A CLASSIFICATION OF SEMEAI WITH APPROACH MOVES

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ABSTRACT

The article analyzes liberty races (semeai) in the game of Go. A rigorous treatment of positions with approach moves, but no ko revealed a class of positions not described in the literature before.

The complete presented classification is simpler and more compact than the one used so far. This was achieved through a classification by the types of liberties and the order they are to be filled instead of a classification by group status.

A principal difference to semeai without approach moves is that in the presence of approach moves and shared liberties and in the absence of eyes, two different strategies are possible because the un-orthodox filling of shared liberties instead of outer liberties may be a better strategy.

1. MOTIVATION

In the game of Go, liberty races (semeai) belong to the few classes of positions that can be analyzed without performing a tree search. Roughly speaking, the reason is that the best moves of both sides do not overlap and thus can be counted separately, apart from a possibly coinciding capturing move in the shared region but that move can be counted like other moves because it is the last move.

Compared to numerous publications on the subject of semeai (see references in section 12) in this contribution we perform a complete classification of all semeai that include "plain" approach moves (no fights, no kos) and describe a case that has not been covered in other publications.

Over thirty years ago, Karl-Friedrich Lenz (1982) introduced the Semeai-formula (see section 13) and described the basic principles of how to evaluate semeai. The publications by Richard Hunter (Spring 1996 - Summer 1997, 1998, 2003) are meant for Go players and contain many examples. The second half of Hunter (2003) discusses the appearance of ko in semeai but not the use of approach moves. In Nakamura (2008, 2009), Teigo Nakamura uses Combinatorial Game Theory to add up the number of moves necessary to occupy all outside liberties, but does not consider the case that the capturing move is outside *and* that approach moves are necessary. The publications by Martin Müller (1999, 2001, 2003) classify more general cases of semeai mainly from the point of view of computer Go, but with the consequence that exact relations cannot be formulated. The book by Robert Jasiek (2011) discusses basic semeai including the case that one of the two essential chains is inside an eye of the other essential chain, but does not discuss the case of approach moves. Fianlly, Thomas Redecker (2012) explains theoretical foundations of a special type of Semeai which includes ko.

As a by-product of our treatment of approach moves we are compacting the classification of all cases to one table with one page of instructions. Other classifications list many more cases. For example, the AGA-Newsletter (Dec 2011) quotes Robert Jasiek about his book on semeai ".. there's the five basic types of semeai with 93 possible cases and over 200 principles governing how to determine status and outcome." (even without discussing approach moves). We believe that a merger of classes will not only be more satisfying for the mathematically interested readers and programmers but also for Go-players.

We will characterize the different cases and playing strategies by equality relations, which we call *balance relations*, describing the unsettled situation where strengths are balanced and success depends on who moves first. If

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these equality-relations are not satisfied then the inequality will directly tell who won unconditionally.

In order to minimize the number of cases and to reach a complete classification in the presence of plain approach moves it is important not to do the obvious thing and classify semeai phenomenologically by possible outcomes (kill/kill or kill/seki) but by the source of their different behavior which is the question where the capturing move is. If a chain has an eye then there is no choice, the capturing move of this chain has to be in the eye. But if the chain has no eye then the capturing move can be in the inner region shared with the opponent's essential chain or it can be on one of the outside liberties as will be shown by examples in subsection 7.6.

The advantage of filling inside liberties first and having the capturing move outside is to be able to select that capturing move which avoids as many approach moves as possible (if there is such a choice).

The aricle is organized as follows. After defining basic semeai with approach moves in section 2, we introduce in section 3 the notation and derive *equations of balance* that characterize unsettled positions. Section 4 gives an overview of all cases. Section 5 comments upon the overview. The sections 6 and 7 discuss the consequences of approach moves. Of special interest are the semeai with a shared region and without eyes which are discussed in section 7.

The question when it is beneficial to convert a larger number of approach moves into a smaller number of outside liberties is answered in section 8. Section 9 gives the proper order in which liberties are to be filled. A generalization to multi-purpose approach moves is discussed in section 10. Section 11 concludes the article with a summary.

2. ASSUMPTIONS

This article considers positions in which only two chains (one white and one black) are in a liberty race. Both are called essential chains which are assumed to have several properties.

- None of the chains is under atari[2] with the opponent moving next.
- If one of the essential chains is captured then this settles the race.

  Examples where this is not the case are positions known as snapback, 2-step snapback and 'under the stones' as described in the appendix.

  Another example for positions that are not considered are positions where an essential chain has a nakade shape, i.e., a shape that is sufficiently small and sufficiently compact so that the capture of that chain does not lead to two eyes for the other side (see `http://senseis.xmp.net/?Nakade`).
- Both essential chains are either in direct contact or separated by empty points that are neighbour to both chains.
- Both chains can have up to one eye which must be small enough not to guarantee seki or life for the essential chain (assumption 1) and it must be settled in the sense that it cannot be split into two eyes in one move (assumption 2). Furthermore, the eye does not include an empty point that is not a liberty of the surrounding essential chain (assumption 3). A fourth assumption on the eye is that it does not contain all four points A1, A2, B1, B2 plus another point in a corner. (If the opponent would occupy these four points and get captured then a move on B1 and A2 would give the opponent an eye in A1 which would change the move count in formula (1) below).
- Both essential chains may require approach moves for their outside liberties. But these moves should be "plain" in the following sense. For example, if the essential chain is white then the position should not give White any gain or incentive to prevent Black from performing these approach moves, i.e., there is no ko and no other fight possible that could delay or prevent these moves.

  Furthermore, the sets of black moves to approach different liberties of White should be disjoint. In other words there are no "multi-purpose" approach moves that serve to approach more than one white liberty.

[2] A chain under atari has only one liberty.

To summarize, it should be straight forward to count for each liberty of White how many approach moves by Black are required to occupy that liberty without the chance of the liberty-filling move being back-captured.

Generalizations to this strict definition of approach moves are discussed in sections 8 and 10.

To summarize, all liberties (in the shared region, in the eyes and on the outside) are "plain" and do not involve any fights and thus are counted in integer numbers.

These assumptions are chosen to be rather conservative in order to allow the derivation of explicit formulas in the next section. Minor generalizations are discussed in section 10.

3. RELATIONS CHARACTERIZING UNSETTLED POSITIONS

In this section we derive all relations that characterize unsettled semeai positions satisfying the above criteria. To be able to describe all cases with only two mathematical relations we need to introduce a notation for input variables and derived variables (see 3.1). Then we describe derived variables (see 3.2), relations of balance (see 3.3), and the applicability of the relations of balance (see 3.4).

3.1 Notation

We use Diagram 2 to illustrate definitions that are made to describe the position in Diagram 1.[3] Most of the variables introduced below refer to properties of essential chains which are marked by ⊗ ⊗ in Diagram 2. If the variable has an index then this specifies the colour of the chain. For example, if Z is the size of an eye then Z_B is the size of the black eye and Z_1 is the size of the eye of player 1. (Player 1 will play the role of what is called attacker in the literature in positions without approach moves.) We define six input variables and four derived variables.

Comments enclosed by [ ] in the following definitions refer to Diagram 2.

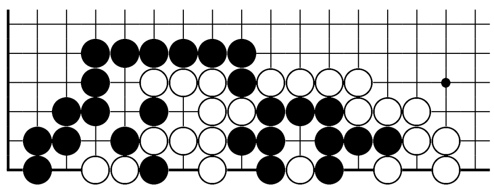


Diagram 1.

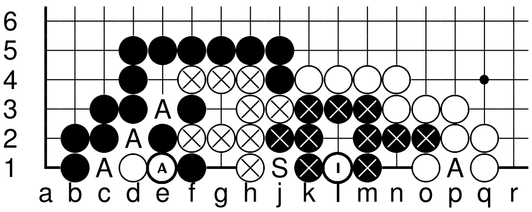


Diagram 2.

