Generalised Species of Structures: Cartesian Closed and Differential Structure

(Preliminary Notes)

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Abstract

We generalise Joyal's notion of species of structures and develop their combinatorial calculus. In particular, we provide operations for their composition, addition, multiplication, pairing and projection, abstraction and evaluation, and differentiation; developing both the cartesian closed and linear structures of species.

Contents

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Species of structures. Joyal species of structures [9] are an abstract categorical formalisation of a type of unlabelled combinatorial structures.

In its basic form, species are defined as functors $\mathbf{B} \longrightarrow \boldsymbol{Set}$ where $\mathbf{B}$ is the category of finite sets and bijections, and $\boldsymbol{Set}$ is the category of sets and functions. Such a species P is to be thought of as providing, for each finite set of tokens T, the set $\mathsf{P}(\mathsf{T})$ of P-structures (*i.e.*, structures of type P) built form tokens in T together with, for each bijective renaming $\sigma : \mathsf{T}_1 \xrightarrow{\cong} \mathsf{T}_2$ between sets of tokens, a bijective correspondence $(\_) \cdot_{\mathsf{P}} \sigma : \mathsf{P}(\mathsf{T}_1) \xrightarrow{\cong} \mathsf{P}(\mathsf{T}_2)$ between the respective sets of P-structures subject to the laws

$$(\_) \cdot_{\mathsf{P}} \mathrm{id}_{\mathsf{T}} = \mathrm{id}_{\mathsf{P}(\mathsf{T})} : \mathsf{P}(\mathsf{T}) \longrightarrow \mathsf{P}(\mathsf{T})$$

and

$$\big((\_) \cdot_{\mathsf{P}} \sigma_1\big) \cdot_{\mathsf{P}} \sigma_2 = (\_) \cdot_{\mathsf{P}} (\sigma_2 \circ \sigma_1) : \mathsf{P}(\mathsf{T}_1) \longrightarrow \mathsf{P}(\mathsf{T}_3)$$

for all $\mathsf{T} \in \mathbf{B}$ and $\mathsf{T}_1 \xrightarrow[\cong]{\sigma_1} \mathsf{T}_2 \xrightarrow[\cong]{\sigma_2} \mathsf{T}_3$ in $\mathbf{B}$.

An important aspect of the theory of species is that it provides a calculus for the structural manipulation of combinatorial structures. Indeed, Joyal provided a calculus of species with operations such as composition, addition, multiplication, derivation, *etc.* subject to the usual algebraic laws. Furthermore, he showed that the calculus is a powerful tool for extracting combinatorial information by associating generating series to species in such a way that the combinatorial operations on species correspond to the usual respective operations on power series. (See [3] for a full treatment.)

Outline. We generalise the notion of species and investigate the resulting calculus.

The basis of the generalisation is the observation that the category of finite sets and bijections **B** is equivalent to the free symmetric strict monoidal category on one generator !**1** (*i.e.*, the category of finite cardinals and bijections). In this view, a species is a functor $!\mathbf{1} \longrightarrow \mathcal{S}et$ and we are naturally led to consider $\mathbb{A}$-species, for $\mathbb{A}$ a small category, as functors $!\mathbb{A} \longrightarrow \mathcal{S}et$. More generally, for small categories $\mathbb{A}$ and $\mathbb{B}$, we introduce $(\mathbb{A}, \mathbb{B})$-species as profunctors $!\mathbb{A} \nrightarrow \mathbb{B}$ (*i.e.*, functors $!\mathbb{A} \longrightarrow \widehat{\mathbb{B}}$), where $!\mathbb{A}$ is the free symmetric strict monoidal completion of $\mathbb{A}$ (and $\widehat{\mathbb{B}}$ is the functor category $[\mathbb{B}^{o}, \mathcal{S}et]$ of $\mathbb{B}$-variable sets). Such a species P provides, for each $!\mathbb{A}$-object of tokens A (given by a finite sequence of $\mathbb{A}$-objects), the $\mathbb{B}$-variable set $P(A)$ of P-structures over A. This notion of species encompasses many of the combinatorial species considered in the literature, including permutationals [9, 2] and partitionals [16]; see also [15].

From this standpoint, the calculus of generalised species is developed within the framework of generalised logic [11]. In particular, we provide operations for the composition, addition, multiplication, pairing and projection, abstraction and evaluation, and differentiation of species. All these operations are shown to satisfy the expected algebraic laws.

The treatment of differentiation necessarily calls for the consideration of both linear and closed structure, as *differentiation operators* are maps

$$\underline{\mathrm{hom}}\,[X, Y] \longrightarrow \underline{\mathrm{hom}}\big[X, \underline{\mathrm{lin}}\,[X, Y]\,\big]$$

where $\underline{\mathrm{hom}}$ and $\underline{\mathrm{lin}}$ respectively represent the full and linear function spaces. This development in the context of generalised species is also included.

1. Categorical background

We recall the basic notions needed throughout the paper.

Monoidal categories. A *monoidal category* is a tuple $(\mathcal{C}, \otimes, I, a, l, r)$ where $\mathcal{C}$ is a category, $\_ \otimes =$ is a bifunctor $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, I is an object of $\mathcal{C}$, and a, l, r are natural isomorphisms with components $a_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$, $l_A : I \otimes A \longrightarrow A$, $r_A : A \otimes I \longrightarrow A$ subject to coherence axioms [10]. We have a *strict monoidal category* when these isomorphisms are equalities. A *monoidal functor* $F : (\mathcal{C}, \otimes, I, a, l, r) \longrightarrow (\mathcal{C}', \otimes', I', a', l', r')$ is a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ equipped with a morphism $I \longrightarrow F(I)$ and a natural transformation with components $F(A) \otimes' F(B) \longrightarrow F(A \otimes B)$ subject to coherence axioms [7, 11]. We have a *strict monoidal functor* if these morphisms are identities.

A *symmetric (strict) monoidal category* is a (strict) monoidal category equipped with a natural isomorphism c, called the symmetry, with components $c_{A,B} : A \otimes B \longrightarrow B \otimes A$ satisfying further coherence axioms [10]. A *symmetric (strict) monoidal functor* between symmetric (strict) monoidal categories is a (strict) monoidal functor that satisfies a further coherence axiom associated to the symmetries [7, 11].

