Famous numbers on a Chessboard

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Abstract

In this article it is shown how famous numbers like Pascal's triangle, the Fibonacci numbers, Catalan's triangle, Delannoy's square array, the Pell numbers and Schröder's triangle can be constructed on a chessboard with a rook, knight, bishop, king or queen. Furthermore, several new triangle sums, which are all named after chess pieces that are leapers and add up numbers according to the way they leap, are introduced. Finally a new theory of how Hipparchus, who lived around 150 BC, might have calculated his two famous numbers with the aid of a 'chessboard' is presented.

1 Introduction

Chess has been a source of inspiration for mathematicians throughout history but the mathematics that rules the movements of the thirty-two pieces on a chessboard remains at large. Wouldn't it be nice to have a book with some formulae that enable us to calculate what to do in a particular position on the chessboard? Of course, no such book exists. We can ask a chess computer what to play in a particular position but, as long as there are more than six pieces on the board, this is only a second best solution. So what can be said about the intersection of the domains of chess and mathematics?

Let's start with a deceptively simple classical example. Leonhard Euler, one of the greatest mathematicians of all time, wrote in 1759: 'I found myself one day in company where, on the occasion of a game of chess, someone proposed this question: to move with a knight through all the squares of a chessboard without ever arriving twice at the same square, and commencing from a given square.' Later that year Euler presented a paper on knight's tours to the members of the Academy of Sciences in Berlin. A brilliant paper but one of his statements, no closed knight's tours are possible on 3 x 2n boards, was wrong. Many authors echoed this statement until Ernest Bergholt exhibited a solution for a 3 x 10 board in the *British Chess Magazine* (1918). The full set of sixteen solutions for a 3 x 10 board was published by Maurice Kraitchik in 1927 and the complete solution for 3 x 2n boards was obtained independently by Donald Knuth and Noam Elkies in 1994 (A070030; for all A-numbers in this article see Neil Sloane's amazing *On-line Encyclopedia of Integer Sequences* at www.oeis.org). So it took some of the best minds in mathematics almost 250 years to solve this simple problem. That doesn't bide well for the general solution of the game of chess.

What other chess related questions could we ask ourselves? There are many but in this article I restrict myself to the question which famous numbers can be constructed on a chessboard with a rook, knight, bishop, king or queen. The first answers to this question turned out to be quite surprising. I will show how four simple questions lead to Pascal's triangle, the Fibonacci numbers, Catalan's triangle, Delannoy's square array, the Pell numbers and Schröder's triangle. En passant I present a new theory of how Hipparchus, who lived around 150 BC, might have calculated his two famous numbers with the aid of a 'chessboard'.

2 Blaise Pascal's triangle

The first problem: how many paths can a rook take from square a1 to square h8 if it can only move to the north and east?

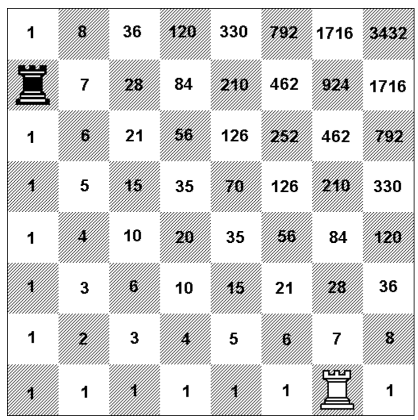

Figure 1: Pascal's triangle.

For the answer to this question see figure 1. If we rotate the board 135 degrees to the right we observe that on the upper half of this board Pascal's triangle appears (A007318). Addition of the numbers in the horizontal rows of this triangle leads to the powers of two (A000079). Addition of the numbers on the chessboard with a knight leads to the Fibonacci numbers (A000045; a1=1; b1=1; c1+a2=2; d1+b2=3; e1+c2+a3=5; f1+d2+b3=8; g1+e2+c3+a4=13; h1+f2+d3+b4=21; etc.).

Pascal's triangle is the most famous of all number triangles. In fact, it was known long before him but Blaise Pascal's *Traité du triangle arithmétique* (1654) is considered to be the most important source of information. An amazing number of relations can be formulated for this triangle and Donald Knuth observed that if someone finds a new identity that no one except the discoverer will get excited about it. So this will probably also be the case for the 'chess sums' that I have added to the OEIS entree for the Pascal triangle (apparently eight of them are new). I named these sums after chess pieces that are leapers and all of them add up numbers on the chessboard according to the way they leap: knight (√5 or 1,2; square root of five), fil (√8 or 2,2), camel (√10 or 3,1), giraffe (√17 or 4,1) and zebra (√13 or 3,2) (A180662). The fil was used in shatranj, the Islamic predecessor of chess, the camel, giraffe and zebra are fairy chess pieces and the knight still plays its classical role in modern chess; see *The Oxford Companion to Chess* (1992) by David Hooper and Kenneth Whyld.

