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Rational Solutions of the Second and the Fourth Painlevé Equations

By

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§ 0. Introduction

0° In this paper, we study rational solutions of the second and the fourth Painlevé equations:

$$P_2(\alpha) \qquad \lambda''=2\lambda^3+t\lambda+\alpha$$

$$P_4(\alpha, \theta) \qquad \lambda''=\frac{1}{2\lambda}(\lambda')^2+\frac{3}{2}\lambda^3+4t\lambda^2+2(t^2-\alpha)\lambda-\frac{2\theta^2}{\lambda},$$

where $'$ denotes d/dt and α, θ are complex constants.

We note that every solution of $P_J (J=2, 4)$ is meromorphic on $\boldsymbol{C}$, and that a rational solution of P_J $(J=2, 4)$ is nothing but a solution which has at most a pole at infinity.

1° On rational solutions of $P_2(\alpha)$, we obtain the following theorem.

Theorem 1. (1) *$P_2(\alpha)$ has a unique rational solution if and only if α is an integer.*

(2) *When $\alpha=0$, the rational solution is identically zero.*

When α is a nonzero integer, the rational solution has the form

$$\lambda=\sum_{j=1}^{q}\frac{\varepsilon_j}{(t-b_j)}=\frac{(-\alpha)(t^{q-1}+\cdots)}{\prod_{j=1}^{q}(t-b_j)}$$

where $\varepsilon_j=1$ or -1 and b_j's denote distinct q complex numbers.

Remark. For any integer α, we can construct the rational solution of $P_2(\alpha)$ from the rational solution of $P_2(0)$ using the transformations T_+ and T_-, the birational transformations between $P_2(\alpha)$ and $P_2(\alpha+1)$, which are introduced in Prop. 2–3 in § 2–1°.

Some examples of the rational solution of $P_2(\alpha)$ are listed in the Table 1.

α	The rational solutions of $P_2(\alpha)$
0	0
1	$\dfrac{-1}{t}$
2	$\dfrac{-2(t^3-2)}{t(t^3+4)}$
3	$\dfrac{-3t^2(t^6+8t^3+160)}{(t^3+4)(t^6+20t^3-80)}$

Table 1.

2° In order to state the theorem on rational solutions of $P_4(\alpha, \theta)$, we must define the subsets X, Y, Z, A_1, A_2, $\cdots$ in $\boldsymbol{C}^2$ as follows.

Definition (See Fig. 0–1, 0–2).

$$X=\{(2k, \pm\tfrac{1}{3}+2m), (2k+1, \pm\tfrac{2}{3}+2m) \mid k, m \in \boldsymbol{Z}\}$$
$$Y=\{(2k, 2m+1), (2k+1, 2m) \mid k, m \in \boldsymbol{Z}\}$$
$$Z=\{(\alpha, \theta) \mid (\alpha, \theta) \in Y, \theta \neq 0\}$$
$$A_1=\{(k, (1+2n)+k) \mid k, n \in \boldsymbol{Z}, n \geq 0, k \geq -n\}$$
$$A_2=\{(\alpha, \theta) \mid (\alpha, -\theta) \in A_1\}$$
$$B_1=\{(k, (1+2n)+k) \mid k, n \in \boldsymbol{Z}, n \leq -1, k \geq -2n\}$$
$$B_2=\{(\alpha, \theta) \mid (-\alpha, -\theta) \in B_1\}$$
$$C_1=\{(\alpha, \theta) \mid (-\alpha, \theta) \in B_1\}$$
$$C_2=\{(\alpha, \theta) \mid (\alpha, -\theta) \in B_1\}$$
$$D_1=\{(2k+1, 0) \mid k \in \boldsymbol{Z}, k \geq 0\}$$
$$D_2=\{(\alpha, \theta) \mid (-\alpha, \theta) \in D_1\}$$

Note that

$$X \cap Y=\phi,$$
$$Z=A_1 \cup A_2 \cup B_1 \cup B_2 \cup C_1 \cup C_2,$$
$$Y=Z \cup D_1 \cup D_2.$$

On rational solutions of $P_4(\alpha, \theta)$, we obtain the following theorem.

Theorem 2. (1) *$P_4(\alpha, \theta)$ has a unique rational solution if and only if (α, θ) belongs to $X \cup Z$.*

(2) *According to (α, θ), the rational solution of $P_4(\alpha, \theta)$ has the following form:*

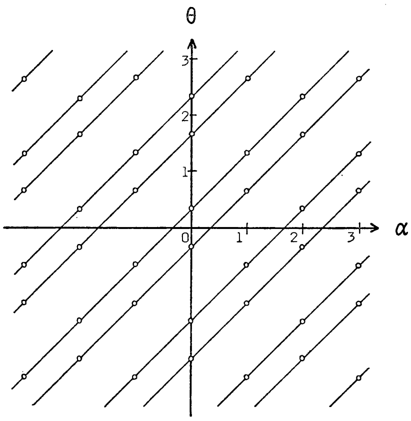

Fig. 0–1 (the set X)

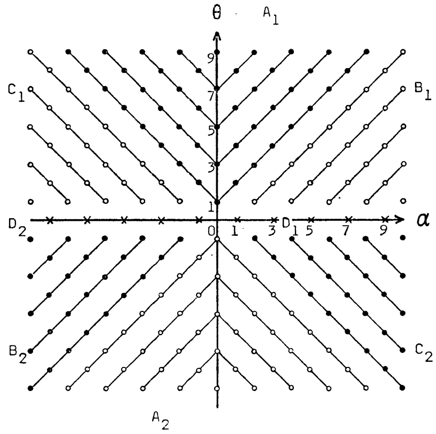

Fig. 0–2 (the set Y)

$(\alpha, \theta) \in X$	$\lambda=-\frac{2}{3}t+\sum_{j=1}^{q}\frac{\varepsilon_j}{(t-b_j)}=-\frac{2}{3}t+\frac{P}{Q}$ (IV–1) $\begin{cases} P=\sum_{j=1}^{q}(\varepsilon_j\prod_{k\neq j}(t-b_j)) \\ \quad =\alpha t^{q-1}+(\alpha A)t^{q-2}+\left[\alpha B-\frac{1}{4}(-9\theta^2+3\alpha^2+1)\right]t^{q-3}+\cdots \\ Q=\prod_{j=1}^{q}(t-b_j) \\ \quad =t^q+At^{q-1}+Bt^{q-2}+\cdots \\ \left(\varepsilon_j=1 \text{ or } -1,\ A=-\sum_{j=1}^{q}b_j,\ B=\sum_{j\neq k}b_jb_k\right) \end{cases}$ $\left(\text{When } (\alpha,\theta)=\left(0, \pm\frac{1}{3}\right),\ \lambda=-\frac{2}{3}t\right)$
$(\alpha, \theta) \in A_1\cup A_2$	$\lambda=-2t+\sum_{j=1}^{q}\frac{\varepsilon_j}{(t-b_j)}=-2t+\frac{P}{Q}$ (IV–2) $\begin{cases} P=(-\alpha)t^{q-1}+(-\alpha A)t^{q-2}+\left[(-\alpha B)+\frac{1}{4}(-\theta^2+3\alpha^2+1)\right]t^{q-3}+\cdots \\ \varepsilon_j, A, B, Q \text{ are the same ones as in (IV–1).} \end{cases}$ (When $(\alpha,\theta)=(0,\pm 1)$, $\lambda=-2t$)
$(\alpha, \theta) \in B_1\cup B_2$	$\lambda=\sum_{j=1}^{q}\frac{\varepsilon_j}{(t-b_j)}=\frac{P}{Q}$ (IV–3) $\begin{cases} P=\theta t^{q-1}+(\theta A)t^{q-2}+\theta\left(B+\frac{\alpha}{2}-\theta\right)t^{q-3}+\cdots \\ \varepsilon_j, A, B, Q \text{ are the same ones as in (IV–1).} \end{cases}$

$(\alpha, \theta) \in C_1 \cup C_2$	$\lambda=\sum_{j=1}^{q} \frac{\varepsilon_j}{(t-b_j)}=\frac{P}{Q}$ (IV–4) $\begin{cases} P=(-\theta)t^{q-1}+(-\theta A)t^{q-2}-\theta\left(B+\frac{\alpha}{2}+\theta\right)t^{q-3}+\cdots \\ \varepsilon_j, A, B, Q \text{ are the same ones as in (IV–1).} \end{cases}$

Remark 1. The unique rational solution of $P_4(-n, n+1)$ is a function $-2t+H_n'/H_n$, where n denotes a positive integer and H_n denotes the Hermite polynomial of order n (cf. Airault [11], Remark, p 47).

Remark 2. Poles of the rational solution of $P_4(\alpha, \theta)$ are either $\{\pm b_1, \cdots, \pm b_q\}$ or $\{0, \pm b_1, \cdots, \pm b_q\}$, where every b_j is not zero and $\{\pm b_j\}\cap\{\pm b_k\}=\phi$ if $j\neq k$.

Remark 3. For any (α, θ) in X, we can construct the rational solution of $P_4(\alpha, \theta)$ from the rational solution $-(2/3)t$ of $P_4(0, \pm 1/3)$, while for any (α, θ) in Z, we can construct it from the rational solution $-2t+1/t$ of $P_4(-1, \pm 2)$. For that purpose, we use the transformations W, W_+, W_-, T_+, T_-, T_1 and T_2 which are the birational transformations between $P_4(\alpha, \theta)$ and $P_4(\alpha_1, \theta_1)$. These transformations are introduced in Prop. 2–6, 2–7, 2–8 in § 2–2°.

Examples of rational solutions of $P_4(\alpha, \theta)$ (For the transformations W, W_+, etc., see the above Remark 3):

$$(1) \qquad \left(0, \frac{1}{3}\right), -\frac{2}{3}t \begin{cases} \xrightarrow[W]{} \left(0, -\frac{1}{3}\right), & -\frac{2}{3}t \\ \xrightarrow[W_\pm]{} \left(0, \frac{1}{3}\right), & -\frac{2}{3}t \\ \xrightarrow[T_+]{} \left(2, \frac{1}{3}\right), & -\frac{2}{3}t+\frac{2t}{t^2+3/2} \\ \xrightarrow[T_-]{} \left(-2, \frac{1}{3}\right), & -\frac{2}{3}t+\frac{2t}{t^2-3/2} \\ \xrightarrow[T_1]{} \left(2, -\frac{1}{3}\right), & -\frac{2}{3}t+\frac{2t}{t^2+3/2} \\ \xrightarrow[T_2]{} \left(1, \frac{2}{3}\right), & -\frac{2}{3}t+\frac{1}{t}, \end{cases}$$

where (0, 1/3), etc. denote the values of the parameter (α, θ).

$$
(2) \qquad
\begin{array}{ll}
(1, 2), & -2t-\dfrac{1}{t} \\
\big\downarrow T_1 & \\
(4, 1), & \dfrac{t^2+1/2}{t(t^2-1/2)} \\
\big\downarrow T_+ & \\
(6, 1), & \dfrac{t^4+3/4}{t(t^2-1/2)(t^2-3/2)} \\
\big\downarrow W_+ & \\
(-6, 1), & \dfrac{-(t^4+3/4)}{t(t^2+1/2)(t^2+3/2)},
\end{array}
$$

where (1, 2), etc. denote the values of the parameter (α, θ).

