Gossips and Telephones*

Brenda Baker and Robert Shostak†

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The following problem has circulated lately among mathematicians. Other solutions have been given independently by R. T. Bumby and by A. Hajnal, E. C. Milner and E. Szemerédi.

The problem. There are n ladies, and each of them knows some item of gossip not known to the others. They communicate by telephone, and whenever one lady calls another, they tell each other all that they know at that time. How many calls are required before each gossip knows everything?

Answer. Let $f(n)$ be the minimum number of calls needed for n people. It is easily shown that $f(1) = 0$, $f(2) = 1$, $f(3) = 3$ and $f(4) = 4$. For $n > 4$, $2n - 4$ calls are sufficient according to the following procedure: one of four "chief" gossips first calls each of the remaining $n-4$ gossips, then the four learn each other's (and hence everyone's) information in 4 calls (as $f(4) = 4$), and finally one of the four chiefs calls each of the other $n - 4$ gossips.

Theorem. $f(n) = 2n - 4$ for $n > 4$.

Proof. Suppose to the contrary that for some $n > 4$, $f(n) \leq 2n-5$. Let m be the least such n and let S be any calling arrangement among m gossips requiring at most $2m - 5$ calls. We will obtain a contradiction based upon the following lemma.

Lemma. No gossip may hear her own information from another in S.

Proof. If gossip G can hear her own information in S, there is a sequence of calls $(G - G_1)(G_1 - G_2) \cdots (G_r - G)$, listed in temporal order. By omiting gossip G, we obtain from S a new arrangement T of calls among $m - 1$ gossips as follows:

Omit from S calls $(G - G_1)$ and $(G_r - G)$. In addition, for each gossip P who makes a call $(P - G)$ (other than $(G - G_1)$ and $(G_r - G)$ of the above sequence) in S, let the G_i's transmit P's information in T as follows: Let t be the least i such that $(G_i - G_{i+1})$ occurs after $(P - G)$ in S, if such a call exists, and r otherwise. Replace $(P - G)$ in S by a call $(P - G_t)$ in T, preserving the temporal ordering of the calls. The arrangement T will contain at most $2(m - 1) - 5$ calls and it is easily verified that all ladies learn all of the gossip in T if they do so in S. The lemma follows by minimality of m.

Conclusion of the proof of the Theorem. By the Lemma, a call is either the final call for both parties to the call in S or it is the final call for neither (for after receiving gossip B's final call, gossip A knows everything and would violate the Lemma by later calling someone else). Also, a call is either the initial call for both parties or for neither (otherwise, if A makes her first call to B after B calls C, then information from C would propagate with A's until it came back to C, contradicting the Lemma).

Thus initial and final calls account for m calls. (Clearly a call cannot be both initial and final.) Let the remaining calls be described as intermediate calls and let I be a graph with m nodes representing gossips and edges representing intermediate calls. Since there are at most $m - 5$ intermediate calls and since $m - 1$ edges are needed for a graph of m nodes to be connected, the graph I must contain at least five disjoint connected components. Information from a given gossip G can propagate into only two components (hers und her initial caller's) before any final calls are made. Similarly, after the initial calls have been made, information may be transmitted to her through the calls of only two components (hers and her final caller's). Thus the calls of at least two components of intermediate calls play no part in propagating her information or informing her. For a gossip G, let $c(G)$ be the number of calls which are not used in transmitting information to her or from her.

At least $n - 1$ calls are required to inform a given gossip completely and $n - 1$ are required to transmit her information. By the Lemma the only calls which can do both are those that she herself takes part in – otherwise she would hear her information from another. Thus at least $2n - 2 - v(G)$ calls are required to convey gossip G's information and inform her, where $v(G)$ is the number of calls in which she participates. From $2n - 5 \geq 2n - 2 - v(G) + c(G)$ we conclude that $v(G) \geq 3 + c(G) \geq 3$. Since $v(G) \geq 3$ for each gossip G, every connected component of intermediate calls contains an edge – otherwise some gossip would make only an initial and a final call. According to the previous paragraph, the calls of at least two components are not used in transmitting information to or from G. Hence $c(G) \geq 2$ and $v(G) \geq 5$ for all gossips G, resulting in more than 2n calls altogether.