Leonardo of Pisa, who is better known as Fibonacci, introduced the Hindu-Arabic number system in Europe with his book *Liber Abaci* (1202). It caused a revolution in the way we calculate. Nowadays Fibonacci's fame rests primarily on a sequence which appears in this book and was named after him by Édouard Lucas. An expression for the general term of this sequence was found some five hundred years later by Abraham de Moivre in 1730 who linked it to the golden ratio (A001622).

De Moivre was a respected scientist with a passion for chess. He met Isaac Newton for the first time just after Newton's *Philosophiae Naturalis Principia Mathematica* (1687) appeared, a book that is regarded as one of the most important works in the history of science. They became friends and spent their evenings together debating philosophical matters. De Moivre promptly mastered the new mathematics of the *Principia* with the result that Newton is said to have referred persons asking him about his work to De Moivre 'who knows all that better than I do'; see *Maty's biography of Abraham de Moivre, translated, annotated and augmented* (2007) by David Bellhouse and Christian Genest.

De Moivre was a regular customer of Slaughter's Coffee House in London where he met fellow Huguenots, would give advice on risk and must have played chess with other customers. A solution of the knight's tour from his hand appeared in *Récréations mathématiques et physiques* (1725) by his teacher Jacques Ozanam and in *The Doctrine of Chances* (1718) he included an engraving on which a chessboard that has been cast aside can be seen. It is possible that De Moivre met François-André Danican Philidor when he played a match with Phillip Stamma at Slaughter's in 1747. Shortly afterwards Philidor's *L'analyze des échecs* (1749) was published in London, a book that did for chess what the *Principia* did for physics.

According to legend the first numbers that appeared on the very first chessboard were the powers of two. Grand Vizier Sissa ben Dahir, who invented chess for the Indian king Shirham around the year 600 AD, proposed for his reward the doubling game with grains of wheat. The doubling game implies that the king had to put one grain on a1, two on b1, four on a2, eight on c1, sixteen on b2, thirty-two on a3, etc. 'And is that all you wish, Sissa, you fool?' the astonished king must have shouted only to discover that there wasn't enough wheat around to fulfil Sissa's request.

3 Eugène Catalan's triangle

The second problem: how many paths can a rook take from square a1 to square h8 if it can only move to the north and east but cannot move below the a1-h8 diagonal?

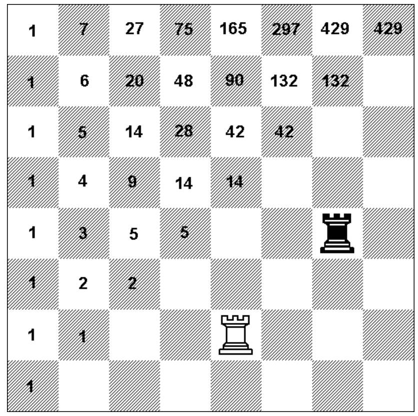

Figure 2: Catalan's triangle.

Catalan's triangle answers this question (see figure 2; A009766). The Catalan numbers appear on the a1-h8 diagonal (A000108). You find these numbers again when you add the numbers in the horizontal ranks or rows. A bishop, which can only move to the north-west and north-east, also leads to this triangle (via A053121).

In *Catalan numbers with Applications* (2009) Thomas Koshy attributes the relation between these rook paths and the Catalan triangle to Henry G. Forder who published his findings in 1961. On the front page of Koshy's book one possible rook path is depicted. Eugène Catalan discovered 'his' numbers in 1838. They have been rediscovered many times by others not only after but also before Catalan. The number of combinatorial interpretations that Richard Stanley gives on his website is an incredible 190, dd. 21-08-10. Occasionally Catalan numbers turn up in situations that are related to chess. In *Queue problems revisited* (2005) Stanley presents the number of solutions of a series-helpmate problem by Eero Bonsdorff and Kauko Väisänen which appears to be C7 = 429 and he shows how to extend this number of solutions to C17 = 129644790.

4 Henri Delannoy's square array

The third problem: how many paths can a king (queen) take from square a1 to square h8 if (s)he can only move to the north, east and north-east?

Delannoy's square array answers this question (see figure 3; A008288). Adding the numbers in the triangle rows, after rotating the board 135 degrees to the right, leads to the Pell numbers (A000129) and adding the numbers of the square array with a knight leads to the tribonacci numbers (A000073). The numbers on the a1-h8 diagonal are the central Delannoy numbers (A001850).