3° Algebraic solutions (particularly, rational solutions) of P_J ($J=2$–6) have been studied by Yablonskii [1], Vorob'ev [2], Lukashevich [3], [4], [5], [6], [7], Gromak [8], [9], [10], and Airault [11], [12].

However, in spite of their investigations, it was very hard to determine all rational solutions of P_J ($J=2$–6).

The main cause lay in the difficulty of finding the transformations between Painlevé equations of the same type with different values of the parameters. But recently, Okamoto [14] has succeeded in deducing many such transformations for P_J ($J=2$–6) and in determining their group structures by a systematic study of Painlevé equations.

So, in this paper, using these transformations (under weaker conditions than in [14]), rational solutions of P_J ($J=2, 4$) are determined.

In § 1, Painlevé systems S_2 and S_4 are introduced. In § 2, several transformations for P_2 and P_4 are presented. These sections are the preliminaries for the proofs of our theorems. In § 3, Th. 1 is proved. In § 4 and § 5, Th. 2 is proved. In Appendix, Th. 1 and Th. 2 are algebraically interpreted in relation to the transformation groups of P_J ($J=2, 4$).

§ 1. Painlevé Systems

1° Equations P_2 and S_2

$P_2(\alpha)$ is equivalent to the Painlevé system $S_2(\alpha)$ which is a Hamiltonian system with a polynomial Hamiltonian H_2 ([13]):

$$P_2(\alpha) \quad \lambda''=2\lambda^3+t\lambda+\alpha$$

$$S_2(\alpha)\quad \begin{cases} \lambda' = \lambda^2 + \mu + \dfrac{t}{2} \\ \mu' = -2\lambda\mu + \left(\alpha - \dfrac{1}{2}\right) \end{cases}$$

$$H_2 = \frac{1}{2}\mu^2 + \left(\lambda^2 + \frac{t}{2}\right)\mu - \left(\alpha - \frac{1}{2}\right)\lambda,$$

where α denotes a complex constant.

Remark. $S_2(1/2)$ admits $\mu \equiv 0$ and λ which is a solution of the Riccati equation R_2: $\lambda' - \lambda^2 - t/2 = 0$. So any solution of R_2 is a solution of $P_2(1/2)$. We note that R_2 is equivalent to the Airy's equation: $u'' - xu = 0$ with the change of the variables:

$$u = \exp\int(-\lambda)dt, \qquad x = \left(\sqrt[3]{-\frac{1}{2}}\right)t.$$

2° Equations P_4, S_4 and E_4

$P_4(\alpha, \theta)$ is equivalent to the Painlevé system $S_4(\theta_0, \theta_\infty)$ which is a Hamiltonian system with a polynomial Hamiltonian H_4 ([13]):

$$P_4(\alpha, \theta)\qquad \lambda'' = \frac{1}{2\lambda}(\lambda')^2 + \frac{3}{2}\lambda^3 + 4t\lambda^2 + 2(t^2 - \alpha)\lambda - \frac{2\theta^2}{\lambda}$$

$$S_4(\theta_0, \theta_\infty)\quad \begin{cases} \lambda' = 4\lambda\mu - (\lambda^2 + 2t\lambda + 2\theta_0) \\ \mu' = -2\mu^2 + (2\lambda + 2t)\mu - \theta_\infty \end{cases}$$

$$H_4 = 2\lambda\mu^2 - (\lambda^2 + 2t\lambda + 2\theta_0)\mu + \theta_\infty\lambda,$$

where $\theta_0 = \theta$, $\theta_\infty = 1/2(\alpha + \theta - 1)$.

Besides, $S_4(\theta_0, \theta_\infty)$ is equivalent to the following equation ([13]):

$$E_4(x_0, x_1, x_2)\quad (h'')^2 - 4(th' - h)^2 + 4(h' + 2x_0)(h' + 2x_1)(h' + 2x_2) = 0,$$

where x_0, x_1 and x_2 are complex constants satisfying $x_0 + x_1 + x_2 = 0$.

Exactly speaking, if $(\lambda(t), \mu(t))$ is a solution of $S_4(\theta_0, \theta_\infty)$ and $h(t) = H_4(t, \lambda(t), \mu(t)) - 2x_0 t$ satisfies $h'(t) + 2x_1 \not\equiv 0$, $h(t)' + 2x_2 \not\equiv 0$, then $h(t)$ is a solution of $E_4(x_0, x_1, x_2)$, where $x_0 = -(\theta_0 + \theta_\infty)/3$, $x_1 = (2\theta_0 - \theta_\infty)/3$, $x_2 = (-\theta_0 + 2\theta_\infty)/3$.

Conversely, if a solution $h(t)$ of $E_4(x_0, x_1, x_2)$ satisfies $h''(t) \not\equiv 0$, then $(\lambda(t), \mu(t))$ is a solution of $P_4(\theta_0, \theta_\infty)$, where

$$\lambda(t) = \frac{h'' - 2(th' - h)}{2(h' + 2x_2)}, \qquad \mu(t) = \frac{h'' + 2(th' - h)}{4(h' + 2x_1)},$$

$$\theta_0 = x_1 - x_0, \qquad \theta_\infty = x_2 - x_0.$$

We note that solutions of P_4, S_4 and E_4 are meromorphic on $\boldsymbol{C}$.

Remark. $S_4(\theta_0, 0)$ admits $\mu \equiv 0$ and λ which is a solution of the Riccati equation R_4: $\lambda' + \lambda^2 + 2t\lambda + 2\theta_0 = 0$. So, if a solution of R_4 is not identically zero, $P_4(1-\theta_0, \theta_0)$ admits it. Since R_4 is equivalent to the Hermite's equation: $u'' - 2tu' + 2(\theta_0 - 1) = 0$, with the change of the dependent variable: $u = \exp \int (\lambda + 2t) dt$, we get a rational solution $-2t + H'_n/H_n$ (H_n: the Hermite polynomial of order n) of $P_4(-n, n+1)$, where n is a positive integer. From these, Remark 1 after Th. 2 is obtained.

§ 2. Trnasformations for P_2 and P_4

1° Transformations for S_2 and P_2

We have the following propositions.

Proposition 2-1. (1) *Assume that* (λ, μ) *is a solution of* $S_2(\alpha)$, *then*

$$T(\lambda, \mu) = (-\lambda, -2\lambda^2 - \mu - t)$$

is a solution of $S_4(-\alpha)$.

(2) *Let* (λ, μ) *be a solution of* $S_2(\alpha)$, *then* $T(T(\lambda, \mu)) = (\lambda, \mu)$.

Proposition 2-2 (Okamoto [14]). (1) *Assume that* (λ, μ) *is a solution of* $S_2(\alpha)$ *and that* $2\lambda^2 + \mu + t \neq 0$, *then*

$$T_+(\lambda, \mu) = \left(-\lambda - \frac{\alpha + 1/2}{2\lambda^2 + \mu + t}, -2\lambda^2 - \mu - t\right)$$

is a solution of $S_4(\alpha + 1)$.

(2) *Assume that* (λ, μ) *is a solution of* $S_2(\alpha)$ *and that* $\mu \neq 0$, *then*

$$T_-(\lambda, \mu) = \left(-\lambda + \frac{\alpha - 1/2}{\mu}, -\mu - 2\left(\lambda - \frac{\alpha - 1/2}{\mu}\right)^2 - t\right)$$

is a solution of $S_2(\alpha - 1)$.

(3) *If a solution* (λ, μ) *of* $S_2(\alpha)$ *satisfies* $2\lambda^2 + \mu + t \neq 0$, *then the solution* $T_+(\lambda, \mu) = (\lambda_+, \mu_+)$ *of* $S_2(\alpha + 1)$ *satisfies* $\mu_+ \neq 0$. *And* $T_-(T_+(\lambda, \mu)) = (\lambda, \mu)$. *Similarly,* $T_+(T_-(\lambda, \mu)) = (\lambda, \mu)$ *holds under the condition* $\mu \neq 0$.

Remark. The above propositions are checked by direct calculations. Here we note that the condition $2\lambda^2 + \mu + t \neq 0$ in Prop. 2–2, (1) and the condition $\mu \neq 0$ in Prop. 2–2, (2) are weaker than the conditions in [14].

In this paper, to prove Th. 1, we need the next proposition which is easily obtained from Prop. 2–2.

Proposition 2-3 (Transformations T_+ and T_- for $P_2(\alpha)$).

(1) *Assume that* λ *is a solution of* $P_2(\alpha)$ *and that* $\lambda' + \lambda^2 + t/2 \neq 0$, *then*

$$T_+(\lambda)=-\lambda-\frac{\alpha+1/2}{\lambda'+\lambda^2+t/2}$$

is a solution of $P_2(\alpha+1)$.

(2) *Assume that* λ *is a solution of* $P_2(\alpha)$ *and that* $\lambda'-\lambda^2-t/2\neq 0$, *then*

$$T_-(\lambda)=-\lambda+\frac{\alpha-1/2}{\lambda'-\lambda^2-t/2}$$

is a solution of $P_2(\alpha-1)$.

(3) *If a solution* λ *of* $P_2(\alpha)$ *satisfies* $\lambda'+\lambda^2+t/2\neq 0$, *then the solution* $T_+(\lambda)$ *of* $P_2(\alpha+1)$ *satisfies* $T'_+(\lambda)-T_+(\lambda)^2-t/2\neq 0$, *and* $T_-(T_+(\lambda))=\lambda$. *Similarly,* $T_+(T_-(\lambda))=\lambda$ *holds under the condition* $\lambda'-\lambda^2-t/2\neq 0$.

Remark. Transformations T_+ and T_- in this proposition are those referred to in Remark after Th. 1. A rational solution λ of $P_2(\alpha)$ is apparently transformed into a rational solution by T_+ (or T_-) if it satisfies $\lambda'+\lambda^2+t/2\neq 0$ (or $\lambda'-\lambda^2-t/2\neq 0$).

2° Transformations for S_4 and P_4

In order to state Okamoto's results on the transformations for $S_4(\theta_0, \theta_\infty)$, we will rewrite the parameters θ_0, θ_∞ of S_4 into the parameters x_0, x_2, x_2:

$$x_0=-\frac{1}{3}(\theta_0+\theta_\infty),\quad x_1=\frac{1}{3}(2\theta_0-\theta_\infty),\quad x_2=\frac{1}{3}(-\theta_0+2\theta_\infty).$$

The correspondence between $(\theta_0, \theta_\infty)\in C^2$ and $(x_0, x_1, x_2)\in H=\{(x_0, x_1, x_2)\in C^3 \mid x_0+x_1+x_2=0\}$ being one-to-one, we can refer to $S_4(\theta_0, \theta_\infty)$ as $S_4(x_0, x_1, x_2)$.