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† Both authors are NSF-Pre-Doctoral Fellows at Harward University

On A Telephone Problem

R. Tijdeman*

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The problem due to A. Boyd is as follows:

There are n ladies, and each of them knows some item of scandal which is not known to any of the others. They communicate by telephone, and whenever two ladies make a call, they pass on to each other, as much scandal as they know at that time. How many calls are needed before all ladies know all the scandal?

Professor E. A. J. M. Wirsing remarked that this problem is equivalent to the following one:

A $n \times n$ telephone matrix $(a_{ij})_{i,j=1}^{n}$ is a matrix of the following form: $a_{ii} = 1$ for $i = 1, \ldots, n$, there exists a pair $i_0 \neq j_0$ such that $a_{i_0 j_0} = a_{j_0 i_0} = 1$ and $a_{ij} = 0$ for all other values of i and j. How many $n \times n$ telephone matrices are needed in order that the product matrix has no entry equal to zero?

This problem has been solved independently by A. Hajnal, E. C. Milner and E. Szemerédi, by R. T. Bumby and by the author. All three solutions are quite different. In the first mentioned solution only the order in which calls are made is changed. The second uses the idea of a proxy. The third is done by identification and interchange of ladies and is given here. I thank Dr. A. J. Jones and Dr. H. L. Montgomery for the help they have given me in solving the problem and preparing this paper.

1. Let $f(n)$ be the number of calls needed for n ladies to know all the scandal. It is easy to see that $f(1) = 0$, $f(2) = 1$, $f(3) = 3$, $f(4) = 4$. Denoting ladies by A, B, C and D a solution for $n = 4$ is given by $A - B$, $C - D$, $A - C$, $B - D$.

An ordered list of telephone calls is called *complete* if after all calls have been made each person knows all the information. Let L be a complete list for n ladies $A_1, \ldots, A_n$. A complete list for these n ladies and a new lady A_{n+1} is given by the list $A_1 - A_{n+1}, L, A_1 - A_{n+1}$. Hence $f(n+1) \leq f(n) + 2$. Since $f(4) = 4$, this implies $f(n) \leq 2n - 4$ for all $n \geq 4$. The object is to prove

Theorem. $f(n) = 2n - 4$ for $n \geq 4$.

This seems rather wasteful, since the same result can be attained by $2n - 2$ letters (or "polarized telephones").

2. Ladies will be given by A, B, A_1, A_2 etc., their initial piece of information by a, b, a_1, a_2 etc., lists of telephone calls by L, L', etc. Let $\mathfrak{L}_n$ be the set of all complete lists on n people of length $f(n)$.

An *identification of A and B in a list* L is a modification of the list as follows: Calls between A and B are omitted. A and B are replaced in the list by a single person denoted by $AB(= BA)$ whose initial information is $ab(= ba)$.

An interchange of A and B (from some point onwards) in a list L is a modification of L as follows: From the point indicated to the end of the list the letter A is replaced by the letter B amd vice–versa.

The abbreviations i.b. and i.a. will be used to mean *immediately before* (some call of the list) or *immediately after*, respectively.

To prove the theorem we assume that $N > 4$, that the theorem is true for $4 \leq n < N$ and that $f(N) < 2N - 4$. We start with some lemmas.

3. **Lemma 1.** If in L the ladies A and B are identified to form a list L' then at any point in L' AB knows at least all of what A and B knew separately at the corresponding point in L. Also, any other person in L' knows at least as much as she knows at the same time in L. In particular the list L' is complete if L is complete.

Proof. By induction on the calls of L.

Lemma 2. If $L \in \mathfrak{L}_N$ then any two ladies call each other at most once.