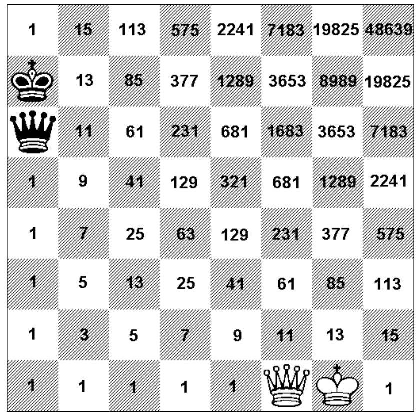

Figure 3: Delannoy's square array.

Our third problem appeared in *Emploi de l'échiquier pour la résolution de divers problèmes de probabilité* (1889) by Henri Delannoy. Charles-Ange Laisant suggested the queen walk, Delannoy found the solution and Édouard Lucas called the number array 'Delannoy's arithmetical square' and that name stuck. Édouard Lucas liked games. He was the author of the four volume book Récréations Mathématiques (1881-1894). Delannoy and Laisant were two of the editors. In his first book the eight-queen problem can be found. This problem was first published by Max Bezzel in the Deutsche Schachzeitung (1848). You have to place eight queens on an 8 x 8 chessboard in such a way that no queen is attacked by another and determine the number of such positions. Carl Friedrich Gauss, the 'Prince of Mathematics', saw this problem two years later in a newspaper and solved it with some difficulty. Lucas describes among others the method used by Gauss and proudly presents the complete set of 92 solutions. He expresses his interest in the solutions for n non-attacking queens on an n x n chessboard for n=9, 10, 11 and 12 and comes back to this question in note IV at the end of the first book. Lucas mentions that for n=9 Pieter Hendrik Schoute found 352 solutions and that for n=10 his devoted friend Henri Delannoy found 724 solutions. Both sets are complete. Nowadays this problem has been solved with the aid of computers for values of n up to 26 (A000170).

The Pell equation plays an important role in finding good rational approximations for the square root of positive integers; see *Number Theory* (1984) by André Weil. The Pell equation got its name around 1765 from Euler and it is generally believed that he made a mistake. A better name would certainly be the BBB equation after Brahmagupta (635), Bhaskara (1150) and William Brouncker (1658) who developed the classical methods for solving the 'Pell equation', but attempts to change the terminology introduced by Euler have always proved futile.

The Pell numbers P(n) can be linked with the positive and negative Pell equations of 'the square root of two' and it can be shown that the formula [1+P(n)/P(n+1)] gives good rational approximations for the square root of two. This formula leads to a curious observation: we have seen that Delannoy's square array gives us the Pell numbers so we might say that just by looking at how a king, or a queen for that matter, moves on a chessboard allows us to calculate the square root of two.

Eric Temple Bell placed in *Men of Mathematics* (1937) Newton, Gauss and Archimedes at the top of his list of the greatest mathematicians of all time but some believe that he should have included Euler. We are all indebted to Euler, one of the most prolific mathematicians ever, of whom Pierre Simon Laplace said 'c'est notre maître à tous'. We have already met three of them so let's make a short detour and meet the fourth, Archimedes, a man with the appearance and mind of a chess player.

In *Parallel Lives: The Life of Marcellus* (75) the Greek historian Plutarch wrote the following words about Archimedes: 'He placed his whole affection and ambition in those purer speculations where there can be no reference to the vulgar needs of life.' and 'Oftentimes his servants got him against his will to the baths, to wash and anoint him, and yet being there, he would ever be drawing with his fingers lines upon his naked body, so far was he taken from himself and brought into trance, with the delight he had in the study of geometry.' Apparently quite a few chess players live parallel lives. A nice example of one of his purer speculations is Archimedes' cattle problem. A manuscript with this highly original problem was discovered in 1773 by Gotthold Ephraim Lessing, the librarian of the Wolfenbüttel library. Apparently one of his competitors had dared to suggest that he could handle large numbers better than the master himself and this required an appropriate reaction. Archimedes presented the cattle problem in a short poem around 250 BC. In his poem he asked his friends to determine the size of the herd of the Sun god that grazes at Sicily. It goes without saying that the Sun god has a large herd. In order to calculate the number of white, black, dappled and brown bulls and cows you have to solve two problems. Solving the first part shows that you are not ignorant and solving the second part wins you the supreme wisdom prize. The latter part requires that you solve a difficult version of Pell's equation and leads to a total number of cattle requiring no less than 206545 digits (A096151); see *Solving the Pell equation* (2002) by Hendrik Willem Lenstra Jr.

