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Studies on the Painlevé Equations IV. Third Painlevé Equation $\mathbf{P}_{III}$

By

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The present article deals with the third Painlevé equation $\mathbf{P}_{III}$; we consider instead the equation $\mathbf{P}_{III'}$, equivalent to the former. By defining the Painlevé system $\mathscr{H}$, we consider the group $\mathbf{G}_*$ of birational canonical transformations of $\mathscr{H}$; $\mathbf{G}_*$ is isomorphic to the affine Weyl group of the root system of the type B_2. A sequence of solutions of $\mathscr{H}$ is obtained from that of τ-functions, satisfying the Toda equation and vice versa. We consider also particular solutions of $\mathscr{H}$ written in terms of the cylinder function.

Contents

Introduction

The present article concerns the third Painlevé equation:

$$\mathbf{P}_{III} \qquad \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{1}{t}(\alpha q^2+\beta) + \gamma q^3 + \frac{\delta}{q},$$

for which we make the assumption, $\gamma\delta \neq 0$, throughout this paper. This equation is well-known and investigated by many authors (for example, [3], [5], [6], [7]), while we consider in the following mainly the equation:

$$\mathbf{P}_{III'} \qquad \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{q^2}{4t^2}(\gamma q+\alpha) + \frac{\beta}{4t} + \frac{\delta}{4q},$$

instead. These two equations are equivalent each other. In fact, by replacing in $\mathbf{P}_{III'}$, t by t^2 and q by tq, we obtain $\mathbf{P}_{III}$. Therefore a result on $\mathbf{P}_{III'}$ can be translated immediately to that of $\mathbf{P}_{III}$. We do not repeat results one by one. We sum up in §1.1 known facts about the equation $\mathbf{P}_{III'}$ or $\mathbf{P}_{III}$. As for the origin of the equation $\mathbf{P}_{III'}$, refer to [8], [10].

The Hamiltonian associated with $\mathbf{P}_{III}$ is:

$$\mathbf{H}_{III} \qquad \frac{1}{t}[2q^2p^2-\{2\eta_\infty tq^2+(2\theta_0+1)q-2\eta_0 t\}p+\eta_\infty(\theta_0+\theta_\infty)tq],$$

where the constants η_Δ, θ_Δ ($\Delta=0, \infty$) are connected to $\alpha, \beta, \gamma, \delta$ of the equation as follows:

$$(0.1) \qquad \alpha = -4\eta_\infty\theta_\infty, \quad \beta = 4\eta_0(\theta_0+1), \quad \gamma = 4\eta_\infty^2, \quad \delta = -4\eta_0^2.$$

By the assumption, we have $\eta_\Delta \neq 0$; moreover we set $\eta_\Delta=1$ without loss of generality. On the other hand, the Hamiltonian associated with $\mathbf{P}_{III'}$ is:

$$\mathbf{H}_{III'} \qquad \frac{1}{t}\left[q^2p^2-\{\eta_\infty q^2+\theta_0 q-\eta_0 t\}p+\frac{1}{2}\eta_\infty(\theta_0+\theta_\infty)q\right].$$

These two Hamiltonians are connected mutually through the canonical transformation ϕ:

$$(0.2) \qquad q \longrightarrow tq, \quad p \longrightarrow t^{-1}p, \quad t \longrightarrow t^2,$$

$$H_{III'} \longrightarrow \frac{1}{2t}\left(H_{III}+\frac{1}{t}qp\right).$$

The Painlevé system $\mathscr{H}_{III'}$ (resp. $\mathscr{H}_{III}$) associated with the equation $\mathbf{P}_{III'}$ (resp. $\mathbf{P}_{III}$) is by definition the quartet:

$$(q, p, H, t)$$

such that $H=H_{III'}(t; q, p)$ (resp. $H=H_{III}(t; q, p)$). We consider in this paper mainly the Painlevé system $\mathscr{H}_{III'}$. All of results obtained for $\mathscr{H}_{III'}$ can be translated to what concerns the other $\mathscr{H}_{III}$ through the canonical transformation:

$$\phi; \quad \mathscr{H}_{III'} \longrightarrow \mathscr{H}_{III}$$

given by (0.2).

Let $\mathscr{H}$ be the Painlevé system associated with $\mathbf{P}_{III'}$. A solution (q, p) of the system of differential equations:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dq}{dt} = -\frac{\partial H}{\partial q}, \tag{0.3}$$

with $H=H_{III'}(t; q, p)$ is called simply a *solution of* $\mathscr{H}$. The *τ-function* related to (q, p) is defined by:

$$H = \frac{d}{dt} \log \tau, \tag{0.4}$$

where H is the Hamiltonian function:

$$H = H(t) = H(t; q(t), p(t)).$$

In a similar manner to [9], we have the

Proposition 0.1. *The τ-function of* $\mathscr{H}$ *is holomorphic on the universal covering surface* $\mathfrak{B}$ *of* $\mathbf{C}$-$\{0\}$. *All of zeros of* $\tau(t)$ *is simple.*

The proof of this proposition is given in §3.2.

Let $\mathbf{V}$ be the two dimensional complex vector space. We regard $\mathbf{V}$ as the space of parameters of the Painlevé system through

$$v_1 = \theta_0, \quad v_2 = \theta_\infty. \tag{0.5}$$

The Painlevé system at $\mathbf{v}=(v_1, v_2)$ is written as $\mathscr{H}(\mathbf{v})$. The Painlevé system $\mathscr{H}$ is provided with the structure of a fiber space over the base space $\mathbf{V}$ such that the fiber on a point $\mathbf{v}$ of $\mathbf{V}$ is the Painlevé system $\mathscr{H}(\mathbf{v})$ at $\mathbf{V}$. Let σ be a canonical transformation of $\mathscr{H}$. The restriction $\sigma_{\mathbf{v}}$ of σ to $\mathscr{H}(\mathbf{v})$ is denoted also by σ in the following of this paper. If there exists a transformation g of $\mathbf{V}$ such that

$$\sigma: \mathscr{H}(\mathbf{v}) \longrightarrow \mathscr{H}(g(\mathbf{v}))$$

for any $\mathbf{v}$, then we write $\sigma=g_*$. We say g_* is associated with g. In Section 2, we will consider the group $\mathbf{G}$, isomorphic to the affine Weyl group of the root system of the type B_2 and show that there exists for any g of $\mathbf{G}$ the canonical transformation g^* associated with g. The homomorphism:

$$\rho: \mathbf{G} \longrightarrow \mathbf{G}_*,$$

thus obtained is called, for short, the *nonlinear representation of* **G** *on the Painlevé system*, where $\mathbf{G}_*$ is the group generated by g^*'s.

Let $H(\mathbf{v})=H(t; \mathbf{v})$ be a Hamiltonian function related to a solution $(q, p)=(q(\mathbf{v}), p(\mathbf{v}))$ of $\mathscr{H}(\mathbf{v})$, We will define the auxiliary Hamiltonian functions:

$$h = tH + \frac{1}{4} v_1^2 - \frac{1}{2} t \tag{0.6}$$

and show it satisfies the differential equation:

$$\left(t \frac{d^2h}{dt^2}\right)^2 + v_1 v_2 \frac{dh}{dt} - \left\{4\left(\frac{dh}{dt}\right)^2 - 1\right\}\left(h - t\frac{dh}{dt}\right) - \frac{1}{4}(v_1^2+v_2^2) = 0. \tag{0.7}$$

There is the one-to-one correspondence Γ from a particular solution $h=h(\mathbf{v})$ of (0.7) to a solution (q, p) of $\mathscr{H}(\mathbf{v})$: see Proposition 1.8. In particular, we have

$$\frac{dh}{dt} = p - 1, \tag{0.8$_1$}$$

$$t \frac{d^2h}{dt^2} = -2qp(p-1) + v_1 p - \frac{1}{2}(v_1+v_2). \tag{0.8$_2$}$$

We can compute the explicit forms of various birational canonical transformations by means of the correspondence Γ.