Proof. If A and B call each other twice, then identify A and B. Then by Lemma 1 L' is a complete list on $N - 1$ ladies, containing $f(N) - 2 < 2N - 6 = 2(N - 1) - 4$ calls thereby contradicting the minimality of N.

Corollary. For $L \in \mathfrak{L}_N$ we may speak of *the* call between A and B, which we shall denote by $\gamma(A, B)(= \gamma(B, A))$.

Lemma 3. If two ladies in a complete list L are interchanged at a time when they have precisely the same information, then the new list L' is also complete.

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Proof. Clear, since at the time of the interchange the two ladies are indistinguishable.

Lemma 4. If $\gamma(A, B)$ is a call of $L \in \mathcal{L}_N$ then i.b. $\gamma(A, B)$ A and B have no common information.

Proof. Suppose A and B have common information c. Originally this information was known only to C. Working backwards through the list L either we can construct two sequences

$$\begin{cases} A = A_0 \text{ learned } c \text{ from } A_1,\ A_1 \text{ from } A_2, \ldots, A_{k-1} \text{ from } A_k,\ A_k \text{ from } D, \\ B = B_0 \text{ learned } c \text{ from } B_1,\ B_1 \text{ from } B_2, \ldots, B_{l-1} \text{ from } B_l,\ B_l \text{ from } D, \end{cases} \tag{1}$$

where by hypothesis D is the only person common to both sequences, or we construct one sequence

$$A = A_0 \text{ learned } c \text{ from } A_1,\ A_1 \text{ from } A_2, \ldots, A_{k-1} \text{ from } A_k,\ A_k \text{ from } B = D \tag{2}$$

(where it may be necessary to interchange A and B from the beginning of L), and $A_i \neq A_j$ $(0 \leq i < j \leq k)$.

If (1) holds we may, without loss of generality, assume that $\gamma(A_k, D)$ occurs before $\gamma(B_l, D)$. Interchange B_l and D i.a. $\gamma(B_l, D)$, subsequently interchange B_{l-1} and D i.a. $\gamma(B_{l-1}, D)$ and so forth. Finally we interchange B and D i.a. $\gamma(B, D)$. Notice that this series of interchanges has not affected any of the $\gamma(A_j, A_{j+1})$ calls of the list.

In both cases (1) and (2) we subsequently put $E = A_k$ and interchange A_{k-1} and E i.a. $\gamma(A_{k-1}, E)$, A_{k-2} and E i.a. $\gamma(A_{k-2}, E), \ldots, A_1$ and E i.a. $\gamma(A_1, E)$, A and E i.a. $\gamma(A, E)$.

Since the initial list L was complete, after each interchange, by Lemma 3, each new list is complete. Moreover since the process of interchange leaves the number of calls and the number of people invariant the final list is in $\mathcal{L}_N$. However by construction the final list contains two calls between D and E which contradicts Lemma 2.

Lemma 5. If $\gamma(A, B)$ is a call of $L \in \mathcal{L}_N$, and if $\gamma(A, B)$ is the last call of A then it is also the last call of B.

Proof. After $\gamma(A, B)$ B knows everything, since $\gamma(A, B)$ is the last call for A. If subsequently B calls C then i.b. $\gamma(B, C)$ B and C have common information c, which contradicts Lemma 4.

4. We can now prove that $\mathcal{L}_N$ is empty. Let $L \in \mathcal{L}_N$. Each person has a last call in L. Denote the set of last calls in L by P. Let $\gamma(A_1, A_2)$ be the final call of L which is not in P (existence obvious). Suppose the last call of A_1 is $\gamma(A_1, A_3) \in P$, of A_2 is $\gamma(A_2, A_4) \in P$. Of course A_1, A_2, A_3 and A_4 are all distinct. It is immaterial whether $\gamma(A_1, A_3)$ is before of after $\gamma(A_2, A_4)$. By Lemma 4 the information known by A_1 and A_2 i.a. $\gamma(A_1, A_2)$ is just complementary to the information known by A_3 and A_4 at that moment. Hence the information known to A_3 and A_4 at that moment is identical.