Proposition 2-4 (Okamoto [14]). (1) *Let* $\mathfrak{S}_3$ *be the symmetric group of degree* 3 *on the set* $\{0, 1, 2\}$ *and let* $\sigma\in\mathfrak{S}_3$. *Assume that* (λ, μ) *is a solution of* $S_4(x_0, x_1, x_2)$ *which satisfies the condition*

$$(C)\quad \lambda\mu\neq 0,\quad \lambda\mu+x_0-x_1\neq 0,\quad \lambda\mu+x_0-x_2\neq 0.$$

Then, $T_\sigma(\lambda, \mu)=(\lambda_\sigma, \mu_\sigma)$ *is a solution of* $S_4(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)})$, *where*

$$\lambda_\sigma=\lambda\frac{\lambda\mu+x_0-x_2}{\lambda\mu+x_0-x_{\sigma(2)}}$$

$$\mu_\sigma=\mu\frac{\lambda\mu+x_0-x_1}{\lambda\mu+x_0-x_{\sigma(1)}}.$$

(2) *If a solution* (λ, μ) *of* $S_4(x_0, x_1, x_2)$ *satisfies the condition* (C), *then the solution* $T_\sigma(\lambda, \mu)$ *of* $S_4(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)})$ *satisfies the condition* (C). *And for any* $\tau\in\mathfrak{S}_3$, $T_\tau(T_\sigma(\lambda, \mu))=T_{\tau\sigma}(\lambda, \mu)$.

Proposition 2-5 (Okamoto [14]). (1) *Assume that* (λ, μ) *is a solution of* $S_4(x_0, x_1, x_2)$ *which satisfies the condition*

$$(C_+)\quad \begin{cases} \lambda\bar{\mu}\neq 0, \quad \lambda\bar{\mu}+x_0-x_1\neq 0, \quad \lambda\bar{\mu}-x_1+x_2+1\neq 0, \\ \text{where } \bar{\mu}=\mu-\dfrac{1}{2}\lambda-t. \end{cases}$$

Then, $T_+(\lambda, \mu)=(\lambda_+, \mu_+)$ *is a solution of* $S_4(x_0-1/3, x_1-1/3, x_2+2/3)$, *where*

$$\lambda_+=2\bar{\mu}\frac{\lambda\bar{\mu}+x_0-x_1}{\lambda\bar{\mu}-x_1+x_2+1}$$

$$\mu_+=-\frac{1}{2}\lambda\frac{\lambda\bar{\mu}-x_1+x_2+1}{\lambda\bar{\mu}}.$$

(2) *Assume that* (λ, μ) *is a solution of* $S_4(x_0, x_1, x_2)$ *which satisfies the condition*

$$(C_-)\quad \lambda\mu\neq 0, \quad \lambda\mu+x_0-x_1\neq 0, \quad \lambda\mu+x_0-x_2\neq 0.$$

Then, $T_-(\lambda_-, \mu_-)$ *is a solution of* $S_4(x_0+1/3, x_1+1/3, x_2-2/3)$, *where*

$$\lambda_-=-2\mu\frac{\lambda\mu+x_0-x_1}{\lambda\mu+x_0-x_2}$$

$$\mu_-=\frac{\lambda(\lambda\mu+x_0-x_2)}{2\lambda\mu}-\frac{\mu(\lambda\mu+x_0-x_1)}{\lambda\mu+x_0-x_2}+t$$

(3) *If a solution* (λ, μ) *of* $S_4(x_0, x_1, x_2)$ *satisfies the condition* (C_+), *then the solution* $T_+(\lambda, \mu)$ *of* $S_4(x_0-1/3, x_1-1/3, x_2+2/3)$ *satisfies the condition* (C_-), *and* $T_-(T_+(\lambda, \mu))=(\lambda, \mu)$. *Similarly,* $T_+(T_-(\lambda, \mu))=(\lambda, \mu)$ *holds under the condition* (C_-).

Remark. Prop. 2–4 and Prop. 2–5 are checked by direct calculations. And we note that the conditions (C), (C_+), (C_-) are weaker than those in [14].

In order to prove Th. 2, we must rewrite Prop. 2–4, Prop. 2–5 and introduce other three transformations.

First, we note that the transformation group $\{T_\sigma \mid \sigma\in\mathfrak{S}_3\}$ is generated by T_{σ_1} and T_{σ_2}, where $\sigma_1=(1, 2)$, $\sigma_2=(0, 2)$. So, it is sufficient to consider $T_1(=T_{\sigma_1})$ and $T_2(=T_{\sigma_2})$ of all T_σ's.

From Prop. 2–4, we obtain the following proposition.

Proposition 2-6 (Transformations T_1 and T_2 for $P_4(\alpha, \theta)$).

(1) *Assume that* λ *is a solution of* $P_4(\alpha, \theta)$ *which satisfies the condition*

$$(D)\quad \lambda'+\lambda^2+2t\lambda\neq\pm 2\theta, \qquad \lambda'+\lambda^2+2t\lambda\neq 2(\alpha-1).$$

Then,

$$T_1(\lambda)=\lambda\frac{(\lambda'+\lambda^2+2t\lambda)-2(\alpha-1)}{(\lambda'+\lambda^2+2t\lambda)-2\theta}$$

is a solution of $P_4(3(\theta+1)-\alpha)/2,\ (\alpha+\theta-1)/2$.

(2) *Assume that* λ *is solution of* $P_4(\alpha, \theta)$ *which satisfies the condition* (D). *Then,*

$$T_2(\lambda)=\lambda\frac{(\lambda'+\lambda^2+2t\lambda)-2(\alpha-1)}{(\lambda'+\lambda^2+2t\lambda)+2\theta}$$

is a solution of $P_4(-3(\theta-1)-\alpha)/2,\ (-\alpha+\theta+1)/2$.

(3) *If a solution* λ *of* $P_4(\alpha, \theta)$ *satisfies the condition* (D), *then the solution* $T_1(\lambda)$ *of* $P_4(\bar{\alpha}, \bar{\theta})$ *satisfies the same condition* (D), *where* $(\bar{\alpha}, \bar{\theta})=(3(\theta+1)-\alpha)/2,\ (\alpha+\theta-1)/2$. *So* T_1 *and* T_2 *are applicable to* $T_1(\lambda)$. *In particular,* $T_1(T_1(\lambda))=\lambda$. *Similarly,* T_1 *and* T_2 *are applicable to* $T_2(\lambda)$ *under the condition* (D) *of* λ, *and particularly* $T_2(T_2(\lambda))=\lambda$ *holds.*

Next, we can derive the following proposition from Prop. 2–5.

Proposition 2-7 (Transformations T_+ and T_- for $P_4(\alpha, \theta)$).

(1) *Assume that* λ *is a solution of* $S_4(\alpha, \theta)$ *which satisfies the condition*

$$(D_+)\quad \lambda'-\lambda^2-2t\lambda\fallingdotseq\pm 2\theta,\quad \lambda'-\lambda^2-2t\lambda\fallingdotseq -2(\alpha+1).$$

Then,

$$T_+(\lambda)=\frac{1}{2\lambda}\frac{(\lambda'-\lambda^2-2t\lambda)^2-4\theta^2}{(\lambda'-\lambda^2-2t\lambda)+2(\alpha+1)}$$

is a solution of $P_4(\alpha+2, \theta)$.

(2) *Assume that* λ *is a solution of* $P_4(\alpha, \theta)$ *which satisfies the condition*

$$(D_-)\quad \lambda'+\lambda^2+2t\lambda\fallingdotseq\pm 2\theta,\quad \lambda'+\lambda^2+2t\lambda\fallingdotseq 2(\alpha-1).$$

Then,

$$T_-(\lambda)=-\frac{1}{2\lambda}\frac{(\lambda'+\lambda^2+2t\lambda)^2-4\theta^2}{(\lambda'+\lambda^2+2t\lambda)-2(\alpha-1)}$$

is a solution of $P_4(\alpha-2, \theta)$.

(3) *If a solution* λ *of* $P_4(\alpha, \theta)$ *satisfies the condition* (D_+), *then the solution* $T_+(\lambda)$ *of* $P_4(\alpha+2, \theta)$ *satisfies the same condition* (D_+) *and* $T_-(T_+(\lambda))=\lambda$. *Similarly,* $T_+(T_-(\lambda))=\lambda$ *holds under the condition* (D_-).

In addition to T_1, T_2, T_+ and T_-, we need the following three transformations W, W_+, W_-.

Proposition 2-8 (Transformations W, W_+ and W_- for $P_4(\alpha, \theta)$).

(1) *If $\lambda(t)$ is a solution of $P_4(\alpha, \theta)$, then $y(t)=\lambda(t)$ is a solution of $P_4(\alpha, -\theta)$:*

$$W:\ P_4(\alpha, \theta) \longrightarrow P_4(\alpha, -\theta), \quad \lambda \longrightarrow y=\lambda.$$

(2) *If $\lambda(t)$ is a solution of $P_4(\alpha, \theta)$, then $x=\sqrt{-1}\,t$ and $y=\sqrt{-1}\,\lambda$ satisfy $P_4(-\alpha, \theta)$:*

$$W_+:\ P_4(\alpha, \theta) \longrightarrow P_4(-\alpha, \theta), \quad (t, \lambda) \longrightarrow (x, y)=(\sqrt{-1}\,t, \sqrt{-1}\,\lambda).$$

(3) *If $\lambda(t)$ is a solution of $P_4(\alpha, \theta)$, then $x=(-\sqrt{-1})t$ and $y=(-\sqrt{-1})\lambda$ satisfy $P_4(-\alpha, \theta)$:*

$$W_-:\ P_4(\alpha, \theta) \longrightarrow P_4(-\alpha, \theta), \quad (t, \lambda) \longrightarrow (x, y)=((-\sqrt{-1})t, (-\sqrt{-1})\lambda).$$

(4) $W \circ W = W_- \circ W_+ = W_+ \circ W_- = id.$

The proof is easy, so we omit it.

Remark. Seven transformations in Prop. 2-6, 2–7, 2–8 are those referred to in Remark 3 after Th. 2 in § 0–2°. We note that each of the transformations W, $W_\pm$, $T_\pm$, T_1 and T_2 transforms a rational solution λ of $P_4(\alpha, \theta)$ into a rational solution $\bar{\lambda}$ of $P_4(\bar{\alpha}, \bar{\theta})$ if λ satisfies the condition of application.

§ 3. Proof of Theorem 1

1° We begin with the investigations of the Laurent series expansions of solutions of $P_2(\alpha)$.

Since the following two propositions are easily verified, we omit the proofs.

Proposition 3-1. *If a solution λ of $P_2(\alpha)$ has at most a pole at $t=\infty$, then λ must have the following Tayler expansion:*

$$\lambda(t)=\frac{-\alpha}{t}+\frac{2\alpha(\alpha^2-1)}{t^4}+O\left(\frac{1}{t^5}\right) \qquad (\textit{around } t=\infty).$$

Proposition 3-2. *If a solution λ of $P_2(\alpha)$ has a pole at $t=b \in \mathbf{C}$, then λ must have the following Laurent expansion:*

$$\lambda(t)=\frac{\varepsilon}{(t-b)}+O(t-b),$$

where ε denotes 1 *or* -1.

The next simple lemma will be used both in this section and in § 4.