The differential equation (0.7) admits of a singular solution, which is characterized by:

$$\frac{d^2h}{dt^2} = \frac{dp}{dt} = 0.$$

It follows from $(0.8)_2$ that

$$p \equiv 0 \quad \text{or} \quad p \equiv 1,$$

corresponding to respectively

$$v_1 + v_2 = 0 \quad \text{or} \quad v_1 - v_2 = 0.$$

These two lines of **V** are walls of the Weyl chamber of the Weyl group **W** of the type B_2, and connected each other through the transformation:

$$s_2(\mathbf{v}) = (v_1, -v_2). \tag{0.9}$$

We will see in Proposition 1.6 the canonical transformation $(s_2)_*$ associated with

(0.9) is given by the replacement:

$$q \longrightarrow -q, \quad p \longrightarrow 1-p,$$
$$H \longrightarrow 1-H, \quad t \longrightarrow -t.$$

Therefore we consider only the case $v_1+v_2=0$. By means of the Hamiltonian system (0.3), q satisfies the Riccati equation:

(0.10) $$t\frac{dq}{dt} = -q^2 - \theta_0 q + t \quad (\theta_0 = v_1),$$

which can be linearized by:

(0.11) $$q = -\frac{1}{2}\theta_0 + t\frac{d\mathfrak{z}}{dt}.$$

We obtain in fact:

(0.12) $$\frac{d^2\mathfrak{z}}{dt^2} + \frac{1}{t}\frac{d\mathfrak{z}}{dt} - \left[\frac{1}{t} + \left(\frac{\theta_0}{2t}\right)^2\right]\mathfrak{z} = 0,$$

hence

$$\mathfrak{z} = Z_\nu(2\sqrt{-t}), \quad \nu = \theta_0.$$

Here $Z_\nu(r)$ is the cylinder function, that is, a solution of the linear equation:

(0.13) $$\frac{q^2\mathfrak{z}}{dr^2} + \frac{1}{r}\frac{d\mathfrak{z}}{dr} + \left(1 - \frac{\nu^2}{r^2}\right)\mathfrak{z} = 0:$$

the Bessel function $J_\nu(r)$, the Hankel functions $H_\nu^{(i)}(r)$ and so on.

A solution of the Painlevé system $\mathscr{H}$, of the form (0.11)$^{-}$(0.12), is called a *classical solution* of $\mathscr{H}$. A birational canonical transformation of $\mathscr{H}$ can be extended even in the case when the auxiliary function is reduced to a linear function of t, namely, a singular solution of (0.7). We will show in Section 4 that the Painlevé system $\mathscr{H}(\mathbf{v})$ has a classical solution if

$$v_1 \pm v_2 = 2m,$$

m being integers. Consider the contiguity relations of the cylinder function:

$(0.14)_1$ $$Z_{\nu+1}(r) = -\frac{dZ_\nu}{dr} + \nu r^{-1}Z_\nu,$$

$(0.14)_2$ $$Z_{\nu-1}(r) = \frac{dZ_\nu}{dr} + \nu r^{-1}Z_\nu.$$

It is known ([12]) that the functions:

$$\tilde{Z}_\nu(r) = r^{\nu^2} \exp\left(\frac{1}{4} r^2\right) Z_\nu(r) \quad (\nu \in \mathbf{C}),$$

satisfy the equation

$$\left(r \frac{d}{dr}\right)^2 \log \tilde{Z}_\nu(r) = \frac{\tilde{Z}_{\nu-1}(r)\tilde{Z}_{\nu+1}(r)}{\tilde{Z}_\nu(r)^2}. \tag{0.15}$$

We gain thus the sequence of solutions of $\mathscr{H}$, of the form (0.10), connected to the Toda equation (0.15).

A canonical transformation g_*, associated with g of the group $\mathbf{G}$, yields in a natural manner the correspondence between the two τ-functions, $\tau(\mathbf{v})$ and $\tau(g(\mathbf{v}))$. We will write it also as $\tau(g(\mathbf{v})) = g_*\tau(\mathbf{v})$. Given a τ-function τ, we say that a function τ_1 is equivalent to τ, if

$$\frac{d}{dt} \log \tau_1 - \frac{d}{dt} \log \tau$$

is a rational function of t; τ_1 is called also a τ-function. Let $\tau = \tau(\mathbf{v})$ be a τ-function of $\mathscr{H}(\mathbf{v})$. Starting from τ, we obtain the sequence of τ-functions:

$$\mathfrak{T}(g) = \{\tau_m;\ m \in \mathbf{Z}\}. \tag{0.16}$$

such that:

$$\tau_0 = \tau, \quad \tau_{m+1} = g_*\tau_m.$$

Note (0.16) is determined uniquely by τ_0, up to multiplicative constants of τ_m. We call (0.16) the *τ-sequence* with respect to g. By replacing τ_m by the equivalent one, τ_m^0, in the suitable manner, we will show, for the certain parallel transformation ℓ of $\mathbf{V}$, the τ-sequence

$$\mathfrak{T}^0(\ell) = \{\tau_m^0;\ m \in \mathfrak{Z}\}$$

is subject to the Toda equation:

$$\delta^2 \log \tau_m^0 = \frac{\tau_{m-1}^0 \tau_{m+1}^0}{(\tau_m^0)^2}, \tag{0.17}$$

where $\delta = t \dfrac{d}{dt}$.

In Section 1 we define at first the Painlevé system $\mathscr{H}_{III'}$ associated with the differential equation $\mathbf{P}_{III'}$. We show that the auxiliary Hamiltonian function (0.6) satisfies the nonlinear differential equation (0.7) and that there exists the one-to-one correspondence from a particular solution of (0.7) to a solution of $\mathscr{H}_{III'}$. Moreover some birational canonical transformations are derived from the symmetry of the Hamiltonian $\mathscr{H}_{III'}$.

The transformation group of $\mathcal{H}_{III'}$ is the subject of the second section. We construct the nonlinear representation of the affine Weyl group of the type B_2 on $\mathcal{H}_{III'}$, as birational canonical transformations (Theorem 1). We give the explicit forms of the various canonical transformations.

In Section 3 we study the τ-function of $\mathcal{H}_{III'}$. We show firstly the τ-function of $\mathcal{H}_{III'}$ is holomorphic on the universal covering surface of $\boldsymbol{C}\setminus\{0\}$ (see Proposition 0.1). The Painlevé transcendental function q, that is, a solution of the equation $\mathbf{P}_{III'}$, is written as the logarithmic derivative of the quotient of τ-functions (Proposition 3.2). Moreover we obtain the Toda equation (0.17) for the τ-sequence with respect to the parallel transformation: $\ell(\mathbf{v})=\mathbf{v}+(1, 1)$ (Theorem 2).

The final section is devoted to the studies on classical solutions. We consider the canonical transformations also in the degenerate case.

§1. Painlevé system

1.1. Painlevé equation $\mathbf{P}_{III'}$

In this paragraph we give a summary of results on the differential equation $\mathbf{P}_{III'}$ or $\mathbf{P}_{III}$, which we need later. First of all, it is easy to see:

Proposition 1.1. *A transformation of the form*:

(i) $t \mapsto -t$,

(ii) $q \mapsto -q$,

(iii) $q \mapsto tq$

yields in $\mathbf{P}_{III'}$ *only the change of constants*:

(i) $\beta \to -\beta$,

(ii) $\alpha \to -\alpha$, $\beta \to -\beta$,

(iii) $\alpha \to -\beta$, $\beta \to -\alpha$, $\gamma \to -\delta$, $\delta \to -\gamma$,

respectively.

Here we mean by $z \mapsto \phi(z)$ that one puts $z=\phi(z')$ and then rewrites z' as z. Moreover we have the

Proposition 1.2. $\mathbf{P}_{III'}$ *remains invariant under the replacement*:

$$q \mapsto \lambda q, \quad t \mapsto \mu t,$$

except for the change of the parameters:

$$\alpha \longrightarrow \lambda\alpha, \quad \beta \longrightarrow \mu\lambda^{-1}\beta, \quad \gamma \longrightarrow \lambda^2\gamma, \quad \delta \longrightarrow \mu^2\lambda^{-2}\delta,$$

λ, μ *being constants*.

