Mathematical Miniatures

Gottfried Helms - Univ Kassel *since 12 - 2006*

Bell-Numbers

Alternating sums of Bell-numbers and of Stirling-numbers 2nd kind

Abstract: This article has two motivations:

an exercise in translation of another number-theoretical aspect into my matrix-concept
and -following from this- an approach to a special matrix-based summation-method using a concrete example.

I consider the definition of Bell-numbers based on the article of E.T. Bell (1938). The matrix approach shows a very elegant and concise notion of those Bell-numbers and their higher (integer) orders, and finally of their generalization to fractional/ continuous orders which can be described very concisely using the matrix-logarithm.

Derived from that I show an approach to sum the alternating series of Bell-numbers. This series is strongly divergent, so that Euler-summation is not sufficient to assign a value to it. The result of this is compared with a serial summation using the idea of the Riesz-method (see: [Knopp]). An implicite effect is a summation-concept for the alternating sums of columns of the matrix of Stirling-numbers 2nd kind (again divergent, but can be Euler-summed); this result, however, is already known.

The derivations given here use the concept of matrix-operators, acting on formal powerseries. I'd like to promote this concept due to its concise and versatile notation, although it lacks widely formal proofs. I'm trying to collect a compilation of basic proofs independent of this series of articles.

For more basic and general introduction see "intro"[1] of my "binomial-matrices"-project.

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[1] *Intro/notation* http://go.helms-net.de/math/binomial/intro.pdf

1. Definition of Bell-numbers

1.1. Definition

The concept of the "Bell"-numbers was developed by E. T. Bell around 1938; I refer here to the article "Iterate exponential integers" [Bell38], where he used the name "ξ-number" for his integers. He derived the ξ-numbers initially using $exp(exp(x)-1)$ as exponential generation function (*"e.g.f."*). Then he introduces iteration, defining the inner function $E^{(1)} = exp(1)-1$ and its iterates $E^{(m+1)}=exp(E^{(m)})-1$ and generalizes the ξ-numbers to $\xi^{(m)}$-numbers using the m'th iterate in the expression $exp(E^{(m)})$ as e.g.f. for $\xi^{(m)}$

From this, the infinite sequences $\xi^{(m)}$ of order m are defined

$$\xi^{(m)} = sequence_{n=0..inf} (\xi^{(m)}_n) \text{ generated by e.g.f } exp(E^{(m)})$$
$$\xi^{(0)} = sequence_{n=0..inf} (\xi^{(m)}_n) \text{ ; e.g.f } exp(E^{(0)}) = exp(1)$$

The sequences $\xi^{(m)}$ for consecutive m can be written as matrix of infinite size $\boldsymbol{\xi}$, where the m'th column contains the m'th sequence (the matrix-indices begin at zero as well), so column 0 contains all 1, and column 1 contains $\xi^{(1)}_n = (1,1,2,5,...)$

$$\boldsymbol{\xi} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 5 & 12 & 22 & 35 & 51 & 70 & 92 \\ 1 & 15 & 60 & 154 & 315 & 561 & 910 & 1380 \\ 1 & 52 & 358 & 1304 & 3455 & 7556 & 14532 & 25488 \\ 1 & 203 & 2471 & 12915 & 44590 & 120196 & 274778 & 558426 \\ 1 & 877 & 19302 & 146115 & 660665 & 2201856 & 5995892 & 14140722 \end{bmatrix}$$

1.2. Relation to the matrix S2 of Stirling-numbers 2nd kind.

Using the function $E^{(1)} = exp(x)-1$ as e.g.f gives the second column of the matrix of Stirling-numbers 2nd kind as coefficients. This matrix begins

$$\mathbf{S2} = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1 & 3 & 1 & . & . \\ 0 & 1 & 7 & 6 & 1 & . \\ 0 & 1 & 15 & 25 & 10 & 1 \end{bmatrix}$$

(infinitely continued)

Using $exp(E^{(1)})$ as e.g.f for its formal parameter $\boldsymbol{E^{(1)}}$, we get the sequence $(1,1,1,..)$ as cofactors for the powers $(E^{(1)})^0, (E^{(1)})^1 ,(E^{(1)})^2, ...)$; expanding these powers into series in x and collecting like powers of x gives then the intended coefficients $\xi^{(1)}_n = (1,1,2,5,...)$ as cofactors at $x^0, x^1, x^2, ...$

Now, the *c*'th column of ***S2*** contains just the required coefficients for the powers $(E^{(1)})^c$ (as for instance given in [A&S],pg. 824), such that

$$Xf\sim * S2 = Yf\sim$$

Example:

$$Xf\sim * S2 = Yf\sim \qquad \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1 & 3 & 1 & . & . \\ 0 & 1 & 7 & 6 & 1 & . \\ 0 & 1 & 15 & 25 & 10 & 1 \end{bmatrix} \; S2$$

$$[\ 1/0! \quad x/1! \quad x^2/2! \quad x^3/3! \quad x^4/4! \quad x^5/5!\ ] * S2 = [\ 1/0! \quad y/1! \quad y^2/2! \quad y^3/3! \quad y^4/4! \quad y^5/5!\ ]$$

where y is $y=E^{(1)}=exp(x)-1$. The small f at Xf and Yf indicate the scaling of the X and the Y-vector by the reciprocal factorials and the ~-symbol the transposition of a matrix/vector (see below).

Repeat: while the coefficents at consecutive powers of y are $(1,1,1,1,..)$ if we use the e.g.f $exp(y)$, the $\xi^{(1)}$-numbers are defined by the e.g.f. of $exp(exp(x)-1)$ as the coefficients of powers of x.