We may suppose without loss of generality that the penultimate call of both A_3 and A_4 is $\gamma(A_3, A_4)$. For if the penultimate calls differ and the last of these is, say, $\gamma(A_3, A_5)$ then A_3, A_4 and A_5 have exactly the same knowledge i.a. $\gamma(A_3, A_5)$. We now interchange A_4 and A_5 i.a. $\gamma(A_3, A_5)$ and in so doing obtain a new list L', complete by Lemma 3, $L' \in \mathcal{L}_N$, and having the required property.

To summarize we can suppose there are calls of L $\gamma(A_1, A_3) \in P$, $\gamma(A_2, A_4) \in P$ with the previous call of A_1 and A_2 being $\gamma(A_1, A_2) \notin P$ and the previous call of A_3 and A_4 being $\gamma(A_3, A_4) \notin P$. A moment's thought reveals there is no loss of generality in supposing that $\gamma(A_3, A_4)$, $\gamma(A_1, A_2)$, $\gamma(A_1, A_3)$, $\gamma(A_2, A_4)$ are the last four calls of L.

Neither $\gamma(A_3, A_4)$ nor $\gamma(A_1, A_2)$ are the first calls of one of them in L, since $N > 4$ and so there would be no opportunity for them to learn the remaining information. In view of Lemmas 2 and 4 the first calls of A_1, A_2, A_3 and A_4 are made with new people, say B_1, B_2, B_3 and B_4. It is easy to see that B_1, B_2, B_3 and B_4 are distinct and hence $N \geq 8$ (although this is not crucial to the argument). Identify A_1 and B_1, subsequently A_2 and B_2, A_3 and B_3, A_4 and B_4. Then by Lemmas 1 and 2 we have a complete list of calls for $N - 4$ ladies with $f(N) - 4$ calls. Moreover in this new list, by Lemma 1, A_1B_1 etc. knows at least as much as A_1 and B_1 together i.a. the corresponding call in L. But this means that i.b. the last four calls everyone in the new list knows everything, since at that moment B_1, B_2, B_3 and B_4 know everything in L. Hence the last four calls in the new list are superfluous. Thus we have obtained a complete list of calls for $N - 4$ persons with $f(N) - 8 < 2(N - 4) - 4$ calls thereby contradicting the minimality of N.

Remark due to Wirsing. It is an immediate consequence of the second form of the problem that a complete list reversed is a complete list and furthermore the reverse L^{-1} of $L \in \mathcal{L}_N$ also belongs to $\mathcal{L}_N$. A list in which each pair of ladies who have a common call have no information i.b. this call (compare Lemma 4) has a corresponding product matrix only consisting of entries 1.

A Cure For The Telephone Disease

A. Hajnal, E. C. Milner and E. Szemerédi*

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The following problem due to A. Boyd, has enjoyed a certain popularity in recent months with several mathematicians. A different solution to the one given here has been given independently by R. T. Bumby and J. Spencer. (Since this paper was written we have received another solution from R. Tijdeman.)

The Problem. There are n ladies, and each of them knows an item of scandal which is not known to any of the others. They communicate by telephone, and whenever two ladies make a call, they pass on to each other, as much scandal as they know at that time. How many calls are needed before all ladies know all the scandal?

If $f(n)$ is the minimum number of calls needed, then it is easy to verify that $f(1) = 0$, $f(2) = 1$, $f(3) = 3$ and $f(4) = 4$. It is also easy to see that $f(n+1) \leq f(n) + 2$, for the $(n+1)$-th lady first calls one of the others and someone calls her back after the remaining n ladies have communicated all the scandal to each other. It follows that $f(n) \leq 2n - 4$ $(n \geq 4)$. We will prove that

$$f(n) = 2n - 4 \qquad (n \geq 4) \tag{1}$$

We shall represent the n ladies by the set of vertices, V, of a multigraph. A sequence of calls

$$c(1), c(2), \ldots, c(t) \tag{2}$$

between them can be represented by the edges of the multigraph labelled according to the order in which the calls are made.

