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# Integer parts of powers of rational numbers

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**Abstract.** We prove that the sequence $[\xi(5/4)^n]$, $n=1,2,\dots$, where $\xi$ is an arbitrary positive number, contains infinitely many composite numbers. A corresponding result for the sequences $[(3/2)^n]$ and $[(4/3)^n], n=1,2,\dots$, was obtained by Forman and Shapiro in 1967. Furthermore, it is shown that there are infinitely many positive integers $n$ such that $([\xi(5/4)^n], 6006) > 1$, where $6006 = 2\cdot 3\cdot 7\cdot 11\cdot 13$. Similar results are obtained for shifted powers of some other rational numbers. In particular, the same is proved for the sets of integers nearest to $\xi(5/3)^n$ and to $\xi(7/5)^n$, $n\in\mathbb{N}$. The corresponding sets of possible divisors are also described.

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## 1. Introduction

There are many unsolved problems concerning the distribution of the fractional parts of powers of a rational number $a > 1$. The sequence $\{a^n\}$, $n = 1, 2, \dots$, and, more generally, the sequence $\{\xi a^n\}$, $n = 1, 2, \dots$, where $\xi$ is a fixed positive number, was studied by Mahler [11], Vijayaraghavan [14], Flatto, Lagarias and Pollington [7], Bugeaud [3] and others. Even the simplest case of $a = 3/2$ is far from being understood; its importance is usually motivated by a remarkable connection between the distribution of $\{(3/2)^n\}$, $n = 1, 2, \dots$, and Waring's problem. (See, for instance, [13].)

In this paper, we are interested in a problem for which the cases $a = 3/2$ and $a = 4/3$ were successfully treated almost forty years ago by Forman and Shapiro [8], but no progress has been made since then. This at first glance simple problem can be stated as follows: prove that for every rational $a > 1$ the sequence of integer parts $[a^n]$ contains infinitely many composite numbers. (This problem is trivial for integer $a$.)

Some metrical results are well-known. Koksma [10] proved that the sequence $\{\xi a^n\}$, $n = 1, 2, \ldots$, where $\xi > 0$, is uniformly distributed in $[0, 1]$ for almost all $a > 1$. This implies that, for almost all $a > 1$, $[a^n]$ are composite for infinitely many $n$ (see [8]). Baker and Harman [2] obtained other metrical results in this direction. Unfortunately, it is impossible to apply these results to rational numbers, because the set of rational numbers is of measure zero. Also very few explicit irrational $a > 1$ producing infinitely many composite numbers are known. Cass [4] proved that the set $[a^n]$, $n \in \mathbb{N}$, contains infinitely many composite numbers if $a > 1$ is a unit in a real quadratic number field. This result was extended by the first named author [5] to all Pisot and Salem numbers $a$. (Every real quadratic unit is a Pisot number of degree 2. See also [6] for a generalization.). Some explicit transcendental $a > 1$ for which $[a^n]$ are composite infinitely often were constructed in [1].

Clearly, if one could prove that the distance between the largest limit point of $\{(\xi/2)a^n\}$, $n \in \mathbb{N}$, and the smallest one is greater than $1/2$, then the smallest limit point is smaller than $1/2$. This would imply that there are infinitely many even numbers among $[\xi a^n]$, $n \in \mathbb{N}$. However, it is only known [7] that this distance is $\geqslant 1/b$, where $b$ is the numerator of $a = b/c \in \mathbb{Q}$, $b > c > 1$, $(b, c) = 1$, which, although being a remarkable result itself, is of no use to the problem initiated in [8].

Finally, there is not too much information about $a > 1$ for which $[a^n]$ is prime for infinitely many integers $n$. See, for instance, [1], [2], [11], [15] for some existence results in this direction. According to a conjecture of Whiteman (see Problem E19 in [9]) the sequence $[a^n]$, $n = 1, 2, \ldots$, where $a > 1$ is a rational noninteger number, contains infinitely many primes. However no results are known to confirm this statement.

We begin with our main theorem. Set $\mathcal{P}(2) = \{2\}$, $\mathcal{P}(3) = \mathcal{P}(4) = \{2, 3\}$, $\mathcal{P}(6) = \mathcal{P}(4/3) = \{2, 3, 5\}$, $\mathcal{P}(3/2) = \{2, 5, 7, 11\}$, $\mathcal{P}(5/4) = \{2, 3, 7, 11, 13\}$.