Lemma. *Suppose b_j's ($j=1, \cdots, q$) are distinct q complex numbers and ε_j is* 1 *or* -1. *Let*

$$\sum_{j=1}^{q} \frac{\varepsilon_j}{(t-b_j)} = \frac{P(t)}{Q(t)},$$

where

$$P(t) = \sum_{j=1}^{q} (\varepsilon_j \prod_{k \neq j} (t-b_k)), \qquad Q(t) = \prod_{j=1}^{q} (t-b_j).$$

Then, $P(t)$ and $Q(t)$ have the following expansions:

$$P(t) = mt^{q-1} + (mA+C)t^{q-2} + (mB+AC+D)t^{q-3} + \cdots,$$
$$Q(t) = t^q + At^{q-1} + Bt^{q-2} + \cdots,$$

where

$$m = \sum_{j=1}^{q} \varepsilon_j, \quad A = -\sum_{j=1}^{q} b_j, \quad B = \sum_{j \neq k} b_j b_k, \quad C = \sum_{j=1}^{q} \varepsilon_j b_j, \quad D = \sum_{j=1}^{q} \varepsilon_j b_j^2.$$

Moreover, $P(t)/Q(t)$ is expressed as

$$\frac{P(t)}{Q(t)} = \frac{m}{t} + \frac{C}{t^2} + \frac{D}{t^3} + O\left(\frac{1}{t^4}\right) \qquad (\textit{around } t=\infty).$$

We omit the proof.

2° From the above two propositions and Lemma, we can derive some information on the rational solutions of $P_2(\alpha)$.

First we note that any rational function is uniquely decomposed into partial fractions.

Let λ be a rational solution of $P_2(\alpha)$.

If λ is a polynomial, then, by Prop. 3–1, it must be expanded as follows:

$$\lambda(t) = \frac{-\alpha}{t} + O\left(\frac{1}{t^4}\right) \qquad (\text{around } t=\infty).$$

So, $\alpha=0$ and $\lambda \equiv 0$.

If λ is not a polynomial, then, from Prop. 3–1 and 3–2, it turns out that

$$\lambda(t) = \sum_{j=1}^{q} \frac{\varepsilon_j}{(t-b_j)},$$

where $\varepsilon_j = 1$ or -1 and b_j's are distinct q complex numbers.

In addition, by Lemma and Prop. 3–1,

$$\lambda(t) = \frac{P(t)}{Q(t)} = \frac{m}{t} + \frac{C}{t^2} + \frac{D}{t^3} + O\left(\frac{1}{t^4}\right)$$
$$= \frac{-\alpha}{t} + O\left(\frac{1}{t^4}\right).$$

So,

$$\alpha=-m=-\sum_{j=1}^{q}\varepsilon_j\in \boldsymbol{Z}.$$

Therefore, we have seen that *if $P_2(\alpha)$ has a rational solution, then α must be an integer.*

Moreover, the following proposition is obtained.

Proposition 3-3. *$\lambda\equiv 0$ is a unique rational solution of $P_2(0)$.*

Proof. Assume that $P_2(0)$ admits a nonzero rational solution $\lambda=P/Q$, where $\deg P=p$ and $\deg Q=q$. By Lemma and the above consideration, $q\geq 2$ and $p<q-1$.

Since

$$\lambda'=\frac{P'Q-PQ'}{Q^2},$$

$$\lambda''=\frac{(P''Q-PQ'')Q-2(P'Q-PQ')Q'}{Q^3},$$

we get

$$(P''Q-PQ'')Q-2(P'Q-PQ')Q'=2P^3+tPQ^2.$$

On the other hand, we have

$$\deg (P''Q-PQ'')Q\leq 2q+p-2,$$
$$\deg (P'Q-PQ')Q'\leq 2q+p-2,$$
$$\deg P^3=3p,$$

and

$$\deg tPQ^2=2q+p+1.$$

This is a contradiction, because

$$2q+p+1>2q+p-2>3p. \qquad \text{Q.E.D.}$$

3° Now, we can prove Th. 1.

Proof of Th. 1. We will recall the transformations T_+ and T_- defined in Prop. 2–3. We can apply T_+ and T_- to $\lambda\equiv 0$, the unique rational solution of $P_2(0)$, because $\lambda'+\lambda^2+t/2=t/2\not\equiv 0$ and $\lambda'-\lambda^2-t/2=-t/2\not\equiv 0$. We can also apply them to a rational solution $\lambda=P/Q$, if it exists, because

$$\deg Q>\deg P$$

and

$$\lambda' \pm \lambda^2 \pm \frac{t}{2} = \frac{P'Q - PQ'}{Q^2} \pm \frac{P^2}{Q^2} \pm \frac{tP^2/2}{Q^2}$$
$$= \frac{1}{Q^2}\left[(P'Q - PQ') \pm P^2 \pm \frac{t}{2}P^2\right] \not\equiv 0.$$

So, if we start from the rational solution $\lambda \equiv 0$ of $P_2(0)$ and use T_+ and T_- successively, we can construct a rational solution of $P_2(\alpha)$ for any integer α. Moreover, by the properties of T_+ and T_-: $T_- \circ T_+ = T_+ \circ T_- = id$ (Prop. 2–3, (3)), and the uniqueness of the rational solution of $P_2(0)$ (Prop 3–3), for any $\alpha \in \boldsymbol{Z}$, $P_2(\alpha)$ cannot have more than one rational solution. Thus, Th. 1, (1) has been proved.

By Prop. 3–3, $\lambda \equiv 0$ is the rational solution of $P_2(0)$. And by the consideration before Prop. 3–3, the rational solution of $P_2(\alpha)$ ($\alpha \in \boldsymbol{Z} - \{0\}$) is expressed as

$$\sum_{j=1}^{q} \frac{\varepsilon_j}{(t-b_j)} = \frac{(\sum_{j=1}^{q} \varepsilon_j)t^{q-1} + \cdots}{\prod_{j=1}^{q}(t-b_j)} = \frac{(-\alpha)t^{q-1} + \cdots}{\prod_{j=1}^{q}(t-b_j)},$$

where $\varepsilon_j = 1$ or -1 and b_j's denote distinct q complex numbers. Therefore, Th. 1, (2) has been proved. Q.E.D.

§ 4. Proof of Theorem 2 (Necessary Conditions)

0° We prove Th. 2, (1) in the following form:

(A) *If $P_4(\alpha, \theta)$ has a rational solution, then (α, θ) belongs to $X \cup Y$.*

(B) *If (α, θ) belongs to $X \cup Z$, then $P_4(\alpha, \theta)$ has a unique rational solution. If (α, θ) belongs to $D_1 \cup D_2$, then $P_4(\alpha, \theta)$ does not admit a rational solution.*

X, Y, Z, D_1 and D_2 are the same sets as in §0–2°.

In this section, (A) will be proved, while in the next section, (B) and Th. 2, (2) will be proved.

In order to prove (A), we will discuss in the following way. We begin with the investigations of the Laurent expansions of solutions of $P_4(\alpha, \theta)$ (Prop. 4–1, Cor., Prop. 4–2), and, using these results and Lemma in § 3, we deduce the necessary conditions on α, θ for $P_4(\alpha, \theta)$ to have a rational solution, and those on the forms of rational solutions (Prop. 4–3). Next, we investigate the effects of the seven transformations W, W_+, W_-, T_+, T_-, T_1 and T_2 on rational solutions (Prop. 4–4 – Prop. 4–8). From these, we derive the detailed necessary conditions on α, θ for $P_4(\alpha, \theta)$ to have a rational solution (Prop. 4–9). Lastly,we prove (A).

1° The next propositions and corollary are easily shown. So we omit the proofs.

Proposition 4-1. (1) *If a solution λ of $P_4(\alpha, \theta)$ has a pole at $t = \infty$, then the Laurent expansion of λ at $t = \infty$ must be*

(a) $$\lambda(t)=-\frac{2}{3}t+\frac{\alpha}{t}-\frac{1}{4}(-9\theta^2+3\alpha^2+1)\frac{1}{t^3}+\cdots,$$

or

(b) $$\lambda(t)=-2t+\frac{-\alpha}{t}+\frac{1}{4}(-\theta^2+3\alpha^2+1)\frac{1}{t^3}+\cdots.$$

(2) *If a solution λ of $P_4(\alpha, \theta)$ is holomorphic at $t=\infty$, then the Taylor expansion of λ at $t=\infty$ must be*

(a) $$\lambda(t)=\frac{\theta}{t}+\frac{(\alpha-2\theta)\theta}{2}\frac{1}{t^3}+\cdots,$$

or

(b) $$\lambda(t)=\frac{-\theta}{t}+\frac{-(\alpha+2\theta)\theta}{2}\frac{1}{t^3}+\cdots,$$

where $\theta\neq 0$.

Corollary. (1) *$\lambda=-(2/3)t$ is a solution of $P_4(\alpha, \theta)$ if and only if $(\alpha, \theta)=(0, \pm 1/3)$.*

(2) *$\lambda=-2t$ is a solution of $P_4(\alpha, \theta)$ if and only if $(\alpha, \theta)=(0, \pm 1)$.*

Proposition 4-2. *If a solution λ of $P_4(\alpha, \theta)$ has a pole at $t=b\in \mathbf{C}$, then λ must have the following Laurent expansion:*

$$\lambda(t)=\frac{\varepsilon}{(t-b)}+c_0+c_1(t-b)+\cdots,$$

where $\varepsilon=1$ or -1.

2° Let us study rational solutions of $P_4(\alpha, \theta)$ according to these results.

First, as in § 3, we note that a rational function is uniquely decomposed into partial fractions.

Let λ be a rational solution of $P_4(\alpha, \theta)$. If λ is a polynomial, by Prop. 4–1 and Cor., either $\lambda=-(2/3)t$ with $(\alpha, \theta)=(0, \pm 1/3)$ or $\lambda=-2t$ with $(\alpha, \theta)=(0, \pm 1)$. If λ is not a polynomial,

$$\lambda=-\frac{2}{3}t+\sum_{j=1}^{q}\frac{\varepsilon_j}{(t-b_j)}$$

or

$$\lambda=-2t+\sum_{j=1}^{q}\frac{\varepsilon_j}{(t-b_j)}$$

or

$$\lambda=\sum_{j=1}^{q}\frac{\varepsilon_j}{(t-b_j)},$$

where $\varepsilon_j=1$ or -1 and b_j's are distinct for different indices.

By more detailed investigations, we get the following.

Proposition 4-3 *When b_j's $(j=1,\cdots,q)$ are distinct complex numbers and ε_j is 1 or -1, we write*

$$\sum_{j=1}^{q}\frac{\varepsilon_j}{(t-b_j)}=\frac{P}{Q},$$

where

$$P=\sum_{j=1}^{q}(\varepsilon_j\prod_{k\neq j}(t-b_j))\ \text{and}\ Q=\prod_{j=1}^{q}(t-b_j).$$

And we set

$$A=-\sum_{j=1}^{q}b_j,\qquad B=\sum_{j\neq k}b_jb_k.$$

Let λ be a rational solution of $P_4(\alpha,\theta)$, then λ is one of the following six types of rational solutions.

(1) $(\alpha,\theta)=\left(0,\frac{1}{3}\right)$ *or* $\left(0,-\frac{1}{3}\right)$, $\lambda=-\frac{2}{3}t$.

(2) $(\alpha,\theta)=(0,1)$ *or* $(0,-1)$, $\lambda=-2t$.

