14 POLYOMINOES

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INTRODUCTION

A *polyomino* is a finite, connected subgraph of the square-grid graph consisting of infinitely many unit cells matched edge-to-edge, with pairs of adjacent cells forming edges of the graph. Polyominoes have a long history, going back to the start of the 20th century, but they were popularized in the present era initially by Solomon Golomb, then by Martin Gardner in his *Scientific American* columns "Mathematical Games," and finally by many research papers by David Klarner. They now constitute one of the most popular subjects in mathematical recreations, and have found interest among mathematicians, physicists, biologists, and computer scientists as well.

14.1 BASIC CONCEPTS

GLOSSARY

Cell: A unit square in the Cartesian plane with its sides parallel to the coordinate axes and with its center at an integer point (u, v). This cell is denoted $[u, v]$ and identified with the corresponding member of $\mathbb{Z}^2$.

Adjacent cells: Two cells, $[u, v]$ and $[r, s]$, with $|u - r| + |v - s| = 1$.

Square-grid graph: The graph with vertex set $\mathbb{Z}^2$ and an edge for each pair of adjacent cells.

Polyomino: A finite set S of cells such that the induced subgraph of the square-grid graph with vertex set S is connected. A polyomino of size n, that is, with exactly n cells, is called an ***n-omino***. Polyominoes are also known as ***animals*** on the square lattice.

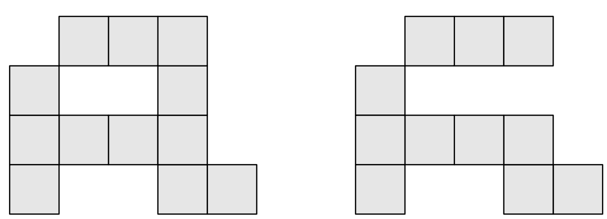

FIGURE 14.1.1
Two sets of cells: the set on the left is a polyomino, the one on the right is not.

[1]This is a revision, by G. Barequet, of the chapter of the same title originally written by the late D.A. Klarner for the first edition, and revised by the late S.W. Golomb for the second edition.

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14.2 EQUIVALENCE OF POLYOMINOES

Notions of equivalence for polyominoes are defined in terms of groups of affine maps that act on the set $\mathbb{Z}^2$ of cells in the plane.

GLOSSARY

Translation by (r,s): The mapping from $\mathbb{Z}^2$ to itself that maps $[u,v]$ to $[u+r, v+s]$; it sends any subset $S \subset \mathbb{Z}^2$ to its *translate* $S+(r,s) = \{[u+r, v+s] : [u,v] \in S\}$.

Translation-equivalent: Sets S, S' of cells such that S' is a translate of S.

Fixed polyomino: A translation-equivalence class of polyominoes; $\boldsymbol{t(n)}$ denotes the number of fixed n-ominoes. (($A(n)$ is also widely used in the literature.)

Representatives of the six fixed 3-ominoes are shown in Figure 14.2.1.

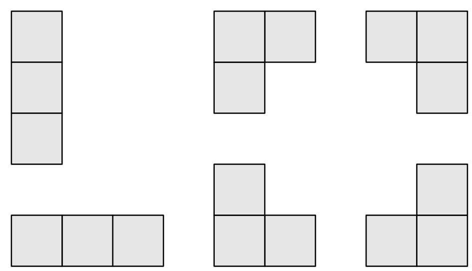

FIGURE 14.2.1
The six fixed 3-ominoes.

The ***lexicographic cell ordering*** $\prec$ on $\mathbb{Z}^2$ is defined by: $[r,s] \prec [u,v]$ if $s < v$, or if $s = v$ and $r < u$.

Standard position: The translate $S-(u,v)$ of S, where $[u,v]$ is the lexicographically minimum cell in S.

A finite set $S \subset \mathbb{Z}^2$ is in standard position if and only if $[0,0] \in S$, $v \geq 0$ for all $[u,v] \in S$, and $u \geq 0$ for all $[u,0] \in S$.

Rotation-translation group: The group $\mathcal{R}$ of mappings of $\mathbb{Z}^2$ to itself of the form $[u,v] \mapsto [u,v]\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^k + (r,s)$. (The matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which is denoted by R, maps $[u,v]$ to $[v,-u]$ by right multiplication, hence represents a clockwise rotation of 90°.)

Rotationally equivalent: Sets S, S' of cells with $S' = \rho S$ for some $\rho \in \mathcal{R}$.

Chiral polyomino, or ***handed polyomino:*** A rotational-equivalence class of polyominoes; $\boldsymbol{r(n)}$ denotes the number of chiral n-ominoes.

The top row of 5-ominoes in Figure 14.2.2 consists of the set of cells $F = \{[0,-1], [-1,0], [0,0], [0,1], [1,1]\}$, together with FR, FR^2, and FR^3. All four of these 5-ominoes are rotationally equivalent. The bottom row in Figure 14.2.2 shows these same four 5-ominoes reflected about the x-axis. These four 5-ominoes are rotationally equivalent as well, but none of them is rotationally equivalent to

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any of the 5-ominoes shown in the top row. Representatives of the seven chiral 4-ominoes are shown in Figure 14.2.3.

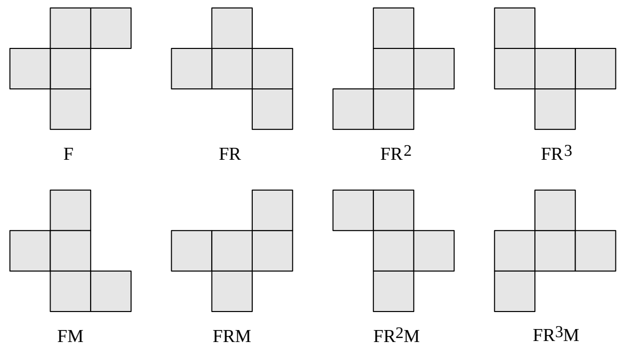

FIGURE 14.2.2
The 5-ominoes in the top row are rotationally equivalent, and so are their reflections in the bottom row, but the two sets are rotationally distinct.

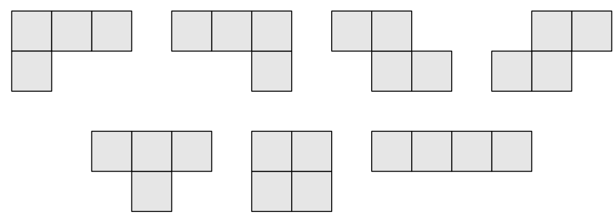

FIGURE 14.2.3
The seven chiral 4-ominoes.

Congruence group: The group $\mathcal{S}$ of motions generated by the matrix $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (reflection in the x-axis) and the rotation-translation group $\mathcal{R}$. (A typical element of $\mathcal{S}$ has the form $[u, v] \mapsto [u, v]R^k M^i + (r, s)$, for some $k = 0, 1, 2,$ or 3, some $i = 0$ or 1, and some $r, s \in \mathbb{Z}$.)