Therefore, assuming $\gamma\delta \neq 0$, we can put, for example, (1.1) $$\gamma = 4, \quad \delta = -4,$$

without loss of generality. The Painlevé equations $\mathbf{P}_{III'}$ and $\mathbf{P}_{III}$ depend essentially on the *two* parameters α, β. As we have mentioned in the second part of this series of papers, it is known that

Proposition 1.3 ([4], 10]). *In the case $\gamma\delta \neq 0$, $\mathbf{P}_{III'}$ is equivalent to the fifth equation $\mathbf{P}_V$ with $\delta = 0$.*

When $\gamma\delta = 0$, we have:

Proposition 1.4 ([10]). *The equation $\mathbf{P}_{III'}$ with $\gamma = \delta = 0$ is transformed into the equation with $\alpha = \beta = 0$, $\gamma\delta \neq 0$.*

Proposition 1.5. *If $\beta = \delta = 0$, then $\mathbf{P}_{III'}$ can be solved by quadratures.*

Therefore, $\mathbf{P}_{III'}$ is soluble also in the case $\alpha = \gamma = 0$, by means of the transformation (iii) of Proposition 1.1. We give below a sketch of a proof of Proposition 1.5. In fact, the equation $\mathbf{P}_{III'}$ with $\beta = \delta = 0$:

(1.2) $$\frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{q^2}{4t^2}(\alpha + \gamma q)$$

possesses the integral:

(1.3) $$\left(t\frac{dq}{dt}\right)^2 - \frac{\gamma}{4}q^4 - \frac{\alpha}{2}q^3 = \lambda^2 q^2$$

λ denoting an integration constant. It follows that, if $\lambda \neq 0$, then

(1.4) $$q = \frac{2\lambda^2 F}{(F-1)\left(\left(\frac{1}{4}\alpha - \varepsilon\right)F - \left(\frac{1}{4}\alpha + \varepsilon\right)\right)},$$

$$F = \mu t^\lambda, \quad \varepsilon = \frac{1}{2}\sqrt{\gamma}\lambda,$$

and if $\lambda = 0$, then

(1.4)′ $$q = \frac{2\alpha}{\frac{1}{4}\alpha F^2 - \gamma},$$

$$F = \log t + \mu,$$

where μ is an arbitrary constant.

1.2. Painlevé system $\mathscr{H}_{III'}$

Let $\mathscr{H} = (q, p, H, t)$ be the Painlevé system $\mathscr{H}_{III'}$ associated with the equation $\mathbf{P}=\mathbf{P}_{III'}$. Consider the canonical transformation of $\mathscr{H}$:

(1.5) $$\psi(\lambda, \mu): (q, p, H, t) \longrightarrow (\lambda^{-1}q, \lambda p, \mu H, \mu^{-1}t),$$

λ, μ being non zero constants. Remark that, if $\lambda=-1$ (resp. $\lambda\mu=-1$), then $\psi(\lambda, \mu)$ yields the alternation of the sign of η_∞ (resp. η_0). In the following of this paper, we normalize as

$$\eta_0 = \eta_\infty = 1,$$

since by the assumption $\eta_0\eta_\infty \neq 0$. Namely the Hamiltonian of $\mathscr{H}$ is:

(1.6) $$H(t; q, p) = \frac{1}{t}\left[q^2p^2 - \{q^2+\theta_0 q - t\} p + \frac{1}{2}(\theta_0+\theta_\infty)q\right].$$

The Hamiltonian system of the differential equation:

$(1.7)_1$ $$t\frac{dq}{dt} = 2q^2p - q^2 - \theta_0 q + t,$$

$(1.7)_2$ $$t\frac{dp}{dt} = -2qp^2 + (2q+\theta_0)p - \frac{1}{2}(\theta_0+\theta_\infty)$$

is equivalent to $\mathbf{P}_{III'}$ with

(1.8) $$\alpha = -4\theta_\infty, \quad \beta = 4(\theta_0+1), \quad \gamma = 4, \quad \delta = -4.$$

Example 1.1. The Hamiltonian of (1.2) is written as:

$$H''(t; q, p) = \frac{1}{t}\left[q^2p^2 - q^2p + \frac{1}{2}\theta_\infty q\right].$$

For the solution (1.4), we have

$$p = \frac{1}{4\lambda}\left[(\lambda+\theta_\infty)F + \frac{\lambda-\theta_\infty}{F}\right] + \frac{1}{2}, \quad F = \mu t^\lambda,$$

and for (1.4)′

$$p = -\frac{1}{8}F + \frac{1}{2}, \quad F = \log t + \mu.$$

In any case, the Hamiltonian system has the first integral:

$$tH''(t; q, p) = \frac{1}{4}\lambda^2.$$

1.3. Symmetry of the Hamiltonian $\mathscr{H}_{III'}$

The transformations (i), (ii), (iii) of Proposition 1.1 are extended to the canonical transformations of the Painlevé system $\mathscr{H}=\mathscr{H}_{III'}$. To verify this fact, consider the following change of constants of the Hamiltonian:

(i) $s_2: \theta_\infty \to -\theta_\infty$,

(ii) $s: \theta_0 \to -\theta_0 - 2$,

(iii) $x: \theta_0 \to \theta_\infty - 1,\ \theta_\infty \to \theta_0 + 1$.

We prove:

Proposition 1.6. *There exists the canonical transformation of* $\mathscr{H}$, *representing each transformation of* (i), (ii), (iii).

In fact, consider the canonical transformation

(1.9) $$\pi': \mathscr{H} \longrightarrow \mathscr{H}' = (q, \bar{p}', \bar{H}', t)$$

such that

(1.10) $$\bar{p}' = p - 1, \quad \bar{H}' = H - 1.$$

Also the system $\bar{\mathscr{H}}'$ is associated with **P** and we have:

$$\bar{H}' = \frac{1}{t}\left[q^2(\bar{p}')^2 - \{-q^2 + \theta_0 q - t\}\bar{p}' + \frac{1}{2}(-\theta_0 + \theta_\infty)q\right].$$

Then the canonical transformation $\psi(-1, -1)\cdot\pi'$ keeps H invariant except for the change s_2 of constants; we denote it by $(s_2)_*$:

(i) $(s_2)_* = \psi(-1, -1)\cdot\pi'$

Set in the Hamiltonian (1.6):

(1.11) $$p = \bar{p} + \frac{\theta_0 + 1}{q} - \frac{t}{q^2}, \quad H = \bar{H} - \frac{1}{q} + \frac{\theta_0 + 1}{t} + 1;$$

we obtain:

(1.12) $$\bar{H} = \frac{1}{t}\left[q^2\bar{p}^2 - \{q^2 - (\theta_0 + 2)q + t\}\bar{p} + \frac{1}{2}(-\theta_0 - 2 + \theta_\infty)q\right].$$

It is easy to verify the transformation:

$$\pi: \mathscr{H} \longrightarrow \bar{\mathscr{H}} = (q, \bar{p}, \bar{H}, t)$$

is canonical and H remains invariant under the transformation $\psi(1, -1)\cdot\pi$ except for the change s. Consequently we have:

(ii) $s_* = \psi(1, -1)\cdot\pi$.

The transformation associated with x is given by:

(1.13) $$q = \frac{t}{q_1}, \quad p = \frac{1}{t}\left\{\frac{1}{2}(\theta_\infty)q_1 - q_1^2 p_1\right\},$$

$$H = H_1 - \frac{1}{t}q_1 p_1 + \frac{1}{4t}(\theta_\infty^2 - \theta_0^2).$$

We obtain in fact,

$$H_1 = \frac{1}{t}\left[q_1^2 p_1^2 - \{q_1^2 + (\theta_\infty - 1)q_1 - t\}p_1 + \frac{1}{2}(\theta_0 + \theta_\infty)q_1\right].$$

We write (1.13) also in the form:

(iii) $x_*: (q, p, H, t) \to (q_1, p_1, H_1, t)$,

$$q_1 = \frac{t}{q}, \quad p_1 = \frac{1}{t}\left\{\frac{1}{2}(\theta_0 + \theta_\infty)q - q^2 p\right\},$$

$$H_1 = H - \frac{1}{t}\left\{qp - \frac{1}{4}(\theta_0 + \theta_\infty)(\theta_0 + 2 - \theta_\infty)\right\}.$$

Remark 1.1. We have for (i), (ii), (iii) the relation:

$$x \cdot s_2 = s \cdot x,$$

and then the relation of the canonical transformations:

$$x_*(s_2)_* = s_* x_*.$$

1.4. Auxiliary Hamiltonian function

Let $H = H(t)$ be a Hamiltonian function related to a solution $(q, p) = (q(t), p(t))$ of $\mathscr{H} = \mathscr{H}_{III'}$. We define by:

(1.14) $$h = tH + \frac{1}{4}\theta_0^2 - \frac{1}{2}t,$$

the auxiliary Hamiltonian function $h = h(t)$ of H. Since

$$\frac{dh}{dt} = p - \frac{1}{2},$$

by virtue of (1.6), we obtain from (1.7),

$$2(-p^2 + p)q + \theta_0 p - \frac{1}{2}(\theta_0 + \theta_\infty) - t\frac{d^2h}{dt^2} = 0.$$

Hence, we have

$(1.15)_1$ $$q = -\frac{t\dfrac{d^2h}{dt^2} - \theta_0\dfrac{dh}{dt} + \dfrac{1}{2}\theta_\infty}{2\left(\dfrac{dh}{dt} - \dfrac{1}{2}\right)\left(\dfrac{dh}{dt} + \dfrac{1}{2}\right)},$$

$(1.15)_2$ $$p = \frac{dh}{dt} + \frac{1}{2}.$$

On the other hand, from (1.6) and $(1.15)_2$, it results that:

$$h - t\frac{dh}{dt} = \left(qp - \frac{1}{2}\theta_0\right)^2 - q\left(qp - \frac{1}{2}(\theta_0 + \theta_\infty)\right).$$

Therefore we arrive at the proposition:

Proposition 1.7. *h satisfies the differential equation*:

$$\mathbf{E}_{III'} \quad \left(t\frac{d^2h}{dt^2}\right)^2 + \theta_0\theta_\infty\frac{dh}{dt} - \left\{4\left(\frac{dh}{dt}\right)^2 - 1\right\}\left(h - t\frac{dh}{dt}\right) - \frac{1}{4}(\theta_0^2 + \theta_\infty^2) = 0.$$

Inversely, for a solution $h = h(t)$ of $\mathbf{E}_{III'}$, we define a pair (q, p) of functions by (1.15). Then (q, p) is actually a solution of the system (1.7), provided that

$$(1.16) \qquad \frac{d^2h}{dt^2} \not\equiv 0.$$

Consequently, we obtain the

Proposition 1.8. *There exists the one-to-one correspondence*:

$$(1.17) \qquad \Gamma(h) = (q, p)$$

from a particular solution of $\mathbf{E}_{III'}$ *to a solution of* $\mathscr{H}$.

The equation $\mathbf{E}_{III'}$ admits of a singular solution of the form:

$$(1.18) \qquad h = \lambda t + \mu,$$

$$\theta_0\theta_\infty\lambda - (4\lambda^2 - 1)\mu - \frac{1}{4}(\theta_0^2 + \theta_\infty^2) = 0.$$

1.5. Painlevé system $\mathscr{H}_{III}$

In this paragraph we state the results on the Painlevé system $\mathscr{H}_{III'}$ derived immediately from Propositions 1.7 and 1.8 by the canonical transformation ϕ of the form (0.2). Let $H = H(t)$ be a Hamiltonian function of $\mathscr{H}_{III}$ related to a solution $(q, p) = (q(t), p(t))$ of $\mathscr{H}_{III}$. We have:

Proposition 1.9. *The auxiliary Hamiltonian function*:

$$h = tH + \frac{1}{4}(2\theta_0 + 1)^2$$

satisfies the equation:

$$\mathbf{E}_{III} \quad \left[\left(t\frac{d^2h}{dt^2}\right)^2 - 4\left(h - t\frac{dh}{dt}\right)f\right]^2 - 096(\theta_0 - \theta_\infty + 1)^2\left(h - t\frac{dh}{dt}\right)^3 = 0,$$

$$f = \left(\frac{dh}{dt}\right)^2 + 16\left(h - t\frac{dh}{dt}\right) - 16\left(\theta_0 + \frac{1}{2}\right)\left(\theta_\infty - \frac{1}{2}\right).$$

A particular solution of $\mathbf{E}_{III}$ *is connected to a solution* (q, p) *as follows*:

$$q = 4\frac{A+\left(\theta_0+\frac{1}{2}\right)\sqrt{A}}{B},$$

$$p = \frac{B}{8\sqrt{A}},$$

$$A = h - t\frac{dh}{dt},$$

$$B = -t\frac{d^2h}{dt^2} + \frac{dh}{dt}\sqrt{A}.$$

Moreover we can verify:

Proposition 1.10. *The function* $\bar{h}$ *defined by*:

$$\bar{h} = tH + qp,$$

is a solution of the equation

$$\left(t\frac{d^2\bar{h}}{dt^2} - \frac{d\bar{h}}{dt}\right)^2 - 4\left\{\theta_0\frac{d\bar{h}}{dt} - 2(\theta_0+\theta_\infty)t\right\}^2 + \frac{d\bar{h}}{dt}\left(\frac{d\bar{h}}{dt} - 8t\right)\left(2t\frac{d\bar{h}}{dt} - 4\bar{h}\right) = 0.$$

Inversely (q, p) *is given by*

$$q = -2\cdot\frac{t\frac{d^2\bar{h}}{dt^2} - (2\theta_0+1)\frac{d\bar{h}}{dt} + 8(\theta_0+\theta_\infty)t}{\frac{d\bar{h}}{dt} - 8},$$

$$p = \frac{1}{4}\frac{d\bar{h}}{dt}.$$

We do not enter into details of verification of these propositions.

§2. Transformation group of $\mathscr{H}_{III'}$

2.1. Root system

Let $\mathbf{e}_1$, $\mathbf{e}_2$ be the canonical basis of the two dimensional complex vector space $\mathbf{V}$; we write a vector of $\mathbf{V}$ as $\mathbf{v}=v_1\mathbf{e}_1+v_2\mathbf{e}_2=(v_1, v_2)$. Recall $\mathbf{v}=(v_1, v_2)$ is regarded as parameters of the Painlevé system by means of

$$\theta_0 = v_1, \quad \theta_\infty = v_2 .$$

Consider in $\mathbf{V}$ the vectors:

$$\mathbf{a}_1 = \mathbf{e}_1 - \mathbf{e}_2, \quad \mathbf{a}_2 = \mathbf{e}_2 ,$$
$$\tilde{\mathbf{a}} = \mathbf{e}_1 + \mathbf{e}_2 .$$

Let $\mathbf{R}=\mathbf{R}_{III'}$ be the root system of the type B_2. Then $\mathbf{a}_i$ ($i=1, 2$) are the fundamental roots of $\mathbf{R}$ and $\tilde{\mathbf{a}}$ is the highest root ([1]). Denote by s_i the reflection of $\mathbf{V}$ with respect to the line:

$$(\mathbf{a}_i \mid \mathbf{v}) = 0,$$

and by s_0 the reflection with respect to

$$(\tilde{\mathbf{a}} \mid \mathbf{v}) = -1,$$

where $(\mathbf{v} \mid \mathbf{v}')$ is the symmetric bilinear form in $\mathbf{V}$ such that $(\mathbf{e}_i \mid \mathbf{e}_j)=(\mathbf{e}_j \mid \mathbf{e}_i)=\delta_{ij}$. We have:

$$s_1 : \mathbf{v} \longmapsto (v_2, v_1),$$
$$s_2 : \mathbf{v} \longmapsto (v_1, -v_2),$$
$$s_0 : \mathbf{v} \longmapsto (-1-v_2, -1-v_1).$$

Let $\mathbf{G}$ be the group generated by s_1, s_2 and s_0 and W the group generated by s_1 and s_2. Then $\mathbf{G}$ is isomorphic to the affine Weyl group $W_a(\mathbf{R})$ and $\mathbf{W}$ is the Weyl group $W(\mathbf{R})$. Now we state the theorem:

Theorem 1. *There exists the nonlinear representation of* $\mathbf{G}$ *on the Painlevé system* $\mathscr{H}$, *as the group* $\mathbf{G}_*$ *of birational canonical transformations.*

To establish this theorem, it suffices to determine the birational canonical transformations $(s_i)_*$, $(s_0)_*$. We will do this in the rest of this section.