We write $\mathcal{C}at$ for the category of small categories and functors, and $\mathcal{SMC}at$ for the category of small symmetric strict monoidal categories and strict monoidal functors. The forgetful functor $\mathcal{SMC}at \longrightarrow \mathcal{C}at : (\mathcal{C}, \otimes, I) \longmapsto \mathcal{C}$ has a left adjoint $!(\_) : \mathcal{C}at \longrightarrow \mathcal{SMC}at$ that maps a small category into its *symmetric strict monoidal completion*. An explicit description of $!\mathbb{C}$ is given by the category with objects consisting of finite sequences $\langle\!|c_i|\!\rangle_{i=1,n}$ $(n \in \mathbb{N})$ of objects of $\mathbb{C}$ with $!\mathbb{C}\big[\langle\!|c_i|\!\rangle_{i=1,k}, \langle\!|d_j|\!\rangle_{j=1,\ell}\big] = \emptyset$ iff $k \neq \ell$ and morphisms $\langle\!|c_i|\!\rangle_{i=1,n} \longrightarrow \langle\!|c'_i|\!\rangle_{i=1,n}$ given by pairs $\big(\sigma, \langle\!|f_i|\!\rangle_{i=1,n}\big)$ consisting of a permutation $\sigma \in \mathfrak{S}_n$ and a sequence of maps $\langle\!|f_i : c_i \longrightarrow c'_{\sigma i}|\!\rangle_{i=1,n}$ in $\mathbb{C}$. (Composition is essentially given pointwise modulo permutation

$$\big(\sigma', \langle\!|f'_i|\!\rangle_{i=1,n}\big) \circ \big(\sigma, \langle\!|f_i|\!\rangle_{i=1,n}\big) = \big(\sigma' \circ \sigma, \langle\!|f'_{\sigma i} \circ f_i|\!\rangle_{i=1,n}\big)$$

and identities are given pointwise.) The symmetric strict monoidal structure of $!\mathbb{C}$ is given by concatenation with unit the empty sequence and the obvious symmetry.

The symmetric strict monoidal completion comes equipped with canonical natural coherent equivalences as follows

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{\cong} & !\mathbf{0} \\ () & \longmapsto & \langle\!|\,|\!\rangle \end{array}$$

$$\begin{array}{ccc} !\mathbb{C}_1 \times !\mathbb{C}_2 & \xrightarrow[\simeq]{\oplus} & !(\mathbb{C}_1 + \mathbb{C}_2) \\ (C_1, C_2) & \longmapsto & !\amalg_1(C_1) \otimes !\amalg_2(C_2) \end{array}$$

Presheaves. For a small category $\mathbb{C}$, we write $\widehat{\mathbb{C}}$ for the functor category $[\mathbb{C}^o, \boldsymbol{Set}]$ of *presheaves* on $\mathbb{C}$ and natural transformations between them, and let $y_{\mathbb{C}}$ denote the *Yoneda embedding* $\mathbb{C} \hookrightarrow \widehat{\mathbb{C}} : c \longmapsto \mathbb{C}[\_, c]$.

For $(\mathbb{C}, \otimes, I)$ a (symmetric) monoidal category, the presheaf category $\widehat{\mathbb{C}}$ acquires a (symmetric) monoidal structure via *Day's tensor product* construction [5, 8] given, for $X_1, X_2 \in \widehat{\mathbb{C}}$, as

$$X_1 \widehat{\otimes} X_2 \;=\; \int^{c_1, c_2 \in \mathbb{C}} X_1(c_1) \times X_2(c_2) \times y_{\mathbb{C}}(c_1 \otimes c_2)$$

where $\int$ is the coend construction whose definition and basic properties can be found in [12, Chapter X]. The unit for Day's tensor product $\widehat{\otimes}$ is $y_{\mathbb{C}}(I)$.

For small categories $\mathbb{A}$ and $\mathbb{B}$, an $(\mathbb{A}, \mathbb{B})$-*profunctor*, indicated as $\mathbb{A} \mathrel{-\!\!\!+\!\!\!\!\rightarrow} \mathbb{B}$, is a functor $\mathbb{A} \longrightarrow \widehat{\mathbb{B}}$. Small categories, profunctors, and natural transformations between them form a bicategory [1]. The profunctor composition $V \circ U : \mathbb{A} \mathrel{-\!\!\!+\!\!\!\!\rightarrow} \mathbb{C}$ of $U : \mathbb{A} \mathrel{-\!\!\!+\!\!\!\!\rightarrow} \mathbb{B}$ and $V : \mathbb{B} \mathrel{-\!\!\!+\!\!\!\!\rightarrow} \mathbb{C}$ is given by

$$(V \circ U)(a)(c) \;=\; \int^{b \in \mathbb{B}} V(b)(c) \times U(a)(b) \qquad (1)$$

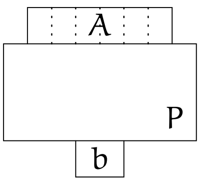
with identities $y_{\mathbb{C}} : \mathbb{C} \mathrel{-\!\!\!+\!\!\!\!\rightarrow} \mathbb{C}$.

For more on the bicategory of profunctors see [1, 11, 4].

2. The calculus of generalised species

For small categories $\mathbb{A}$ and $\mathbb{B}$, an $(\mathbb{A}, \mathbb{B})$-*species of structures* is a profunctor $!\mathbb{A} \mathrel{-\!\!\!+\!\!\!\!\rightarrow} \mathbb{B}$. In particular, $(\mathbb{A}, \mathbf{1})$-species are referred to as $\mathbb{A}$-species. The notation $P : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ is used to indicate that P is an $(\mathbb{A}, \mathbb{B})$-species.

Structures in $P(A)(b)$, for a species $P : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$, are pictorially represented as follows



Concrete examples of *combinatorial species* abound in the literature.

- *Joyal's* k*-sorted species* [9, 3] are $\left(\sum_{i=1}^{k} \mathbf{1}\right)$-species.
- *Permutationals* [9, 2] are **CP**-species for **CP** the groupoid of finite cyclic permutations.
- *Partitionals* [16] are $\mathbf{B}^*$-species for $\mathbf{B}^*$ the groupoid of non-empty finite sets.