(3) $\alpha\in\boldsymbol{Z}$,

$$\lambda=-\frac{2}{3}t+\sum_{j=1}^{q}\frac{\varepsilon_j}{(t-b_j)}=-\frac{2}{3}t+\frac{P}{Q},$$

$$\begin{cases}P=\alpha t^{q-1}+(\alpha A)t^{q-2}+\left[\alpha B-\frac{1}{4}(-9\theta^2+3\alpha^2+1)\right]t^{q-3}+\cdots,\\ Q=t^q+At^{q-1}+Bt^{q-2}+\cdots.\end{cases}$$

(4) $\alpha\in\boldsymbol{Z}$,

$$\lambda=-2t+\sum_{j=1}^{q}\frac{\varepsilon_j}{(t-b_j)}=-2t+\frac{P}{Q},$$

$$\begin{cases}P=(-\alpha)t^{q-1}+(-\alpha A)t^{q-2}+\left[(-\alpha B)+\frac{1}{4}(-\theta^2+3\alpha^2+1)\right]t^{q-3}+\cdots,\\ Q=t^q+At^{q-1}+Bt^{q-2}+\cdots.\end{cases}$$

(5) $\theta\in\boldsymbol{Z}-\{0\}$,

$$\lambda=\sum_{j=1}^{q}\frac{\varepsilon_j}{(t-b_j)}=\frac{P}{Q},$$

$$\begin{cases} P=\theta t^{q-1}+(\theta A)t^{q-2}+\theta\left(B+\frac{\alpha}{2}-\theta\right)t^{q-3}+\cdots, \\ Q=t^q+At^{q-1}+Bt^{q-2}+\cdots. \end{cases}$$

(6) $\theta \in Z-\{0\}$,

$$\lambda=\sum_{j=1}^{q}\frac{\varepsilon_j}{(t-b_j)}=\frac{P}{Q},$$

$$\begin{cases} P=(-\theta)t^{q-1}+(-\theta A)t^{q-2}-\theta\left(B+\frac{\alpha}{2}+\theta\right)t^{q-3}+\cdots, \\ Q=t^q+At^{q-1}+Bt^{q-2}+\cdots. \end{cases}$$

Remark. Hereafter, we shall refer to a rational solution of the type (3) ((4), (5), (6)) in Prop. 4–3 as a rational solution of the type $-(2/3)t+P/Q$ ($-2t+P/Q$, P/Q, P^*/Q, resp.).

Using Lemma in § 3 and Prop. 4–1, one can check this proposition.

3° We have obtained necessary conditions for $P_4(\alpha, \theta)$ to have a rational solution in Prop. 4–3. For further discussions, we need the transformations W, $W_\pm$, $T_\pm$, T_1 and T_2 defined in Prop. 2–6, 2–7, 2–8.

As is noted in Remark after Prop. 2–8, these transform a rational solution into a rational solution under the conditions of applications. So, we will see the effects of these transformations on rational solutions.

Proposition 4-4. *We can apply W, W_+, W_-, T_+, T_1 and T_2 to $-(2/3)t+P/Q$, and*

$$-\frac{2}{3}+\frac{P}{Q}\xrightarrow[W,W_\pm,T_\pm,T_1,T_2]{}-2t+\frac{P_1}{Q_1}.$$

That is, any rational solution of the type $-(2/3)t+P/Q$ is transformed into a rational solution of the same type $-(2/3)t+P/Q$ by each transformation.

Proposition 4-5.

(1) $-2t+\dfrac{P}{Q}\xrightarrow[W,W_\pm]{}-2t+\dfrac{P_1}{Q_1}.$

(2) *When* $\theta\neq\pm(\alpha+1)$,

$$-2t+\frac{P}{Q}\xrightarrow[T_+]{}-2t+\frac{P_1}{Q_1}.$$

(3) *When* $\theta\neq\pm(\alpha-1)$,

$$-2t+\frac{P}{Q}\xrightarrow[T_-]{}-2t+\frac{P_1}{Q_1},$$

$$-2t+\frac{P}{Q}\xrightarrow[T_1]{}\frac{P_1}{Q_1},$$

and

$$-2t+\frac{P}{Q}\xrightarrow[T_2]{}\frac{P_1^*}{Q_1}.$$

Proposition 4-6.

(1) $\dfrac{P}{Q}\xrightarrow[W,W_\pm]{}\dfrac{P_1^*}{Q_1}.$

(2) *When* $\theta\neq\alpha+1$,

$$\frac{P}{Q}\xrightarrow[T_+]{}\frac{P_1}{Q_1}.$$

(3) *When* $\theta\neq\alpha-1$,

$$\frac{P}{Q}\xrightarrow[T_-]{}\frac{P_1}{Q_1},$$

$$\frac{P}{Q}\xrightarrow[T_1]{}-2t+\frac{P_1}{Q_1},$$

and

$$\frac{P}{Q}\xrightarrow[T_2]{}\frac{P_1}{Q_1}.$$

Proposition 4-7.

(1) $\dfrac{P^*}{Q}\xrightarrow[W,W_\pm]{}\dfrac{P_1}{Q_1}.$

(2) *When* $\theta\neq-\alpha-1$,

$$\frac{P^*}{Q}\xrightarrow[T_+]{}\frac{P_1^*}{Q_1}.$$

(3) *When* $\theta\neq-\alpha+1$,

$$\frac{P^*}{Q}\xrightarrow[T_-]{}\frac{P_1^*}{Q_1},$$

$$\frac{P^*}{Q}\xrightarrow[T_1]{}\frac{P_1^*}{Q_1},$$

and

$$\frac{P^*}{Q}\xrightarrow[T_2]{}-2t+\frac{P_1}{Q_1}.$$

Proposition 4-8. (1) *The rational solution* $-(2/3)t$ *of* $P_4(0, \pm 1/3)$ *is transformed into a rational solution of the type* $-(2/3)t+P/Q$ *by* $T_\pm$, T_1 *and* T_2.

(2) *The solution* $-2t$ *of* $P_4(0, \pm 1)$ *does not satisfy the conditions* (D_+), (D_-) *in Prop.* 2–7, (D) *in Prop.* 2–6, *and so,* $T_\pm$, T_1 *and* T_2 *are not applicable to* $-2t$. *Consequently, any rational solution of* $P_4(\alpha, \theta)$ *with* $(\alpha, \theta)\neq(0, \pm 1)$ *is not transformed into* $-2t$ *of* $P_4(0, \pm 1)$ *by the seven transformations.*

Since these propositions are proved by Prop. 2–6, 2–7, 2–8 and Prop. 4–3, we shall only show Prop. 4–5, (2).

Proof of Prop. 4–5, (2). Let λ be a rational solution of $P_4(\alpha, \theta)$ of the type $-2t+P/Q$. By Prop 4–3, (4),

$$P=(-\alpha)t^{q-1}+(-\alpha A)t^{q-2}+[(-\alpha B)+\tfrac{1}{4}(-\theta^2+3\alpha^2+1)]t^{q-3}+\cdots,$$
$$Q=t^q+At^{q-1}+Bt^{q-2}+\cdots,$$

where q denotes a certain positive integer and A, B are the same ones as in Prop. 4–3.

By Prop. 2–7, (1), we can apply T_+ to λ only when $\lambda'-\lambda^2-2t\lambda \neq \pm 2\theta$, $\lambda'-\lambda^2-2t\lambda \neq -2(\alpha+1)$.

Since

$$\lambda'-\lambda^2-2t\lambda=-2+\frac{P'Q-PQ'}{Q^2}-\left(-2t+\frac{P}{Q}\right)^2-2t\left(-2t+\frac{P}{Q}\right)$$
$$=\frac{1}{Q^2}\left[-2Q^2+2tPQ+(P'Q-PQ')-P^2\right],$$

replacing Q and P by the above expanded expression,

$$\lambda'-\lambda^2-2t\lambda=\frac{1}{Q^2}\Big[-2(\alpha+1)t^{2q}-4A(\alpha+1)t^{2q-1}$$
$$+\left\{-4B(\alpha+1)-2A^2(\alpha+1)+\frac{1}{2}((\alpha+1)^2-\theta^2)\right\}t^{2q-2}+\cdots\Big].$$

On the other hand,

$$\pm 2\theta=\frac{1}{Q^2}(\pm 2\theta)Q^2$$
$$=\frac{1}{Q^2}[(\pm 2\theta)t^{2q}+(\pm 4\theta)At^{2q-1}+(\pm 2\theta)(2B+A^2)t^{2q-2}+\cdots],$$
$$-2(\alpha+1)=\frac{1}{Q^2}[-2(\alpha+1)t^{2q}-4A(\alpha+1)t^{2q-1}-2(\alpha+1)(2B+A^2)t^{2q-2}+\cdots].$$

Therefore, if $\theta \neq \pm(\alpha+1)$, then $\lambda'-\lambda^2-2t\lambda \neq \pm 2\theta$, $\lambda'-\lambda^2-2t\lambda \neq -2(\alpha+1)$, and we can apply T_+ to λ.

Moreover, from

$$(\lambda'-\lambda^2-2t\lambda)^2-4\theta^2=\frac{1}{Q^4}[4((\alpha+1)^2-\theta^2)t^{4q}+\cdots],$$
$$(\lambda'-\lambda^2-2t\lambda)+2(\alpha+1)=\frac{1}{Q^2}\left[\frac{1}{2}((\alpha+1)^2-\theta^2)t^{2q-2}+\cdots\right],$$

and

$$\frac{1}{2\lambda}=\frac{Q}{2(-2tQ+P)},$$

it turns out that

$$\begin{aligned}T_+(\lambda)&=\frac{1}{2\lambda}\,\frac{(\lambda'-\lambda^2-2t\lambda)^2-4\theta^2}{(\lambda'-\lambda^2-2t\lambda)+2(\alpha+1)}\\&=\frac{1}{2Q(-2tQ+P)}\,\frac{4((\alpha+1)^2-\theta^2)t^{4q}+\cdots}{(1/2)((\alpha+1)^2-\theta^2)t^{2q-2}+\cdots}\\&=\frac{1}{-4t^{2q+1}+\cdots}\,\frac{8t^{4q}+\cdots}{t^{2q-2}+\cdots}\\&=-2t+\frac{\cdots}{t^{4q-1}+\cdots}.\end{aligned}$$

That is to say, $T_+(\lambda)$ is a rational solution of the type $-2t+P/Q$. Q.E.D.

4° From Prop. 4–3, 4–4, 4–5, 4–6 and 4–7, we can deduce the more detailed necessary conditions of α, θ for $P_4(\alpha, \theta)$ to have a rational solution.

Proposition 4-9. *As in Def. in* § 0–2°, *let*

$$X=\{(2k, \pm\tfrac{1}{3}+2m), (2k+1, \pm\tfrac{2}{3}+2m)\,|\,k, m\in \boldsymbol{Z}\},$$
$$Y=\{(2k, 2m+1), (2k+1, 2m)\,|\,k, m\in \boldsymbol{Z}\}.$$

(1) *If $P_4(\alpha, \theta)$ has a rational solution of the type $-(2/3)t+P/Q$ (or $-(2/3)t$), then $(\alpha, \theta)\in X\cup Y$.*

(2) *If $P_4(\alpha, \theta)$ has a rational solution of the type $-2t+P/Q$ (or $-2t$), then $(\alpha, \theta)\in Y$.*

(3) *If $P_4(\alpha, \theta)$ has a rational solution of the type P/Q or of the type P^*/Q, then $(\alpha, \theta)\in Y$ and $\theta\neq 0$.*

Proof. We will only prove (2). Others are shown in the same way.

If $P_4(\alpha, \theta)$ admits $-2t$, then $(\alpha, \theta)=(0, \pm 1)\in Y$ (Cor. after Prop. 4–1).

Let λ be a rational solution of $P_4(\alpha, \theta)$ of the type $-2t+P/Q$.