Congruent: Sets S and S' of cells such that $S' = \sigma(S)$ for some $\sigma \in \mathcal{S}$.

Free polyomino: A congruence class of polyominoes; $\boldsymbol{s(n)}$ denotes the number of free n-ominoes. The twelve free 5-ominoes are shown in Figure 14.2.4.

THEOREM 14.2.1 *Embedding Theorem*

For each n, let U_n consist of the $n^2 - n + 1$ cells of the form $[u, v]$, where $\begin{cases} 0 \le u \le n, & \textit{for } v = 0 \\ |u| + v \le n, & \textit{for } v > 0 \end{cases}$. (See Figure 14.2.5 for the case $n = 5$.) *Then, all n-ominoes in standard position are edge-connected subsets of U_n that contain* $[0, 0]$.

COROLLARY 14.2.2

The number of fixed n-ominoes is finite for each n.

The same result can be obtained by a simple argument due to Eden [Ede61]: Every polyomino P of size n can be built according to a set of $n-1$ "instructions" taken from a superset of size $3(n-1)$. Starting with a single square, each instruction tells us how to choose a lattice cell c, neighboring a cell already in P, and add c to P.

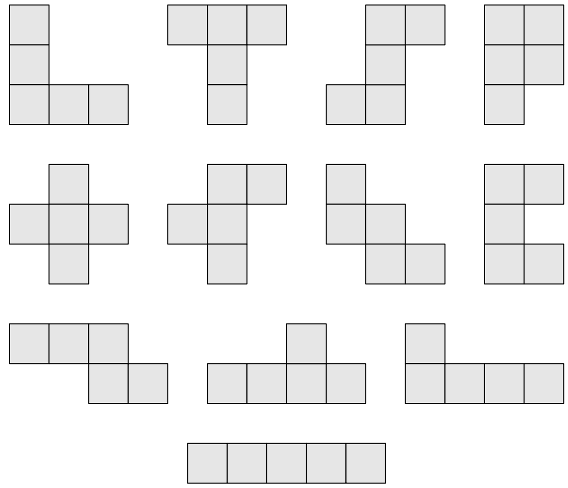

FIGURE 14.2.4
The twelve free 5-ominoes.

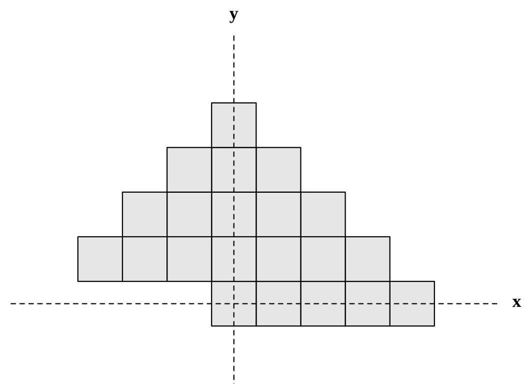

FIGURE 14.2.5
A set of n^2-n+1 cells (for $n=5$) that contains every n-omino in standard position.

(Some of these instruction sets are illegal, and some other sets produce the same polyominoes.) Hence, the number of polyominoes of size n is less than $\binom{3(n-1)}{n-1}$.

14.3 HOW MANY n-OMINOES ARE THERE?

Table 14.3.1, calculated by Redelmeier [Red81], indicates the values of $t(n)$, $r(n)$, and $s(n)$ for $n = 1, \ldots, 24$. Jensen and Guttmann [JG00, Jen01] and Jensen [Jen03] extended the enumeration of polyominoes up to $n = 56$. See also sequence A001168 in the OEIS (On-line Encyclopedia of Integer Sequences) [oeis].

It is easy to see that for each n, we have

$$\frac{t(n)}{8} \le s(n) \le r(n) \le t(n).$$

The values of $t(n)$ seem to be growing exponentially, and indeed they have exponential bounds.

THEOREM 14.3.1 [Kla67]
$\lim_{n\to\infty}(t(n))^{1/n} = \lambda$ *exists.*

TABLE 14.3.1 The number of fixed, chiral, and free n-ominoes for $n \leq 24$.

n	$t(n)$	$r(n)$	$s(n)$
1	1	1	1
2	2	1	1
3	6	2	2
4	19	7	5
5	63	18	12
6	216	60	35
7	760	196	108
8	2725	704	369
9	9910	2500	1285
10	36446	9189	4655
11	135268	33896	17073
12	505861	126759	63600
13	1903890	476270	238591
14	7204874	1802312	901971
15	27394666	6849777	3426576
16	104592937	26152418	13079255
17	400795844	100203194	50107909
18	1540820542	385221143	192622052
19	5940738676	1485200848	742624232
20	22964779660	5741256764	2870671950
21	88983512783	22245940545	11123060678
22	345532572678	86383382827	43191857688
23	1344372335524	336093325058	168047007728
24	5239988770268	1309998125640	654999700403

This constant (often referred to in the literature as the "growth constant" of polyominoes) has since then been called "Klarner's constant." Only three decades later, Madras proved the existence of the asymptotic growth ratio.

THEOREM 14.3.2 [Mad99]

$\lim_{n\to\infty} t(n+1)/t(n)$ *exists (and is hence equal to* λ*).*

The currently best known lower [BRS16] and upper [KR73] bounds on the constant λ are 4.0025 and 4.6496, respectively. The proof of the lower bound uses an analysis of polyominoes on *twisted cylinders* with the help of a supercomputer, and the proof of the upper bound uses a composition argument of so-called *twigs*. The currently best (unproved) *estimate* of λ, 4.0625696 ± 0.0000005, is by Jensen [Jen03].

ALGORITHMS

Redelmeier

Considerable effort has been expended to find a formula for the number of fixed n-ominoes (say), with no success. Redelmeier's algorithm, which produced the entries in Table 14.3.1, generates *all* fixed n-ominoes one by one and counts them. The recursive algorithm searches G, the underlying cell-adjacency graph of the square lattice, and counts all connected subgraphs of G (up to a predetermined size) that contain some canonical vertex, say, $(0, 0)$, which is assumed to always correspond to the leftmost cell in the bottom row of the polyomino. (This prevents multiple counting of translations of the same polyomino.) See more details in Section 14.4. Although the running time of the algorithm is necessarily exponential, it takes only $O(n)$ space. Redelmeier's algorithm was extended to other lattices, parallelized, and enhanced further; see [AB09a, AB09b, LM11, Mer90, ML92].