Further examples that fit into generalised species are *coloured species and permutationals* [14], and *species on graphs and digraphs* [13].

Basic general examples of species follow.

- Presheaves on $\mathbb{C}$ are essentially species $\mathbf{0} \mathrel{!\!\!\longrightarrow} \mathbb{C}$, whilst presheaves on $!\mathbb{C}$ also correspond to species $\mathbb{C} \mathrel{!\!\!\longrightarrow} \mathbf{1}$.
- The Yoneda embedding $y_{!\mathbb{C}}$ is a $\mathbb{C} \mathrel{!\!\!\longrightarrow} !\mathbb{C}$ species.
- The species $\epsilon_{\mathbb{C}} : !\mathbb{C} \mathrel{!\!\!\longrightarrow} \mathbb{C}$ is defined as $\epsilon_{\mathbb{C}}(C) = !!\mathbb{C}\left[\,\{\!|\,\{\!|\_|\!\}\,|\!\}, C\,\right]$.
- The species $S_{\mathbb{C}} : \mathbb{C} \mathrel{!\!\!\longrightarrow} \mathbb{C}$ is defined as

$$S_{\mathbb{C}}(C) \;=\; \sum_{c \in C} y_{\mathbb{C}}(c) \qquad (2)$$

- The species $E_{\mathbb{A},\mathbb{B}} : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ is defined by $E_{\mathbb{A},\mathbb{B}}(A) = 1$.

2.1. The bicategory of species

We introduce the bicategory $\boldsymbol{\mathcal{ES}}$ (*Espèces de Structures*) of generalised species of structures.

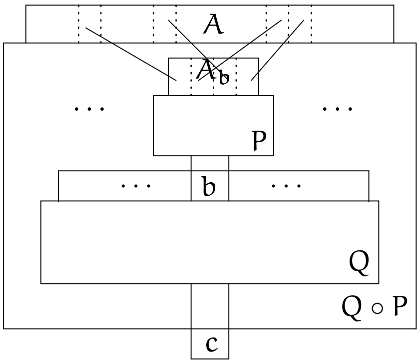
Composition. For species $P : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ and $Q : \mathbb{B} \mathrel{!\!\!\longrightarrow} \mathbb{C}$, the *composition* $Q \circ P : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{C}$ is defined as

$$(Q \circ P)(A)(c) \;=\; \int^{B \in !\mathbb{B}} Q(B)(c) \times P^{\#}(A)(B)$$

where

$$\begin{aligned} & P^{\#}(A)(B) \\ &= \int^{A_b \in !\mathbb{A}\ (b \in B)} \left(\textstyle\prod_{b \in B} P(A_b)(b)\right) \times !\mathbb{A}\left[\textstyle\bigotimes_{b \in B} A_b, A\right] \end{aligned}$$

One can visualise the structures in $(Q \circ P)(A)(c)$ as follows



We give explicit descriptions of sample pre- and post-compositions with a species $P : \mathbb{A} \mathrel{!\!\!\rightarrow} \mathbb{B}$.

- For $b \in \mathbb{B}$, the composite species $\mathbb{A} \overset{P}{\mathrel{!\!\!\rightarrow}} \mathbb{B} \xrightarrow{y_{!\mathbb{B}}\langle\!|b|\!\rangle} \mathbf{1}$ is isomorphic to the species $P_b : \mathbb{A} \mathrel{!\!\!\rightarrow} \mathbf{1}$ defined as

$$P_b(A)() \;=\; P(A)(b) \tag{3}$$

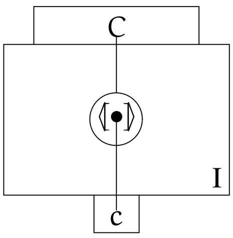
- For $X \in \widehat{\mathbb{A}}$, the composite $\mathbf{0} \overset{X}{\mathrel{!\!\!\rightarrow}} \mathbb{A} \overset{P}{\mathrel{!\!\!\rightarrow}} \mathbb{B}$, is as follows

$$(P \circ X)\langle\!|\,|\!\rangle(b) \cong \int^{A \in !\mathbb{A}} P(A)(b) \times \widehat{\mathbb{A}}\left[S_{\mathbb{A}}A, X\langle\!|\,|\!\rangle\right]$$

where $S_{\mathbb{A}} : !\mathbb{A} \longrightarrow \widehat{\mathbb{A}}$ is as in (2).

Identities. The *identity* species $I_{\mathbb{C}} : \mathbb{C} \mathrel{!\!\!\rightarrow} \mathbb{C}$ is defined as

$$I_{\mathbb{C}}(C)(c) \;=\; !\mathbb{C}\left[\langle\!|c|\!\rangle, C\right]$$



For $P : \mathbb{A} \mathrel{!\!\!\rightarrow} \mathbb{B}$, $Q : \mathbb{B} \mathrel{!\!\!\rightarrow} \mathbb{C}$, and $R : \mathbb{C} \mathrel{!\!\!\rightarrow} \mathbb{D}$, we have canonical natural coherent isomorphisms as follow

$$\boxed{\begin{array}{c}(R \circ Q) \circ P \cong R \circ (Q \circ P) \\ P \circ I_{\mathbb{A}} \cong P \cong I_{\mathbb{B}} \circ P\end{array}}$$

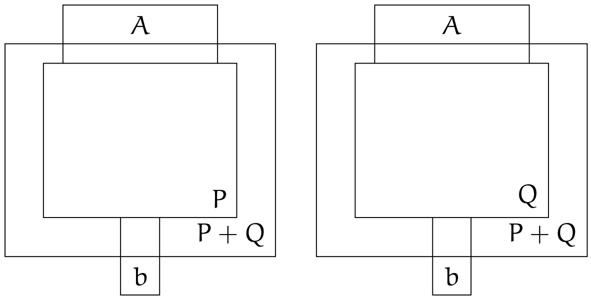
establishing the associativity of composition and the unit laws of identities.