Input variables

- Z si**Z**e of an eye (number of points inside the eye whether occupied or not) of an essential chain ($Z = 0$ if no eye) [$Z_W = 0$ (no eye), $Z_B = 2$ (l1,l2)]
- I number of opponent stones **I**nside the eye ($I = 0$ if no eye) [$I_W = 0$ (no eye), $I_B = 1$ (stone I)]
- O number of external (**O**utside) liberties of an essential chain [$O_W = 3$ (e4, g3, g1), $O_B = 1$ (n1)]
- A total number of **A**pproach moves to all outside liberties of an essential chain, A is an integer as we consider only "plain" approach moves [$A_W = 4, A_B = 1$ (points marked A and A)]
- V number of approach moves to an outside liberty that can be a**V**oided when the capturing move is on an outside liberty, i.e., V is zero if the chain has an eye, else V is the maximal number of approach moves to any one of the outside liberties of the essential chain [$V_W = 3$ (c1, d2, e1), $V_B = 0$ (because Black has an eye)]
- S number of **S**hared liberties [$S = 1$ (point marked S)]

Derived variables

- E the difference of the number of moves played by both sides inside an **E**ye of size Z, with already I opponent stones inside in order to capture the chain with this eye

[3] Strictly speaking, the position in Diagram 1 does not fully satisfy the condition that approach moves to all liberties are "plain". White could play on g2 and have the strategy to defend the new liberty f1 through ko. But because White has no eye and has only one such liberty, our theory also applies to this position.

X the number of moves of the opponent to occupy safely all e**X**clusive liberties of a chain, i.e., the number of all outside liberties and all their approach moves and moves to fill the eye, if necessary, repeatedly minus the $Z-3$ own moves inside the own eye of size Z if $Z>3$ to defend the eye, in other words X is the number of moves to capture a chain without the moves on shared liberties, under the assumption that the capturing move is in the shared region

R number of moves to capture a chain without the moves on shared space, all under the assumption that the capturing move is NOT in the shared region (a liberty count **R**educed by the number of approach moves to the outside capturing move)

P_i **P**layers P_1, P_2, P_I, P_{II} = Black/White

In the following we show relations between the symbols introduced above.

3.2 Derived Variables

After determining input variables through a straight forward counting of liberties and stones, the other variables are determined as follows.

The formula for the number E of moves to fill an eye repeatedly until it can be captured minus the number of own moves to defend the eye is[4]

$$E = \begin{cases} \text{for } Z=0,1: & Z \\ \text{for } Z>1: & (Z-1)(Z-2)/2+2-I. \end{cases} \tag{1}$$

Applying (1) to Diagram 2 we obtain $E_W=0$ because $Z_W=0$ (White has no eye) and $E_B=(2-1)(2-2)/2+2-1=1$.

If a chain is captured and the capturing move is on an outside liberty then one can choose that outside liberty as the capturing move which requires the maximal number of approach moves because these approach moves need not be made. The number V of approach moves that can be avoided this way is defined as

$$V = \begin{cases} \text{for } Z\neq 0: & 0 \\ \text{for } Z=0: & \text{max number of approach moves of any one of the outside liberties} \end{cases} \tag{2}$$

i.e., if the chain has an eye then the capturing move will be in the eye, so $V=0$. For Diagram 2 we obtain $V_W=3$ (the moves of Black on c1, d2, e1 are needed to prepare Black on g1 whereas Black on g3 needs only one approach move on e3), and $V_B=0$ because of $Z_B\neq 0$.

The formula for the number X of all moves needed to occupy all exclusive liberties of a chain (i.e., all liberties other than the shared liberties) safely, i.e., including all approach moves if none of these moves is the capturing move, minus the own moves to defend the eye (if there is one) is

$$X = E+O+A \tag{3}$$

where E is given by (1). Applying (3) to Diagram 1 we obtain $X_W=0+3+4=7$ and $X_B=1+1+1=3$.

If one of the moves on an outside liberty is the capturing move, then V moves less than counted under X have to be played and the resulting number is

$$R = X-V \tag{4}$$

with the values $R_W=7-3=4,\ R_B=3$ in Diagram 2.

The values X, R and S are used in the following to formulate relations that characterize unsettled semeai positions.

[4] We find it easier to remember for a given Z to multiply $Z-1$ and $Z-2$ rather than the other form of the formula: $(Z^2-3Z+6)/2-I$ that appears more often in the literature. Go players may want to learn the values $E(Z)=Z$ for $Z\leq 3$ and $=5,8,12,17$ for $Z=4..7$.

3.3 Relations of Balance

To derive relations that characterize unsettled positions we consider as the first case, the case that in an unsettled position one such optimal and successful sequence of moves has its last move - the capturing move - in the shared region. Let us assume White was successful in this sequence.

Because White did the capturing move in the shared region, White has to occupy all outer liberties of Black and perform all approach moves to these liberties before. This means, up to the move before the capturing move White had to do $E_B+O_B+A_B+S-1 = X_B+S-1$ moves.[5] To be successful and do the capturing move, White had to make the first move in the sequence. But because the position is unsettled and White played optimally, if Black would have started, White should not be successful. The only way for White not to be successful is that White itself is under atari one move before White's capturing move in the shared region. Because one empty point in the shared region is left, this is a liberty of White and it is therefore the last liberty of White. That means in the last $X_B + S - 1$ moves of Black, Black must have occupied all outside liberties of White and made all the approach moves to these liberties but no moves in the shared region because they would have helped White as White's capturing move was in the shared region. Because the position is unsettled, the computed number of moves by White and Black must be equal so that whoever started and can do the next (capturing) move wins. This means $X_W \geq X_B$ and $X_W = X_B + S - 1$. In general, the form of this relation of balance is[6]

$$X_I = X_{II} + S - 1 \quad \text{(FOF)} \tag{5}$$

where the inequality $(X_I \geq X_{II})$ identifies players I and II. We call this type FOF (**F**ill **O**utside liberties **F**irst) because before the capturing move all outside liberties are filled.

The second case that none of the capturing moves of the two winning sequences of Black and White is located in the shared region is easily characterized. If an essential chain has no eye then optimal play will select as capturing move that outer liberty which needs the most approach moves because these approach moves need not be played if that liberty is the capturing move.

Thus the second case is characterized by replacing in relation (5) the $S-1$ through S because all S shared liberties are occupied before the capturing move and by replacing X through $X - V = R$ because not all approach moves are necessary if the chain to be captured does not have an eye. Thus the relation of balance is

$$R_1 = R_2 + S \quad \text{(FIF)} \tag{6}$$

where the inequality $R_1 \geq R_2$ identifies players 1 and 2.[7] The (FIF) relation (6) applies also if player 1 or 2 or both have an eye because then $V = 0$ for that player. We call this case FIF (**F**ill **I**nside liberties **F**irst)[8] because before the capturing move *all* inside liberties have to be filled by one player which is player P_1.

To apply these relations to a position like Diagram 2 we first have to identify players I and II based on $X_I \geq X_{II}$ before checking (5) and players 1 and 2 based on $R_1 \geq R_2$ before checking (6). For Diagram 2 we obtain $7 = X_W > X_B = 3$ and thus

$$7 = X_W > X_B + S - 1 = 3 \tag{7}$$

for the FOF scenario and $4 = R_W > R_B = 3$ and thus

$$4 = R_W = R_B + S = 4 \tag{8}$$

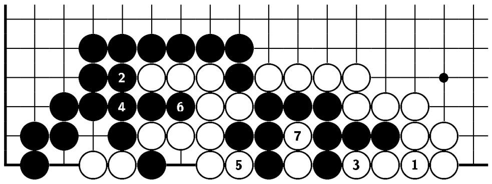
for the FIF approach. Relation (7) tells us as expected that Black would be unsuccessful in occupying at first all exclusive liberties of White by 7 moves, whereas relation (8) shows an equality, i.e., the position is unsettled, both sides can kill when moving first.