2.2. Weyl group W

We construct $(s_i)_*$ ($i=1, 2$). The differential equation $\mathbf{E}_{III'}$:

$$\left(t\frac{d^2h}{dt^2}\right)^2 + v_1v_2\frac{dh}{dt} - \frac{1}{4}(v_1^2+v_2^2) - \left\{4\left(\frac{dh}{dt}\right)^2 - 1\right\}\left(h - t\frac{dh}{dt}\right) = 0$$

is invariant under the change s_1 of parameters. Hence we obtain $(s_1)^*$ from the schema:

$$\begin{array}{ccc} & \mathbf{E}(\mathbf{v}) = \mathbf{E}(s_1(\mathbf{v})) & \\ {}^{r}\swarrow & & \searrow^{r} \\ (q(\mathbf{v}), p(\mathbf{v})) & \cdots\cdots\rightarrow & (q(s_1(\mathbf{v})), p(s_1(\mathbf{v}))), \end{array}$$

where $\mathbf{E}(\mathbf{v})$ denotes the equation $\mathbf{E}_{III'}$ with the parameters $\mathbf{v}$. In fact, if h_1 is a solution of $\mathbf{E}(s_1(\mathbf{v}))$, then (q_1, p_1) given by:

$$(2.1)_1 \qquad q_1 = -\frac{t\dfrac{d^2h_1}{dt^2} - v_2\dfrac{dh_1}{dt} + \dfrac{1}{2}v_1}{2\left(\dfrac{dh_1}{dt} - \dfrac{1}{2}\right)\left(\dfrac{dh_1}{dt} + \dfrac{1}{2}\right)},$$

$$(2.1)_2 \qquad p_1 = \frac{dh_1}{dt} + \frac{1}{2}$$

is a solution of the Painlevé system $\mathscr{H}(s_1(\mathbf{v}))=(q_1, p_1, H_1, t)$ at $s_1(\mathbf{v})$ (see (1.15)). By putting

$$h = h_1,$$

we obtain from (1.15) and (2.1)

$$(2.2)_1 \qquad q_1 = q + \frac{\frac{1}{2}(v_2 - v_1)}{p-1}, \quad p_1 = p,$$

$$(2.2)_2 \qquad H_1 = H - \frac{1}{4t}(v_2^2 - v_1^2).$$

It is clear $(2.2)_1$, $(2.2)_2$ define a canonical transformation of the Painlevé system. We have thus $(s_1)_*$, while the transformation $(s_2)_*$ has been constructed already in Proposition 1.6 (the case (i)).

Remark 2.1. It is not difficult to realize the transformation s_0 as *the transformation of the Painlevé equation* $\mathbf{P}_{III'}$. In fact, the change of the variable:

$$(2.3) \qquad q \longrightarrow -\frac{t}{q},$$

keeps $\mathbf{P}_{III'}$ invariant except for the change of constants:

$$s_0 \colon \theta_0 \longrightarrow -\theta_\infty - 1, \quad \theta_\infty \longrightarrow -\theta_0 - 1.$$

Therefore we will have $(s_0)_*$ by extending (2.3) to a canonical transformation of $\mathscr{H}$.

2.3. Involution of E

The differential equation $\mathbf{E}=\mathbf{E}_{III'}$ is invariant under the involution of $\mathbf{V}$:

$$y \colon \mathbf{v} \longrightarrow -\mathbf{v}.$$

By using this fact, we obtain the transformation:

$$y_* \colon \mathscr{H}(\mathbf{v}) \longrightarrow \mathscr{H}(-\mathbf{v}).$$

In fact, since, by (1.15),

$$h = tH + \frac{1}{4}v_1^2 - \frac{1}{2}t$$
$$= tH_- + \frac{1}{4}(-v_1)^2 - \frac{1}{2}t = h_-,$$

$$q_- = -\frac{t\dfrac{d^2h}{dt^2} + v_1\dfrac{dh}{dt} - \dfrac{1}{2}v_2}{2\left(\dfrac{dh}{dt} - \dfrac{1}{2}\right)\left(\dfrac{dh}{dt} + \dfrac{1}{2}\right)},$$

$$p_- = \frac{dh}{dt} + \frac{1}{2},$$

it follows that:

$$q_- = q - \frac{v_1 p - \frac{1}{2}(v_1+v_2)}{p(p-1)},$$

$$p_- = p,$$

$$H_- = H.$$

Here we write $\mathscr{H}(-\mathbf{v}) = (q_-, p_-, H_-, t)$.

2.4. Auxiliary functions

Let $h = h(\mathbf{v})$ be a solution of the equation $\mathbf{E}(\mathbf{v})$. We define the auxiliary function $\mathbf{g} = \mathbf{g}(t; \mathbf{v})$ by

$$\mathbf{g} = h + \frac{1}{4}(2v_1+1) - X, \tag{2.4}$$

$$X = q(p-1). \tag{2.5}$$

We prove the proposition:

Proposition 2.1. *The function* $\mathbf{g}$ *satisfies the equation*:

$$\left(t\frac{d^2\mathbf{g}}{dt^2}\right)^2 + (v_1+1)(v_2+1)\frac{d\mathbf{g}}{dt} - \frac{1}{4}(v_1+1)^2 - \frac{1}{4}(v_2+1)^2 - \left\{4\left(\frac{d\mathbf{g}}{dt}\right)^2 - 1\right\}\left(\mathbf{g} - t\frac{d\mathbf{g}}{dt}\right) = 0. \tag{2.6}$$

Inversely, for a particular solution $\mathbf{g}$ *of* (2.6), *the solution* $(q, p) = (q(\mathbf{v}), p(\mathbf{v}))$ *of* $\mathscr{H}(\mathbf{v})$ *is given as follows*:

$$(2.7)_1 \qquad X = \frac{t\dfrac{d^2\mathbf{g}}{dt^2} + (v_1+1)\dfrac{d\mathbf{g}}{dt} - \dfrac{1}{2}(v_2+1)}{2\left(\dfrac{d\mathbf{g}}{dt} + \dfrac{1}{2}\right)},$$

$$(2.7)_2 \qquad q = \frac{t\left(\dfrac{d\mathbf{g}}{dt} - \dfrac{1}{2}\right)}{X - \dfrac{1}{2}(v_1-v_2)}.$$

Proof. We obtain from (1.7):

$$t\frac{dX}{dt} = -qX + \frac{1}{2}(v_1-v_2)q + t(p-1),$$

and then

$$(2.8) \qquad t\frac{d\mathbf{g}}{dt} = qX - \frac{1}{2}(v_1-v_2)q + \frac{1}{2}t.$$

On the other hand, since, by (2.4)

$$(2.9) \qquad \mathbf{g} - t\frac{d\mathbf{g}}{dt} = \left(X - \frac{1}{2}(v_1+1)\right)^2 + t(p-1),$$

we have

$$(2.10) \qquad t\frac{d^2\mathbf{g}}{dt^2} = 2X\left(\frac{d\mathbf{g}}{dt} + \frac{1}{2}\right) - (v_1+1)\frac{d\mathbf{g}}{dt} + \frac{1}{2}(v_2+1),$$

by differentiating (2.9) with respect to t. From (2.10) and (2.8) it results $(2.7)_1$ and $(2.7)_2$. The differential equation (2.6) follows immediately from (2.7), (2.8).

Remark 2.2. The function X defined by (2.5) is written also in the following form:

$$(2.11) \qquad X = -\frac{t\dfrac{d^2h}{dt^2} - v_1\dfrac{dh}{dt} + \dfrac{1}{2}v_2}{2\left(\dfrac{dh}{dt} + \dfrac{1}{2}\right)},$$

where h is the auxiliary Hamiltonian function (1.14).

Besides the function $\mathbf{g}$, we define by:

$$(2.12) \qquad \bar{\mathbf{g}} = h + \frac{1}{2}v_1 + \frac{1}{4} - Y,$$

$$(2.13) \qquad Y = qp$$

the other auxiliary function $\bar{\mathbf{g}} = \bar{\mathbf{g}}(t;\mathbf{v})$. Then we can verify in a manner similar to Proposition 2.1 the

Proposition 2.2. $\bar{\mathbf{g}}$ *is a solution of the differential equation*

$$\left(t\frac{d^2\bar{\mathbf{g}}}{dt^2}\right)^2 + (v_1+1)(v_2-1)\frac{d\bar{\mathbf{g}}}{dt} - \frac{1}{4}((v_1+1)^2 + (v_2-1)^2) - \left\{4\left(\frac{d\bar{\mathbf{g}}}{dt}\right)^2 - 1\right\}\left(\bar{\mathbf{g}} - t\frac{d\bar{\mathbf{g}}}{dt}\right) = 0, \tag{2.14}$$

and is connected to a solution of $\mathscr{H}(\mathbf{v})$ *as follows:*

$$Y = \frac{t\dfrac{d^2\bar{\mathbf{g}}}{dt^2} + (v_1+1)\dfrac{d\bar{\mathbf{g}}}{dt} - \dfrac{1}{2}(v_2-1)}{2\left(\dfrac{d\bar{\mathbf{g}}}{dt} - \dfrac{1}{2}\right)}, \tag{2.15$_1$}$$

$$q = -\frac{t\left(\dfrac{d\bar{\mathbf{g}}}{dt} + \dfrac{1}{2}\right)}{Y - \dfrac{1}{2}(v_1+v_2)}. \tag{2.15$_2$}$$

We omit the proof.