**Theorem 1.** *Let $\xi > 0$ be a real number and let $a \in \{2, 3, 4, 6, 3/2, 4/3, 5/4\}$. Then the set $[\xi a^n]$, $n \in \mathbb{N}$, contains infinitely many elements divisible by at least one number of the set $\mathcal{P}(a)$.*

In the above mentioned paper [8] Forman and Shapiro proved that the sets $[(3/2)^n]$ and $[(4/3)^n]$, $n \in \mathbb{N}$, contain infinitely many composite numbers. Their proof extends without change to $[\xi(3/2)^n]$ and $[\xi(4/3)^n]$ with arbitrary $\xi > 0$. However, we will give a proof for $a = 4/3$ once again, because (after a small preparation) we will be able to do this in just few lines in contrast to eight lemmas used in [8]. This will serve as a warm-up for the proof of a corresponding result for $a = 5/4$. A slight difference between our approach and that of [8] is that we are seeking a contradiction with Lemma 2 below which is obtained using fractional parts rather than a similar lemma for the integer parts of powers as in [8]. The main advantage in doing this is that we are able not only to prove that there are infinitely many composite numbers among integer parts, but also describe some explicit (unavoidable) finite sets for possible divisors. (We show, however, that such unavoidable sets do not exist for some rational $a$; see the proposition in Section 5.) The proof of Theorem 1 for $a = 3/2$ is given by combining our Lemma 2 with the main result of [8]. In [11] Mahler asked a question equivalent to the following: is there a positive number $\xi$ such that the numbers $[\xi(3/2)^n]$, $n \in \mathbb{N}$, are *all* even? Mahler's question remains unsolved. Theorem 1 shows that, for *every* $\xi > 0$, the set $[\xi(3/2)^n]$, $n \in \mathbb{N}$, contains infinitely many elements divisible by one of the numbers $2, 5, 7, 11$.

Note that the number $a = 5$ is missing in Theorem 1. The only reason for this is that, for $a = 5$, such universal explicit (unavoidable) set for divisors of the elements of the set $[\xi 5^n]$, $n \in \mathbb{N}$, cannot be given. We will prove a corresponding proposition in Section 5. However, we are able to prove that the set $[\xi 5^n]$, $n \in \mathbb{N}$, contains infinitely many composite numbers. This is a simple consequence of the following theorem.

**Theorem 2.** *Let* $\xi > 0$ *be a real number. If* $\xi \neq (4k+3)/(2 \cdot 5^r)$, *where* $k, r$ *are nonnegative integers, then the set* $[\xi 5^n]$, $n \in \mathbb{N}$, *contains infinitely many elements divisible by* $2, 3$ *or* $5$. *If* $\xi = (4k+3)/(2 \cdot 5^r)$, *where* $(4k+3, 5^r) = 1$, *then the set* $[\xi 5^n]$, $n \in \mathbb{N}$, *contains infinitely many elements divisible by* $10k+7$.

At the moment, we cannot prove that the integer parts $[\xi a^n]$, $n = 1, 2, \ldots,$ where $\xi$ is an arbitrary positive number and $a \geqslant 7$ is an integer, are composite for infinitely many $n \in \mathbb{N}$. Also, we cannot extend Theorem 1 to any rational non-integer number other than $3/2, 4/3, 5/4$ even for $\xi = 1$, although at first glance the case $a = 6/5$ may seem simpler than the case $a = 5/4$. However, by slightly changing the problem, we can include some new rational numbers.

**Theorem 3.** *Let* $\xi > 0$ *be a real number. Then each of the sets* $[\xi(5/2)^n] - 1$ *and* $[\xi(6/5)^n] - 1$, *where* $n \in \mathbb{N}$, *contains infinitely many elements divisible by at least one number of the set* $\{2, 3, 5\}$.

One can also consider the nearest integers to powers instead of integer parts. We define the nearest integer to $z$ as $[z + 1/2]$. Some new numbers can be obtained again.

**Theorem 4.** *Let* $\xi > 0$ *be a real number. Then*

$(i)$ *the set* $[\xi 7^n + 1/2]$, $n \in \mathbb{N}$, *contains infinitely many composite numbers,*
$(ii)$ *the set* $[\xi(5/3)^n + 1/2]$, $n \in \mathbb{N}$, *contains infinitely many elements divisible by* $2$ *or* $3$,
$(ii)$ *the set* $[\xi(7/5)^n + 1/2]$, $n \in \mathbb{N}$, *contains infinitely many elements divisible by at least one of the numbers* $2, 3, 5, 11$.