Jensen

A faster transfer-matrix algorithm for counting polyominoes was described by Jensen [Jen03], but its running time is still exponential in the size of counted polyominoes. This algorithm does not produce all polyominoes. Instead, it maintains all possible polyomino *boundaries* (see Figure 14.3.1), which are the possible configurations of

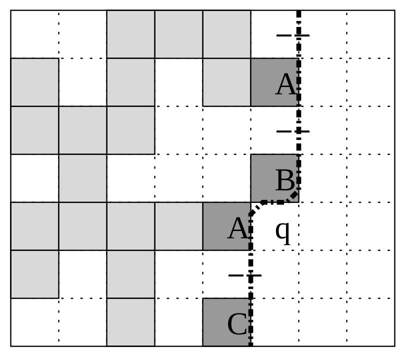

FIGURE 14.3.1
A partially built 21-cell polyomino with boundary `-A-BA-C`.

the right sides of the polyominoes, associated with information about the connectivity between cells of the boundary through polyomino cells found to the left of the boundary. Polyominoes are "built" column by column from left to right, and in every column, cells are considered from top to bottom. The algorithm maintains a database indexed by the boundaries, where for each possible boundary σ, the database keeps only the *counts* of "partial" polyominoes of all possible sizes, which can have σ as their boundary. ("Partial" in the sense that polyominoes may still be invalid in this stage due to having more than one component, but they may become valid (connected) later when more cells are added to the polyomino.) The figure shows a polyomino with the boundary `-A-BA-C`, where letters represent components of the polyominoes corresponding to this boundary. When considering the next cell q (just below the "kink" in the boundary), there are two options: either to add q to the polyomino, or to not add it. Choosing either option will change the boundary from σ to σ' (possibly $\sigma' = \sigma$), and the algorithm will then update the contents of the entry with index σ' in the database. (Note again that the database does not keep the polyominoes, but only counts of polyominoes for each possible boundary.) The algorithm counts all connected polyominoes produced in this process, up to a prescribed size, and for all possible heights (7 in the figure). Analysis of the algorithm [BM07] reveals that the major factor that influences the performance of the algorithm (in terms of both running time and memory consumption) is the number of possible boundaries. It turns out that the number of possible boundaries of length b is proportional to 3^b, up to a small polynomial factor. Due to symmetry, the algorithm needs to consider boundaries of length up to only $\lceil n/2 \rceil$ for counting polyominoes of size n. Hence, the time complexity of the algorithm is roughly $3^{n/2} \approx 1.73^n$, which is significantly less than the total number of polyominoes (about 4.06^n).

UNSOLVED PROBLEMS

PROBLEM 14.3.3

Can $t(n)$ be computed in time polynomial with n?

A related problem concerns the constant λ defined above:

PROBLEM 14.3.4

Is there a polynomial-time algorithm to find, for each n, an approximation λ_n of λ satisfying

$$10^{-n} < |\lambda_n - \lambda| < 10^{-n+1} \text{ ?}$$

PROBLEM 14.3.5

Define some decreasing sequence $\beta = (\beta_1, \beta_2, \dots)$ that tends to λ, and give an algorithm to compute β_n for every n.

Define the two sequences $\tau_1(n) = (t(n))^{1/n}$ and $\tau_2(n) = t(n+1)/t(n)$. A folklore polyomino-concatenation argument shows that for all n we have $(t(n))^2 \leq t(2n)$, hence, $(t(n))^{1/n} \leq (t(2n))^{1/(2n)}$, which implies that $(t(n))^{1/n} \leq \lambda$ for all n, that is, $\tau_1(n)$ approaches λ from below (but is not necessarily monotone). However, it seems, given the first 56 elements of $t(n)$, that both $\tau_1(n)$ and $\tau_2(n)$ are monotone increasing. This gives two more unsolved problems:

PROBLEM 14.3.6

Show that $\tau_1(n) < \tau_1(n+1)$ for all n.

PROBLEM 14.3.7

Show that $\tau_2(n) < \tau_2(n+1)$ for all n.

14.4 GENERATING POLYOMINOES

The algorithm we describe to generate all n-ominoes, which is essentially due to Redelmeier [Red81], also provides a way of encoding n-ominoes. Starting with all n-ominoes in standard position, with each cell and each neighboring cell numbered, it constructs without repetitions all numbered $(n+1)$-ominoes in standard position.

GLOSSARY

Border cell of an n-omino S: A cell $[u, v]$, with $v \geq 0$ or with $v = 0$ and $u \geq 0$, adjacent to some cell of S. The set of all border cells, which is denoted by $B(S)$, can be shown by induction to have no more than $2n$ elements.

The algorithm, illustrated in Figure 14.4.1 for $n = 1$, 2, and 3, begins with cell 1 in position $[0, 0]$, with its border cells marked 2 and 3, and then adds these—one at a time and in this order—each time numbering *new* border cells in their lexicographic order. Whenever a number used for a border cell is not larger than the largest internal number, it is circled, and the corresponding cell is *not* added at the next stage.

FIGURE 14.4.1

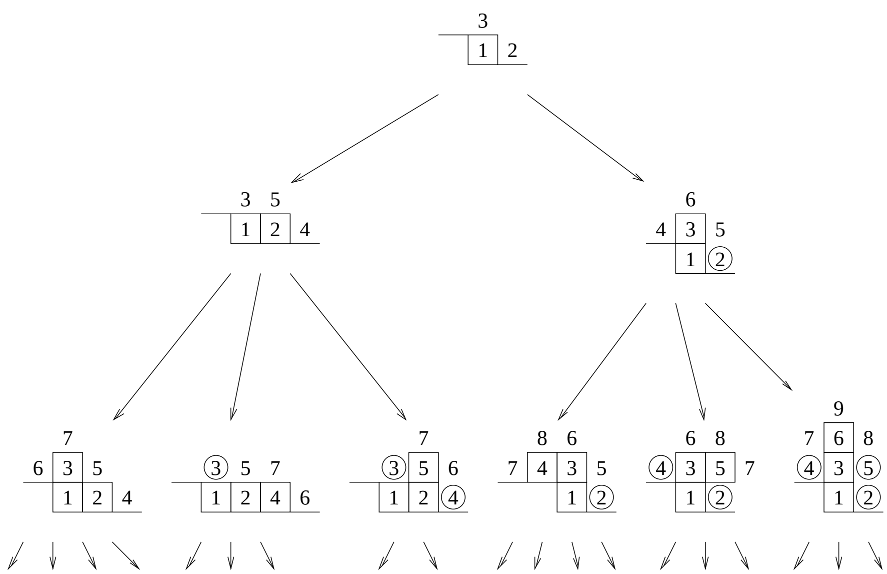

Figure 14.4.2 shows all the 4-ominoes produced in this way, with their border cells marked for the next step of the algorithm.

This process assigns a unique set of positive integers to each n-omino S, also illustrated in Figure 14.4.2. The *set character functions* for these integer sets, in turn, truncated after their final 1's, provide a binary codeword $\chi(S)$ for each n-omino S. For example, the code words for the first three 4-ominoes in Figure 14.4.2 would be 1111, 11101, and 111001.

PROBLEM 14.4.1

Which binary strings arise as codewords for n-ominoes?