Remark 2.3. We obtain from (1.15):

$$Y = -\frac{t\dfrac{d^2h}{dt^2} - v_1\dfrac{dh}{dt} + \dfrac{1}{2}v_2}{2\left(\dfrac{dh}{dt} - \dfrac{1}{2}\right)}; \tag{2.16}$$

compare it with (2.11) and (2.15).

2.5. Parallel transformation ℓ

Set:

$$\ell = y\cdot s_0\cdot s_1 \;:\; \mathbf{v} \longrightarrow (v_1+1,\, v_2+1). \tag{2.17}$$

If we denote by $h(\mathbf{v})$ a particular solution of the differential equation $\mathbf{E}(\mathbf{v})$, the auxiliary function $\mathbf{g}$, defined by (2.4), is a solution of $\mathbf{E}(\ell(\mathbf{v}))$:

$$\mathbf{g} = h(\ell(\mathbf{v})). \tag{2.18}$$

Therefore we obtain the birational canonical transformation ℓ_* associated with ℓ by the use of the following diagram:

$$\begin{array}{ccc} \mathbf{E}(\mathbf{v}) & \longrightarrow & \mathbf{E}(\ell(\mathbf{v})) \\ r\big\downarrow & & \big\downarrow r \\ \ell_*: \mathscr{H}(\mathbf{v}) & \longrightarrow & \mathscr{H}(\ell(\mathbf{v})). \end{array}$$

In fact, we have from (2.11) and (2.18)

$$X_+ = -\frac{t\dfrac{d^2\mathbf{g}}{dt^2} - (v_1+1)\dfrac{d\mathbf{g}}{dt} + \dfrac{1}{2}(v_2+1)}{2\left(\dfrac{d\mathbf{g}}{dt} + \dfrac{1}{2}\right)},$$

where we write $\mathscr{H}(\ell(\mathbf{v}) = (q_+, p_+, H_+, t)$, $X_+ + q_+(p_+ - 1)$. It follows from $(2.7)_1$ that:

$$(2.19)_1 \qquad X_+ + X = v_1 + 1 - \frac{v_1+v_2+2}{2\left(\dfrac{d\mathbf{g}}{dt} + \dfrac{1}{2}\right)},$$

while we obtain:

$$(2.19)_2 \qquad \begin{aligned} \frac{d\mathbf{g}}{dt} + \frac{1}{2} &= p_+ \\ &= \frac{q}{t}\left[X - \frac{1}{2}(v_1 - v_2))\right] + 1, \end{aligned}$$

by means of $(1.15)_2$ and (2.8). The explicit forms of ℓ_* and ℓ_*^{-1} are given by $(2.19)_{1,2}$. In particular, (2.18) implies the relation:

$$(2.20) \qquad H_+ = H - \frac{1}{t}X.$$

Remark 2.4. We have, besides (2.17), the relations:

$$\ell = s_2 \cdot s_1 \cdot x \cdot s_2 = s_2 \cdot s_1 \cdot s \cdot x,$$

with respect to the transformations s, x, considered in §1.3. $(2.19)_{1,2}$ follow also from the expressions:

$$\ell_* = (s_2)_*(s_1)_*x_*(s_2)_* = (s_2)_*(s_1)_*s_*x_*.$$

Consider the parallel transformation:

$$\tilde{\ell} = s_1 \cdot x : \mathbf{v} \longrightarrow \mathbf{v} + (1, -1).$$

The canonical transformation $\tilde{\ell}_*$ associated with $\tilde{\ell}$ is given by $\tilde{\ell}_* = (s_1)_* \cdot x_*$ or by Proposition 2.2. In fact, by setting $\mathscr{H}(\tilde{\ell}(\mathbf{v})) = \tilde{\ell}_*\mathscr{H}(\mathbf{v}) = (\tilde{q}, \tilde{p}, \tilde{H}, t)$ and $\tilde{Y} = \tilde{q}\tilde{p}$, we obtain from Proposition 2.2:

$$(2.21) \qquad \bar{\mathbf{g}} = h(\tilde{\ell}(\mathbf{v})),$$

and then, by $(2.15)_{1,2}$ and (2.16),

$$Y + \tilde{Y} = v_1 + 1 - \frac{v_1 - v_2 + 2}{\frac{d\bar{g}}{dt} - \frac{1}{2}},$$

$$\frac{d\bar{g}}{dt} - \frac{1}{2} = \tilde{p} - 1$$

$$= -\frac{q}{t}\left[Y - \frac{1}{2}(v_1 + v_2)\right] - 1.$$

Moreover we have

$$H_+ = H - \frac{1}{t}Y \tag{2.22}$$

by means of (2.21) and (2.12).

2.6. Realization of s_0 as the the canonical transformation

We compute the birational canonical transformation $(s_0)_*$ by using the relation

$$s_0 = x \cdot s \cdot s_2 .$$

In fact, for $\mathscr{H}(s_0(\mathbf{v})) = (q_0, p_0, H_0, t)$, we have

$$q_0 = -\frac{t}{q}, \tag{2.23$_1$}$$

$$p_0 = \frac{q}{t}\left[X - \frac{1}{2}(v_1 - v_2 + 2)\right] + 1, \tag{2.23$_2$}$$

and moreover

$$X_0 + X = \frac{1}{2}(v_1 - v_2 + 2),$$

where $X = q(p-1)$, $X_0 = q_0(p_0 - 1)$; refer to (2.3). $(s_0)_*$ is given by (2.23) together with:

$$H_0 = H - \frac{X}{t} + \frac{1}{4t}(v_1 - v_2)(v_1 + v_2 + 2).$$

The proof of Theorem 1 is completed.

§3. Toda equation and τ-functions

3.1. τ-function

Let $H(\mathbf{v})$ be a Hamiltonian function of the Painlevé system $\mathscr{H}(\mathbf{v})$ at $\mathbf{v}$. The

τ-function $\tau=\tau(\mathbf{v})$ related to $H(\mathbf{v})$ is by definition:

$$H = \frac{d}{dt}\log\tau. \tag{3.1}$$

The canonical transformation

$$g_*: \mathscr{H}(\mathbf{v}) \longrightarrow \mathscr{H}(g(\mathbf{v}))$$

extends to the mapping from the τ-functions $\tau(\mathbf{v})$ to $\tau(g(\mathbf{v}))$, uniquely up to multiplicative constants. We will write it also as:

$$g_*\tau(\mathbf{v}) = \tau(g(\mathbf{v})).$$

For example, the auxiliary function $h(\mathbf{v})$ is invariant with respect to s_1, so that we have

$$\tau(s_1(\mathbf{v})) = (s_1)_*\tau(\mathbf{v}) = t^a\cdot\tau(\mathbf{v}),$$

$$a = \frac{1}{4}(v_1^2-v_2^2);$$

see §2.2.

Example 3.1. Consider the Hamiltonian $H''(t;\, q,\, p)$ given in Example 1.1. Since a Hamiltonian function is written in the form:

$$H = \frac{\lambda^2}{4t},$$

the τ-function related to it is:

$$\tau = \text{const.}\ t^{1/4\lambda^2}.$$

3.2. Proof of Proposition 0.1

We verify now the result stated in Proposition 0.1; we obtain the

Proposition 3.1. *If a Hamiltonian function $H(t)$ has a pole at $t=t_0$ ($t_0\neq 0, \infty$), it can be written as*

$$H(t) = \frac{1}{T}[1+O(T)], \tag{3.2}$$

in a neighbourhood of $t=t_0$, where T is the local parameter, $T=t-t_0$, the Landau notation $O(T^k)$ denoting a convergent series in T of powers higher than k.