2.2. Addition and multiplication

Each hom-category $\mathcal{ES}\,[\mathbb{A}, \mathbb{B}]$ acquires a commutative rig (= ri*n*g without *n*egatives) structure given by the addition and multiplication of species.

Addition. For $P, Q : \mathbb{A} \mathrel{!\!\!\rightarrow} \mathbb{B}$, the *addition* $P + Q : \mathbb{A} \mathrel{!\!\!\rightarrow} \mathbb{B}$ is defined by

$$(P + Q)(A)(b) \;=\; P(A)(b) + Q(A)(b)$$



More generally, for $X_i \in \widehat{\mathbb{B}}$ and $P_i : \mathbb{A} \mathrel{!\!\!\rightarrow} \mathbb{B}$ $(i \in I)$, the *linear combination* $\sum_{i \in I} X_i P_i : \mathbb{A} \mathrel{!\!\!\rightarrow} \mathbb{B}$ is defined by

$$\Big(\sum_{i \in I} X_i P_i\Big)(A)(b) \;=\; \sum_{i \in I} X_i(b) \times P_i(A)(b)$$

Addition together with the species $\underline{0} : \mathbb{A} \mathrel{!\!\!\rightarrow} \mathbb{B}$ defined as

$$\underline{0}(A) \;=\; 0$$

satisfy commutative monoid laws:

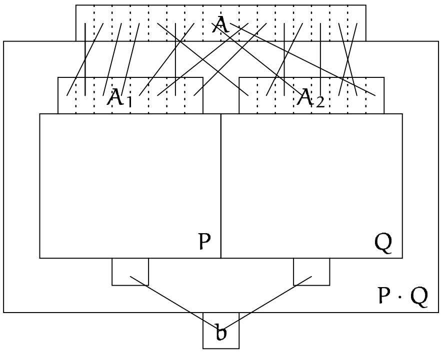
$$\boxed{\begin{array}{cc}\multicolumn{2}{c}{(P + Q) + R \cong P + (Q + R)} \\ P + \underline{0} \cong P & P + Q \cong Q + P\end{array}}$$

for $P, Q, R : \mathbb{A} \mathrel{!\!\!\rightarrow} \mathbb{B}$. Further, for $P, Q : \mathbb{A} \mathrel{!\!\!\rightarrow} \mathbb{B}$ and $R : \mathbb{C} \mathrel{!\!\!\rightarrow} \mathbb{A}$, we have

$$\boxed{(P + Q) \circ R \cong (P \circ R) + (Q \circ R)}$$

Multiplication. For $P, Q : \mathbb{A} \mathrel{!\!\!\rightarrow} \mathbb{B}$, the *multiplication* $P \cdot Q : \mathbb{A} \mathrel{!\!\!\rightarrow} \mathbb{B}$ is defined by

$$\begin{aligned}&(P \cdot Q)(A)(b) \\ &= \int^{A_1, A_2 \in !\mathbb{A}} P(A_1)(b) \times Q(A_2)(b) \times !\mathbb{A}\,[A_1 \otimes A_2, A]\end{aligned} \tag{4}$$



That is, using (3),

$$(P \cdot Q)_b \;=\; P_b \mathbin{\widehat{\otimes}} Q_b$$

for $b \in \mathbb{B}$.

Multiplication together with the species $\underline{1} : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ defined by

$$\underline{1}(A)(b) \;=\; !\mathbb{A}\left[\{\!|\,|\!\}, A\right]$$

satisfy commutative monoid and distributive laws:

$$\boxed{\begin{array}{c} (P \cdot Q) \cdot R \cong P \cdot (Q \cdot R) \\ P \cdot \underline{1} \cong P \qquad\qquad P \cdot Q \cong Q \cdot P \\ P \cdot \underline{0} \cong \underline{0} \qquad P \cdot (Q + R) \cong (P \cdot Q) + (P \cdot R) \end{array}}$$

for $P, Q, R : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$. Further, for $P, Q : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ and $R : \mathbb{C} \mathrel{!\!\!\longrightarrow} \mathbb{A}$, we have

$$\boxed{(P \cdot Q) \circ R \cong (P \circ R) \cdot (Q \circ R)}$$

2.3. Linear structure

We refer to a $\mathbb{C} \mathrel{!\!\!\longrightarrow} \mathbb{A}^o \times \mathbb{B}$ species as a ($\mathbb{C}$-parameterised) $\mathbb{A} \times \mathbb{B}$-*matrix*. The *transpose* of an $\mathbb{A} \times \mathbb{B}$-matrix $U : \mathbb{C} \mathrel{!\!\!\longrightarrow} \mathbb{A}^o \times \mathbb{B}$ is the $\mathbb{B}^o \times \mathbb{A}^o$-matrix $U^t : \mathbb{C} \mathrel{!\!\!\longrightarrow} (\mathbb{B}^o)^o \times \mathbb{A}^o$ defined as

$$U^t(C)(b, a) \;=\; U(C)(a, b)$$

More generally, for a species $P : \mathbb{C} \mathrel{!\!\!\longrightarrow} \prod_{i=1}^n \mathbb{A}_i$ we define the *transposition* $P^\sigma : \mathbb{C} \mathrel{!\!\!\longrightarrow} \prod_{i=1}^n \mathbb{A}_{\sigma i}$ according to the permutation $\sigma \in \mathfrak{S}_n$ by

$$P^\sigma(C)(a_1, \ldots, a_n) \;=\; P(C)(a_{\sigma 1}, \ldots, a_{\sigma n})$$

Matrix multiplication. The *matrix multiplication* (or *linear composition*) of the matrices $U : \mathbb{K} \mathrel{!\!\!\longrightarrow} \mathbb{A}^o \times \mathbb{B}$ and $V : \mathbb{K} \mathrel{!\!\!\longrightarrow} \mathbb{B}^o \times \mathbb{C}$ is the matrix $V \bullet_{\mathbb{B}} U : \mathbb{K} \mathrel{!\!\!\longrightarrow} \mathbb{A}^o \times \mathbb{C}$ defined by

$$\begin{aligned} &(V \bullet_{\mathbb{B}} U)(K)(a, c) \\ &= \int^{b \in \mathbb{B},\, K_1, K_2 \in \mathbb{K}} V(K_1)(b, c) \times U(K_2)(a, b) \times !\mathbb{K}\,[K_1 \otimes K_2, K] \end{aligned}$$

(Compare with the composition of profunctors (1) and the multiplication of species (4).) Using (3), we obtain the familiar formula for matrix multiplication

$$\left(V \bullet_{\mathbb{B}} U\right)_{(a,c)} \;=\; \int^{b \in \mathbb{B}} V_{(b,c)} \cdot U_{(a,b)}$$

for $a \in \mathbb{A}$ and $c \in \mathbb{C}$.