[5] E_B is not needed in this formula because if the capturing move is in the shared region then Black has no eye, but in that case $E_B = 0$, so having E_B in the formula is not wrong.

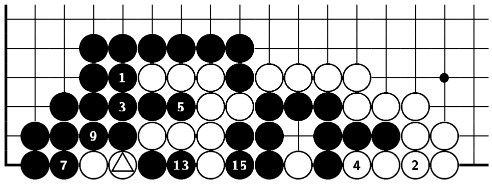
[6] We call it relation and not equation because equations can be used to compute one variable in terms of all others and also because in our relations the $=$ sign may be replaced by $>$ or $<$.

[7] In the (FIF) relation (6) we use 1, 2 to label players whereas in (5) we label them with I, II because P_1 need not be the same as P_I.

[8] More precise would be FILBCM (**F**ill **I**nside **L**iberties **B**efore the **C**apturing **M**ove) but that would be hard to memorize.



○ captures
Diagram 3.



● captures
⓫ @ △, ⑥, ⑧, ⑩, ⑫, ⑭ pass
Diagram 4.

3.4 Applicability of Relations of Balance

Because the first (FOF-)relation (5) requires at least one capturing move of a winning sequence in the shared region, this relation is not applicable if there is no shared region and also not if both sides have an eye, because then capturing moves can only be moves in the eyes.

But even if only one side, say Black, has an eye such as in Diagram 1, the FOF-relation (5) does not apply. In that case a move capturing Black can only occur in Black's eye, i.e., for White to have a chance it needs to occupy all shared liberties. If Black manages to occupy all but one outside liberty of White (❺ in Diagram 4) (in $X_W - V_W - 1 = 7 - 3 - 1 = 3$ moves) before White occupies the last shared liberty (in $X_B + S - 1 = 3 + 1 - 1 = 3$ moves which is equal to $X_B + S - 1 - V_B$ because $V_B = 0$ due to the black eye) then White can occupy only $S - 1 = 1 - 1 = 0$ shared liberties before putting itself under atari (i.e., without taking its last but one liberty and allowing to be captured instantly). That means $X_W - V_W - 1 = X_B + S - 1 - V_B$, i.e., the (FIF-)relation (6) applies and is the only one to apply if one side has an eye and the other not.

A comment about Diagram 4: The mathematical correctness of the used formula is demonstrated in the above paragraph but one may wonder why is the formula called FIF (Fill Inside First before capturing) if the capturing move ⓯ is inside and not outside. Answer: If the only purpose is to win the liberty race then one of the best moves for ⑥ is to play at ⓯. This move leads to White's capture but all other moves would ultimately too. At least ⑥ on ⓯ is needed for White to have any chance to win. The FIF sequence is not the only best sequence of White, another one also leading to capture is to pass, but FIF is one of the best sequences and therefore it can be used to formulate a balance relation and determine the outcome of the race. If White does not play FIF (in order to save the move as a ko-threat for later) then Black is allowed to change the reply too and play approach moves first before capturing inside as shown in Diagram 4, or, even better, to pass too and not to waste moves. But all of that does not change the fact that purely for winning the liberty race, ⑥ on ⓯ is one of the best possible moves for White and therefore suitable to establish a balance relation for determining the status of the position.

In the remaining case of an existing shared region and no eyes, both relations of type FOF and FIF are relevant as discussed in section 7 further below.

Section 4 gives an overview of all cases and provides all instructions to determine the status.

4. OVERVIEW

Table 1 gives a complete overview of all cases of semeai that involve approach moves and otherwise satisfy the assumptions of section 2. We repeat both balance relations (5) and (6) to have them close to the table:

$$X_I = X_{II} + S - 1 \quad \text{(FOF)} \qquad (9)$$

$$R_1 = R_2 + S \quad \text{(FIF).} \qquad (10)$$

case	characterization of cases	def. of P_i	rel.	win of P_1/P_2
(A)	$S = 0$	arbitrary	(10)	kill/kill
(B)	$S \neq 0,\ Z_W = Z_B = 0$	(B)$_{\mathrm{FOF}}$: $X_I \geq X_{II}$	(9)	kill/kill ($S = 1$)
				kill/seki ($S > 1$)
		(B)$_{\mathrm{FIF}}$: $R_1 \geq R_2$	(10)	kill/kill (played only if P_1 can kill)
(C)	$S \neq 0,\ (0 = Z_1 < Z_2)$ or $(0 < Z_1 < Z_2 > 3)$	$Z_1 < Z_2$	(10)	kill/kill
(D)	$S \neq 0,\ ((0 < Z_1 = Z_2)$ or $(0 < Z_1, Z_2 \leq 3))$	$X_1 \geq X_2$	(10)	kill/seki

Table 1: Overview of all cases.

Steps to determine the status using Table 1

Step 1: For a given position at first decide the case: Case (A) has no shared liberties and case (B) has no eyes. In case (C) there is either only one eye or there are two eyes of different size of which the bigger one has a size ≥ 4.[9] Case (D) has two eyes, either both of size ≤ 3 or both of *equal* size (≥ 4).

Step 2: Then identify players 1,2 based on inequalities in column 2.

- A distinction between players 1 and 2 is only needed for the allocation of shared liberties in the balance relation, i.e., in case (A) without shared liberties the distinction between player 1 and 2 is irrelevant.
- In case (C) player 2 is identified as the one with an eye (if the other player has no eye) or, as the player with the bigger eye of size ≥ 4. In case (D) player 1 has a higher or at least equally high sum of exclusive liberties plus approach moves compared to player 2.
- In case (B) the player I identified through $X_I \geq X_{II}$ does not need to be the same as player 1 identified through $R_1 \geq R_2$! This case may need two steps. If player 1 identified through $R_1 \geq R_2$ can kill based on relation (10) then this settles the status. Otherwise player I has to be identified, now on the basis of $X_I \geq X_{II}$ with possibly a different result, and relation (9) is to be used to determine the status. For example, in Diagram 15 (further below) is $3 = R_B > R_W = 1$ and therefore $P_1 =$ Black whereas $4 = X_W > X_B = 3$ would give $P_I =$ White. Detailed explanations are given in section 7.

Step 3: The balance relations in the 3$^{\mathrm{rd}}$ column 'rel.' describe unsettled positions for each case where the outcome of the race shown in the last column depends on who moves first. If the $=$ is replaced by $>$ then player 1 wins always and if $=$ is replaced by $<$ then player 2 always wins. If the difference between both sides of the relation is n then the loser is n moves too slow.

Step 4: Optimal play is described in section 9.

5. COMMENTS

The following comments concern the new case (B)$_{\mathrm{FIF}}$, they relate known facts about semeai to our Table 1 and the relation of our terminology to definitions used in the literature.

5.1 The new Case

From the two cases under (B) it is the case (B)$_{\mathrm{FOF}}$ that occurs most frequently in games. Case (B)$_{\mathrm{FIF}}$ has even not been described in the literature yet as far as the author knows. In section 7 it is shown that a necessary condition for being able to kill through FIF when reaching only seki under FOF is $V_2 - V_1 > 1$ and when being killed under FOF is $V_2 - V_1 > 2S + 1$ where players 1,2 are identified through $R_1 > R_2$. This shows that case

[9] In the literature eyes of size ≤ 3 are called *small eyes* and eyes of size ≥ 4 are called *big eyes*. The difference between both types is that for size $n \geq 4$ capturing an inside opponent chain with $n - 1$ stones costs one move which lets the opponent allow to make a move without cost but still ≥ 2 liberties are left which is more than the one liberty before the capture. In other words, for size ≥ 4 capture buys time.