In fact, we gain the following table of local expansions of $(q(t), p(t))$:

(i) $q(t)=T\left(1-\frac{\theta_0}{2t_0}T+O(T^2)\right)$.

$p(t)$: holomorphic,
$H(t)$: holomorphic,

(ii) $q(t) = -T\left(1+\frac{\theta_0+2}{2t_0}T+O(T^2)\right)$,
$p(t) = -t_0T^{-2}(1+O(T^2))$,
$H(t)$: of the form (3.2),

(iii) $q(t) = t_0T^{-1}\left(1+\frac{\theta_\infty+1}{2t_0}T+O(T^2)\right)$,
$p(t) = \frac{\theta_0+\theta_\infty}{2t_0}T(1+O(T))$,
$H(t)$: holomorphic,

(iv) $q(t) = -t_0T^{-1}\left(1-\frac{\theta_\infty-1}{2t_0}T+O(T^2)\right)$,
$p(t) = 1-\frac{\theta_0-\theta_\infty}{2t_0}T+O(T^2)$,
$H(t)$: holomorphic.

We do not enter into details of computation.

3.3. Painlevé transcendental function and τ-function

Consider the parallel transformations of $\mathbf{V}$:

$$\ell : \mathbf{v} \longrightarrow \mathbf{v} + (1, 1),$$
$$\tilde{\ell} : \mathbf{v} \longrightarrow \mathbf{v} + (1, -1).$$

Let $\tau(\mathbf{v})$ be the τ-function related to a solution $(q, p) = (q(\mathbf{v}), p(\mathbf{v}))$ of $\mathscr{H}(\mathbf{v})$. We obtain from (2.20) and (3.1):

$$X = t\frac{d}{dt}\log\frac{\tau(\mathbf{v})}{\tau(\ell(\mathbf{v}))}\quad X = q(p-1), \tag{3.3}$$

while from (2.22)

$$Y = t\frac{d}{dt}\log\frac{\tau(\mathbf{v})}{\tau\tilde{\ell}((\mathbf{v}))}, \quad Y = qp. \tag{3.4}$$

From (3.3) and (3.4) it results that:

Proposition 3.2. *The Painlevé transcendental function $q(\mathbf{v})$ is written as*

$$q(\mathbf{v}) = t\frac{d}{dt}\log\frac{\tau(\ell(\mathbf{v}))}{\tau(\tilde{\ell}(\mathbf{v}))} = t\frac{d}{dt}\log\frac{\tau(t; v_1+1, v_2+1)}{\tau(t; v_1+1, v_2-1)}. \tag{3.5}$$

3.4. Toda equation

For an arbitrary fixed $\mathbf{v}$ of $\mathbf{V}$ and for $m \in \boldsymbol{Z}$, set:

$$\mathbf{v}_0 = \mathbf{v}, \quad \mathbf{v}_m = \ell^m(\mathbf{v}),$$

$$\tilde{\mathbf{v}}_0 = \mathbf{v}, \quad \tilde{\mathbf{v}}_m = \tilde{\ell}^m(\mathbf{v}),$$

that is,

$$\mathbf{v}_m = (v_1+m,\ v_2+m), \quad \tilde{\mathbf{v}}_m = (v_1+m,\ v_2-m).$$

Moreover we write as:

$$\mathscr{H}_0 = \tilde{\mathscr{H}}_0 = \mathscr{H}(\mathbf{v})$$
$$\mathscr{H}_m = \ell_*^m \mathscr{H}_0 = (q_m, p_m, H_m, t),$$
$$\tilde{\mathscr{H}}_m = \tilde{\ell}_*^m \tilde{\mathscr{H}}_0 = (\tilde{q}_m, \tilde{p}_m, \tilde{H}_m, t).$$

Let τ_m (resp. $\tilde{\tau}_m$) be the τ-function of $\mathscr{H}_m$ (resp. $\tilde{\mathscr{H}}_m$) such that $\ell_*\tau_m = \tau_{m+1}$ (resp. $\tilde{\ell}_*\tilde{\tau}_m = \tilde{\tau}_{m+1}$) and define the τ-sequence

(3.6) $$\mathfrak{T}^0(\ell) = \{\tau_m^0;\ m \in \boldsymbol{Z}\},$$
$$(\text{resp.}\ \mathfrak{T}^0(\tilde{\ell}) = \{\tilde{\tau}_m^0;\ m \in \boldsymbol{Z}\}),$$

by

(3.7) $$t\frac{d}{dt}\log \tau_m^0 = t\frac{d}{dt}\log \tau_m + \frac{1}{2}m^2,$$
$$\left(\text{resp}\cdot t\frac{d}{dt}\log \tilde{\tau}_m^0 = t\frac{d}{dt}\log \tilde{\tau}_m + \frac{1}{2}m^2 - t\right).$$

We prove:

Theorem 2. $\mathfrak{T}^0(\ell)$ *and* $\mathfrak{T}^0(\tilde{\ell})$ *satisfy the Toda equation*

(3.8) $$\delta^2 \log \tau_m^0 = \frac{\tau_{m-1}^0 \tau_{m+1}^0}{(\tau_m^0)^2},$$

where $\delta = t\dfrac{d}{dt}$.

3.5. Proof of Theorem 2

Setting:

$$X_m = q_m(p_m - 1), \quad Y_m = q_m p_m,$$

we obtain from (3.3), (3.4):

(3.9) $$X_m = t\frac{d}{dt}\log \frac{\tau_m}{\tau_{m+1}}, \quad Y_m = t\frac{d}{dt}\log \frac{\tilde{\tau}_m}{\tilde{\tau}_{m+1}}.$$

On the other hand, if we define the auxiliary functions by:

$$h_m = tH_m + \frac{1}{4}(v_1+m+1)^2 - \frac{1}{2}t,$$

$$\tilde{h}_m = t\tilde{H}_m + \frac{1}{4}(v_1+m+1)^2 - \frac{1}{2}t,$$

it follows from $(2.7)_1$, (2.11) that

$$(3.10)_1 \qquad X_m = -\frac{t\dfrac{d^2h_m}{dt^2} - (v_1+m)\dfrac{dh_m}{dt} + \dfrac{1}{2}(v_2+m)}{2\left(\dfrac{dh_m}{dt} + \dfrac{1}{2}\right)},$$

$$(3.10)_2 \qquad X_{m-1} = \frac{t\dfrac{d^2h_m}{dt^2} + (v_1+m)\dfrac{dh_m}{dt} - \dfrac{1}{2}(v_2+m)}{2\left(\dfrac{dh_m}{dt} + \dfrac{1}{2}\right)},$$

and from $(2.15)_1$, (2.16)

$$(3.11)_1 \qquad Y_m = -\frac{t\dfrac{d^2\tilde{h}_m}{dt^2} - (v_1+m)\dfrac{d\tilde{h}_m}{dt} + \dfrac{1}{2}(v_2-m)}{2\left(\dfrac{d\tilde{h}_m}{dt} - \dfrac{1}{2}\right)},$$

$$(3.11)_2 \qquad X_{m-1} = \frac{t\dfrac{d^2\tilde{h}_m}{dt^2} + (v_1+m)\dfrac{d\tilde{h}_m}{dt} - \dfrac{1}{2}(v_2-m)}{2\left(\dfrac{d\tilde{h}_m}{dt} - \dfrac{1}{2}\right)}.$$

Therefore, since by (3.10)

$$X_{m-1} - X_m = t\frac{d}{dt}\log\left(\frac{dh_m}{dt} + \frac{1}{2}\right),$$

we obtain from (3.9)

$$\frac{dh_m}{dt} + \frac{1}{2} = c(m)\frac{\tau_{m-1}\tau_{m+1}}{\tau_m^2},$$

and then by the definition of the τ-function,

$$(3.12) \qquad \frac{d}{dt}t\frac{d}{dt}\log\tau_m = c(m)\frac{\tau_{m-1}\tau_{m+1}}{\tau_m^2}.$$

Moreover since by (3.11)

$$\frac{d\tilde{h}_m}{dt} - \frac{1}{2} = \tilde{c}(M)\frac{\tilde{\tau}_{m-1}\tilde{\tau}_{m+1}}{\tilde{\tau}_m^2},$$

it follows from the definition that

(3.13) $$\frac{d}{dt}t\frac{d}{dt}\log\tilde{\tau}_m - 1 = \tilde{c}(m)\frac{\tilde{\tau}_{m-1}\tilde{\tau}_{m+1}}{\tilde{\tau}_m^2}.$$

Here $c(m)$ and $\tilde{c}(m)$ denote nonzero constants. Theorem 2 is an immediate consequence of (3.12) and (3.13).