The following is easy to see:

THEOREM 14.4.2

$t(n + 1) = \sum n + |B(S)| - |\chi(S)|$, *where the sum extends over all n-ominoes S in standard position, and $|\chi(S)|$ is the number of bits in the codeword of S.*

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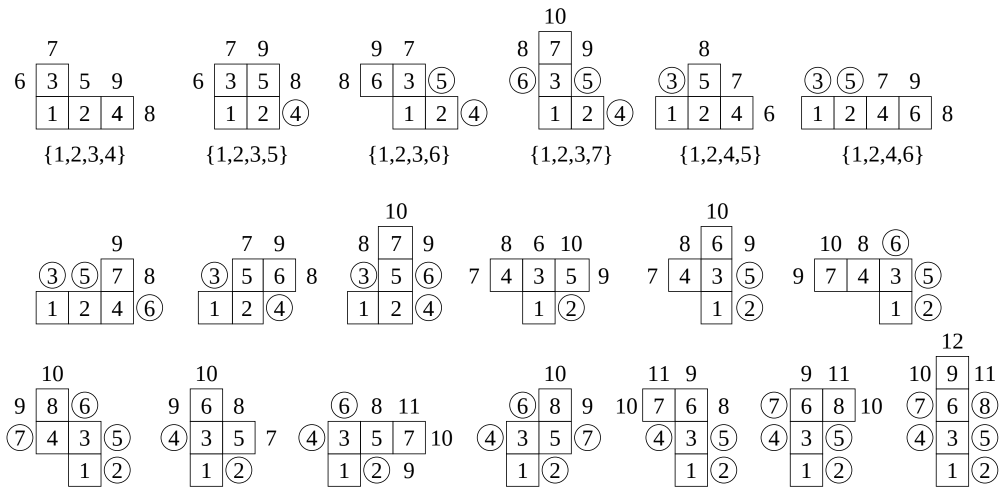
FIGURE 14.4.2



PROBLEM 14.4.3

Is the generating function $T(z) = \sum_{n=1}^{\infty} t(n)z^n$ *a rational function? Is* $T(z)$ *even algebraic?*

14.5 SPECIAL TYPES OF POLYOMINOES

Particular kinds of polyominoes arise in various contexts. We will look at several of the most interesting ones. See more details in a survey by Bousquet-Mélou and Brak [Gut09, §3].

GLOSSARY

A ***composition*** of n with k parts is an ordered k-tuple $(p_1, \ldots, p_k)$ of positive integers with $p_1 + \cdots + p_k = n$.

A ***row-convex*** polyomino: One each of whose horizontal cross-sections is continuous.

A ***column-convex*** polyomino: One each of whose vertical cross-sections is continuous.

A ***convex*** polyomino: A polyomino which is both row-convex and column-convex.

Simply connected polyomino: A polyomino without holes. (Golomb calls these nonholey polyominoes ***profane***.)

A ***width-k*** polyomino: One each of whose vertical cross-sections fits in a $k \times 1$ strip of cells.

A ***directed*** polyomino is defined recursively as follows: Any single cell is a directed polyomino. An $(n+1)$-omino is directed if it can be obtained by adding a new cell immediately above, or to the right of, a cell belonging to some directed n-omino.

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COMPOSITIONS AND ROW-CONVEX POLYOMINOES

There is a natural 1-1 correspondence between compositions of n and a certain class of n-ominoes in standard position, as indicated in Figure 14.5.1 for the case $n = 4$.

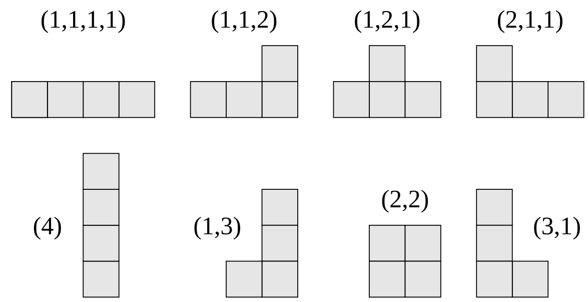


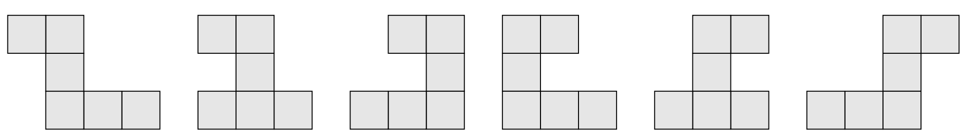
FIGURE 14.5.1
Compositions of 4 *corresponding to certain* 4*-ominoes.*

Let us, instead, assign to each composition $(a_1, \dots, a_k)$ of n an n-omino with a *horizontal* strip of a_i cells in row i. This can be done in many ways, and the results are all the row-convex n-ominoes. Since there are $m + n - 1$ ways to form an $(m+n)$-omino by placing a strip of n cells atop a strip of m cells, it follows that for each composition $(a_1, \dots, a_k)$ of n into positive parts, there are

$$(a_1 + a_2 - 1)(a_2 - a_3 - 1) \cdots (a_{k-1} - a_k - 1)$$

n-ominoes having a strip of a_i cells in the ith row for each i (see Figure 14.5.2 for an example arising from the composition $6 = 3 + 1 + 2$).

FIGURE 14.5.2
The 6 *row-convex* 6*-ominoes corresponding to the composition* $(3, 1, 2)$ *of* 6*.*



It follows that if $b(n)$ is the number of row-convex n-ominoes, then

$$b(n) = \sum (a_1 + a_2 - 1)(a_2 - a_3 - 1) \cdots (a_{k-1} - a_k - 1),$$

where the sum extends over all compositions $(a_1, \dots, a_k)$ of n into k parts, for all k. It is known that $b(n)$, and the generating function $B(z) = \sum_{n=1}^{\infty} b(n) z^n$, are given by

THEOREM 14.5.1 [Kla67]

$b(n + 3) = 5b(n + 2) - 7b(n + 1) + 4b(n)$, and $B(z) = \dfrac{z(1 - z)^3}{1 - 5z + 7z^2 - 4z^3}$.

COROLLARY 14.5.2

$\lim_{n\to\infty}(b(n))^{1/n} = \beta$, *where* β *is the largest real root of* $z^3 - 5z^2 + 7z - 4 = 0$; $\beta \approx 3.20557$.

CONVEX POLYOMINOES

The existence of a generating function for $c(n)$ with special properties [KR74], enabled Bender to prove the following asymptotic formula:

THEOREM 14.5.3 [Ben74]

$c(n) \sim kg^n$, where $k \approx 2.67564$ and $g \approx 2.30914$.

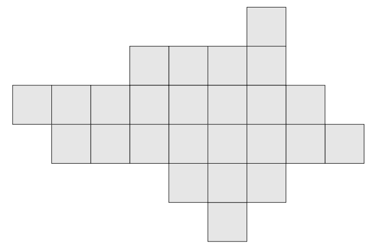


FIGURE 14.5.3
A typical convex polyomino.