The coefficients $\xi^{(1)}=sequence(\xi^{(1)}_0,\xi^{(1)}_1,\xi^{(1)}_2,...)$ are found after y^k are expanded into powers of x and like powers of x are collected.

Such an expansion is tedious when done sequentially, but is easy when written in matrix-notation.

Some helpful notational definitions

To express this all in a more consistent form than that of the above ad-hoc matrix-sketch I first want to introduce few formalisms to get consistency with the general style of my matrix-formulae around.

All vectors and matrices are assumed with infinite size.

define

$V(x) := columnvector(1,x,x^2,x^3,...)$

to be a formal "vandermonde vector" of a free parameter x

${}^dV(x) := diag(1,x,x^2,x^3,...)$

using V(x) as a diagonal-matrix

define

$F := columnvector(0!,1!,2!,3!,...)$

${}^dF := diag(0!,1!,2!,3!,...)$

define the diagonal-matrix for shortness in matrix-formulae

$J := \quad J_{k,k} = (-1)^k$

so $J = {}^dV(-1)$

define "~" being the "transpose"-symbol

(as used in the algebra-program Pari/GP)

Sometimes I also use the abbreviation for a similarity-scaling

$$fS2F = {}^dF^{-1} * S2 * {}^dF$$

to keep matrix-expressions simpler in notation.

With this, the e.g.f-relation of *exp(y)* and the according coefficients may be written as

*Yf~ * V(1) = exp(y)*

Example:

*Yf~ * V(1) = exp(y)*

$$\begin{bmatrix} 1/0! & y/1! & y^2/2! & y^3/3! & y^4/4! & y^5/5! \end{bmatrix} * \begin{bmatrix} 1\\1\\1\\1\\1\\1 \end{bmatrix} = exp(y)$$

But ***Yf*** is also composed by

*Xf~ *S2 = Yf~*

Example:

*Xf~ *S2 = Yf~*

$$\begin{bmatrix} 1/0! & x/1! & x^2/2! & x^3/3! & x^4/4! & x^5/5! \end{bmatrix} * \underbrace{\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1 & 3 & 1 & . & . \\ 0 & 1 & 7 & 6 & 1 & . \\ 0 & 1 & 15 & 25 & 10 & 1 \end{bmatrix}}_{S2} = \begin{bmatrix} 1/0! & y/1! & y^2/2! & y^3/3! & y^4/4! & y^5/5! \end{bmatrix}$$

as shown above.

So the full matrix-equation is

*Xf ~ *S2 *V(1) = exp(exp(x)-1)*

Example:

*Xf ~ *S2 *V(1) = exp(exp(x)-1)*

$$\begin{bmatrix} 1/0! & x/1! & x^2/2! & x^3/3! & x^4/4! & x^5/5! \end{bmatrix} * \underbrace{\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1 & 3 & 1 & . & . \\ 0 & 1 & 7 & 6 & 1 & . \\ 0 & 1 & 15 & 25 & 10 & 1 \end{bmatrix}}_{S2} * \begin{bmatrix} 1\\1\\1\\1\\1\\1 \end{bmatrix} = exp(exp(x)-1)$$

To get the coefficients at the powers of *x*, we reorder the summation in the formula

$$\begin{aligned} exp(exp(x)-1) &= Xf\sim * S2 * V(1) \\ &= Xf\sim * (S2 * V(1)) \\ &= Xf\sim * \xi^{(1)} \end{aligned}$$

$$\Rightarrow \quad \xi^{(1)} = S2 * V(1)$$

where the numbers $\xi^{(1)}{}_n$ occur simply as rowsums of the ***S2***-matrix and we have the definition of $\xi^{(1)}$ by the Stirling-numbers 2nd kind:

$$S2 * V(1) = \xi^{(1)}$$

Example:

$$S2 * V(1) = \xi^{(1)}$$

$$\begin{matrix} & & * \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1 & 3 & 1 & . & . \\ 0 & 1 & 7 & 6 & 1 & . \\ 0 & 1 & 15 & 25 & 10 & 1 \end{bmatrix} & S2 & = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \\ 15 \\ 52 \end{bmatrix} \end{matrix}$$

and then

$$Xf \sim * \xi^{(1)} = exp(exp(x)-1)$$

Example:

$$Xf \sim * \xi^{(1)} = exp(exp(x)-1)$$

$$\begin{matrix} & * \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \\ 15 \\ 52 \end{bmatrix} \\ \begin{bmatrix} 1/0! & x/1! & x\text{^}2/2! & x\text{^}3/3! & x\text{^}4/4! & x\text{^}5/5! \end{bmatrix} & = exp(exp(x)-1) \end{matrix}$$

1.3. Iteration

The $\xi^{(2)}$ coefficients are found by the e.g.f. for

$$exp(z) = exp(exp(y)-1) = exp(exp(exp(x)-1)-1)$$

the same way just by iteration, where y takes the place of the previous x and z takes the place of the previous y:

$$\begin{aligned} Zf\sim &= Yf\sim * S2 \\ &= Xf\sim * S2 * S2 \\ &= Xf\sim * S2^2 \\ exp(\ exp(exp(x)-1)-1) &= Xf\sim * S2^2 * V(1) \\ &= Xf\sim * \xi^{(2)} \end{aligned}$$

$$\Rightarrow \qquad \xi^{(2)} \qquad = S2^2 * V(1)$$

where the numbers $\xi^{(2)}{}_n$ occur as rowsums of $\mathbf{S2^2}$.