The interchange rule. Suppose (2) is a given sequence of calls, and suppose that the a calls $c(i)$, $c(i+1)$, $\ldots$, $c(i+a-1)$ are vertex disjoint from the succeeding b calls $c(i+a)$, $c(i+a+1)$, $\ldots$, $c(i+a+b-1)$. Then we can interchange the order of these two blocks of a and b calls, i.e. if we make the same calls as in (2) but in the order

$$c(1), \ldots, c(i-1), c(i+a), \ldots, c(i+a+b-1), c(i), \ldots, c(i+a-1), c(i+a+b), \ldots, c(t)$$

then the total information conveyed is exactly the same as for the sequence (2). If $c'(1)$, $\ldots$, $c'(t)$ is a rearrangement of the sequence (2) obtained by a number of interchanges of adjacent blocks of vertex disjoint calls of the kind just described, we say that c' is an *equivalent* calling system and write $c' \sim c$.

Let (2) be a given sequence of calls. A vertex x of the graph will be called an F-point if the corresponding lady knows everything after the t calls have been made. Obviously, if $c' \sim c$, then the sequence of calls $c'(1)$, $\ldots$, $c'(t)$ gives the same F-points as c. In order that there be any F-points at all, the graph G, with vertex set V and edge set (2), must be connected. Consequently, we have

Lemma 1. There are no F-points after $n-2$ calls.

In order to prove (1) it is enough to prove

Lemma 2. After $n+k-4$ calls there are at most k F-points.

Proof. We shall actually prove the following stronger assertion $P(k)$:

- If $c(1)$, $\ldots$, $c(n+k-4)$ is a sequence of $n+k-4$ calls, then there are at most k F-points.
- Further, if there are k F-points, then there is an equivalent calling sequence $c' \sim c$ in which the last k calls
$$c'(n-3), c'(n-2), \ldots, c'(n+k-4)$$
are all between F-points.

The first part of $P(k)$ follows from Lemma 1 if $k = 0$, 1, or 2 and for these values of k the second part of $P(k)$ is satisfied vacuously. We now assume that $k > 2$ and use induction on k.

Suppose there are $k+1$ F-points after the $n+k-4$ calls. Since the last call $c(n+k-4)$ can produce at most two F-points, it follows from the induction hypothesis that there must be $k-1$ F-points $\{x_1, \ldots, x_{k-1}\}$ after the first $n+k-5$ calls and the last call $c(n+k-4)$ is between two additional F-points $\{x_k, x_{k+1}\}$. By the second part of $P(k-1)$, we can assume that the last $k-1$ calls of the sequence $c(1)$, $\ldots$, $c(n+k-5)$ are between the F-points $\{x_1, \ldots, x_{k-1}\}$. By the interchange rule, the last call $c(n+k-4)$ could be made before $c(n-3)$, $\ldots$, $c(n+k-5)$. It follows that after the $n-3$ calls

$$c(1), c(2), \ldots, c(n-4), c(n+k-4)$$

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there would by two F-points $\{x_k, x_{k+1}\}$ contrary to Lemma 1. This shows that there can be at most k F-points.

To complete the proof we must show that the second part of the inductive statement $P(k)$ holds.

