performance dashboard apps that will, in real time and instantaneously, enable inference about the performance of a system of interest via any of a wide range of methods we have been (and will be further) developing over decades. (A note for those reading this in a few decades: Currently, it often takes a few researchers performing such analysis over a few months or even years! It also took over a year to write and fine-tune this paper.)

To conclude, productivity analysis is a very important research area and many models and estimators have been developed already, with fundamental contributions from Färe and Grosskopf, among others. As some colleagues pointed out, some of these contributions have generated new streams of literature in our field due to "flaws" (or I would rather say imperfections, typically present in anyone's work) left in those papers that inspired others to try to correct or perfect them. Here, I would respond with the phrase attributed to Albert

Einstein: "*Anyone who has never made a mistake has never tried anything new.*" Indeed, Rolf Färe and Shawna Grosskopf not only tried new concepts, ideas and methods, they also discovered many of them and, to my knowledge and personal experience, appreciated and encouraged others trying to improve upon them. And, importantly, it is this type of spirit of appreciation and encouragement from them and others in our great community (ISEAPA) that, I think, promises that many more interesting advancements are yet to come in our field of research, especially from the younger generation, standing on the shoulders of their teachers and our teachers, where Färe and Grosskopf play a prominent role. So, let's apply these skills and knowledge the best we can to help improve this fundamentally important aspect for an economic development. Let's do so based on solid theories from economics, statistics, operations research, as well as from the new areas of research (CI, ML, AI, among others)—for the benefit of the entire world! After all, and paraphrasing Paul Krugman: productivity (and efficiency) is almost everything...

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Appendix

Table 1: Top cited papers (as in Figure 1)

Research	Citations	Source
Färe et al. (1994c)	2530	American Economic Review
Olley and Pakes (1996)	2325	Econometrica
Yachi and Loreau (1999)	1753	Proceedings of the National Academy of Sciences of the USA
Javorcik (2004)	1563	American Economic Review
Chung et al. (1997a)	1558	Journal of Environmental Management
Färe et al. (1989b)	1226	Review of Economics and Statistics
Siegel et al. (2003)	964	Research Policy
Färe et al. (1992b)	701	Journal of Productivity Analysis
Emrouznejad et al. (2008)	697	Socio-Economic Planning Sciences
Banker and Natarajan (2008)	479	Operations Research
Hailu and Veeman (2001)	445	American Journal of Agricultural Economics
Hollingsworth (2008)	445	Health Economics
Färe and Grosskopf (1996b)	383	Economics Letters
Delgado et al. (2002)	346	Journal of International Economics
Gillen and Lall (1997)	312	Transportation Research Part E: Logistics and Transportation Review
Coelli and Rao (2005)	247	Agricultural Economics
Färe et al. (2001)	241	Journal of Regional Science
Li and Lin (2016)	239	Applied Energy
Boyd and Pang (2000)	238	Energy Policy
Chambers et al. (1996a)	224	Pacific Economic Review

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