The following problem concerns polyominoes radically different from convex ones.

PROBLEM 14.5.4

Find the smallest natural number n_0 such that there exists an n_0-omino with no *row or column consisting of just a single strip of cells.* (An example of a 21-omino with this property is shown in Figure 14.5.4.)

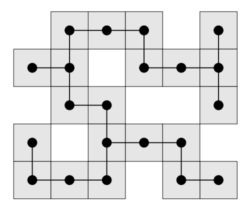


FIGURE 14.5.4
A 21-omino with no row or column a single strip of cells.

PROBLEM 14.5.5

How many polyominoes of size $n \geq n_0$ with the above property exist?

SIMPLY-CONNECTED POLYOMINOES

Simply-connected (profane) polyominoes are the interiors of *self-avoiding* polygons. Counts of these polygons (measured by area) are currently known up to $n = 42$ [Gut09, p. 475]. Let $t^*(n)$, $s^*(n)$, and $r^*(n)$ denote the numbers of profane fixed, free, and chiral n-ominoes, respectively. It is easy to see that $(t^*(n))^{1/n}$, $(s^*(n))^{1/n}$, and $(r^*(n))^{1/n}$ all approach the same limit, λ^*, as $n \to \infty$, and that $\lambda^* \leq \lambda$ $(= \lim_{n\to\infty}(t(n))^{1/n}$ as defined in Section 14.3). Van Rensburg and Whittington [RW89, Thm. 5.6] showed that $\lambda^* < \lambda$.

WIDTH-k POLYOMINOES

A typical width-3 polyomino is shown in Figure 14.5.5.

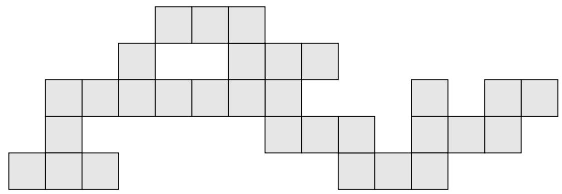


FIGURE 14.5.5
A width-3 polyomino.

THEOREM 14.5.6 [Rea62]

Let $t(n,k)$ be the number of fixed width-k n-ominoes, and $T_k(z) = \sum_{n=1}^{\infty} t(n,k)z^n$. Then $T_k(z) = P_k(z)/Q_k(z)$ for some polynomials $P_k(z), Q_k(z)$ with integer coefficients, no common zeroes, and $Q_k(0) = 1$. Equivalently, the sequence $t(n,k)$, $n = 1, 2, \ldots$, satisfies a linear, homogeneous difference equation with constant coefficients for each fixed k; the order of the equation is roughly 3^k. Furthermore, the sequence $(t(n,k))^{1/n}$ converges to a limit τ_k as $n \to \infty$, and $\lim_{k\to\infty} \tau_k = \lambda$ (see Section 14.3).

For example, for the fixed width-2 n-ominoes (shown in Figure 14.5.6 for small values of n), we have

$$T_2(z) = \frac{z}{1 - 2z - z^2} = z + 2z^2 + 5z^3 + 12z^4 + \ldots,$$

and $t(n+2,2) = 2t(n+1,2) + t(n,2)$ for $n \geq 1$.

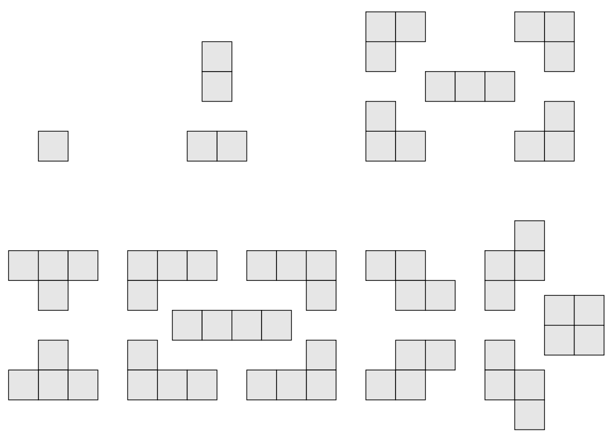


FIGURE 14.5.6
Width-2 n-ominoes for $n = 1, 2, 3, 4$.

DIRECTED POLYOMINOES

A portion of the family tree for directed polyominoes, constructed similarly to the one in Figure 14.4.1, is shown in Figure 14.5.7. As in Section 14.4, codewords can be defined for directed polyominoes, and converted into binary words. Let $\mathcal{V}$ be the language formed by all of these.

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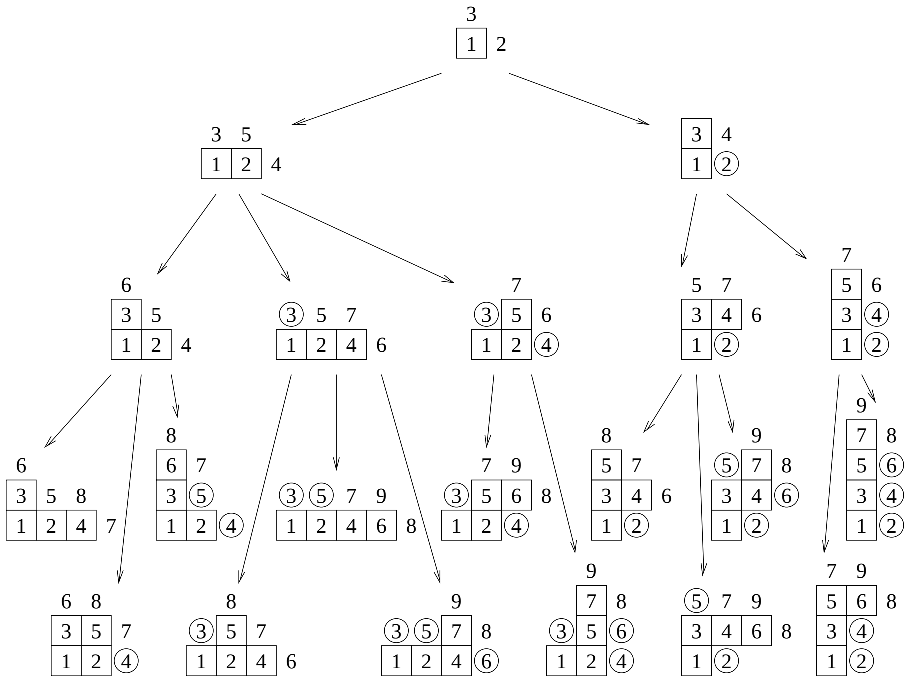
FIGURE 14.5.7
A family tree for fixed directed polyominoes.



PROBLEM 14.5.7

Characterize the words in $\mathcal{V}$. In particular, is $\mathcal{V}$ an unambiguous context-free language?