$S2^2 * V(1) = \xi^{(2)}$

Example:

$$S2^2 * V(1) = \xi^{(2)}$$

$$\begin{matrix} & & * \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 2 & 1 & . & . & . \\ 0 & 5 & 6 & 1 & . & . \\ 0 & 15 & 32 & 12 & 1 & . \\ 0 & 52 & 175 & 110 & 20 & 1 \end{bmatrix} & S2\text{^}2 & = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 12 \\ 60 \\ 358 \end{bmatrix} \end{matrix}$$

and then

$$Xf \sim * \xi^{(2)} = exp(exp(exp(x)-1) -1)$$

Example:

$$\begin{matrix} & * \begin{bmatrix} 1 \\ 1 \\ 3 \\ 12 \\ 60 \\ 358 \end{bmatrix} \\ Xf \sim * \xi^{(2)} = exp(exp(exp(x)-1) -1) & \\ \begin{bmatrix} 1/0! & x/1! & x^2/2! & x^3/3! & x^4/4! & x^5/5! \end{bmatrix} & = exp(exp(exp(x)-1)-1) \end{matrix}$$

Obviously this can be generalized to any integer index m:

$$\xi^{(m)} \quad = S2^{\ m} * V(1)$$

1.4. some recursive properties

From the simple matrix-relation also some simple recursion-schemes can be derived:

$$\xi^{(0)} = V(1)$$
$$\xi^{(m)} = S2 * \xi^{(m-1)}$$
$$\xi^{(m)} = S2^{-1} * \xi^{(m+1)} = S1 * \xi^{(m+1)} \qquad \text{// S1: Stirling-numbers of 1st kind}$$

This says, in scalar expression

$$\xi_r^{(m)} = \sum_{c=0}^{r} \left(S2_{r,c} \xi_c^{(m-1)} \right) = \sum_{c=0}^{r} \left(S1_{r,c} \xi_c^{(m+1)} \right)$$

where $S2_{r,c}$ are the Stirling-numbers 2nd kind and $S1_{r,c}$ that of the 1st kind, at row r and col c of the resp. triangle.

Also, since the premultiplication with the Pascal-/Binomialmatrix P the matrix $S2$ is shifted by one row/column, we get a shifting of the index for the $\xi^{(1)}$ –numbers if they are binomially summed; the $\xi^{(1)}$-number $\xi^{(1)}{}_{r+1}$ can be computed by the binomially weighted previous ones, which can also be used as a recursive definition:

$$\xi^{(1)}{}_{r+1} = \sum_{k=0}^{r} \left(\binom{r}{k} \xi_k^{(1)} \right)$$

Example:

$$\xi^{(1)}{}_{r+1} = \sum_{k=0}^{r} \left(\binom{r}{k} \xi_k^{(1)} \right) \qquad * \begin{bmatrix} 1 & . & . & . & . \\ 0 & 1 & . & . & . \\ 0 & 1 & 1 & . & . \\ 0 & 1 & 3 & 1 & . \\ 0 & 1 & 7 & 6 & 1 \end{bmatrix}_{S2} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1 & 2 & 1 & . & . \\ 1 & 3 & 3 & 1 & . \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}_{P} = \begin{bmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1 & 3 & 1 & . & . \\ 1 & 7 & 6 & 1 & . \\ 1 & 15 & 25 & 10 & 1 \end{bmatrix}_{S2-} \Rightarrow \begin{bmatrix} 1 \\ 2 \\ 5 \\ 15 \\ 52 \end{bmatrix}$$

Arbitrarily we may find more of such relations and recursive (re-) definitions, whose general numbertheoretical importance, however, must be shown.

1.5. Generalization to fractional (and complex) orders m

Fractional orders *m* can be defined in the obvious way: by finding an interpolation-formula for each *r*'th row of the list of $\xi^{(m)}$, so *interpolate*($\xi^{(0)}_r$, $\xi^{(1)}_r$, $\xi^{(2)}_r$, ... $\xi^{(m)}_r$,...) gives polynomials in *m*, for which then fractional (or even complex) orders can be defined. E.T. Bell gives the first few such polynomials as

$$\xi^{(m)}_1 = 1$$
$$\xi^{(m)}_2 = m + 1$$
$$\xi^{(m)}_3 = 1/2(m + 1)(3m+2)$$
$$\xi^{(m)}_4 = 1/2(m + 1)(2m+1)(3m+2)$$
$$\xi^{(m)}_5 = 1/6(m + 1)(45m^3 + 70m^2 +35m+6)$$

(...)

and derives these formulae by considering the differences $\xi^{(m+1)}_r$ - $\xi^{(m)}_r$ and expanding into polynomials.

Using the matrix-notation, we may refer to the formula

$$\xi^{(m)} \quad = S2^{\,m} * V(1)$$

and ask for the fractional *m*'th power of ***S2***, where I now use "*s*" for the fractional or complex generalization of *m*.

The most simple approach is to use the matrix-logarithm and restate:

$$\xi^{(s)} \quad = \exp(s*\log(S2)) * V(1)$$

Since the diagonal of ***S2*** contains only the unit, the matrix *(**S2** – **I**)*, which is needed for the computation of *log(**S2**)* by the powerseries expansion of the log-function, is nilpotent for each number of rows/columns considered and so the logarithm-function reduces –for any finite approximation - to a finite sum of scaled powers of ***(S2 – I)***. Increasing the size does not affect the coefficients, so any top-left segment found by this is exact in rational arithmetic.