$(B)_{FIF}$ can only be relevant in the presence of approach moves and only if the winner under FOF has an outside liberty that needs at least 2 approach moves more than any outside liberty of the loser under FOF, or if one player can win through FIF and FOF but FIF allows more passes (see subsection 7.4).

5.2 Shared Liberties

In cases $(B)_{FIF}$, (C) and (D) player 2 obtains all S shared liberties in relation (10) and in case $(B)_{FOF}$ player 2 obtains only $S - 1$ shared liberties in the corresponding relation (9).

In case (D) the player 2 with fewer exclusive liberties X_2 obtains all shared liberties in the FIF formula because for player 2 to be killed, player 1 needs to occupy all shared liberties, i.e., player 1 needs at least S more exclusive liberties.

5.3 Seki for Different Eye Sizes

If both sides have an eye of size ≤ 3 then the status can still be seki even if both eyes are of different size. The reason is that for size 3 the catching of a 2-stone opponent chain inside the eye does not buy time because the opponent answers instantly and the eye has still only one liberty inside. The situation is different when a 3-stone chain is caught inside an eye of size 4. Even if the opponent puts a stone inside the eye immediately, the eye has at least temporarily 2 inner liberties. This might be sufficient to occupy a shared liberty and put the opponent under atari and win the semeai.

5.4 Relation to other Publications

- In Table 1 the positions with $S = 1, Z_1 = Z_2 = 0$ are listed under case (B) and not as usually in the literature under case (A). One could consider them under case (A) but case (A) contains eyes of all sizes whereas positions with $S = 1, Z_1 = Z_2 = 0$ have no eyes. More importantly, these positions have the capturing move in the shared region and in the presence of approach moves they are characterized completely like case (B) positions. An example is shown in Diagrams 5 to 9 and discussed in section 7.6.

- In positions with shared liberties the literature on semeai identifies players as *attacker* and *defender*. In Müller (1999) and Jasiek (2011) player 1 is called attacker and player 2 is the defender, i.e., defender is a synonym for the side which obtains the shared liberties in the liberty count.[10] In Nakamura (2008) on page 179, each external region has its defender (the owner of the essential chain) and its attacker (the other side), for example, in the external region of White, the defender is White and the attacker is Black. But then on page 180, the words attacker and defender are used like in all other publications.

  Players are also identified as *favourite* and *underdog*. In Jasiek (2011), Hunter (Spring 1996 - Summer 1997, 1998, 2003) the site with advantages (an eye versus no eye, or a bigger eye of size ≥ 4 versus a smaller eye, or in the case of no eyes then the side with more outer liberties) is called favourite and the other underdog. In kill/kill positions (C) player 2 is the favourite and obtains the shared liberties due to its higher strength, in kill/seki positions $(B)_{FOF}$, (D) the underdog obtains the shared liberties because the underdog only wants to obtain a seki and does not try to kill. Case $(B)_{FIF}$ is not described in the literature.

  To summarize: What we call player 1 in cases (C),(D) is called *attacker* or *underdog* in other publications and what we call player 1 in case $(B)_{FOF}$ is called *attacker* or *favourite* elsewhere.

  In this article we do not identify players as attacker or defender because this would imply that one side can be identified as *the* attacker and the other as *the* defender which is not the case anymore in the presence of approach moves. For example, in Diagram 17 further below Black plays the FOF approach and is the defender (i.e., shared liberties count for Black) whereas if Black plays the FIF approach in the same position as shown in Diagram 18 then shared liberties count for White and thus Black would now have to be called attacker. Also the identification as favourite or underdog becomes problematic when in the absence of eyes one side has more exclusive physical liberties and the other requires more outside approach moves.

[10] Calling a player 'defender' who attacks as well, only from the outside, does not seem very obvious but that is what is used in the literature.

- If one writes the balance equations in the form

$$\Delta_{\text{FOF}} \quad := \quad X_I - X_{II} = S - 1 \qquad (11)$$
$$\Delta_{\text{FIF}} \quad := \quad R_1 - R_2 \ = S \qquad (12)$$

and defines what is called *forced liberties* in Müller (1999)

$$F := \begin{cases} S-1 & \text{for FOF} \\ S & \text{for FIF} \end{cases}$$

then in the absence of approach moves both relations (11), (12) become the Semeai-formula

$$\Delta = F \qquad (13)$$

by Karl-Friedrich Lenz (1982), also shown in Müller (1999) and Jasiek (2011).

6. APPROACH MOVES IN CASES (A), (C) AND (D)

As already explained in subsections 3.3 and 3.4 the generalization to approach moves is straightforward in cases (A), (C) and (D). In these cases if a chain has no eye then the capturing move is on an outside liberty and then an outside liberty can be chosen as capturing move which otherwise would require the most approach moves from all outside liberties.

By defining the variable V as the number of approach moves that can be avoided (which is zero if the chain has an eye, and otherwise is the maximal number of approach moves to any outside liberty) and introducing $R = X - V$, we can use one and the same balance relation (10) for all cases (A), (C), (D) whether chains have eyes or not.

Because case (B) is more rich, a whole section is devoted to it.

7. APPROACH MOVES IN CASE (B)

In case (B) the situation is different as there are two different types of play which both could be successful. One play has the capturing move inside, i.e., it **F**ills all **O**utside liberties **F**irst (FOF) - the standard play, and the other has the capturing move outside and therefore **F**ills **I**nside liberties **F**irst (FIF) - the novel play.

The following subsections discuss all aspects of case (B) and answer the following questions.

- What are necessary conditions for FIF to be better than FOF?
- What are possible outcomes of both plays?
- Can a win by FIF be enforced?
- What if FIF and FOF can both kill?
- Which steps decide whether to do FIF or FOF?
- What are rules for optimal play?

We start with deriving necessary conditions for success with FIF if FOF is sub-optimal.

7.1 Necessary Conditions for FIF to be more useful than FOF

This section is not necessary for any decision process but it provides surprisingly short necessary conditions for a player to be more successful playing FIF than FOF.

If FIF is played as described by formula (10) then player 1 has to occupy all inside liberties first. Player 1 would do this only if he can win, otherwise he would play more defensively on outside liberties first because in this case (B) both players have the option to occupy outside liberties first. Therefore, if we assume in this subsection that for one player it is better to play FIF than FOF then this must be player 1 identified through $R_1 > R_2$.

There are two cases: Under FOF P_1 has seki (in the following case 1) or gets killed (case 2).

Case 1:
If we assume that P_1 moves first then we have

$$R_1 \geq R_2 + S \quad (14)$$
$$X_1 < X_2 + S - 1 \quad (15)$$

where (14) follows from P_1 being successful under FIF and (15) follows from P_1 not being able to kill under FOF.

This includes the two subcases that player 1 identified through $R_1 > R_2$ is the same as either player I or player II identified through $X_I \geq X_{II}$. All that is assumed is that player 1 cannot kill through FOF which implies (15).

Using at first (14) together with $R_i = X_i - V_i$ and then (15) gives

$$X_1 - V_1 = R_1 \geq R_2 + S = X_2 - V_2 + S = (X_2 + S - 1) - V_2 + 1 > (X_1) - V_2 + 1 \quad (16)$$

and therefore

$$V_2 - V_1 > 1. \quad (17)$$

If P_2 would move first then $\geq$ would become $>$ in (14) and $<$ would become $\leq$ in (15) and similarly in (16) but (17) would not be affected.

For example, in Diagram 10 below we have P_1 = White and $V_1 = 0, V_2 = 1$, i.e., (17) is not satisfied. White can kill using FIF in Diagram 14 *and* using FOF in Diagram 12, so FIF is not better.

Differently in Diagram 15 where we have P_1 = Black and $V_1 = 0, V_2 =$, i.e., (17) is satisfied. Indeed, Black can kill using FIF in Diagram 18 but reaches only a seki using FOF in Diagram 17.