The associativity of matrix multiplication and the unit laws with respect to the *identity matrix* $\Delta_{\mathbb{A}} : \mathbb{C} \mathrel{!\!\!\longrightarrow} \mathbb{A}^o \times \mathbb{A}$ defined as

$$\Delta_{\mathbb{A}}(C)(a', a) \;=\; !\mathbb{C}\left[\{\!|\,|\!\}, C\right] \times \mathbb{A}\,[a, a']$$

hold

$$\boxed{\begin{array}{c} W \bullet_{\mathbb{B}} (V \bullet_{\mathbb{A}} U) \cong (W \bullet_{\mathbb{B}} V) \bullet_{\mathbb{A}} U \\ U \bullet_{\mathbb{A}} \Delta_{\mathbb{A}} \cong U \cong \Delta_{\mathbb{B}} \bullet_{\mathbb{B}} U \end{array}}$$

where $U : \mathbb{K} \mathrel{!\!\!\longrightarrow} \mathbb{A}^o \times \mathbb{B}$, $V : \mathbb{K} \mathrel{!\!\!\longrightarrow} \mathbb{B}^o \times \mathbb{C}$, and $W : \mathbb{K} \mathrel{!\!\!\longrightarrow} \mathbb{C}^o \times \mathbb{D}$. Further, for $U_i : \mathbb{K} \mathrel{!\!\!\longrightarrow} \mathbb{A}^o \times \mathbb{B}$ ($i \in I$), and $V_j : \mathbb{K} \mathrel{!\!\!\longrightarrow} \mathbb{B}^o \times \mathbb{C}$ ($j \in J$), we have

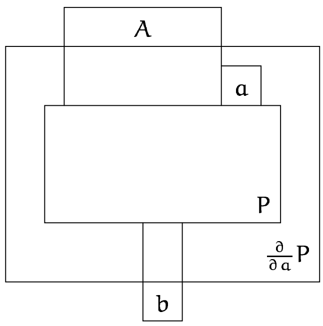
$$\boxed{\left(\textstyle\sum_{j \in J} V_j\right) \bullet_{\mathbb{B}} \left(\textstyle\sum_{i \in I} U_i\right) \cong \textstyle\sum_{(j,i) \in J \times I} V_j \bullet_{\mathbb{B}} U_i}$$

2.4. Differential structure

We introduce differentiation in the context of generalised species and establish its basic properties. Higher-order differential operators are further considered in Subsection 2.7.

Differentiation. For $P : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ and $a \in \mathbb{A}$, the *partial derivative* $\frac{\partial}{\partial a} P : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ is defined as

$$\left(\tfrac{\partial}{\partial a} P\right)(A)(b) \;=\; P(A \otimes \{\!|a|\!\})(b)$$



For all $P, Q : \mathbb{A} \mathrel{!\!\!\rightarrow} \mathbb{B}$ and $X \in \widehat{\mathbb{B}}$, we have the following basic properties

$$\boxed{\begin{gathered} \tfrac{\partial}{\partial a}\big(\tfrac{\partial}{\partial a'}P\big) \cong \tfrac{\partial}{\partial a'}\big(\tfrac{\partial}{\partial a}P\big) \\ \tfrac{\partial}{\partial a}(P+Q) \cong (\tfrac{\partial}{\partial a}P) + (\tfrac{\partial}{\partial a}Q) \\ \tfrac{\partial}{\partial a}(XP) = X\big(\tfrac{\partial}{\partial a}P\big) \\ \tfrac{\partial}{\partial a}(\mathrm{E}) = \mathrm{E} \end{gathered}}$$

and the *Leibniz's rule*

$$\boxed{\tfrac{\partial}{\partial a}(P \cdot Q) \cong \big(\tfrac{\partial}{\partial a}P\big) \cdot Q + P \cdot \big(\tfrac{\partial}{\partial a}Q\big)}$$

Further, for $P : \mathbb{A} \mathrel{!\!\!\rightarrow} \mathbb{B}$ and $Q : \mathbb{B} \mathrel{!\!\!\rightarrow} \mathbb{C}$, we have the *chain rule*

$$\boxed{\Big(\tfrac{\partial}{\partial a}(Q \circ P)\Big)_c \;\cong\; \int^{b \in \mathbb{B}} \Big(\tfrac{\partial}{\partial b}(Q) \circ P\Big)_c \cdot \Big(\tfrac{\partial}{\partial a}P\Big)_b}$$

where $a \in \mathbb{A}$ and $c \in \mathbb{C}$.

The *differential* application (or *Jacobian* matrix) $dP : \mathbb{A} \mathrel{!\!\!\rightarrow} \mathbb{A}^o \times \mathbb{B}$ of $P : \mathbb{A} \mathrel{!\!\!\rightarrow} \mathbb{B}$ is defined as

$$(dP)(A)(a,b) \;=\; \big(\tfrac{\partial}{\partial a}P\big)(A)(b)$$

The basic properties of partial derivatives translate in terms of differentials; in particular, the *chain rule* amounts to the identity

$$\boxed{d(Q \circ P) \cong \big(d(Q) \circ P\big) \bullet_{\mathbb{B}} dP}$$

For a species $P : \sum_{i \in I} \mathbb{A}_i \mathrel{!\!\!\rightarrow} \mathbb{B}$, one may introduce j-differentials $d_j P : \sum_{i \in I} \mathbb{A}_i \mathrel{!\!\!\rightarrow} {\mathbb{A}_j}^o \times \mathbb{B}$ ($j \in I$) as follows

$$(d_j P)(A)(a,b) \;=\; \big(\tfrac{\partial}{\partial \amalg_j(a)}P\big)(A)(b)$$

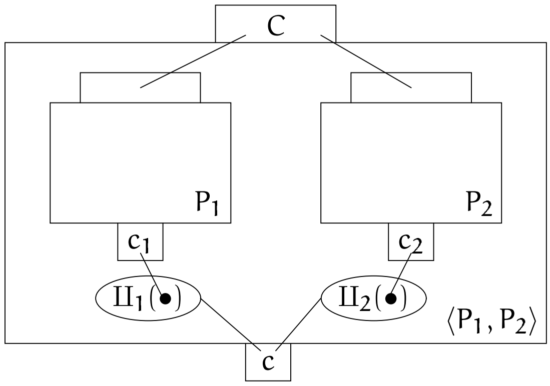
However, as we show below, these are derivable.