The matrix-logarithm ***S2L*** = *log(**S2**)* has its top-left segment as

$$S2L = \log(S2) = \begin{bmatrix} 0 & . & . & . & . & . \\ 0 & 0 & . & . & . & . \\ 0 & 1 & 0 & . & . & . \\ 0 & -1/2 & 3 & 0 & . & . \\ 0 & 1/2 & -2 & 6 & 0 & . \\ 0 & -2/3 & 5/2 & -5 & 10 & 0 \end{bmatrix}$$

Then we can find the symbolic description of the *s*'th power $\boldsymbol{S2}^{\,2}$ by matrix-exponentiation $\boldsymbol{S2}^s = \exp(s*\boldsymbol{S2L})$:

Example:

$$S2^s = \exp(s*S2L) = \begin{bmatrix} 1 & . & . & . & . \\ 0 & 1 & . & . & . \\ 0 & s & 1 & . & . \\ 0 & 3/2*s\hat{}2-1/2*s & 3*s & 1 & . \\ 0 & 3*s\hat{}3-5/2*s\hat{}2+1/2*s & 9*s\hat{}2-2*s & 6*s & 1 \end{bmatrix}$$

whose top-left segment is again exact for any size of the matrix (by the same arguments as above).

The fractional *s*'th order of the ξ-numbers are given by the rowsums:

$$\xi^{(s)} = S2^{s} * V(1)$$

Example:

$$\xi^{(s)} = S2^{s} * V(1) \qquad \begin{bmatrix} 1 \\ 1 \\ s+1 \\ 3/2*s^2+5/2*s+1 \\ 3*s^3+13/2*s^2+9/2*s+1 \\ 15/2*s^4+115/6*s^3+35/2*s^2+41/6*s+1 \end{bmatrix}$$

(where we find $\xi^{(s)}{}_n$ if we evaluate the *n*'th row) which is the same as the solution given by E.T. Bell using the interpolation formula.

1.6. Short Summary

The above matrix-notation gives some concise formulae for the ξ –numbers.

definition by translation into matrix-language:

$$\xi^{(m)} = S2^{m} * V(1)$$

recursion over orders

$$\xi^{(0)} = V(1)$$
$$\xi^{(m+1)} = S2 * \xi^{(m)}$$
$$\xi^{(m-1)} = S1 * \xi^{(m)} \qquad \text{// not seen before}$$

recursion by index shift

$$\text{shift}(\xi^{(1)}) = P * \xi^{(1)} \qquad \text{// computes } \xi^{(1)}{}_{n+1} \text{ from } \xi^{(1)}{}_0 \ldots \xi^{(1)}{}_n$$

fractional / general orders

$$\xi^{(s)} = \exp(s*\log(S2)) * V(1)$$

alternating sums of $\xi^{(1)}$ *and* $\xi^{(2)}$ *(see next chapter)*

$$s_B = V(-1)* \xi^{(1)} = 1- \exp(-1)$$
$$s_{B2} = V(-1)* \xi^{(2)} = (Ei(1)- Ei(1/e))* \exp(-1)$$

(to be continued)

1.7. Conclusion

In his article E. T. Bell discusses the ξ –numbers in much detail, mainly with the focus on the modularity wrt. to prime-numbers, but also on the relations between ξ –numbers of different orders and/or index, including also binomials as cofactors and so on. The basic $\xi^{(1)}$ –numbers are named after him as "Bell-numbers" because his intense investigations and finding of theorems (I found[2] this naming-convention attributed to John Riordan).

From here on I'll refer to them as Bell-numbers now, use B instead $\xi^{(1)}$ and b_r instead of $\xi^{(1)}{}_n$

The higher orders $\xi^{(m)}$, $m>1$, have no special name yet; in the following I may refer to them as "Bell-numbers of higher orders" and denote them for simpliness as $B^{(m)}{}_r$ and the order-parameter as "m" (in my other articles which deal with iteration I use "h" for iteration-height, but the "m" in this context keeps consistency for the reader)

The top part of the array of $B^{(0)}..B^{(7)}$ was already shown in the first chapter:

Example:

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 5 & 12 & 22 & 35 & 51 & 70 & 92 \\ 1 & 15 & 60 & 154 & 315 & 561 & 910 & 1380 \\ 1 & 52 & 358 & 1304 & 3455 & 7556 & 14532 & 25488 \\ 1 & 203 & 2471 & 12915 & 44590 & 120196 & 274778 & 558426 \\ 1 & 877 & 19302 & 146115 & 660665 & 2201856 & 5995892 & 14140722 \end{bmatrix}$$

There is a vast amount of literature concerning the Bell-numbers, which is due to their combinatorical relevance. For a start the reader may follow the link at $c=1$ below in the "Online encyclopedia of Sequences" OEIS to find a long list of references.

The first few columns of that array are –for instance- known in OEIS:

$c=0$ http://www.research.att.com/~njas/sequences/A000012
$c=1$ *Bell or exponential numbers:*
ways of placing n labeled balls into n indistinguishable boxes
Number of partitions of a set of n labeled elements
http://www.research.att.com/~njas/sequences/A000110
$c=2$ *Number of 3-level labeled rooted trees with n leaves.*
http://www.research.att.com/~njas/sequences/A000258
$c=3$ *Number of 4-level labeled rooted trees with n leaves.*
http://www.research.att.com/~njas/sequences/A000307
$c=4$ *Number of 5-level labeled rooted trees with n leaves.*
http://www.research.att.com/~njas/sequences/A000357
$c=5$ *Number of 6-level labeled rooted trees with n leaves.*
http://www.research.att.com/~njas/sequences/A000405

with lots of comments and further links, especially for $c=1$ and $c=2$.

In the following chapter(s) I'll add some more observations / explorations which I've not seen elsewhere and I'll update this article as I get them settled for this article.

The first of these is the consideration of alternating sum of Bell-numbers and of the Stirling-numbers 2nd kind.

[2] according to Pat Ballew; see "links" in OEIS at $c=1$ and follow to P.B. homepage, or [Pbal]

2. Further interesting properties of the Bell/Stirling-numbers

2.1. Alternating sums of Bell-numbers

The Bell-numbers form a strongly divergent sequence; B_n is, very roughly, of order

$\ln(B_n)/n \sim \ln(n) - \ln\ln(n) + \ldots$

or $B_n \sim n^n / (\ln n)^n$

(due to a formula in "mathworld", see [Weiss]) which shows a stronger growth than a geometric sequence and, for instance, even their alternating sum would not be summable by conventional Euler-summation.