Case 2:
In the second case we assume that P_1 does even not reach seki under FOF (i.e., $P_1 = P_{II}$ because only P_{II} can get killed under FOF) but P_1 can kill under FIF. If P_1 moves first then these two assumptions give

$$R_1 \geq R_2 + S \quad (18)$$
$$X_2 > X_1 + S - 1 \quad (19)$$

and using at first (18) and then (19) we arrive at

$$X_1 - V_1 = R_1 \geq R_2 + S = (X_2) - V_2 + S > (X_1 + S - 1) - V_2 + S \quad (20)$$

and therefore

$$V_2 - V_1 > 2S - 1. \quad (21)$$

This includes the two subcases that player 1 identified through $R_1 > R_2$ is the same as either player I or player II identified through $X_I \geq X_{II}$. All that is assumed is that player 1 cannot reach seki, i.e., gets killed using FOF which implies (19).

If P_2 would move first then $\geq$ would become $>$ in (18) and $>$ would become $\geq$ in (19) and similarly in (20) but (21) would not be affected.

For example, in Diagram 27 we have P_1 = Black and $V_1 = 0, V_2 = 2, S = 1$, i.e., (21) is satisfied. Indeed, Black can kill playing FIF in Diagram 30 even though it dies playing FOF in Diagram 29.

To summarize:
It is possible to be able to kill with FIF when playing FOF does even not give seki (case 2) but it requires a bigger difference of the maximal number of approach moves $V_2 - V_1$ in (21) than in (17) for case 1 when playing FOF at least gives seki. (In both inequalities P_1 is the winner of FIF.) In other words, playing FIF becomes attractive if the opponent has an outside liberty that requires many approach moves and if oneself has only outside liberties that require no or only few approach moves.

7.2 Possible Outcomes of FIF and FOF and their Enforcement

If all internal liberties are occupied before a capture happens then there is no reason not to continue making moves; therefore FIF cannot result in seki.

Differently, when occupying outside liberties first, then unsettled positions are of type kill/seki if $S > 1$ (i.e., if there is more than one inner liberty).

If player P_1 can kill through FIF then the other player cannot prevent this. In other words, a FIF kill by P_1 overwrites any FOF result, there is no competition in filling all inside liberties first (FIF). Only one player can have an interest in FIF which is player P_1 with $R_1 > R_2$. If P_2 would start filling all inner liberties then this would speed up the defeat of P_2 and would be welcomed by P_1.

7.3 Sensitivity of the decision between FIF and FOF

As established above, only player P_1, i.e., the player with higher R value has the option to play FIF. Player P_1 will play FIF if that wins the race. But whether P_1 wins may depend on who moves first. For example, in Diagram 10 below, player P_1 is White. But White can win with FIF only if playing first, therefore if White plays first then White kills playing FIF as in Diagram 14. If Black plays first then White plays FOF as in Diagram 11 and reaches seki.

The decision whether player P_1 plays FIF or FOF may not only depend on the position but also on the play of player P_2. For example, in Diagram 31 player P_1 is Black because $R_B = 2 > 1 = R_W$. If White, i.e., P_2 plays first then Black's play depends on White's first move. If White decreases X_B with ① in Diagram 32 then Black should play FOF and if White increases V_W as in Diagram 34 then Black should play FIF.

7.4 The Case that FIF and FOF can both kill

If player P_1 can kill through FIF and player P_2 can kill through FOF[11] then P_1 can enforce the win as argued above. But what if P_1 can kill through FIF and FOF (i.e., $P_I = P_1$)?

If player P_1 can kill with both approaches then P_1 would want to pursue the more stable one, i.e., the approach where P_1 can pass as often as possible. Pursuing FIF, P_1 can pass $R_1 - (R_2 + S)$ times according to (10) and pursuing FOF, P_1 can pass $X_1 - (X_2 + S + 1)$ times according to (9) (with $P_1 = P_I$). Using $R_i = X_i - V_i$ the difference between both passing numbers is $D := V_2 - V_1 - 1$. That means if D is positive then P_1 should FIF, if D is negative then P_1 should FOF and if $D = 0$ then ko-fights that are ongoing or may arise elsewhere on the board come into play.

If P_1 plays FOF then losing the semeai due to losing a ko-fight will still get seki whereas pursuing FIF and losing the semeai due to ko would be a total loss. In contrast, it takes P_1 only $R_2 + S$ moves to capture through FIF but $X_2 + S + 1$ moves using FOF, i.e., pursuing FOF takes $V_2 + 1$ moves more. This is the number of extra ko-threats that P_2 gets if P_1 plays FOF. Thus, for P_1 to win a potential ko when playing FIF takes fewer but larger ko-threats whereas winning a potential ko when playing FOF requires more but lower value ko-threats. That means the decision whether to play FIF or FOF depends on the number of available ko-threats and their value in comparison to the difference in killing and getting at least seki, and of course it depends on whether P_1 can afford to play safe or not.

7.5 Summary of Steps to choose between FIF and FOF

For optimal play, both balance relations need to be checked. If one player is successful in FIF this already decides the status, but for practical play one still would have to check FOF as argued above. Because success in killing through FIF is rare it is more efficient to check at first FOF. The complete decision algorithm is given through the following steps.

- Determine X_B, X_W and player P_I from $X_I \geq X_{II}$. Check the preliminary status based on FOF (9): if $X_I > X_{II} + S - 1$ or $X_I = X_{II} + S - 1$ and P_I plays first then P_I doing FOF can kill otherwise if $S > 1$ then it is a seki, if $S \leq 1$ then P_I gets killed.
- Determine V_B, V_W. If $V_B, V_W < 2$ then stop, the FOF result is final.

[11]i.e., $P_I = P_2$ because only P_I can kill in FOF, not P_{II}

- Compute $R_B = X_B - V_B,\ R_W = X_W - V_W$ and determine player P_1 from $R_1 \geq R_2$.
- If $R_1 < R_2 + S$ or $R_1 = R_2 + S$ and P_2 plays first then stop, the FOF result is final. (Independent of $P_1 = P_I$ or $P_1 = P_{II}$, it is for both players not advantageous to play FIF. Whoever plays inside liberties first gets killed.)
- In the remaining cases P_1 kills. If P_1 cannot kill if playing FOF then it should play FIF.
- In the remaining cases P_1 kills under FIF and FOF, i.e., $P_1 = P_I$ (because only P_I has a chance to kill under FOF). If possible, P_1 should play so to be able to pass and still win (if $V_2 - V_1 - 1 > 0$ then FIF and if $V_2 - V_1 - 1 < 0$ then FOF).
- If P_1 cannot pass *and* still win (i.e., $V_2 - V_1 - 1 = 0$) then whether playing FIF or FOF does matter only for possible ko-fights. Playing FOF is less risky for P_1 (the outcomes are kill/seki instead of kill/kill) but gives P_2 more ko-threats because it takes P_1 more moves to catch P_2 under FOF.

7.6 Examples

The following examples illustrate the difference between cases (B)$_{\mathrm{FIF}}$ and (B)$_{\mathrm{FOF}}$. Examples for cases (A), (C), (D) are well known from the literature.

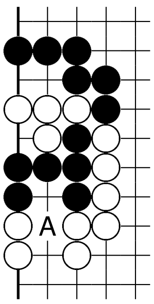


Diagram 5.

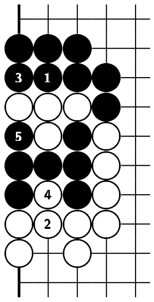


Diagram 6.
● kills.

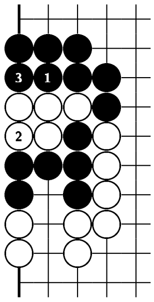


Diagram 7.
● kills.