THEOREM 14.5.8 [Dha82]; see also [Bou94]

If $d(n)$ is the number of directed n-ominoes in standard position, and $D(z) = \sum d(n)z^n$, then

$$D(z) = \frac{1}{2}\left(\sqrt{\frac{1+z}{1-3z}} - 1\right).$$

COROLLARY 14.5.9

$$d(n) = \sum_{k=0}^{n-1} \binom{k}{\lfloor k/2 \rfloor}\binom{n-1}{k},$$

and $d(n)$ satisfies the recurrence relation

$$d(n) = 3^{n-1} - \sum_{k=1}^{n-1} d(k)d(n-k),$$

which can be represented also as

$$d(n) = (3(n-2)d(n-2) + 2nd(n-1))/n \quad \textit{with} \quad d(1) = 1,\ d(2) = 2.$$

14.6 TILING WITH POLYOMINOES

We consider the special case of the tiling problem (see Chapter 3) in which the space we wish to tile is a set S of cells in the plane and the tiles are polyominoes. Usually S will be a rectangular set.

GLOSSARY

π-type: If S is a finite set of cells, $\mathcal{C}$ a collection of subsets of S, $\pi = (S_1, \ldots, S_k)$ a partition (or cover) of S, and $T \subset S$, the π-type of T is defined as

$$\tau(\pi, T) = (|S_1 \cap T|, \ldots, |S_k \cap T|).$$

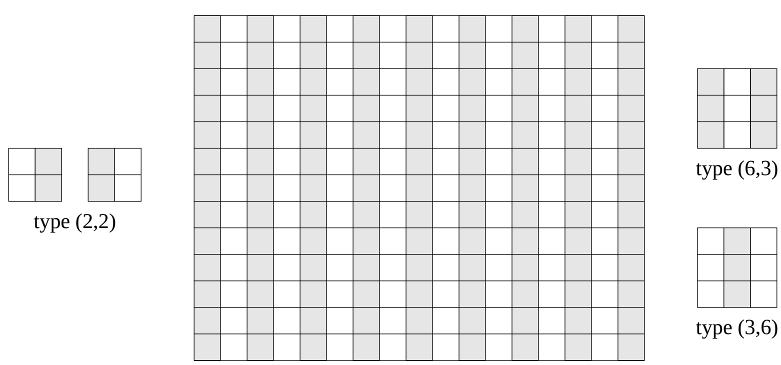
Basis: If every rectangle in a set R can be tiled with translates of rectangles belonging to a finite subset $B \subset R$, and if B is minimal with this property, B is called a basis of R.

THEOREM 14.6.1 [Kla70]

Suppose S is a finite set and $\mathcal{C}$ a collection of subsets of S. Then, $\mathcal{C}$ tiles S if and only if, for every partition (or cover) π of S, $\tau(\pi, S)$ is a non-negative integer combination of the types $\tau(\pi, T)$ where T ranges over $\mathcal{C}$.

For example, one can use this to show that a 13×17 rectangular array of squares cannot be tiled with 2×2 and 3×3 squares: Let π be the partition of the 13 array S into "black" and "white" cells shown in Figure 14.6.1, and $\mathcal{C}$ the set of all 2×2 and 3×3 squares in S.

FIGURE 14.6.1
A coloring of the 13×17 rectangle.



Then, each 2×2 square in $\mathcal{C}$ has type $(2, 2)$, while the 3×3 squares have types $(6, 3)$ and $(3, 6)$. If a tiling were possible, with x 2×2 squares, and with y_1 and y_2 3×3 squares of types $(6, 3)$ and $(3, 6)$ (respectively), then we would have

$$(9 \cdot 13, 8 \cdot 13) = x(2, 2) + y_1(6, 3) + y_2(3, 6),$$

which gives $13 = 3(y_1 - y_2)$, a contradiction.

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THEOREM 14.6.2

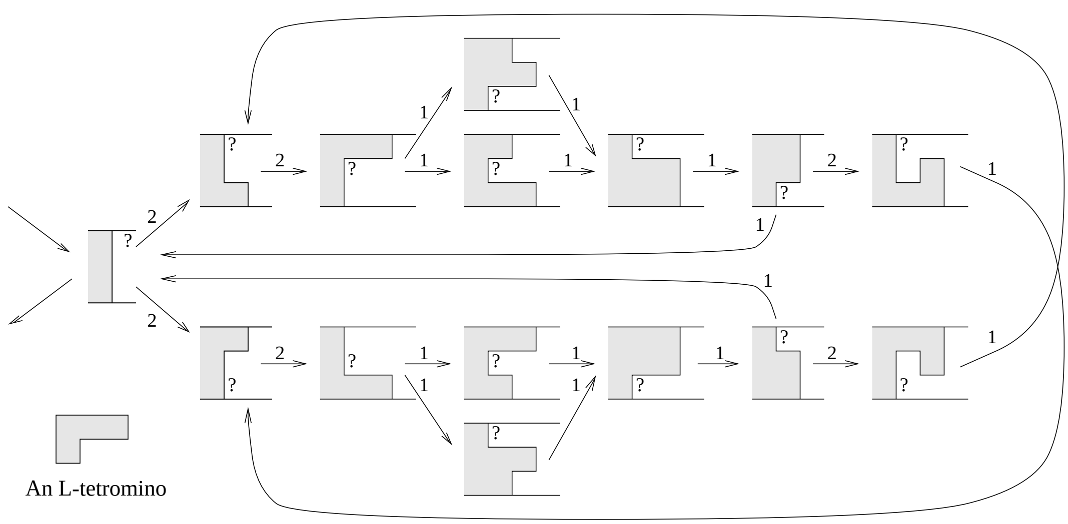
Let $\mathcal{C}$ be a finite union of translation classes of polyominoes, and let w be a fixed positive integer. Then, one can construct a finite automaton that generates all $\mathcal{C}$-tilings of $w \times n$ rectangles for all possible values of n.

COROLLARY 14.6.3

If w is fixed and $\mathcal{C}$ is given, then it is possible to decide whether there exists some n for which $\mathcal{C}$ tiles a $w \times n$ rectangle.

For example, if we want to tile a $3 \times n$ rectangle with copies of the L-tetromino shown in Figure 14.6.2 in all eight possible orientations, the automaton of Figure 14.6.2 shows that it is necessary and sufficient for n to be a multiple of 8.

FIGURE 14.6.2
An automaton for tiling a $3 \times n$ rectangle with L-tetrominoes.



THEOREM 14.6.4 [KG69, BK75]

Let R be an infinite set of oriented rectangles with integer dimensions. Then, R has a finite basis.

(This theorem, which was originally conjectured by F. Göbel, extends to higher dimensions as well [BK75].)

For example, let R be the set of all rectangles that can be tiled with the L-tetromino of Figure 14.6.2, and let $B = \{2 \times 4, 4 \times 2, 3 \times 8, 8 \times 3\} \subset R$. Then, one can show the following three facts:

(a) R is the set of all $a \times b$ rectangles with $a, b > 1$ and $8|ab$;
(b) B is a basis of R;
(c) Each member of B is tilable with the L-tetromino.