So: can the alternating sum of the Bell-numbers get a value assigned by a not too exotic summation method?

We set up the notation for this alternating series:

$$s_B = \sum_{k=0}^{oo} (-1)^k * B_k$$

or in matrix-notation

$V(-1)\sim *B = s_B = ???$

Example:

$V(-1)\sim *B = s_B = ???$

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \\ 15 \\ 52 \end{bmatrix} = s_B$$

This sum is divergent – the quotient of absolute values of consecutive elements even grows; so Euler-summation should not give usable results.

However, with a modification of the Euler-summation-matrix due to the idea of Riesz-sums (see [Knopp], p.487) I got at least an heuristic value

$s_B \sim 0.63212$

as a first empirical approximation.

2.1.1. A useful transformation

Here I introduce a transformation of the Bell-numbers which allows to use formal decomposition (according to their re-definition as sums) via matrix-operations into factors, where only divergent *geometric series* remain – and values for geometric series are well defined even for the divergent cases.

If we use the decomposition of the Bell-numbers into Stirling-numbers 2nd kind as shown in the previous chapter, we may restate the matrix-equation for their infinite alternating sum:

$$
\begin{array}{llll}
V(-1)\sim * & B & = s_B & \\
V(-1)\sim * (S2 * V(1)) & & = s_B & \text{//decomposition} \\
(V(-1)\sim * S2) * V(1) & & = s_B & \text{// changing associativity} \\
G\sim & * V(1) & = s_B & \text{// order of computation}
\end{array}
$$

Example:

$$
V(-1)\sim * (S2 * V(1)) = s_B \qquad
S2 = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1 & 3 & 1 & . & . \\ 0 & 1 & 7 & 6 & 1 & . \\ 0 & 1 & 15 & 25 & 10 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \qquad = s_B
$$

This way we separate the summation of alternating sum of Bell-numbers into two steps: first alternating sum the ***S2***-numbers, then sum the results.

The idea behind this is, that the rate of growth in the columns of the Stirling-matrix is only geometric (composed by constant number of geometric series); and alternating series of geometric growth can be Euler-summed.

Example: for the second column in ***S2*** we get its alternating sum just by inspection

$$\lim_{x\to 1} 0-1x+1x^2-1x^3+1x^4\ldots = 1/2 - 1 = -1/2$$

and for the third column, whose r'th entry is $2^{r-1} - 1^{r-1}$ we get

$$
\begin{aligned}
& \lim_{x\to 1} 0-(2^0-1)x+(2^1-1)x^2-(2^2-1)x^3+\ldots \\
= & \lim_{x\to 1} \; -2^0x + 2^1x^2 - 2^2x^3 + \\
& \qquad\qquad +1x - 1x^2 + 1x^3 - \ldots \\
= & \lim_{x\to 1} \; -2x/(1+2x)/2 \quad + \quad 1x/(1+1x) \\
= & -1/3 + 1/2 = 1/6
\end{aligned}
$$

(This agrees with the practical Euler-summation)

But for the following columns this becomes a tidy task, since the structure of Stirling numbers has many definitions and it is not easy to see, which of the definitions will help us to compute the alternating sum along an arbitrary column in the same or at least comparable easy way as in the first two examples.

If we use the following further binomial composition, by which the Stirling-numbers 2nd kind are sometimes (implicitely) defined (see also [A&S], pg. 824):

$$c^r = \sum_{k=0}^{c} \left(S2_{r,k} * \binom{c}{k} * k! \right)$$

or, as a matrix-expression:

$S2 * {}^dF * P\sim \quad = VZ \qquad // \; VZ := \{c^r\}_{c=0..inf,\; r=0..inf} \; ; \; 0^0:=1$

$S2 \quad = \quad VZ * P^{-1}\sim * {}^dF^{-1}$

the alternating sums:

$V(-1)\sim S2 \quad = V(-1)\sim * VZ \quad * P^{-1}\sim * {}^dF^{-1}$

$\quad = (V(-1)\sim * VZ) * P^{-1}\sim * {}^dF^{-1}$

to sum their transformed values first and retransform the results, then we arrive at simple geometric series (like for ***S2***'s second and third column), because the columns of ***VZ*** provide just the coefficients for such purely geometric series.

Example:

$S2 * {}^dF * P\sim \quad = VZ$

$$P\sim = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 2 & 3 & 4 & 5 \\ . & . & 1 & 3 & 6 & 10 \\ . & . & . & 1 & 4 & 10 \\ . & . & . & . & 1 & 5 \\ . & . & . & . & . & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1 & 3 & 1 & . & . \\ 0 & 1 & 7 & 6 & 1 & . \\ 0 & 1 & 15 & 25 & 10 & 1 \end{bmatrix}}_{S2} * diag\left(\begin{bmatrix} 1 \\ 1 \\ 2 \\ 6 \\ 24 \\ 120 \end{bmatrix}\right) * P\sim = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 4 & 9 & 16 & 25 \\ 0 & 1 & 8 & 27 & 64 & 125 \\ 0 & 1 & 16 & 81 & 256 & 625 \\ 0 & 1 & 32 & 243 & 1024 & 3125 \end{bmatrix}}_{VZ}$$

To describe the explicite structure of the Stirling-numbers by this definition requires then to find the reciprocal of this relation; so we may write:

$S2 = \; VZ * P\sim * {}^dF^{-1}$

Example:

$S2 = \; VZ * P\sim * {}^dF^{-1}$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 4 & 9 & 16 & 25 \\ 0 & 1 & 8 & 27 & 64 & 125 \\ 0 & 1 & 16 & 81 & 256 & 625 \\ 0 & 1 & 32 & 243 & 1024 & 3125 \end{bmatrix}}_{VZ} * \underbrace{\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ . & 1 & -2 & 3 & -4 & 5 \\ . & . & 1 & -3 & 6 & -10 \\ . & . & . & 1 & -4 & 10 \\ . & . & . & . & 1 & -5 \\ . & . & . & . & . & 1 \end{bmatrix}}_{PInv\sim} * diag\left(\underbrace{\begin{bmatrix} 1/0! \\ 1/1! \\ 1/2! \\ 1/3! \\ 1/4! \\ 1/5! \end{bmatrix}}_{f}\right) = \underbrace{\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1 & 3 & 1 & . & . \\ 0 & 1 & 7 & 6 & 1 & . \\ 0 & 1 & 15 & 25 & 10 & 1 \end{bmatrix}}_{S2}$$

and evaluate by expanding the matrix-factors and changing order of operation:

$$
\begin{aligned}
s_b &= V(-1)\sim * \quad B \\
&= V(-1)\sim * (\; S2 * V(1)) \\
&= V(-1)\sim * (VZ * P^{-1} \sim d^{F-1}) * V(1) \\
&= (V(-1)\sim * VZ) * P^{-1} \sim * ({}^{d}F^{-1} * V(1))
\end{aligned}
$$

The sums by ***V***(-1)~ * ***VZ*** are simply determined by the the closed form for divergent geometric series, they are just 1/(1+c) for the c'th column. Let's call the vector of these results ***G***.

V(-1)~ * VZ = G~ // by analytic contin. of div. geom. series
G~ = rowvector(1, 1/2, 1/3, 1/4,....)

Here we have "transferred" the part of divergent summation into the sums of divergent geometric series, getting closed forms for them. All the following involves then only conventionally convergent sums or series.

Example:

V(-1)~ * VZ = G~

$$
\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 4 & 9 & 16 & 25 \\ 0 & 1 & 8 & 27 & 64 & 125 \\ 0 & 1 & 16 & 81 & 256 & 625 \\ 0 & 1 & 32 & 243 & 1024 & 3125 \end{bmatrix} \; VZ
$$

$$
\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \end{bmatrix}
$$

Next step is, that we have to re-transform G by the inverse of ***P***~. This gives the vector G_J, which follows from the simple binomial transform

$$
\begin{aligned}
G\sim * P^{-1} \sim &= G\sim * (J * P\sim * J) \\
&= (G\sim * J * P\sim) * J \\
&= \quad G\sim \quad * J \\
&= \quad Gj \sim
\end{aligned}
$$

saying, for a column c in $\boldsymbol{G_j}$ it comes out

$$
Gj_c = \sum_{k=0}^{c} \left(\frac{1}{k+1} * (-1)^k \binom{c}{k} \right) = \frac{(-1)^c}{k+1}
$$

(for a proof see appendix). So

Gj~ = rowvector(1,-1/2,1/3,-1/4,...)

The last step is then to sum this with the reciprocal factorials to arrive at s_B. The final formula is

$$
\begin{aligned}
s_B &= Gj\sim * ({}^{d}F^{-1} * V(1)) \\
&= \sum_{k=0}^{oo} \left(\frac{(-1)^k}{k+1} * \frac{1}{k!} \right) \\
&= -\sum_{k=1}^{oo} \left(\frac{(-1)^k}{k!} \right) \\
&= -(\exp(-1)-1) \\
&= 1-\exp(-1)
\end{aligned}
$$

$$
s_B = 0.63212056...
$$

So the full decomposition of the summation-process for the alternating sum of Bell-numbers is

$$
\begin{aligned}
s_B &= V(-1)\sim * \quad B \\
&= V(-1)\sim * (\ S2 * V(1)) \\
&= V(-1)\sim * (VZ * P^{-1}\sim {}^dF^{-1}) * V(1) \\
&= (V(-1)\sim * VZ) * P^{-1}\sim * ({}^dF^{-1} * V(1)) \\
&= (\quad G\sim \quad * J\ P\sim)\ J * {}^dF^{-1} * V(1) \\
&= (\quad Gj\sim \quad) * {}^dF^{-1} * V(1) \\
&= -(\exp(-1)-1) \\
&= 1-\exp(-1) \\
s_B &= 0.63212056...
\end{aligned}
$$

which agrees perfectly with the result, which I approximated using the Riesz-summation.

It is interesting, that as part of these derivations we have also determined the alternating sum of the Stirling numbers 2nd kind themselves: they are just the individual terms of the last formula in the previous. Only we should normalize for the sign, since the sums begin at different index ($r=0,1,2,3,...$ for column $c=0,1,2,3,...$) while the index of the negative signs of their coefficients is the same for all columns.