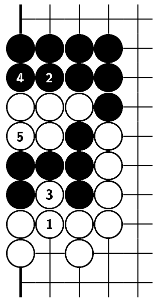


Diagram 8.
○ kills.

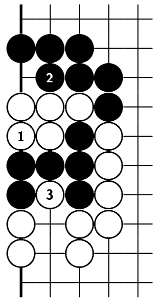


Diagram 9.
○ kills.

In Diagram 5 we have $S = 1$ and $X_W = 2 = X_B$ and thus balance condition (9) gives $X_W = 2 = X_B + S - 1$, thus the position is unsettled, Black moving first, can kill as demonstrated in Diagrams 6 and 7 and White can kill moving first as shown in Diagram 8, both playing FOF. We also have $R_W = 2 > 1 = R_B$ and with FIF equation (10) further $R_W = 2 = R_B + S$. Thus, White should also be successful when moving first and playing FIF by occupying all inner liberties before the outside capturing move which is shown in Diagram 9.

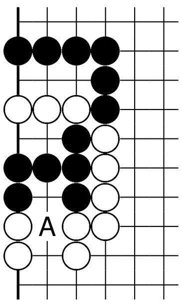


Diagram 10.

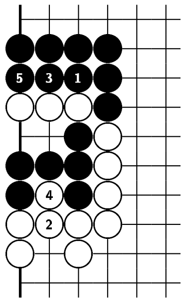


Diagram 11.
● gets seki.

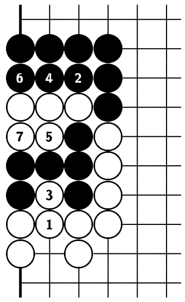


Diagram 12.
○ kills.

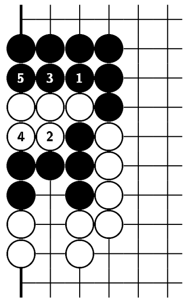


Diagram 13.
● kills.

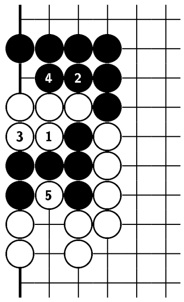


Diagram 14.
○ kills.

Similarly in Diagram 10, now with $S = 2$ and $X_W = 3 > 2 = X_B$, (9) gives: $X_W = 2 = X_B + S - 1$, i.e., the position is unsettled when playing FOF: Black moving first gets seki (Diagram 11) and White moving first can kill (Diagram 12). Here FIF gives $R_W = 3 > R_B = 1$ and $R_W = 3 = R_B + S$, i.e., also an unbalanced position (Diagrams 13 and 14) with the difference that the outcome for White in Diagram 13 is worse than in Diagram 11, i.e., one should play FIF only if one can kill.

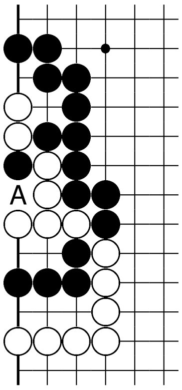


Diagram 15.
● to move.

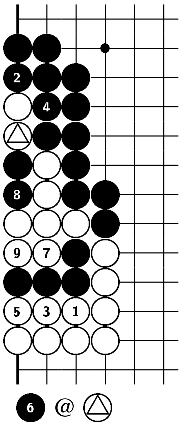


Diagram 16.
○ kills.

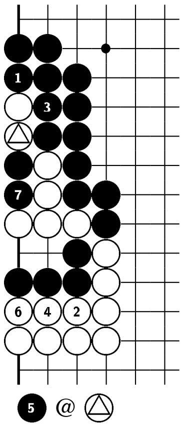


Diagram 17.
● gets seki.

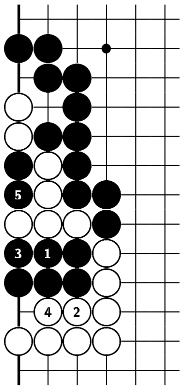


Diagram 18.
● kills.

In Diagram 15 the liberty A needs more than one approach move. This is a necessary condition for FIF to be more useful than FOF for the player that does not have the liberty A, here Black. With $S = 2$ and $X_W = 4 > 3 = X_B$, FOF (9) gives: $X_W = 4 = X_B + S - 1$, i.e., Black playing first can get seki (Diagram 17). The FIF approach gives $R_B = 3 > 1 = R_W$ and with (10): $R_B = 3 = R_W + S$, i.e., Black playing first can kill (Diagram 18). Therefore FIF is here better for Black.

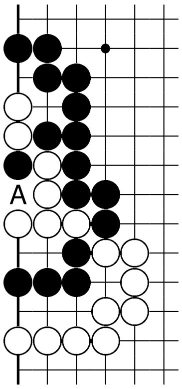


Diagram 19.
● to move.

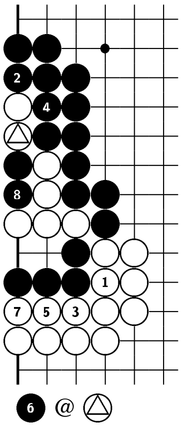


Diagram 20.
○ gets seki.

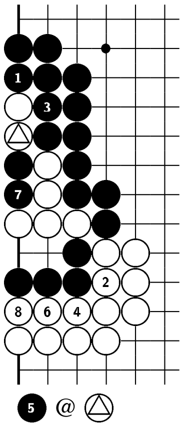


Diagram 21.
● gets seki.

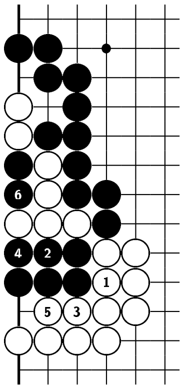


Diagram 22.
○ dies.

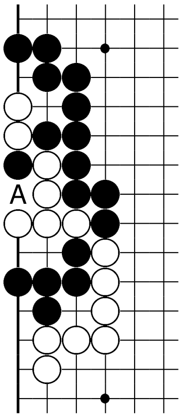


Diagram 23.
● to move.

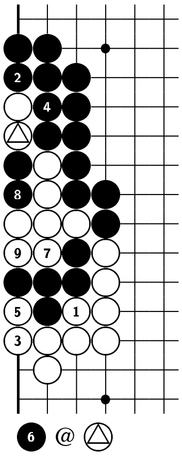


Diagram 24.
○ kills.

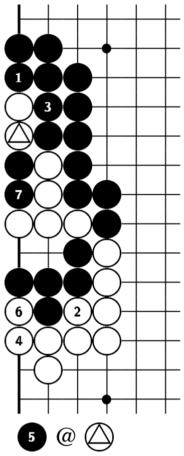


Diagram 25.
● gets seki.

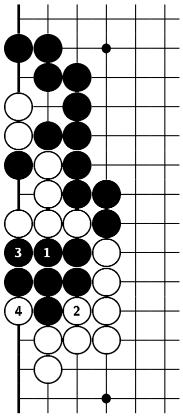


Diagram 26.
● dies.

In Diagram 19 the black essential chain has one liberty more than in Diagram 15 leading to $X_W = 4 = X_B$ and (9): $X_B = 4 < 5 = X_W + S - 1$, i.e., to seki under FOF. We also notice $R_B = 4 > 1 = R_W$, i.e., the FIF relation (10) gives $R_B = 4 > 3 = R_W + S$. Therefore Black can kill through FIF even if White plays first (Diagram 22) whereas under FOF even when moving first, Black can get only seki (Diagram 21).

Diagram 23 shows a position where both sides have approach moves. Black has one outside liberty less and one approach move more, so the same X_B value. With $S = 2$ and $X_W = 4 > 3 = X_B$, FOF (9) gives $X_W = 4 = X_B + S - 1$, i.e., an unsettled position as demonstrated in Diagrams 24 and 25. The FIF approach gives $R_B = 2 > 1 = R_W$ and with (10): $R_B = 2 < 3 = R_W + S$ and therefore Black playing first dies under FIF in Diagram 26.