First, by Prop. 4–3, (4), α must be an integer. Even if we can apply W or $W_\pm$ or $T_\pm$ to λ, we cannot derive a new condition on α, θ (See Prop. 2–7, 2–8, 4–3). But the application of T_1 or T_2 gives us new information. In fact, when $\theta\neq\pm(\alpha-1)$, λ is transformed into a rational solution of $P_4((3(\theta+1)-\alpha)/2, (\alpha+\theta-1)/2)$ of the type P/Q by T_1 (See Prop. 2–6, (1) and Prop. 4–5, (3)). So, by Prop. 4–3, (5),

$$\tfrac{1}{2}(\alpha+\theta-1)=m\in \boldsymbol{Z}-\{0\},$$
$$\alpha+\theta=2m+1.$$

Consequently, the following condition (C) must be satisfied.

$$\text{(C)}\quad \begin{cases} \alpha \text{ and } \theta \text{ are integers and} \\ \text{either } \alpha \text{ is even and } \theta \text{ is odd,} \\ \text{or } \alpha \text{ is odd and } \theta \text{ is even.} \end{cases}$$

Additionally, even if $\theta=\pm(\alpha-1)$, α and θ satisfy the condition (C), which means that (α, θ) belongs to Y. Q.E.D.

5° *Proof of* (A). Compare Prop. 4–3 with Prop. 4–9. Then we find that if $P_4(\alpha, \theta)$ has a rational solution, (α, θ) must belong to $X\cup Y$. Q.E.D.

§ 5. Proof of Theorem 2 (Sufficient Conditions)

1° In this section, we shall prove (B) of § 4 and Th. 2, (2).

Proposition 5-1. (1) $-(2/3)t$ *is a unique rational solution of* $P_4\,(0, \pm 1/3)$.

(2) $P_4(0, \pm 1)$ *have a rational solution* $-2t$, *but they do not admit rational solutions of the type* $-2t+P/Q$.

(3) $-2t+1/t$ *is a rational solution of* $P_4(-1, \pm 2)$. *Except this,* $P_4(-1, \pm 2)$ *do not admit rational solutions of the type* $-2t+P/Q$.

Proof. We will prove (1) only. Proofs of (2) and (3) are similar to that of (1).

By Cor. after Prop. 4–1, $-(2/3)t$ is a solution of $P_4(0, \pm 1/3)$. And by Prop. 4–9, $P_4(0, \pm 1/3)$ cannot have rational solutions of the types $-2t+P/Q$ (nor $-2t$), P/Q, P^*/Q.

Now, let λ be a rational solution of $P_4(0, \pm 1/3)$ of the type $-(2/3)t+P/Q$. Then, it has a Laurent expansion:

$$\lambda(t)=-\frac{2}{3}t+c_0+\frac{c_1}{t}+\frac{c_2}{t^2}+\frac{c_3}{t^3}+\frac{c_4}{t^4}+\cdots$$

at $t=\infty$.

By Prop. 4–1, (1), (a),

$$c_0=c_1=c_2=c_3=0.$$

So, assume that

$$c_1=c_2=\cdots=c_n=0 \qquad (n\geq 3).$$

Since

$$P_4\left(0, \pm\frac{1}{3}\right) \qquad \lambda''=\frac{(\lambda')^2}{2\lambda}+\frac{3}{2}\lambda^3+4t\lambda^2+2t^2\lambda-\frac{2}{9}\frac{1}{\lambda}$$

is equivalent to the equation

$$2\lambda\lambda''=(\lambda')^2+3\lambda^4+8t\lambda^3+4t^2\lambda^2-\frac{4}{9},$$

we substitute

$$\lambda=-\frac{2}{3}t+\frac{c_{n+1}}{t^{n+1}}+O\left(\frac{1}{t^{n+2}}\right)$$

into the latter equation.

Comparing the coefficients of $1/t^{n-2}$, we get

$$[3\times4\times(-\tfrac{2}{3})^3+8\times3\times(-\tfrac{2}{3})^2-\tfrac{16}{3}]c_{n+1}=\tfrac{16}{9}c_{n+1}=0.$$

Therefore, $c_{n+1}=0$.

By the induction, we can conclude that λ is equal to $-(2/3)t$. This is a contradiction. So, $P_4(0, \pm1/3)$ have a unique rational solution $-(2/3)t$. Q.E.D.

To prove the following proposition, we must recall the relations among the equations P_4, S_4 and E_4 (see § 1–2°).

Proposition 5-2. (1) $P_4(1, 0)$ *does not admit a rational solution of the type* $-(2/3)t+P/Q$ *(nor* $-(2/3)t$*).*

(2) $P_4(1, 0)$ *does not admit a rational solution of the type* $-2t+P/Q$ *(nor* $-2t$*).*

Proof. (1) By Cor., (1) after Prop. 4–1, $-(2/3)t$ is not a solution of $P_4(1, 0)$.

Assume that there exist a rational solution λ of $P_4(1, 0)$ of the type $-(2/3)t+P/Q$. Then, by Prop. 4–1, (1), (a), λ has the Laurent expansion:

$$\text{(A)} \qquad \lambda(t)=-\frac{2}{3}t+\frac{1}{t}-\frac{1}{t^3}+O\left(\frac{1}{t^4}\right)$$

at $t=\infty$.

Now since

$$\theta_0=\theta=0, \qquad \theta_\infty=\tfrac{1}{2}(\alpha+\theta-1)=0,$$

(λ, μ) is a rational solution of $S_4(0, 0)$, where

$$\mu(t)=\frac{1}{4\lambda(t)}[\lambda'(t)+\lambda(t)^2+2t\lambda(t)].$$

Therefore,

$$\begin{aligned}h(t)&=H_4(t, \lambda(t), \mu(t))\\&=2\lambda\mu^2-(\lambda^2+2t\lambda)\mu\end{aligned}$$

$$=\mu[2\lambda\mu-(\lambda^2+2t\lambda)]$$
$$=\frac{1}{8\lambda}(\lambda'+\lambda^2+2t\lambda)(\lambda'-\lambda^2-2t\lambda)$$

is a rational function, and is developped as

$$\text{(B)} \qquad h(t)=\frac{4}{27}t^3+\left(-\frac{1}{3}\right)\frac{1}{t}+O\left(\frac{1}{t^2}\right)$$

at $t=\infty$ by (A).

Since $h'(t)+2x_1=h'(t)+2x_2=h'(t)\not\equiv 0$ from (B), we can conclude that $h(t)$ is a non-polynomial rational solution of

$$E_4(0, 0, 0) \qquad (h'')^2-4(th'-h)^2+4h'^3=0,$$

and that it has the Laurent expansion (B) at $t=\infty$.

This fact is a contradiction, since $h(t)$, as a solution of $E_4(0, 0, 0)$, can not have a pole in $\boldsymbol{C}$. Consequently, there does not exist a rational solution of $P_4(1, 0)$ of the type $-(2/3)t+P/Q$.

(2) By Cor., (2) after Prop. 4–1, $-2t$ is not a solution of $P_4(1, 0)$.

Suppose that there exist a rational solution $\lambda=-2t+\sum_{j=1}^{q}\varepsilon_j/(t-b_j)$ of $P_4(1, 0)$, where $\varepsilon_j=1$ or -1 and b_j's are distinct q complex numbers. By Prop. 4–1, (1), (b), λ is developped as

$$\lambda(t)=-2t+\frac{-1}{t}+\frac{1}{t^3}+O\left(\frac{1}{t^4}\right)$$

at $t=\infty$.

As in the proof of (1), $h(t)=H_4(t, \lambda(t), \mu(t))$ is a rational function and $h(t)=O(1/t^4)$ at $t=\infty$.

Moreover,

$$h'(t)=\frac{\partial H_4}{\partial t}+\frac{\partial H_4}{\partial \lambda}\lambda'+\frac{\partial H_4}{\partial \mu}\mu'$$
$$=\frac{\partial H_4}{\partial t}$$
$$=-2\lambda\mu$$
$$=-\frac{1}{2}(\lambda'+\lambda^2+2t\lambda).$$

If $h'(t)\not\equiv 0$, $h(t)$ is a solution of $E_4(0, 0, 0)$ and has a zero at $t=\infty$. But, the direct examination of $E_4(0, 0, 0)$ shows that no solution of $E_4(0, 0, 0)$ has a zero at $t=\infty$. So $h'(t)$ must be identically zero. That is, λ should be a solution of the Riccati equation $\lambda'+\lambda^2+2t\lambda=0$.

We note that this Riccati equation is transformed into the linear equation (L) $u''-2tu'-2t=0$, by the change of the dependent variable: $u=\exp\int(\lambda+2t)dt$.

Therefore,

$$u=\exp\left(\int\sum_{j=1}^{q}\frac{\varepsilon_j}{(t-b_j)}dt\right)$$

should be a soultion of (L). Since any solution of (L) has no pole in $\boldsymbol{C}$, it turns out that $u=A(t-b_1)(t-b_2)\cdots(t-b_q)$, where A denotes a nonzero complex constant. However, by the direct examination of (L), we find that any solution of (L) has no pole at $t=\infty$. This is a contradiction.

After all, there does not exist a rational solution of $P_4(0, 1)$ of the type $-2t+P/Q$. Q.E.D.

2° From now on, we will investigate rational solutions of $P_4(\alpha, \theta)$ according to the types.

(i) On the type $-(2/3)t+P/Q$ and the solution $-(2/3)t$

Proposition 5-3. (1) *When $(\alpha, \theta)\in X$, if $P_4(\alpha, \theta)$ has a rational solution of the type $-(2/3)t+P/Q$, it is transformed into a rational solution of the type $-(2/3)t+P/Q$ of $P_4(\bar{\alpha}, \bar{\theta})$ with $(\bar{\alpha}, \bar{\theta})\in X$, by W, $W_{\pm}$, $T_{\pm}$, T_1 and T_2. And also, $-(2/3)t$ of $P_4(0, \pm 1/3)$ is transformed into a rational solution of the type $-(2/3)t+P/Q$ of $P_4(\alpha, \theta)$ with $(\alpha, \theta)\in X$, by $T_{\pm}$, T_1 and T_2.*

(2) *When $(\alpha, \theta)\in Y$, if $P_4(\alpha, \theta)$ has a rational solution of the type $-(2/3)t+P/Q$, it is transformed into a rational solution of the type $-(2/3)t+P/Q$ of $P_4(\bar{\alpha}, \bar{\theta})$ with $(\bar{\alpha}, \bar{\theta})\in Y$, by the seven transformations.*

Proof. Since it is already shown that a rational solution of the type $-(2/3)t+P/Q$ and the solution $-(2/3)t$ are transformed into a rational solution of the type $-(2/3)t+P/Q$ (See Prop. 4–4 and 4–8, (1)), it is enough to check the changes of α, θ by the seven transformations:

$$(0, \pm\tfrac{1}{3})\in X\longrightarrow(\bar{\alpha}, \bar{\theta})\in X,$$
$$(\alpha, \theta)\in X\longrightarrow(\bar{\alpha}, \bar{\theta})\in X,$$

and

$$(\alpha, \theta)\in Y\longrightarrow(\bar{\alpha}, \bar{\theta})\in Y.$$

We immediately find that $(0, \pm 1/3)\rightarrow(\bar{\alpha}, \bar{\theta})\in X$, from Prop. 2–6 and 2–7. Moreover, by the relationships: $W_+\circ W_-=W_-\circ W_+=id$, $T_+\circ T_-=T_-\circ T_+=id$ and $W^2=T_1^2=T_2^2=id$, it is sufficient to see that $(\alpha, \theta)\in Y\rightarrow(\bar{\alpha}, \bar{\theta})\in Y$. But this is easily verified from Prop. 2–6, 2–7, 2–8. Q.E.D.