From

$$
\begin{aligned}
V(-1)\sim * B &= V(-1)\sim * S2 * V(1) \\
&= \quad Gj\sim * {}^dF^{-1} * V(1)
\end{aligned}
$$

we need only remove the last vectorial summation $V(1)$ to get the vector $\boldsymbol{AS}_{S2}$ of that individual sums, and append one ${}^dJ$-multiplication for sign-correction:

$$
\begin{aligned}
Y_1\sim &= V(-1)\sim * S2 && \text{// the uncorrected matrix-product} \\
&= Gj\sim * {}^dF^{-1} \\
AS_{S2}\sim &= Y_1\sim * {}^dJ && \text{// use J for correction of sign-offsets} \\
&= \text{rowvector}(1,1/2!,1/3!,1/4!,...)
\end{aligned}
$$

or said differently, for a column c of $\boldsymbol{S2}$ we have the following finite value for its divergent alternating sum, if the sum begins always at $+1$:

$$\sum_{r=0}^{oo}\left((-1)^{r+c} * S2_{r,c}\right) = \frac{1}{(c+1)!} \qquad \text{// divergent summation}$$

Generalization

We can even generalize to sum them with a powerseries argument:

$V(1/x)\sim J\ VZ = V(1)*J*[V(0), V(1/x), V(2/x), ...]$
$= [1, 1/(1+1/x), 1/(2/x+1)$ // by geometric sum

Replace $1/x$ by x:

$V(x)\sim *J*VZ = [1, 1/(1+x), 1/(1+2x), 1/(1+3x), ...]$

$V(x)\sim *J*ZV*P^{-1}\sim$

=	[1,	= [1
	1-1/(1+x),	= x/(1+x)
	1-2/(1+x)+1/(1+2x),	= $2x^2/(1+x)(1+2x)$
	1-3/(1+x)+3/(1+2x)-1/(1+3x),	= $6x^3/(1+x)(1+2x)(1+3x)$
	...]	= ...]

Then the reduction of the factorials:

$V(x)\sim *J*ZV*P^{-1}\sim F^{-1} = V(x)\sim * S2$

and thus

$$V(x)\sim * S2 = [1, x/p(x,1), x^2/p(x,2), ... , x^k/p(x,k),...]$$

where $p(x,k) = prod_{j=1..k}(1+j\ x)$

2.2. Alternating sum of $B^{(2)}$-numbers

Using the previous result, we may simply proceed:

$B^{(2)} = S2^2 * V(1)$

$s_{B2} = V(-1)\sim * B^{(2)}$ // its alternating sum
$= V(-1)\sim * S2 * B$
$= V(-1)\sim * S2 * S2 * V(1)$
$= V(-1)\sim * (VZ*P^{-1}\sim*{}^dF^{-1}) * S2 * V(1)$
$= (V(-1)\sim * VZ*P^{-1}\sim) * ({}^dF^{-1} * S2) * V(1)$
$= \quad (Z(1)\sim*J) \quad * \quad ({}^dF^{-1} * S2) * V(1)$

$= \ (\ G_j \sim * {}^dF^{-1} * S2) * V(1)$ // G_j =[1,-1/2,1/3,...]
// from the previous paragraph

Here the product $G_j\sim * {}^dF^{-1} * S2$ implies evaluation of convergent series only, so we can give very well approximations for the first part:

$G_2\sim = G_j\sim * {}^dF^{-1} * S2$

and we get the approximations

$G_2\sim$ = rowvector(1, -0.36788, 0.084046, -0.013983, 0.0018326, -0.00019831, 0.000018284, -0.0000014690,...)

The sum is again convergent, we get by serial summation:

$s_{B2} = G_2 \sim * V(1) = 0.703834423154...$

and thus

$s_{B2} = V(-1)\sim * B^{(2)}$
$= 0.703834423154...$

But there is also a full analytical description possible, using two terms of the exponential-integral-function *EI()* only.

If we proceed as in the previous chapter and apply the $VZ * P^{-1}\sim * {}^dF^{1}$ transformation we get

$$\begin{aligned} s_{B2} &= G_j \sim * {}^dF^{-1} * (VZ * P^{-1}\sim * {}^dF^{-1}) * V(1) \\ &= (G_j \sim * {}^dF^{-1} * VZ) * (P^{-1}\sim * {}^dF^{-1} * V(1)) \\ &= Y_2 \sim * {}^dF^{-1} * 1/e \end{aligned}$$

For Y_2 we get by $G_j = [1, -1/2, 1/3, -1/4, \dots]$

$$Y_2 = [1,\ 1-e^{-1},\ (1-e^{-2})/2,\ (1-e^{-3})/3, \dots]$$

So $(Y_2 * {}^dF^{-1})/e$ leads then to

$$s_{B2} = e^{-1}\left(1 + \sum_{k=1}^{inf} \left(\frac{1-e^{-k}}{k} * \frac{1}{k!} \right)\right)$$

Since this expression is convergent, we may separate the terms to get

$$s_{B2} = e^{-1}\left(1 + \sum_{k=1}^{inf} \left(\frac{1}{k * k!} \right) - \sum_{k=1}^{inf} \left(\frac{e^{-k}}{k * k!} \right)\right)$$

The sum-terms in this agree to the sum-terms in the exponential-integral function *Ei()* (see [A&S], pg.229])

$$Ei(x) = gamma + \ln(x) + \sum_{k=1}^{oo} \frac{x^k}{k * k!} \qquad // x>0$$

such that the sum s_{B2} can be expressed

$$s_{B2} = e^{-1}(1 + (Ei(1) - \ln(1) - gamma) - (Ei(e^{-1}) - \ln(e^{-1}) - gamma))$$

which reduces to

$$s_{B2} = \frac{Ei(1) - Ei(\frac{1}{e})}{e} = 0.703834423154$$ [3]

I didn't derive the sums for higher-order Bell-numbers yet.

[3] in the Pari/Gp-implementation

```
gp > e = exp(1)
gp > EI(x) = (- eint1(-x))        //define EI-function
gp > ( EI(1) - EI(1/e)) /e
%558 = 0.703834423154
```

2.3. Alternating sum of $B^{(m)}$ of same index

To determine the alternating sums of $B^{(m)}$ over all m of a fixes index r is not difficult. The $B^{(m)}{}_r$ grow according to the r'th polynomial and thus Euler- or even Cesaro-summation is sufficient. But it is even simpler: since the related polynomials are just finite linear combinations of $a_n *m^n$, with a_n some coefficients, we may replace this using the alternating zeta-function, the Dirichlet eta-function $\eta()$, at non-positive integer arguments.