2.5. Cartesian closed structure

We describe the cartesian closed structure of species.

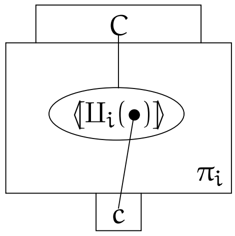
Pairing and projections. There is exactly one species $\mathbb{C} \mathrel{!\!\!\rightarrow} \mathbf{0}$. More generally, for $P_i : \mathbb{C} \mathrel{!\!\!\rightarrow} \mathbb{C}_i$ ($i \in I$), the *pairing* $\langle P_i \rangle_{i \in I} : \mathbb{C} \mathrel{!\!\!\rightarrow} \sum_{i \in I} \mathbb{C}_i$ is defined as

$$\begin{aligned} &\langle P_i \rangle_{i \in I}(C)(c) \\ &\quad = \; \textstyle\sum_{i \in I} \int^{c_i \in \mathbb{C}_i} P_i(C)(c_i) \times \big(\sum_{i \in I} \mathbb{C}_i\big)[c, \amalg_i(c_i)] \\ &\quad \cong \; P_i(C)(c_i) \quad \text{where } c = \amalg_i(c_i) \end{aligned}$$



For $i \in I$, the *projection* species $\pi_i : \sum_{i \in I} \mathbb{C}_i \mathrel{!\!\!\rightarrow} \mathbb{C}_i$ is defined as

$$\pi_i(C)(c) \;=\; !\Big(\textstyle\sum_{i \in I} \mathbb{C}_i\Big)\big[\{\!|\amalg_i(c)|\!\}, C\big]$$



The usual laws of pairing and projections are satisfied up to isomorphism:

$$\boxed{\begin{aligned} \pi_k \circ \langle P_i \rangle_{i \in I} &\cong P_k : \mathbb{C} \mathrel{!\!\!\rightarrow} \mathbb{C}_k \qquad (k \in I) \\ \langle \pi_i \circ P \rangle_{i \in I} &\cong P \;\; : \mathbb{C} \mathrel{!\!\!\rightarrow} \textstyle\sum_{i \in I} \mathbb{C}_i \end{aligned}}$$

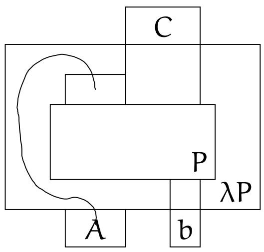
Note that in the presence of cartesian structure the differentials $d_k P : \sum_{i \in I} \mathbb{A}_i \mathrel{!\!\!\rightarrow} \mathbb{A}_k^o \times \mathbb{B}$ ($k \in I$) are derivable from the differential $dP : \sum_{i \in I} \mathbb{A}_i \mathrel{!\!\!\rightarrow} \sum_{i \in I} \mathbb{A}_i^o \times \mathbb{B}$, as

$$\boxed{d_k(P) \cong \pi_k \circ d(P) \qquad (k \in I)}$$

for all $P : \sum_{i \in I} \mathbb{A}_i \mathrel{!\!\!\rightarrow} \mathbb{B}$.

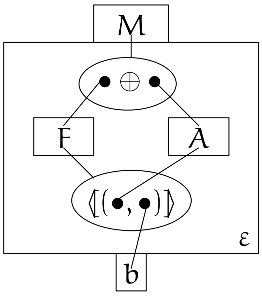
Abstraction and evaluation. For $P : \mathbb{C} + \mathbb{A} \mathrel{!\!\!\rightarrow} \mathbb{B}$, the *abstraction* $\lambda_{\mathbb{A}} P : \mathbb{C} \mathrel{!\!\!\rightarrow} !\mathbb{A}^o \times \mathbb{B}$ is defined as

$$(\lambda_{\mathbb{A}} P)(C)(A,b) \;=\; P(C \oplus A)(b)$$



and the *evaluation* $\varepsilon_{\mathbb{A},\mathbb{B}} : (!\mathbb{A}^o \times \mathbb{B}) + \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ by

$$\varepsilon_{\mathbb{A},\mathbb{B}}(M)(b) = \int^{F \in !(!\mathbb{A}^o \times \mathbb{B}),\, A \in !\mathbb{A}} !(!\mathbb{A}^o \times \mathbb{B})[\,\{\!\!\{(A,b)\}\!\!\}\,, F\,] \times !\big((!\mathbb{A}^o \times \mathbb{B}) + \mathbb{A}\big)[F \oplus A, M] \tag{5}$$



For $P : \mathbb{C} \mathrel{!\!\!\longrightarrow} !\mathbb{A}^o \times \mathbb{B}$, we write $\upsilon_{\mathbb{A}}(P)$ for the composite $\varepsilon \circ \langle P \circ \pi_1, \pi_2 \rangle : \mathbb{C} + \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$. The usual laws of abstraction and evaluation are satisfied up to isomorphism:

$$\boxed{\begin{array}{l} \upsilon(\lambda P) \cong P : \mathbb{C} + \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B} \\ \lambda(\upsilon P) \cong P : \mathbb{C} \mathrel{!\!\!\longrightarrow} !\mathbb{A}^o \times \mathbb{B} \end{array}}$$

We further note the following interesting commutation property between abstraction and linear composition: for σ the permutation $(12)(3)$,

$$\boxed{\big((\lambda_{\mathbb{A}} Q)^\sigma \underset{\mathbb{C}}{\bullet} P\big)^\sigma \cong \lambda_{\mathbb{A}}\big(Q \underset{\mathbb{C}}{\bullet} (P \circ \pi_1)\big)} \tag{6}$$

for all $P : \mathbb{K} \mathrel{!\!\!\longrightarrow} \mathbb{B}^o \times \mathbb{C}$ and $Q : \mathbb{K} + \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{C}^o \times \mathbb{D}$.