A comparison of Diagrams 15 and 23 and their corresponding results shows that a physical liberty and an approach move have the same protection effect under FOF, i.e., both cost the opponent a move. But for an attacking chain a physical liberty (of Black in Diagram 15) is more valuable than an extra approach move (of Black in Diagram 23).

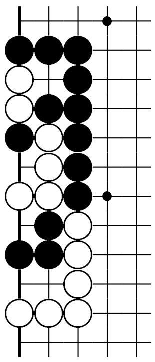


Diagram 27.

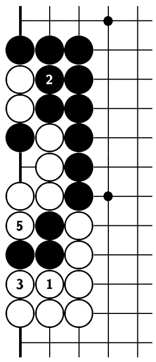
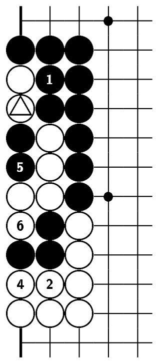


Diagram 28.
○ kills.



❸ @ △
Diagram 29.
● dies.

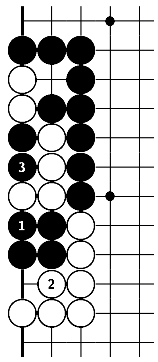


Diagram 30.
● kills.

Diagrams 27 to 30 give examples for the correctness of relations (9), (10) for $S = 1$. With $S = 1$, $X_W = 3 > 2 = X_2$ and FOF relation (9): $X_W = 3 > 2 = X_B + S - 1$ Black cannot win even if moving first (Diagram 29). Under FIF we get $R_B = 2 > 1 = R_W$ and with (10): $R_B = 2 = R_W + S$, i.e., Black can kill if moving first (Diagram 30).

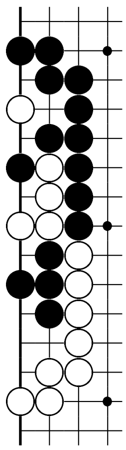


Diagram 31.

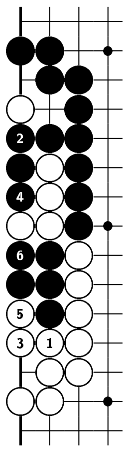


Diagram 32.
Black plays
FOF and kills

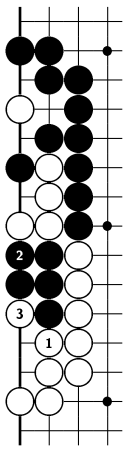


Diagram 33.
Black plays
FIF and dies

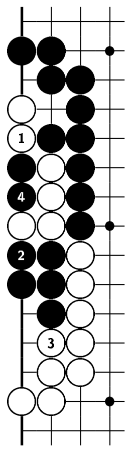
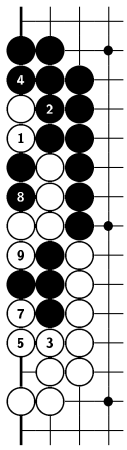


Diagram 34.
Black plays
FIF and kills



❻ @ ①
Diagram 35.
Black plays
FOF and dies

Diagrams 31 to 35 demonstrate that the decision by player P_1 whether to play FOF or FIF is taken when it is player P_1's turn. P_1 is Black, because of $R_B = 2 > 1 = R_W$ in Diagram 31. After ① in Diagram 32 is $X_B = 2 = X_W$ and $S = 1$, i.e., (9) gives: $X_B = X_W + S - 1$ therefore Black moving next can kill with FOF, whereas FIF would not work: (10) gives $R_B = 1 < 2 = 1 + 1 = R_W + S$ which is demonstrated in Diagram 33.

If White plays ① in Diagram 34 then $V_W = 3, X_W = 4, R_W = 1, V_B = 1, X_B = 3, R_B = 2, S = 1$, i.e., $R_B = R_W + S$ and Black playing next can kill with FIF as in Diagram 34 whereas $X_B = 3 < 4 = X_W + S - 1$, i.e., Black playing next dies with FOF as in Diagram 35.

8. CHOOSING BETWEEN APPROACH MOVES AND OUTSIDE LIBERTIES

Diagram 36 shows only the outer region of a white essential chain with some of its stones △. The question is whether the shown part of the board already determines how many outer liberties O_W, approach moves A_W and maximal number V_W of approach moves White has. If White would not move in its own outer region as in all other positions of this article then the values would be $O_W = 1, A_W = 5$ and $V_w = 5$ if White has no eye, else $V_W = 0$. As an alternative, White could link its two chains like in Diagram 37 with the result $O_W = 2, A_W = V_W = 0$. Which of both versions is better for White depends on the hidden remainder of the position. In any case, the formulas introduced in this article can be used to decide which of both alternatives is better for White.

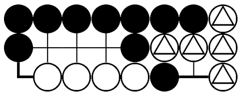


Diagram 36.

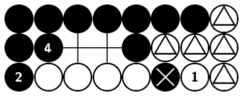


Diagram 37.
③ at ⊗.

If White has an eye then $V_W = 0$ and Diagram 36 is better for White, because the capturing move would have to be in the eye, i.e., FIF is played requiring $O_W + A_W - V_W = 6$ moves of Black to occupy all outside liberties of White.

If White has no eye and either Black has an eye or there are no shared liberties, then FIF is played as well but now $V_w = 5$ in Diagram 36 with $A_W - V_W + Q_W = 1$ move of Black is required to cover all outside liberties of White. In Diagram 37 White reaches $A_W - V_W + Q_W = 2$ which is slightly better in this situation.

Finally, if both sides have no eye (case (B)) in Table 1 and there is at least one shared liberty then FIF and FOF are possible. If FOF is played then Diagram 36 is (5+1)-(0+2)=4 moves better for White than Diagram 37 and if FIF is played then Diagram 37 is 1 move better than Diagram 36. FIF is played only if one side filling inside liberties first can kill.

To decide whether to play as in Diagram 36 or as in 37, White has to check whether $O_W = 1, A_W = 5, V_w = 5$ (Diagram 36) would allow Black to kill with FIF (i.e., whether $R_B (= O_B + A_B - V_B) > R_W + S (= O_W + A_W - V_W + S = 1 + S)$ or $R_B = R_W + S$ and Black moves first) and then the only chance for White is to link as in Diagram 37 and have one more liberty in FIF, i.e., hopefully to have $R_B < R_W + S = 2 + S$ or $R_B = 2 + S$ and White with moving first, to prevent FIF.

To summarize, if a player has the option to convert approach moves into outside liberties then the Table 1 with relations (9) and (10) allow a quick decision about which variation is better for that player.

9. OPTIMAL PLAY

Knowing the status, i.e., the optimal achievable result can already be useful: if the position is settled and it is not urgent to eliminate ko-threats then one can tenuki, i.e., play elsewhere and save moves as ko-threats if they are at least forcing moves. In contrast, if the position is unsettled then one needs extra information about how to reach the optimal result. The following are rules that are to be used in this order for each move in unsettled positions.[12] These rules apply to all cases; only for case (B) in Table 1 one has to decide in criterion 5 based on section 7.5 whether to do FIF or FOF.

0. Capture a chain of the opponent that is enclosing or approaching from the outside if that captures settles the semeai.[13] Equivalently prevent the capture of an own chain that is enclosing or approaching from the

[12]Robert Jasiek is thanked for comments on exceptions for generalized semeai which have been incorporated.

[13]The reason for such a move to be more valuable than capturing the essential chain is that not only the semeai is won but more points and influence to the outside are gained. Strictly speaking, such an opportunity exists only for more general semeai with weak enclosing chains which are not treated in this article. We still mention this type of move for the benefit of the Go-player.

outside if its capture would have settled the semeai.