As we can apply T_+, T_- and T_1 to rational solutions of the type $-(2/3)t+P/Q$ and to the solution $-(2/3)t$ without any restriction, we can define new transformations composed of these transformations.

Definition.

$$U_\pm = T_1 T_\pm T_1 T_\pm : (\alpha, \theta) \longrightarrow (\alpha \pm 1, \theta \pm 1),$$
$$V_\pm = T_1 (T_\pm)^2 T_1 T_\pm : (\alpha, \theta) \longrightarrow (\alpha, \theta \pm 2),$$

where the compositions are done from the right to the left.

By the above transformations, we get the following four sequences of the values of the parameters:

(a) $$\underset{\substack{\cap \\ X}}{(0, \pm\tfrac{1}{3})} \underset{T_\pm}{\longleftrightarrow} \underset{\substack{\cap \\ X}}{(2k, \pm\tfrac{1}{3})} \underset{V_\pm}{\longleftrightarrow} \underset{\substack{\cap \\ X}}{(2k, \pm\tfrac{1}{3}+2m)},$$

(b) $$\underset{\substack{\cap \\ X}}{(0, \tfrac{1}{3})} \underset{U_\pm}{\longleftrightarrow} \underset{\substack{\cap \\ X}}{(-1, -\tfrac{2}{3})} \underset{T_\pm}{\longleftrightarrow} \underset{\substack{\cap \\ X}}{(2k+1, -\tfrac{2}{3})} \underset{V_\pm}{\longleftrightarrow} \underset{\substack{\cap \\ X}}{(2k+1, -\tfrac{2}{3}+2m)},$$

(c) $$\underset{\substack{\cap \\ X}}{(0, -\tfrac{1}{3})} \underset{U_\pm}{\longleftrightarrow} \underset{\substack{\cap \\ X}}{(1, \tfrac{2}{3})} \underset{T_\pm}{\longleftrightarrow} \underset{\substack{\cap \\ X}}{(2k+1, \tfrac{2}{3})} \underset{V_\pm}{\longleftrightarrow} \underset{\substack{\cap \\ X}}{(2k+1, \tfrac{2}{3}+2m)},$$

(d) $$\underset{\substack{\cap \\ Y}}{(1, 0)} \underset{V_\pm}{\longleftrightarrow} \underset{\substack{\cap \\ Y}}{(1, 2m)} \underset{T_\pm}{\longleftrightarrow} \underset{\substack{\cap \\ Y}}{(1+2k, 2m)} \underset{U_\pm}{\longleftrightarrow} \underset{\substack{\cap \\ Y}}{(2(k+1), 2m+1)},$$

where k and m denote integers.

Now, we can derive the next conclusion from the relationships: $T_+ \circ T_- = T_- \circ T_+ = id$, $T_1^2 = id$, and Prop. 4–9, 5–1, (1), 5–2, (1) and the above (a)–(d).

Proposition 5-4. (1) *If (α, θ) belongs to X, then $P_4(\alpha, \theta)$ has a unique rational solution of the type $-(2/3)t+P/Q$. In the cases of $P_4(0, \pm 1/3)$, $-(2/3)t$ is the unique rational solution.*

(2) *If (α, θ) belongs to Y, then $P_4(\alpha, \theta)$ does not admit a rational solution of the type $-(2/3)t+P/Q$ nor the rational solution $-(2/3)t$.*

Proof. (1) By Prop. 5–1, (1), $P_4(0, \pm 1/3)$ have a unique rational solution $-(2/3)t$.

Let $(\alpha, \theta) \in X$, then $(\alpha, \theta) = (2k, \pm 1/3 + 2m)$ or $(2k+1, \pm 2/3 + 2m)$. For example, let $(\alpha, \theta) = (2k, 1/3+2m)$. By the chain (a) of transformations, a rational solution of $P_4(\alpha, \theta)$ of the type $-(2/3)t+P/Q$ can be constructed. Moreover, by $T_+ \circ T_- = T_- \circ T_+ = id$, $T_1^2 = id$, the uniqueness of the rational solution of $P_4(0, 1/3)$ and Prop. 4–9, (2), (3), such a rational solution is the unique one of $P_4(\alpha, \theta)$.

(2) Let $(\alpha, \theta) \in Y$. $P_4(\alpha, \theta)$ does not admit $-(2/3)t$ (Cor., (1) after Prop. 4–1).

If for some (α, θ) $P_4(\alpha, \theta)$ has a rational solution of the type $-(2/3)t+P/Q$, then, by the chain (d), $P_4(1, 0)$ should have a rational solution of the type $-(2/3)t+P/Q$. This is a contradiction (See Prop. 5–2, (1)). Q.E.D.

(ii) On the type $-2t+P/Q$ and the solution $-2t$

Proposition 5-5. *Let A_1, A_2, etc. are the same sets as in Def. in* § 0–2°.

(1) *If $(\alpha, \theta) \in A_1 \cup A_2$, then $P_4(\alpha, \theta)$ has a rational solution of the type $-2t+P/Q$ or the solution $-2t$. Moreover, the rational solution of this type to $P_4(\alpha, \theta)$ is uniquely determined. In particular, $-2t$ is the unique one to $P_4(0, \pm 1)$.*

(2) *If $(\alpha, \theta) \in Y-A_1 \cup A_2$, then $P_4(\alpha, \theta)$ does not admit a rational solution of the type $-2t+P/Q$ nor the solution $-2t$.*

Proof. (1) We note that $P_4(\alpha. \theta)$ admits $-2t$ if and only if $(\alpha, \theta)=(0, \pm 1)$ (See Cor., (2) after Prop. 4–1).

Now, we define the points P_k, Q_k, R_k and S_k on the lines L_1: $\theta=\alpha+1$, L_2: $\theta=\alpha+3$, L_3: $\theta=\alpha-3$, L_4: $\theta=\alpha-5$, by

$$P_k=(k, k+1), \qquad Q_k=(k-1, k+2),$$
$$R_k=(k+4, k+1), \qquad S_k=(k+6, k+1),$$

where k denotes a non-negative integer (Fig. 5–1).

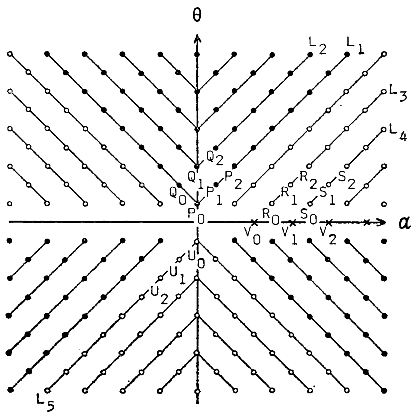

Fig. 5–1

First, by Prop. 5–1, (3), as for the type $-2t+P/Q$, there is a unique rational solution for Q_0.

Next, from Prop. 2–6, 2–7, 4–5, 4–6, we get the following sequence:

$$\begin{array}{ccccc} (\alpha,\theta)=Q_0 & & P_1 & & R_0 \\ -2t+\dfrac{P}{Q} & \underset{T_+}{\longrightarrow} & -2t+\dfrac{P_1}{Q_1} & \underset{T_1}{\longrightarrow} & \dfrac{P_2}{Q_2} \\ & & S_0 & & Q_1 \\ & \underset{T_+}{\longrightarrow} & \dfrac{P_3}{Q_3} & \underset{T_1}{\longrightarrow} & -2t+\dfrac{P_4}{Q_4}. \end{array}$$

From this, for Q_1 also, a rational solution of the type $-2t+P/Q$ exists and is uniquely determined as for this type.

Suppose that for each of $Q_0, Q_1, \cdots, Q_n$ ($n\geq 1$) a rational solution of the type $-2t+P/Q$ exists and that it is uniquely determined concerning this type. Then, by the sequence

$$\begin{array}{ccccc} (\alpha,\theta)=Q_n & & P_{n+1} & & R_n \\ -2t+\dfrac{P}{Q} & \underset{T_+}{\longrightarrow} & -2t+\dfrac{P_1}{Q_1} & \underset{T_1}{\longrightarrow} & \dfrac{P_2}{Q_2} \\ & & S_n & & Q_{n+1} \\ & \underset{T_+}{\longrightarrow} & \dfrac{P_3}{Q_3} & \underset{T_1}{\longrightarrow} & -2t+\dfrac{P_4}{Q_4}, \end{array}$$

it is also true of Q_{n+1}.

Consequently, as for the type $-2t+P/Q$, there is a unique rational solution for any Q_n (Because of Prop. 4–8, (2), we cannot use the solution $-2t$ of $P_4(0, 1)$ for the above induction).

Using T_+ and T_-, we find that a rational solution of the type $-2t+P/Q$ (or $-2t$) exists for every point in A_1 and that it is uniquely determined concerning this type (See Prop. 4–5, (2), (3) and Prop. 5–1, (2)). Moreover, it is also true of every point in A_2 by using W (See Prop. 4–5, (1)).

(2) By Prop. 5–2, (2) and $T_\pm$ (See Prop. 4–5, (2), (3)), we easily find that a rational solution of the type $-2t+P/Q$ does not exist for any point in D_1. By $W_\pm$, it is also true of any point in D_2.

We refer to the point $(-k, -(k+1))$ on the line L_5: $\theta=\alpha-1$ and the point $(2k+3, 0)$ in D_1 as U_k and V_k respectively, where k denotes a non-negative integer.

Then, from the transformation

$$T_2\colon U_k \longleftrightarrow V_k, \quad \frac{P^*}{Q} \longleftrightarrow -2t+\frac{P_1}{Q_1}$$

(See Prop. 4–5, (3) and Prop. 4–7, (3)), we find that there is no rational solution of the type P^*/Q for any U_k. By $T_\pm$, we also find that a rational solution of the type P^*/Q does not exist for any point in A_2. And by W, there does not exist a rational solution of the type P/Q for any point in A_1 (See Prop. 4–6, (1)).

So, by the transformation

$$T_1\colon R_{k-1} \longleftrightarrow P_k, \quad -2t+\frac{P}{Q} \longleftrightarrow \frac{P_1}{Q_1}$$

(See Prop. 4–5, (3) and Prop. 4–6, (3)), there is no rational solution of the type $-2t+P/Q$ for any R_k. Using $T_\pm$, we find that there does not exist a rational solution of the type $-2t+P/Q$ for any point in B_1.

Finally, by W and $W_\pm$, we find that it is also true of any point in B_2, C_1 and C_2. Q.E.D.

(iii) On the type P/Q

Proposition 5-6. (1) *If* $(\alpha, \theta) \in B_1 \cup B_2$, *then* $P_4(\alpha, \theta)$ *has a unique rational solution of the type* P/Q.

(2) *If* $(\alpha, \theta) \in Y - B_1 \cup B_2$, *then* $P_4(\alpha, \theta)$ *does not admit a rational solution of the type* P/Q.