2.6. Graph models of the lambda calculus

It does not take much to construct models of the lambda calculus. Indeed, for a small category $\mathbb{C}$, the free $!(\_)^o \times (\_)$-algebra $\widetilde{\mathbb{C}}$ on $\mathbb{C}$ in $\mathfrak{Cat}$ yields

$$\widetilde{\mathbb{C}}^{\widetilde{\mathbb{C}}} \trianglelefteq \widetilde{\mathbb{C}} \qquad \text{in } \mathcal{ES}$$

Further, the final $!(\_)^o \times (\_)$-coalgebra $\mathbf{U}$ in $\mathfrak{Gpd}$ yields an equivalence

$$\mathbf{U}^{\mathbf{U}} \simeq \mathbf{U} \qquad \text{in } \mathcal{ES}$$

Interestingly, $\mathbf{U}$ has the following explicit description: the objects are given by the class of planar trees described by ω-chains

$$\{0\} \underset{\perp}{\rightleftarrows} O_1 \underset{\perp}{\rightleftarrows} \cdots \underset{\perp}{\rightleftarrows} O_n \underset{\perp}{\rightleftarrows} \cdots \qquad (n \in \mathbb{N})$$

of reflections between finite ordinals, with morphisms given by natural isomorphisms.

2.7. Higher-order differential structure

We relate the linear and cartesian closed structures, and introduce an operator which is shown to satisfy the basic properties of differentiation.

Linear and cartesian closed structure. For a matrix $U : \mathbb{C} \mathrel{!\!\!\longrightarrow} \mathbb{A}^o \times \mathbb{B}$ we define the species $\widetilde{U} : \mathbb{C} + \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ as

$$\widetilde{U}(M)(b) = \int^{a \in \mathbb{A},\, C \in !\mathbb{C},\, A \in !\mathbb{A}} U(C)(a,b) \times !\mathbb{A}\left[\{\!\!\{a\}\!\!\}, A\right] \times !(\mathbb{C} + \mathbb{A})\,[C \oplus A, M]$$

This construction internalises as an embedding of matrices into exponentials as

$$\iota_{\mathbb{A},\mathbb{B}} = \lambda_{\mathbb{A}}(\widetilde{I_{\mathbb{A}^o \times \mathbb{B}}}) : \mathbb{A}^o \times \mathbb{B} \mathrel{!\!\!\longrightarrow} !\mathbb{A}^o \times \mathbb{B}$$

given explicitly by

$$\iota_{\mathbb{A},\mathbb{B}}(U)(A,b) \cong \int^{a \in \mathbb{A}} !(\mathbb{A}^o \times \mathbb{B})\left[\{\!\!\{(a,b)\}\!\!\}, U\right] \times !\mathbb{A}\left[\{\!\!\{a\}\!\!\}, A\right]$$

Indeed, for all $P : \mathbb{C} \mathrel{!\!\!\longrightarrow} \mathbb{A}^o \times \mathbb{B}$, we have that

$$\iota_{\mathbb{A},\mathbb{B}} \circ P \cong \lambda_{\mathbb{A}}(\widetilde{P}) : \mathbb{C} \mathrel{!\!\!\longrightarrow} !\mathbb{A}^o \times \mathbb{B}$$

Further, the embedding commutes with identities and composition; since, for

$$\ell_{\mathbb{A},\mathbb{B},\mathbb{C}} = \pi_2 \underset{\mathbb{B}}{\bullet} \pi_1 : (\mathbb{B}^o \times \mathbb{C}) + (\mathbb{A}^o \times \mathbb{B}) \mathrel{!\!\!\longrightarrow} \mathbb{A}^o \times \mathbb{C}$$

we have that

$$\ell \circ \langle P, Q \rangle \cong P \underset{\mathbb{B}}{\bullet} Q$$

for all $P : \mathbb{K} \mathrel{!\!\!\longrightarrow} \mathbb{A}^o \times \mathbb{B}$ and $Q : \mathbb{K} \mathrel{!\!\!\longrightarrow} \mathbb{B}^o \times \mathbb{C}$, and

$$\iota_{\mathbb{A},\mathbb{A}} \circ \Delta_{\mathbb{A}} \cong \lambda_{\mathbb{A}}(I_{\mathbb{A}}) : \mathbf{0} \mathrel{!\!\!\longrightarrow} !\mathbb{A}^o \times \mathbb{A}$$

$$\iota_{\mathbb{A},\mathbb{B}} \circ \ell_{\mathbb{A},\mathbb{B},\mathbb{C}} \cong \langle \iota_{\mathbb{B},\mathbb{C}} \circ \pi_2, \iota_{\mathbb{A},\mathbb{B}} \circ \pi_1 \rangle \circ m_{\mathbb{A},\mathbb{B},\mathbb{C}}$$

for $m_{\mathbb{A},\mathbb{B},\mathbb{C}} = \lambda_{\mathbb{A}}\big(\varepsilon_{\mathbb{B},\mathbb{C}} \circ \langle \pi_1, \varepsilon_{\mathbb{A},\mathbb{B}} \circ \pi_2 \rangle\big)$ the internal composition $(!\mathbb{B}^o \times \mathbb{C}) + (!\mathbb{A}^o \times \mathbb{B}) \mathrel{!\!\!\longrightarrow} !\mathbb{A}^o \times \mathbb{C}$.