In order to get a feeling for this number let's use the doubling game once again: we put one cow or bull on a1, two on b1, four on a2, eight on c1, sixteen on b2, thirty-two on a3, etc.. To allocate the whole herd we need a chessboard of no less than 828 x 828 squares. A rather large chessboard I admit but remember that we are dealing with the herd of the Sun god. One might argue, as several nineteenth century German scholars did, that there aren't enough bulls and cows on earth and that they might not fit on the isle of Sicily but, as Lessing remarked, the Sun god will have coped with it.

Archimedes must have been keenly aware of the fact that his problem led to an impossible large number and there must have been a mischievous smile on his face when he sent the poem to his friend Eratosthenes who lived in Alexandria where he was the chief librarian at the Mouseion library.

5 Ernst Schröder's triangle

The fourth problem: how many paths can a king (queen) take from square a1 to square h8 if (s)he can only move to the north, east and north-east but with the extra condition that (s)he cannot move below the a1-h8 diagonal?

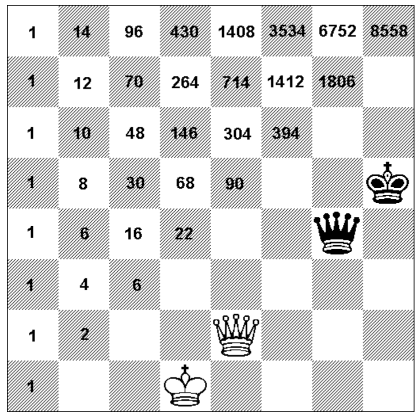

Figure 4: Schröder's triangle.

Schröder's triangle answers this question (see figure 4; A033877). On the a1-h8 diagonal the large Schröder numbers appear (A006318). Addition of the numbers in the ranks or rows leads to the little Schröder numbers (A001003).

While studying the numbers that appear in Schröder's triangle, I noticed that the sum of the numbers in the ninth rank or row is 103049. Quite surprisingly this number appeared around 150 BC in a statement of the Greek astronomer Hipparchus about the number of compound propositions that can be made from only ten simple propositions but it is a mystery how he calculated it. The situation becomes even more mysterious if we use a fil, the predecessor of the bishop, to add up numbers. A fil, Arabic for elephant, moves diagonally and leaps over one square. Walking upwards along the a1-h8 diagonal addition with a fil leads to the sequence: 1, 2, 7 (=6+1), 28 (=22+6), 121 (=90+30+1), 550, 2591, 12536, 61921, 310954, etc. (A010683). The tenth number agrees closely with the second number that was mentioned by Hipparchus in his statement, namely 310952. So the question arises: 'Did Hipparchus know a game that looked like chess?'

In *Hipparchus, Plutarch, Schröder and Hough* (1997) Richard Stanley describes the history of the two numbers given by Hipparchus that were transmitted to us by Plutarch. A scholarly account of the original Greek sources can be found in *On the Shoulders of Hipparchus* (2003) by Fabio Acerbi. It is interesting to note that Acerbi starts his article with the sentence: 'To write about combinatorics in ancient Greek mathematics is to write about an empty subject.' The effect of the two numbers given by Hipparchus is, according to Acerbi, disruptive and the whole issue of ancient Greeks combinatorics must be reconsidered. The possibility that a game that looked like chess played a role isn't considered by Acerbi and Stanley. The latter, whom I asked for his opinion, feels that it is farfetched to think that Hipparchus knew of any game similar to chess. He added that Plutarch states Hipparchus' results in terms of compound propositions that can be made from ten simple propositions; Thus is seems that Hipparchus was thinking in terms of Stoic logic, as discussed by Acerbi. Stanley concluded that there is no reason to believe that a game like chess was involved.

Figure 5: Achilles and Ajax.

Let's look at the situation from a somewhat different perspective. Board games were popular in ancient Greece. Around 530 BC, the potter painter Exekias made a beautiful amphora with Achilles (left) and Ajax (right) playing a board game, see figure 5. What board games did the ancient Greeks play? Roland G. Austin starts his article about *Greek board games* (1940) with the sentence: 'The study of Greek board-games is almost wholly inconclusive, owing to the scanty and extremely imprecise evidence available.' So very little can be said about the games they played. We can, however, speculate that somebody close to Hipparchus had a petteia board and using one of the pieces he asked himself the same question I formulated above. Obviously for a king-like piece the terms in the a-column (file) must all be one and the other terms on the board must be the sums of three terms, i.e. c5=c4+b4+b5 and e5=d4+d5+e4 with e4=0; simple additions that reflect the king's movements. After that he had to add the terms in the rows (ranks) in two ways just like I did and show his results to Hipparchus. Hipparchus, with his knowledge of the intricacies of Stoic logic, would of course have recognized these numbers instantly and must have used this fairly simple method to calculate the larger numbers that reached us through the writings of Plutarch. A mystery solved? It is possible.

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