§4. Cylinder function and Painlevé transcendental function

4.1. Classical solution

A solution of the Painlevé system is said *classical* if it is written in terms of the classical transcendental functions. As we have mentioned in Introduction, $\mathscr{H}_{III'}$ has a solution written by the use of the cylinder function:

(4.1) $$\frac{d^2\mathfrak{z}}{dt^2} + \frac{1}{r}\frac{d\mathfrak{z}}{dt} + \left(1 - \frac{\nu^2}{r^2}\right)\mathfrak{z} = 0.$$

In fact, if $v_1 + v_2 = 0$, then $\mathscr{H}(\mathbf{v})$ possesses a solution of the form

(4.2) $$t\frac{dq}{dt} = -q^2 - \theta_0 q + t, \quad p \equiv 0,$$

for which we have:

$$\tau(\mathbf{v}) = 1.$$

It follows from (3.3) that

$$q = t\frac{d}{dt}\log\tau_1,$$

$$\tau_1 = \tau(\ell(\mathbf{v})),$$

and τ_1 is a solution of the equation:

(4.3) $$t\frac{d^2\tau_1}{dt^2} + (1+\theta_0)\frac{d\tau_1}{dt} - \tau_1 = 0.$$

Since the auxiliary function $h(\ell(\mathbf{v}))$ is not a singular solution of $\mathbf{E}(\ell(\mathbf{v}))$, we can apply the birational canonical transformation ℓ_* sucecssively to (4.2) and then obtain the semi-sequence of τ-functions:

$$\mathfrak{T}_+(\ell) = \{\tau_m;\ m \geqq 0\}.$$

If we determine τ_m^0 by (3.7), then $\tau_0^0 = 1$, $\tau_1^0 = \sqrt{t}\,\tau_1$ and

(4.4)
$$\tau_m^0 = \det\begin{pmatrix} \tau, & \delta\tau ,\dots, & \delta^{m-1}\tau \\ \delta\tau, & \delta^2\tau,\dots, & \delta^m\tau \\ \dots & \dots & \dots \\ \delta^{m-1}\tau, & \delta^m\tau,\dots, & \delta^{2m-2}\tau \end{pmatrix}$$

with $\delta = t\dfrac{d}{dt}$, $\tau=\tau_1^0$ (Darboux's formula: see [2], [11]).

4.2. Transformation ℓ_*^{-1} in the degenerate case

First we compute transformation ℓ_*^{-1} on the solution (4.2); note the auxiliary function h related to it is linear in t. By setting $\mathscr{H}(\ell^{-1}(\mathbf{v}))=(q_-, p_-, H_-, t)$, we obtain from (2.19):

$$X + X_- = v_1,$$

$$q_-(X_- - v_1) = t(p-1),$$

where $X=q(p-1)$, $X_-=q_-(p_- -1)$. Taking the limit: $p\to 0$ we arrive at the expression:

(4.5)
$$q_- = -\frac{t}{q}, \quad X_- = q + v_1.$$

It coincides exactly with the canonical transformation $(s_0)_*$ (see (2.23)). In fact, in the case $v_1+v_2=0$, we have

$$s_0(\mathbf{v}) = \ell^{-1}(\mathbf{v}).$$

The pair of functions (q_-, p_-) defined by (4.5) is actually a solution of $\mathscr{H}(\ell^{-1}(\mathbf{v}))$. The τ-function τ-related to it is given by:

$$q + v_1 = t\frac{d}{dt}\log\tau_-,$$

by means of (3.3) and (4.5). Thus we obtain the

Proposition 4.1. *There exists the semi-sequence*

$$\mathfrak{T}_-^0(\ell) = \{\tau_m^0;\ m\leqq 0\}$$

such that $\tau_0^0=1$, $\tau_-^0=\sqrt{t}\,\tau_-$ *and* τ_{-m}^0 $(m\geqq 2)$ *are given by the Darboux's formula* (4.4).

4.3. Sequence of the cylinder functions

Consider the transformation $\tilde{\ell}$ (see §3.3). If $\mathbf{v}$ is on the line:

(4.6)
$$v_1 + v_2 = 0,$$

then so is the point $\tilde{\ell}(\mathbf{v})$. Therefore the Painlevé system $\tilde{\mathscr{H}}_m=\tilde{\ell}^*\tilde{\mathscr{H}}_0=(\tilde{q}_m, \tilde{p}_m, \tilde{H}_m, t)$ has a solution of the form:

$$t\frac{d\tilde{q}}{dt} = -\tilde{q}^2-(v_1+m)\tilde{q}+t, \quad \tilde{p}=0. \tag{4.7}$$

For a solution q of the Riccati equation (4.7), set:

$$\tilde{q}' = \frac{t}{q+v+m}, \quad \tilde{p}'=0. \tag{4.8}$$

It is easy to verify $(\tilde{q}', \tilde{p}')$ defines a solution of $\tilde{\mathscr{H}}_{m-1}$. Hence the restriction of the canonical transformation $\tilde{\ell}_*$ to (4.7) is given by (4.8).

The τ-function of $\tilde{\mathscr{H}}_m$ related to $(\tilde{q}_m, \tilde{p}_m)$ of the form (4.7) is:

$$\tilde{\tau}_m = 1.$$

If we write as

$$\mathfrak{f}_m = \tau(\ell(\tilde{\mathbf{v}}_m)) = \tau(\ell\cdot\tilde{\ell}^m(\mathbf{v})),$$

then it follows from (3.5) that $\mathfrak{f}_m$ satisfies

$$t\frac{d^2\mathfrak{f}}{dt^2}+(1+v_1+m)\frac{d\mathfrak{f}}{dt}-\mathfrak{f}=0. \tag{4.9}$$

On the other hand, we can verify the

Proposition 4.2. *For the set of solutions of* (4.9): $\{\mathfrak{f}_m;\ m\in\mathfrak{Z}\}$, *we have the contiguity relations*:

$$\mathfrak{f}_{m+1} = \frac{d\mathfrak{f}_m}{dt}, \tag{4.10$_1$}$$

$$\mathfrak{f}_{m-1} = t\frac{d\mathfrak{f}_m}{dt}+(1+v_1+m)\mathfrak{f}_m. \tag{4.10$_2$}$$

The relations $(4.10)_{1,2}$ imply the birational canonical transformation $\tilde{\ell}_*$ of the Painlevé system. In fact, substituting

$$\tilde{q} = t\frac{d}{dt}\log\mathfrak{f}_m,\ \tilde{q}' = t\frac{d}{dt}\log\mathfrak{f}_{m-1},$$

into (4.8), we have the relation:

$$\left(t\frac{d\mathfrak{f}_m}{dt}+(v_1+m)\mathfrak{f}_m\right)\frac{d\mathfrak{f}_{m-1}}{dt} = \mathfrak{f}_m\mathfrak{f}_{m-1}.$$

Moreover if we define $\tilde{\mathfrak{f}}_m$ by:

$$\tilde{\mathfrak{f}}_m = (-t)^{e(m)} e^{-t} \mathfrak{f}_m,$$

$$e(m) = \frac{1}{2}(v_1 + m)^2,$$

then $\{\tilde{\mathfrak{f}}_m;\ m \in \boldsymbol{Z}\}$ satisfies the Toda equation:

$$\delta^2 \log \tilde{\mathfrak{f}}_m = \frac{\tilde{\mathfrak{f}}_{m-1}\tilde{\mathfrak{f}}_{m+1}}{\tilde{\mathfrak{f}}_m^2}. \tag{4.11}$$

Comparing this with (0.15), we have

$$\mathfrak{f}_m(t) = t^{-e'(m)} Z_{v+m}(2\sqrt{-t})4^{-e(m)},$$

$$\tilde{\mathfrak{f}}_m(t) = \tilde{Z}_{v+m}(2\sqrt{-t})4^{-e(m)},$$

where $e'(m) - \frac{1}{2}(v+m)$.

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