However, different from that definition, we need a variant, where the index of the $\eta()$ function begins at zero, denote it as $\eta°()$ with the effect, that its sign is changed except for $k=0$:

$$\text{def:}\quad \eta°(-k) = 0^k-\eta(-k)$$

Then

$$\begin{aligned} s_n &= \sum_{m=0}^{oo}\left((-1)^m b_n^m\right) \\ &= \sum_{m=0}^{oo}\left((-1)^m \sum_{k=0}^{n}\left(a_k * m^k\right)\right) \\ &= \sum_{k=0}^{n}\left(a_k \sum_{m=0}^{oo}\left((-1)^m m^k\right)\right) \\ &= \sum_{k=0}^{n}\left(a_k \eta^o(-k)\right) \end{aligned}$$

Examples. Recall the m-polynomials as given by E.T.Bell, expanded for m:

$$\begin{aligned} \xi^{(m)}{}_1 &= 1 && = 1m^0 \\ \xi^{(m)}{}_2 &= m + 1 && = 1m^1 + 1m^0 \\ \xi^{(m)}{}_3 &= 1/2(m + 1)(3m+2) && = (3m^2 +5m^1 + 2m^0)/2 \\ \xi^{(m)}{}_4 &= 1/2(m + 1)(2m+1)(3m+2) && = (6m^3 + 13m^2 + 9m^1 + 2m^0)/2 \end{aligned}$$

Then for a fixed row n we get the alternating sum by the formal replacement of m^k by $\eta°(-k)$

$$\begin{aligned} s_1 &= 1*\eta°(0) && && = 1/2 \\ s_2 &= 1*\eta°(-1)+1*\eta°(0) && = -1/4+1/2 && = 1/4 \\ s_3 &= 3*\eta°(-2)+5*\eta°(-1)+2*\eta°(0) && = (0-5/4+1)/2 && = -1/8 \\ &\ldots \end{aligned}$$

giving the sequence

$$s = (1/2, 1/2, 1/4, -1/8, -1/4, 19/16, 39/32, -2623/64, 365/8, 258257/64,\ldots)$$

and separated for numerators and denominators we get

$$\begin{aligned} \text{numerators}(s_n) &= (1, 1, 1, -1, -1, 19, 39, -2623, 365, 258257, \ldots) \\ \text{denominators}(s_n) &= (2, 2, 4, 8, 4, 16, 32, 64, 8, 64, \ldots) \end{aligned}$$

Although these sums don't look as of much interest, I'll show the same problem in the chosen matrix-notation.

Since $B^{(k)} = S2^{k} * V(1)$ the alternating sum over $k=0..inf$ of all these vectors is

$$\begin{aligned} AB &= S2^0 * V(1) - S2^1 * V(1) + S2^2 * V(1) - \ldots + \ldots \\ &= (S2^0 - S2^1 + S2^2 - S2^3 + \ldots - \ldots) * V(1) \\ &= (I + S2)^{-1} * V(1) \end{aligned}$$

where in the last row I applied the closed form for the alternating geometric series to the matrix-argument.

The shortness of this formula is impressive. We get the matrix ***S2A*** by

$$S2A = (I + S2)^{-1}$$

Example:

$$S2A = (I + S2)^{-1} \qquad \begin{bmatrix} 1/2 & . & . & . & . & . \\ 0 & 1/2 & . & . & . & . \\ 0 & -1/4 & 1/2 & . & . & . \\ 0 & 1/8 & -3/4 & 1/2 & . & . \\ 0 & 1/4 & 1/2 & -3/2 & 1/2 & . \\ 0 & -19/16 & 25/8 & 5/4 & -5/2 & 1/2 \end{bmatrix}$$

and for the required alternating sums

$$AB = S2A * V(1)$$

we get the vector

$$AB = [1/2, 1/2, 1/4, -1/8, -1/4, 19/16, 39/32, -2623/64, \ldots]$$

where the entries give the same results as before.

Example:

$$AB = S2A * V(1) \qquad \begin{bmatrix} 1/2 \\ 1/2 \\ 1/4 \\ -1/8 \\ -1/4 \\ 19/16 \end{bmatrix}$$

3. Appendix

3.1. Proof: Binomial sum of reciprocals

Proposal:

For an integer n>= 0 there is the equality

$$\frac{1}{1+n} = \sum_{k=0}^{n}\left((-1)^k \frac{C(n,k)}{1+k} \right)$$

Proof:

First, we have: $\frac{C(n,k)}{k+1} = \frac{C(n+1,k+1)}{n+1}$

Then, looking at (1-1)^n, we get

$$0 = \sum_{k=0}^{n}\left((-1)^k C(n,k) \right)$$

Putting these together, we get

$$\begin{aligned}
\sum_{k=0}^{n}\left((-1)^k \frac{C(n,k)}{1+k} \right) \\
&= \sum_{k=0}^{n}\left((-1)^k \frac{C(n+1,k+1)}{n+1} \right) \\
&= \frac{1}{n+1} \sum_{k=1}^{n+1}\left((-1)^{k-1} * C(n+1,k) \right) \\
&= \frac{1}{n+1}\left(C(n+1,0) + \sum_{k=0}^{n+1}\left((-1)^k * C(n+1,k) \right) \right) \\
&= \frac{1}{n+1}(1+0) \\
&= \frac{1}{n+1}
\end{aligned}$$

(Thanks for this proof to Rob Johnson in *news: sci.math*, see: [RJoh])

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The bitmaps for the example are produced with the bitmap-facility of my Pari/GP-GUI
Pari-TTY *http://go.helms-net.de/sw/paritty*

email of author: *helms(at)uni-kassel.de*