1. Capture the opponent essential chain if it is under atari.
2. If the own chain is under atari and has an eye which has an chain of the opponent inside that is under atari and has more than 2 stones then capture that opponent chain.
3. If the opponent just captured a chain in his eye allowing the opponent to get two eyes in the next move then prevent this eye splitting by playing on this point yourself.
4. Pass if the status is settled according to Table 1.
5. Fill or approach an outside liberty of the opponent chain. Exception to this rule only for case $(B)_{\mathrm{FIF}}$: Do not occupy the last outside liberty and its approach moves which is the liberty with the most approach moves from all opponent's outside liberties.
6. Make a move in the eye of the opponent in a way that your throw-in stones have a dead eye shape (nakade shape).
7. Make a move on a shared liberty.

10. GENERALIZATIONS

The following generalizations may change the way how moves are counted but not the balance relations (9), (10) themselves.

The assumption that each approach move is associated with only one outside liberty was only made to be able to determine V as it is defined in subsection 3.1 easily as the maximum number of approach moves to any one outside liberty of the essential chain. If approach moves are not independent or serve to approach different outside liberties then V needs to be determined not through (2) but directly as number of approach moves to an outside liberty that can be avoided if the capturing move is on an outside liberty. All other statements about V remain unchanged, i.e., its definition and its appearance in all relations.

11. SUMMARY

In this article we provided a complete analysis of semeai positions under the assumptions of section 2 which apart from the common restrictions for semeai allows an arbitrary number of approach moves without ko.

We derive relations of balance, i.e., mathematical conditions which characterize positions where strengths are balanced, in other words, positions that are unsettled where each side moving first can be successful. If positions are settled then these relations give the number of moves the losing side is behind which can be useful when fighting kos.

We distinguish between the case that the capturing move is in the shared region (when both players **F**ill **O**utside liberties **F**irst - called 'FOF') and the case that the capturing move is not in the shared region (when one player has to **F**ill **I**nside liberties **F**irst - called 'FIF', i.e., before the capturing move) and derive relations for both cases. By classifying positions according to these two cases we manage to fit the whole classification into only one table.

In this approach we also show that the case of no eyes and a single shared liberty should be taken as a sub-case of the case of shared liberties in order to generalize naturally the treatment to approach moves whereas in the literature the main distinction is made between positions with status kill/kill and those with status kill/seki which merges the case of no shared region with 1-point shared regions.

By performing a complete derivation and not only presenting an ad hoc collection of positions we find a case that has not been analyzed in the literature yet: two chains with a shared region and both without eyes. For these situations we are able to derive rather straightforward necessary conditions for the case that the new FIF approach is successful when the traditional FOF approach gives only seki or even death.

A result, that at least for the author was unexpected, is that for semeai without eyes but with approach moves and with a shared region (even when consisting of only one point), one cannot avoid to check two cases. The check is simple and even simpler necessary conditions are available that help, but nevertheless, two cases need to be considered. This is in contrast to basic semeai without approach moves that can always be decided in a single decision process. In this case of shared liberties and no eyes only the player P_1 with more external liberties plus approach moves minus the highest number of approach moves to one external liberty has the option to play FIF. P_1 will play FIF only if P_1 wins and that may depend on whether P_1 moves first and if the opponent P_2 moves first then whether P_1 can win by playing FIF may depend on the move P_2 plays first. All statements are illustrated in numerous examples. There are three topics that are of interest for further research.

- The generalization to more than two essential chains.
- The merger of the work by Nakamura (2008, 2009) with our work on approach moves.
- The inclusion of ko as a type of approach moves.

ACKNOWLEDGMENT

The author would like to thank Robert Jasiek, Martin Müller, Harry Fearnley and an anonymous referee for the comments that led to the final manuscript.

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13. APPENDICES

Typical for all the positions listed in this appendix is that the white essential chain (on the right of each diagram) although being able to capture the essential black chain on its left, can still not link to the living enclosing white chain on the left.

APPENDIX A: SNAPBACK

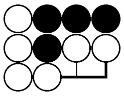
Diagram 38.

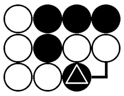
Diagram 39.

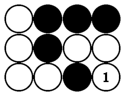
Diagram 40.

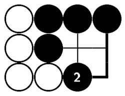
Diagram 41.

Positions known as 'snapback' such as the one in Diagram 38 are not semeai positions because there is no black essential chain. But after the first move ▲ in Diagram 39 the position is at least topologically a semeai. It is listed in this appendix because White has no chance despite catching the black essential chain.

APPENDIX B: 2-STEP SNAPBACK

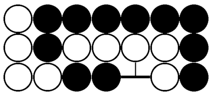
Diagram 42.

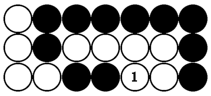
Diagram 43.

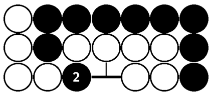
Diagram 44.

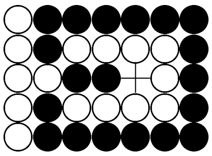
Diagram 45.

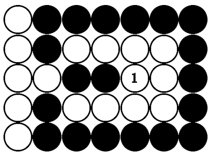
Diagram 46.

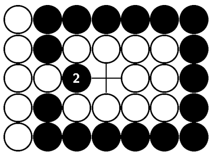
Diagram 47.

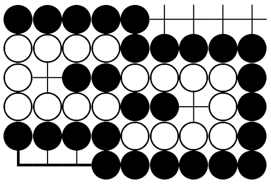
Diagram 48.

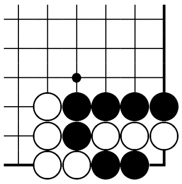
Diagram 49.

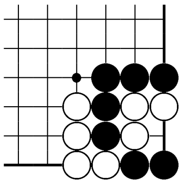
Diagram 50.

In these positions the black essential chain can afford to be captured twice and Black is still able to capture the white capturing chain afterwards. The white chain cannot be linked to its strong white chain on the left. This is possible because the black captured chain includes only 2 stones initially. From the first capture a 'normal' snapback results.
The position in Diagram 50 is special in that when White captures the two black stones at t2, Black plays atari at s1, but then the position is "torazu san moku", such that (a) if Black captures, White can recapture the 2 black stones, and either get an eye or capture another black stone, or (b) if White captures, then a possible ko results. (Diagrams 48 to 50 had been provided by Harry Fearnley.)

APPENDIX C: UNDER THE STONES

The technical term 'Under the Stones' refers to a situation where after the first capture the cutting stone 2 is not captured but can capture White in the next move. A possible characterization is:

- After capture, the capturing chain has only 2 liberties both resulting from the capture (if the capturing chain would have afterwards only one liberty then it would just be a single ko or a snapback).
- If after the capture the captured cross-cutting stone is replaced (by 2 in diagrams 53, 56), it needs to have 2 liberties and it will be able to capture the capturing chain, i.e.,
- after replacing the cross-cutting stone the only remaining liberty of the capturing chain must not be a liberty of another chain so that the capturing chain can be saved by linking.

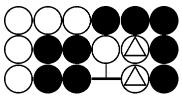
Diagram 51.

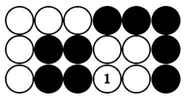
Diagram 52.

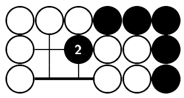
Diagram 53.

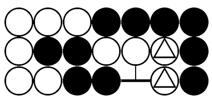
Diagram 54.

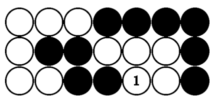
Diagram 55.

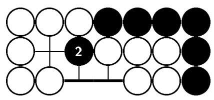
Diagram 56.

(If the stones △ in diagrams 51, 54 would all or individually be black then the positions would not be semeai anymore but re-capture would still be possible.)