Proof. (1) In Prop. 5–5, (1), we showed that there exists a rational solution of the type $-2t+P/Q$ (or $-2t$) for any point in A_1 and that it is uniquely determined. From this and the transformation

$$T_1\colon P_k \longleftrightarrow R_{k-1}, \quad -2t+\frac{P}{Q} \longleftrightarrow \frac{P_1}{Q_1},$$

for every point $R_k (k \geq 0)$, there is a rational solution of the type P/Q and it is uniquely determined concerning this type. By $T_\pm$, we find that a rational solution of the type P/Q uniquely exists for every point in B_1. And by W, $W_\pm$, it is also true of every point in B_2.

(2) In the proof of Prop. 5–5, (2), we found that there does not exist a rational solution of the type P/Q for any point in A_1. Therefore, by $T_\pm$, it is also true of any point in C_1. And so, by W, $W_\pm$, there does not exist a rational solution of the type P/Q for any point in C_2. Using $T_\pm$ again, it is also true of any point in A_2. By Prop. 4–9, (3), of course, there does not exist a rational solution of the type P/Q for any point in D_1 and D_2. Q.E.D.

(iv) On the type P^*/Q

Proposition 5-7. (1) *If* $(\alpha, \theta) \in C_1 \cup C_2$, *then* $P_4(\alpha, \theta)$ *has a unique rational solution of the type* P^*/Q.

(2) *If* $(\alpha, \theta) \in Y - C_1 \cup C_2$, $P_4(\alpha, \theta)$ *does not admit a rational solution of the type* P^*/Q.

Proof. (1) and (2) are proved by the application of W to rational solutions of the type P/Q. Q.E.D.

3° We now prove (B) in § 4 and Th. 2, (2).

Proof of (B). From Prop. 4–3, 4–9, 5–4, 5–5, 5–6, 5–7, we obtain the following conclusion:

(i) If $(\alpha, \theta) \in X$, $P_4(\alpha, \theta)$ has a unique rational solution of the type $-(2/3)t+P/Q$. In the cases of $P_4(0, \pm 1/3)$, $-(2/3)t$ is the unique one.

(ii) If $(\alpha, \theta) \in A_1 \cup A_2$, $P_4(\alpha, \theta)$ has a unique rational solution of the type $-2t+P/Q$. In the cases of $P_4(0, \pm 1)$, $-2t$ is the unique one.

(iii) If $(\alpha, \theta) \in B_1 \cup B_2$, $P_4(\alpha, \theta)$ has a unique rational solution of the type P/Q.

(iv) If $(\alpha, \theta) \in C_1 \cup C_2$, $P_4(\alpha, \theta)$ has a unique rational solution of the type P^*/Q.

(v) If $(\alpha, \theta) \in D_1 \cup D_2$, $P_4(\alpha, \theta)$ does not admit a rational solution.

These imply that if $(\alpha, \theta) \in X \cup Z$ then $P_4(\alpha, \theta)$ has a unique rational solution, and that if $(\alpha, \theta) \in D_1 \cup D_2$ then $P_4(\alpha, \theta)$ does not admit a rational solution.

Q.E.D.

Proof of Th. 2, (2). Comparing Prop. 4–3 with (i)-(v) in the proof of (B), we obtain the table in Th. 2, (2). Q.E.D.

On Remark 2 *after Th.* 2. A rational solution $Ct+P/Q$ of $P_4(\alpha, \theta)$, where C denotes $-2/3$ or -2 or 0, is obtained from the solution $Ct+P_1/Q_1$ of $P_4(-\alpha, \theta)$ both by W_+ and by W_-. So it follows that the set of the poles $\{b_1, \cdots, b_r\}$ of $Ct+P/Q$ has a property that $\{b_1, \cdots, b_r\}=\{-b_1, \cdots, -b_r\}$ (See Prop. 2–8, (2), (3)). From this, Remark 2 is obvious.

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[14] ——, Transformation groups for Painlevé equations, to appear.

Appendix

1° Let T, T_+ and T_- be the same transformations as in Prop. 2–1, 2–2.

Then we can regard these as the transformations on the set $F_2=\{(\alpha, \lambda, \mu) \mid \alpha \in \boldsymbol{C}, (\lambda, \mu) \text{ is a solution of } S_2(\alpha)\}$. And in this sense, T, T_+ generate a transformation group $\tilde{G}_2$ on F_2.

In addition, we define the transformations, τ, τ_+ on the parameter space $\boldsymbol{C}$ as follows:

$$\tau: \boldsymbol{C} \longrightarrow \boldsymbol{C}, \quad \alpha \longmapsto -\alpha$$
$$\tau_+: \boldsymbol{C} \longrightarrow \boldsymbol{C}, \quad \alpha \longmapsto \alpha+1.$$

And we refer to the group generated by τ, τ_+ as G_2.

Then, on these groups, Okamoto [14] indicated the following fact.

Proposition A-1. *$\tilde{G}_2 \simeq G_2 \simeq W_a(A_1)$, where $W_a(A_1)$ is the Affine Weyl group associated with the Lie algebra of type A_1.*

In fact, if we restrict the domain of action of G_2 to $\boldsymbol{R}$, we can regard open intervals $(n, n+1/2)$, $n \in \boldsymbol{Z}$, as Weyl chambers (See Fig. A–1).

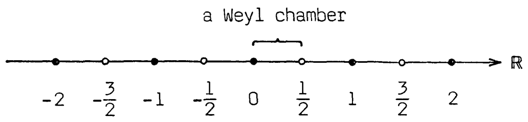
On the other hand, we note that solutions λ, μ of $S_2(\alpha)$ are rational functions if and only if the corresponding solution λ of $P_2(\alpha)$ is a rational function. So, by Th. 1, (1), $S_2(\alpha)$ has a unique rational solution (λ, μ) only when α is an integer.

And, by the successive applications of T_+ and T_- to the Riccati solutions of $S_2(1/2)$ (See Remark in § 1–1°), we find that when $\alpha=n+1/2$ ($n \in \boldsymbol{Z}$) $S_2(\alpha)$ has a one parameter family of solutions which are rational functions of Airy's functions A_i, B_i and their derivatives.

Therefore, $S_2(\alpha)$ has degenerated solutions when α is located in one of the walls of the Weyl chambers of G_2 (Fig. A–1).

2° Let T_σ ($\sigma \in \mathfrak{S}_3$), T_+ and T_- be the same transformations as in Prop. 2–4, 2–5.

As in the case of $S_2(\alpha)$, we regard these as the transformations on the set $F_4=$



● a rational solution
○ a family of solutions expressible by Airy's functions

Fig. A-1

$\{(x_0, x_1, x_2, \lambda, \mu) \mid (x_0, x_1, x_2) \in H, (\lambda, \mu)$ is a solution of $S_4(x_0, x_1, x_2)\}$, and refer to the group generated by T_σ, T_+ as $\tilde{G}_4$. Moreover, we define the transformations τ_σ $(\sigma \in \mathfrak{S})$, τ_+ on the parameter space H as follows:

$$\tau_\sigma: H \longrightarrow H, \quad (x_0, x_1, x_2) \longrightarrow (x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)})$$
$$\tau_+: H \longrightarrow H, \quad (x_0, x_1, x_2) \longrightarrow (x_0 - \tfrac{1}{3}, x_1 - \tfrac{1}{3}, x_2 + \tfrac{2}{3}).$$

And we refer to the group generated by τ_σ, τ_+ as G_4.

On these groups, Okamoto [14] pointed out the following fact.

Proposition A-2. *Let (,) be the Killing form and $\mathfrak{h}$ be a Cartan subalgebra and Δ be the root system of a semisimple complex Lie algebra. We define the group D as follows:*

$$D = \{T_d \mid d \in \mathfrak{h}_R^* \text{ and for any } \lambda \in \sum_{\alpha \in \Delta} \mathbf{Z}\alpha, (d, \lambda) \in \mathbf{Z}\},$$

where $\mathfrak{h}_R^*$ is the dual space of the real part of $\mathfrak{h}$ and

$$T_d: \mathfrak{h}_R^* \longrightarrow \mathfrak{h}_R^*, \quad x \longmapsto x + d.$$

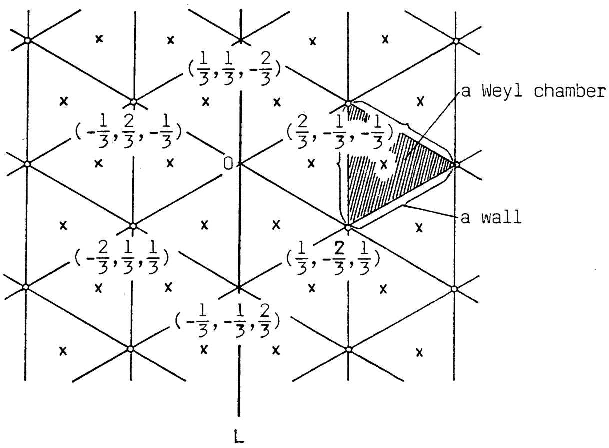
Then, $\tilde{G}_4 \simeq G_4 \simeq W(A_2) \ltimes D$, the semiproduct of $W(A_2)$ and D, where $W(A_2)$ is the Weyl group associated with the simple Lie algebra of type A_2 and D denotes the above parallel translation group for this algebra.

On the other hand, solutions, λ, μ of $S_4(x_0, x_1, x_2)$ are rational functions if and only if the corresponding solution λ of $P_4(\alpha, \theta)$ is a rational function, where $\alpha = -x_0 - x_1 + 2x_2 + 1$, $\theta = x_1 - x_0$.

So, by Th. 2, $S_4(x_0, x_1, x_2)$ has a unique rational solution only when (x_0, x_1, x_2) is located either at one of the vertices of the Weyl chambers of G_4 except for ones on the line L, or one of the barycenters of Weyl chambers (See Fig. A-2).

The rational solution of $P_4(\alpha, \theta)$ which corresponds to $S_4(x_0, x_1, x_2)$ with (x_0, x_1, x_2) at one of the barycenters of Weyl chambers is $-(2/3)t$ or the type $-(2/3)t + P/Q$.

By the applications of T_σ $(\sigma \in \mathfrak{S}_3)$, T_+ and T_- to the Riccati solutions of $S_4(-\theta_0/3, 2\theta_0/3, -\theta_0/3)$ (See Remark in § 1–2°), we also find that $S_4(x_0, x_1, x_2)$ has a one parameter family of solutions which are rational functions of the two independent solutions of the Hermite's equation: $u'' - 2tu' + 2(\theta_0 - 1)u = 0$ and their



× a rational solution of the type $-(2/3)t+P/Q$ or the solution $-(2/3)t$
○ a rational solution of the other type

Fig. A–2

derivatives when the set $\{x_0, x_1, x_2\}$ is equal to the set

$$\{-\tfrac{1}{3}(\theta_0+n),\ \tfrac{1}{3}(2\theta_0-n),\ \tfrac{1}{3}(-\theta_0+2n)\},$$

where $\theta_0 \in \boldsymbol{C}$, $n \in \boldsymbol{Z}$.

In particular, when (x_0, x_1, x_2) is located in one of the walls of Weyl chambers of G_4 (See Fig. A–2), $S_4(x_0, x_1, x_2)$ has such a one parameter family of solutions. When (x_0, x_1, x_2) is at one of the vertices except for ones on the line L, the unique rational solution is included in such a one parameter family of solutions.

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