Differentiation operator. We introduce the *differentiation operator* $\mathrm{D}_{\mathbb{A},\mathbb{B}} : !\mathbb{A}^o \times \mathbb{B} \mathrel{!\!\!\longrightarrow} !\mathbb{A}^o \times \mathbb{A}^o \times \mathbb{B}$ as follows

$$\mathrm{D}_{\mathbb{A},\mathbb{B}}(F)(A,a,b) = !(!\mathbb{A}^o \times \mathbb{B})\left[\{\!\!\{(A \otimes \{\!\!\{a\}\!\!\}, b)\}\!\!\}, F\right]$$

This operator is linear, as

$$D \cong \widetilde{\delta}$$

for δ the $(!\mathbb{A}^o \times \mathbb{B}) \times (!\mathbb{A}^o \times \mathbb{A}^o \times \mathbb{B})$-matrix given by

$$\delta(U, (A, a, b)) \;=\; !\mathbb{A}^o \times \mathbb{B}\left[(A \otimes \{\!|a|\!\}, b), U\right]$$

and internalises differential application since

$$\boxed{d_2 P \cong \upsilon_{\mathbb{A}}(D \circ \lambda_{\mathbb{A}} P) : \mathbb{C} + \mathbb{A} \;!\!\!\longrightarrow \mathbb{A}^o \times \mathbb{B}} \qquad (7)$$

for all $P : \mathbb{C} + \mathbb{A} \;!\!\!\longrightarrow \mathbb{B}$.

Further, it is constant on linear maps, as

$$\boxed{D \circ \iota_{\mathbb{A},\mathbb{B}} \cong \lambda_{\mathbb{A}}(\pi_1) : \mathbb{A}^o \times \mathbb{B} \;!\!\!\longrightarrow\; !\mathbb{A}^o \times \mathbb{A}^o \times \mathbb{B}}$$

It follows that

$$\boxed{d(I_{\mathbb{A}}) \cong \Delta_{\mathbb{A}} : \mathbb{A} \;!\!\!\longrightarrow \mathbb{A}^o \times \mathbb{A}}$$

and we have from (6) and (7) above that, for σ the permutation $(12)(3)$,

$$\boxed{\left((D \circ \lambda_{\mathbb{A}} P)^{\sigma} \underset{\mathbb{A}}{\bullet} U\right)^{\sigma} \cong \lambda_{\mathbb{A}}\left(d_2(P) \underset{\mathbb{A}}{\bullet} (U \circ \pi_1)\right)}$$

for all $P : \mathbb{C} + \mathbb{A} \;!\!\!\longrightarrow \mathbb{B}$ and $U : \mathbb{C} \;!\!\!\longrightarrow \mathbb{D}^o \times \mathbb{A}$. This identity corresponds to the β-rule of the differential lambda calculus [6].

2.8. Operators on generalised Fock space

Annihilation and creation. For $a \in \mathbb{A}$ define the *annihilation* and *creation* operators as the $(!\mathbb{A}^o \times \mathbb{B}) \times (!\mathbb{A}^o \times \mathbb{B})$-matrices α_a and γ_a given by

$$\alpha_a(U, (A, b)) \;=\; \delta(U, (A, a, b))$$

and

$$\gamma_a(U, (A, b)) \\ = \int^{A' \in !\mathbb{A}} !\mathbb{A}^o \times \mathbb{B}\,[(A', b), U] \times !\mathbb{A}\left[A' \otimes \{\!|a|\!\}, A\right]$$

Further, let α and γ be the $(!\mathbb{A}^o \times \mathbb{B}) \times (!\mathbb{A}^o \times \mathbb{B})$-matrices

$$\alpha(U, V) \;=\; \int^{a \in \mathbb{A}} \alpha_a(U, V)$$

and

$$\gamma(U, V) \;=\; \int^{a \in \mathbb{A}} \gamma_a(U, V)$$

For $u, v \in \mathbb{A}$ the following hold

$$\boxed{\begin{array}{c} \alpha_u \bullet \gamma_v \;\cong\; \gamma_v \bullet \alpha_u + \mathbb{A}[v, u]\ \Delta_{!\mathbb{A}^o \times \mathbb{B}} \\ \alpha_u \bullet \alpha_v \;\cong\; \alpha_v \bullet \alpha_u \qquad \gamma_u \bullet \gamma_v \;\cong\; \gamma_v \bullet \gamma_u \end{array}}$$

Further, for non-empty $\mathbb{A}$, we also have that

$$\boxed{\alpha \bullet \gamma \;\cong\; \gamma \bullet \alpha + \Delta_{!\mathbb{A}^o \times \mathbb{B}}}$$

Let $\mathsf{A}_a = \widetilde{\alpha_a}$, $\mathsf{C}_a = \widetilde{\gamma_a}$ and $\mathsf{A} = \widetilde{\alpha}$, $\mathsf{C} = \widetilde{\gamma}$.

For $\mathsf{A}_a, \mathsf{C}_a : !\mathbb{A}^o \times \mathbb{B} \;!\!\!\longrightarrow\; !\mathbb{A}^o \times \mathbb{B}$, we have that

$$\boxed{\mathsf{A}_a(F)(A, b) \cong (\tfrac{\partial}{\partial a}\overline{F})(A)(b)}$$

and

$$\boxed{\mathsf{C}_a(F)(A, b) \cong \left(\overline{F} \cdot \chi_a\right)(A)(b)}$$

where $\overline{F}(A)(b) \;=\; \varepsilon_{\mathbb{A},\mathbb{B}}(F \oplus A)(b)$ and $\chi_a(A)(b) \;=\; I_{\mathbb{A}}(A)(a)$.

Further, for $u, v \in \mathbb{A}$ the following hold

$$\boxed{\begin{array}{c} \mathsf{A}_u \circ \mathsf{C}_v \;\cong\; \mathsf{C}_v \circ \mathsf{A}_u + \mathbb{A}[v, u]\ I_{!\mathbb{A}^o \times \mathbb{B}} \\ \mathsf{A}_u \circ \mathsf{A}_v \;\cong\; \mathsf{A}_v \circ \mathsf{A}_u \qquad \mathsf{C}_u \circ \mathsf{C}_v \;\cong\; \mathsf{C}_v \circ \mathsf{C}_u \end{array}}$$

and, for non-empty $\mathbb{A}$, we also have that

$$\boxed{\mathsf{A} \circ \mathsf{C} \;\cong\; \mathsf{C} \circ \mathsf{A} + I_{!\mathbb{A}^o \times \mathbb{B}}}$$

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