PROBLEM 14.6.5

The smallest rectangle that can be tiled with the Y-pentomino (see Figure 14.6.3) *is 5×10. Find a basis B for the set R of all rectangles that can be tiled with Y-pentominoes.*

Reid [Rei05] showed that the cardinality of the basis of the Y-pentomino (Problem 14.6.5) is 40. In addition, he proved that there exist polyominoes with arbitrarily large bases.

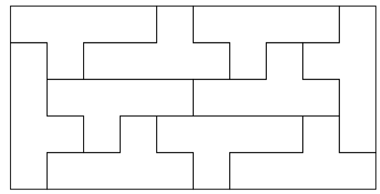


FIGURE 14.6.3
A 5×10 rectangle tiled with Y-pentominoes.

14.7 RECTANGLES OF POLYOMINOES

Here we consider the question of which polyomino shapes have the property that some finite number of copies, allowing all rotations and reflections, can be assembled to form a rectangle. Klarner [Kla69] defined the ***order*** of a polyomino P as the minimum number of congruent copies of P that can be assembled (allowing translation, rotation, and reflection) to form a rectangle. For those polyominoes that will not tile any rectangle, the order is undefined. (A polyomino has order 1 if and only if it is itself a rectangle.)

A polyomino has order 2 if and only if it is "half a rectangle," since two identical copies of it must form a rectangle. This necessarily means that the two copies will be 180° rotations of each other when forming a rectangle. Some examples are shown in Figure 14.7.1.

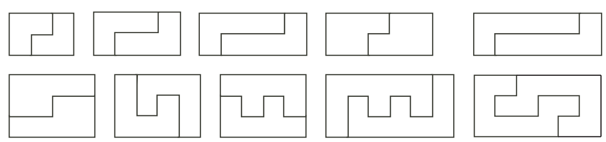


FIGURE 14.7.1
Some polyominoes of order 2.

There are no polyominoes of order 3 [SW92]. In fact, the only way any rectangle can be divided up into three identical copies of a "well-behaved" geometric figure is to partition it into three *rectangles* (see Figure 14.7.2), and by definition a rectangle has order 1.

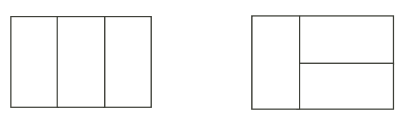


FIGURE 14.7.2
How three identical rectangles can form a rectangle.

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There are various ways in which four identical polyominoes can be combined to form a rectangle. One way, illustrated in Figure 14.7.3, is to have four 90° rotations of a single shape forming a square. Another way to combine four identical shapes to form a rectangle uses the fourfold symmetry of the rectangle itself: left-right, up-down, and 180° rotational symmetry. Some examples of this appear in Figure 14.7.4. More complicated order-4 patterns were found by Klarner [Kla69].

FIGURE 14.7.3
Polyominoes of order 4 under 90° rotation.

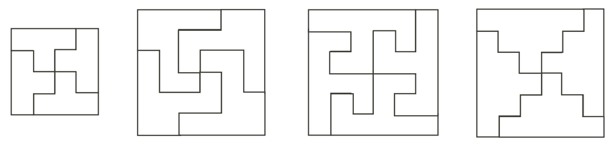
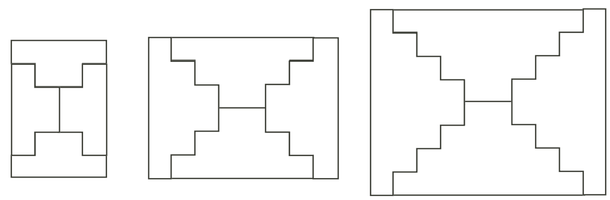
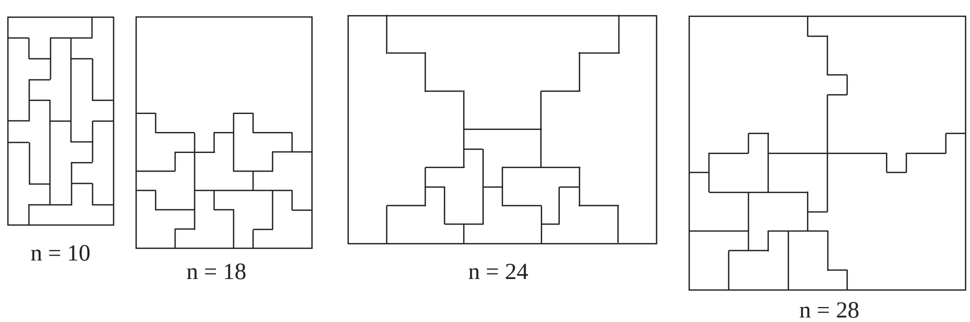


FIGURE 14.7.4
Polyominoes of order 4 under rectangular symmetry.



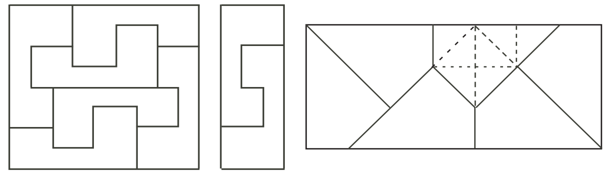
Beyond order 4, there is a systematic construction [Gol89] that gives examples of order $4s$ for every positive integer s. Isolated examples of polyominoes with orders of the form $4s+2$ are also known. Figure 14.7.5 shows examples of order 10 [Gol66] and orders 18, 24, and 28 [Kla69]; see also Marshall [Mar97]. More examples are found in the Polyominoes chapter of the second edition of this handbook.

FIGURE 14.7.5
Four "sporadic" polyominoes of orders 10, 18, 24, and 28, respectively.



No polyomino whose order is an odd number greater than 1 has ever been found, but the possibility that such polyominoes exist (with orders greater than 3) has never been ruled out. The smallest even order for which no example is known is 6. Figure 14.7.6 shows one way in which six copies of a polyomino can be fitted together to form a rectangle, but the polyomino in question (as shown) actually has order 2.

FIGURE 14.7.6
A 12-omino of order 2 that suggests an order-6 tiling, and Michael Reid's order-6 "heptabolo" (a figure made of seven congruent isosceles right triangles). Is there any polyomino *of order 6?*



Yang [Yan14] proved that *rectangular tileability*, the problem of whether or not a given set of polyominoes tiles some (possibly large) rectangle, is undecidable. However, the status of rectangular tileability for a single polyomino is unknown.

PROBLEM 14.7.1

Given a polyomino P, is there a rectangle which P tiles?

14.8 HIGHER DIMENSIONS

GLOSSARY

Polycube: In higher dimensions, the generalization of a polyomino is a d-dimensional *polycube*, which is a connected set of cells in the d-dimensional cubical lattice, where connectivity is through $(d{-}1)$-dimensional faces of the cubical cells.