Suppose there are k F-points after the $n+k-4$ calls

$$c(1), c(2), \ldots, c(n+k-4).$$

Consider the disconnected graph G_0 with vertex set V and edge set $E_0 = \{c(1), \ldots, c(n-2)\}$. Suppose G_0 has an isolated vertex x. There are at least $k-1$ F-points $x_i \neq x$ $(1 \leq i < k)$ and each of these is connected to x by a path from the edge set $E_1 = \{c(n-1), \ldots, c(n+k-4)\}$. This implies that the points x, x_i $(1 \leq i < k)$ are in a single component of the graph G_1 on V with edge set E_1. This is impossible since $|E_1| + 1 < k$. Thus G_0 has no isolated vertex and each component of this graph has at least one edge. By the interchange rule, the first $n-3$ calls can be equivalently re-ordered so that the $(n-3)$-rd call is in a different component of G_0 to $c(n-2)$. Therefore, we may assume that $c(n-3)$ and $c(n-2)$ are disjoint.

Now suppose that the last k calls of the given sequence are not all between F-points. Then there is p, $1 \leq p \leq k$, such that the last $p-1$ calls $c(n+k-p-2), \ldots, c(n+k-4)$ are all between F-points but the preceding call, $c(n+k-p-3)$, is adjacent to at most one F-point. In fact, we can assume that $p < k$. For, if $p = k$ we can, by the last paragraph, consider instead the equivalent calling sequence obtained by interchanging $c(n-3)$ and $c(n-2)$.

If $c(n+k-p-3)$ is not adjacent to any F-point, then by the interchange rule, this call could be made last and then there would be k F-points after only $n+k-5$ calls

$$c(1), \ldots, c(n+k-p-4), c(n+k-p-2), \ldots, c(n+k-4).$$

This contradicts the induction hypothesis and so we can assume that $c(n+k-p-3)$ is adjacent to exactly one F-point.

Consider the graph G_2 on V having the p edges $c(n+k-p-3), \ldots, c(n+k-4)$ and let C be the component of this graph containing the edge $c(n+k-p-3)$. Let $\bar{c}(1) = c(n+k-p-3)$, $\bar{c}(2), \ldots, \bar{c}(r)$ be the edges of C in the order in which these calls are made and let $\hat{c}(1), \hat{c}(2), \ldots, \hat{c}(p-r)$ be the remaining edges of G_2 in order. By the interchange rule, $\hat{c}(1)$ can be made before any of the calls in C and similarly for $\hat{c}(2), \ldots, \hat{c}(p-r)$. Thus the original calling sequence is equivalent to the sequence of calls

$$c(1), c(2), \ldots, c(n+k-p-4), \hat{c}(1), \ldots, \hat{c}(p-r), \bar{c}(1), \ldots, \bar{c}(r). \tag{3}$$

Since $\bar{c}(1)$ is adjacent to only one F-point, the component C contains at most r F-points (C has r edges and at most $r+1$ points). It follows that after the first $n+k-r-4$ calls in the sequence (3), there are at least $k-r$ F-points. Therefore, by the induction hypothesis there must be exactly $k-r$ such F-points (and the component C contains exactly r F-points) and there is an equivalent re-ordering of these $n+k-r-4$ calls so that the last $k-r$ are between the $k-r$ F-points not in C. In this way we obtain an equivalent calling sequence, say

$$c_1(1), \ldots, c_1(n+k-r-4), \bar{c}(1), \ldots, \bar{c}(r). \tag{4}$$

Since the $k-r$ calls $c_1(n-3), \ldots, c_1(n+k-r-4)$ are vertex disjoint from $\bar{c}(1), \ldots, \bar{c}(r)$ (they are between F-points not in C) it follows, again by the interchange rule, that an equivalent sequence is

$$c_1(1), \ldots, c_1(n-4), \bar{c}(1), \ldots, \bar{c}(r), c_1(n-3), \ldots, c_1(n+k-r-4). \tag{5}$$

The first $n-4+r$ calls in the sequence (5) give rise to the r F-points in C. Therefore, by the induction hypothesis, these calls can be rearranged so that the last r calls are between F-points. After re-ordering the first $n+r-4$ calls of (5) in this way we obtain an equivalent calling system $c' \sim c$ in which the last k calls are all between F-points. This completes the proof of Lemma 2.

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