Proper polycube: A polycube P is *proper* in d dimensions if it spans d dimensions, that is, the convex hull of the centers of all cells of P is d-dimensional.

Counts of polycubes were given by Lunnon [Lun75], Gaunt et al. [GSR76, Gau80], and Barequet and Aleksandrowicz [AB09a, AB09b]. See sequences A001931 and A151830–35 for counts of polycubes in dimensions 3 through 9, respectively, in the OEIS [oeis]. The most comprehensive counts to date were obtained by a parallel version of Redelmeier's algorithm adapted to dimensions greater than 2.

Similarly to two dimensions, let λ_d denote the growth constant of polycubes in d dimensions. It was proven [BBR10] that $\lambda_d = 2ed - o(d)$, where e is the base of natural logarithms, and evidence shows that $\lambda_d \sim (2d-3)e + O(1/d)$ as $d \to \infty$. Gaunt and Peard [GP00, p. 7521, Eq. (3.8)]) provide the following *semi-rigorously* proved expansion of λ_d in $1/d$:[2]

$$\lambda_d \sim 2ed - 3e - \frac{31e}{48d} - \frac{37e}{16d^2} - \frac{279613e}{46080d^3} - \frac{325183e}{10240d^4} - \frac{54299845e}{2654208d^5} + O\left(\frac{1}{d^6}\right).$$

PROBLEM 14.8.1

Find a general formula for λ_d or a generating function for the sequence of coefficients of the expansion above.

Following Lunnon [Lun75], let $\mathrm{CX}(n,d)$ denote the number of fixed polycubes

[2] The reference above provides a much more general formula. To obtain the expansion for the model of strongly embedded site animals (see below), one needs to substitute in the cited formula $y := 1$, $z := 0$, and $\sigma := 2d-1$, take the exponent of the formula, and expand the result as a power series of $1/d$, i.e., around infinity.

of size n in d dimensions, and let $\mathrm{DX}(n,d)$ denote the number of those of them that are also proper in d dimensions. Lunnon [Lun75] observed that $\mathrm{CX}(n,d) = \sum_{i=0}^{d} \binom{d}{i} \mathrm{DX}(n,i)$. Indeed, every d-dimensional polycube is proper in some $0 \le i \le d$ dimensions (the singleton cube is proper in zero dimensions!), and then these i dimensions can be chosen in $\binom{d}{i}$ ways. However, a polycube of size n cannot obviously be proper in more than $n-1$ dimensions. Hence, this formula can be rewritten as

$$\mathrm{CX}(n,d) = \sum_{i=0}^{\min(n-1,d)} \binom{d}{i} \mathrm{DX}(n,i).$$

Assume now that the value of n is fixed, and let $\mathrm{CX}_n(d)$ denote the number of fixed d-dimensional polycubes of size n, considered as a function of d only. A simple consequence [BBR10] of Lunnon's formula is that $\mathrm{CX}_n(d)$ is a polynomial in d of degree $n-1$. The first few polynomials are $\mathrm{CX}_1(d) = 1$, $\mathrm{CX}_2(d) = d$, $\mathrm{CX}_3(d) = 2d^2 - d$, $\mathrm{CX}_4(d) = \frac{16}{3}d^3 - \frac{15}{2}d^2 + \frac{19}{6}d$, $\mathrm{CX}_5(d) = \frac{50}{3}d^4 - 42d^3 + \frac{239}{6}d^2 - \frac{27}{2}d$, and so on. It was also shown that the leading coefficient of $\mathrm{CX}_n(d)$ is $2^{n-1}n^{n-3}/(n-1)!$.

Interestingly, there is a pattern in the "diagonal formulae" of the form $\mathrm{DX}(n, n-k)$, for small values of $k \ge 1$. Using Cayley trees, it is easy to show that $\mathrm{DX}(n, n-1) = 2^{n-1}n^{n-3}$ (sequence A127670 in the OEIS [oeis]). Barequet et al. [BBR10] proved that $\mathrm{DX}(n, n-2) = 2^{n-3}n^{n-5}(n-2)(2n^2-6n+9)$ (sequence A171860), and Asinowski et al. [ABBR12] proved that $\mathrm{DX}(n, n-3) = 2^{n-6}n^{n-7}(n-3)(12n^5 - 104n^4 + 360n^3 - 679n^2 + 1122n - 1560)/3$ (sequence A191092). Barequet and Shalah [BS17] provided computer-generated proofs for the formulae for $\mathrm{DX}(n, n-4)$ and $\mathrm{DX}(n, n-5)$, as well as a complex recipe for producing these formulae for any value of k. Formulae for $k = 6$ and $k = 7$ were conjectured by Peard and Gaunt [PG95] and by Luther and Mertens [LM11], respectively.

14.9 MISCELLANEOUS

COUNTING BY PERIMETER

Polyominoes are rarely counted by *perimeter* instead of by area. Delest and Viennot [DV84], and Kim [Kim88], proved by completely different methods that the number of convex polyominoes with perimeter $2(m+4)$ (for $m \ge 0$) is

$$(2m+11)4^m - 4(2m+1)\binom{2m}{m}.$$

OTHER MODELS OF ANIMALS

In the literature of statistical physics, polyominoes are referred to as *strongly embedded site animals*. This terminology comes from considering the dual graph of the lattice. When switching to the dual setting, that is, to the cell-adjacency graph, polyomino cells turn into vertices (sites) and adjacencies of cells turn into edges (bonds) of the graph. In the dual setting, connected sets of sites are called *site animals*. Instead of counting animals by the number of their sites, one can count them by the number of their bonds, in which case they are called *bond animals*. The term "strongly embedded" refers to the situation in which if two neighboring sites belong to the animal, then the bond connecting them must also belong to the animal. If this restriction is relaxed, then *weakly-embedded* animals are considered. These extensions have applications in computational chemistry; see, for example, the book by Vanderzande [Van98].

NONCUBICAL LATTICES

In the plane, one can also consider *polyhexes* and *polyiamonds*, which are connected sets of cells in the hexagonal and triangular lattices, respectively. Algorithms and counts for polyhexes and polyiamonds were given by Lunnon [Lun72], Barequet and Aleksandrowicz [AB09a], and Vöge and Guttmann[VG03]. Counts of polyhexes and polyiamonds are currently known up to sizes 46 and 75, respectively [Gut09, pp. 477 and 479]. See also sequences A001207 and A001420, respectively, in the OEIS [oeis].

14.10 SOURCES AND RELATED MATERIAL

FURTHER READING

An excellent introductory survey of the subject, with an abundance of references, is by Golomb [Gol94]. Another notable book on polyominoes is by Martin [Mar91]. A deep and comprehensive collection of essays on the subject is edited by A.J. Guttmann [Gut09]. Finally, there are many articles, puzzles, and problems concerning polyominoes to be found in the magazine *Recreational Mathematics*.

RELATED CHAPTERS

Chapter 3: Tilings

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