All preparations will be made in Section 2. In Section 3 we will prove Theorem 1 for all $a$, except for $a = 5/4$. The case $a = 5/4$ will be treated in a separate section. In Section 5 we will prove Theorem 2 and show that there are no unavoidable sets of divisors for every integer $a$ of the form $4k+1$, where $k \in \mathbb{N}$. Theorems 3 and 4 will be proved in Section 6.

## 2. Preliminary results

**Lemma 1.** *Suppose that* $\xi > 0$ *and* $\nu$ *are real numbers. If* $a > 1$ *is a rational noninteger number, then the sequence* $\{\xi a^n + \nu\}$, $n = 1, 2, \ldots,$ *is not periodic.*

*Proof.* Indeed, if the the sequence is periodic, then for infinitely many $m \in \mathbb{N}$ we have the equality $\{\xi a^n + \nu\} = \{\xi a^{n+m} + \nu\}$, where $n$ is fixed. This implies that $\xi(a^{n+m} - a^n) = \xi a^n(a^m - 1)$ is an integer. This can only happen if $\xi$ is a rational number. Writing $a = b/c$, where $b > c > 1$ are relatively prime integers, and multiplying the above number by $c^n$ and by the denominator of $\xi$, we obtain that there is a fixed positive integer $g$ such that $g(b^m - c^m)/c^m$ is an integer. For $m$ sufficiently large, this can only happen if $b^m - c^m$ is divisible by $c$, which is impossible. This proves the lemma. □

Write $a = b/c$, where $b > c \geqslant 1$ are relatively prime integers. (Note that $a$ is allowed to be an integer.) Setting $x_n = [\xi a^n + \nu]$ and $y_n = \{\xi a^n + \nu\}$, we obtain the equality $a(x_n + y_n - \nu) = x_{n+1} + y_{n+1} - \nu$. Consequently, $cx_{n+1} = bx_n + by_n - cy_{n+1} + (c - b)\nu$, so $s_n = by_n - cy_{n+1} + (c - b)\nu$ is an integer. It follows that

$$x_{n+1} = (bx_n + s_n)/c, \qquad y_{n+1} = (by_n + (c - b)\nu - s_n)/c.$$

Furthermore, since $0 \leqslant y_n, y_{n+1} < 1$, we deduce that $-c + (c - b)\nu < s_n < b + (c - b)\nu$. Let throughout $S(a, \nu)$ be the set of integers which belong to $(-c + (c - b)\nu, b + (c - b)\nu)$. Of course, $s_n$ can take only finitely many values $|S(a, \nu)|$.

**Lemma 2.** *Suppose that $a > 1$ is a rational noninteger number. Then the sequence $s_1, s_2, s_3, \dots$ is not periodic.*

*Proof.* Assume it is periodic, of period $\ell$. Then the sequence $((c - b)\nu - s_n)/c$, $n = 1, 2, \dots$, is periodic too. So there is a positive integer $n$, and, for every $j = 0, 1, \dots, \ell - 1$, there are two real numbers $\zeta = \zeta(j) > 1$ and $\omega = \omega(j)$ such that $y_{n+j+t\ell} = \zeta y_{n+j+(t-1)\ell} + \omega$ for every $t \in \mathbb{N}$. Fix $j$. Each fractional part in the subsequence $y_{n+j+t\ell}$, $t = 1, 2, \dots$, is obtained from the preceding one by the formula $y \to \zeta y + \omega$. For $z = y + \omega/(\zeta - 1)$ this transformation can be written as $z \to \zeta z$. We claim that $y_{n+j} = -\omega/(\zeta - 1)$. Indeed, if $y_{n+j} > -\omega/(\zeta - 1)$ then $y_{n+j+t\ell} + \omega/(\zeta - 1) \to \infty$ as $t \to \infty$. This is impossible, because every fractional part is bounded above by 1. Similarly, if $y_{n+j} < -\omega/(\zeta - 1)$ then $y_{n+j+t\ell} + \omega/(\zeta - 1) \to -\infty$ as $t \to \infty$, which is also impossible. By the recurrent formula, it follows that $y_{n+j+t\ell} = -\omega/(\zeta - 1) = -\omega(j)/(\zeta(j) - 1)$ for every $t \in \mathbb{N}$. The same is true for every $j$ in the range $0 \leqslant j \leqslant \ell - 1$. Hence the sequence $y_n, y_{n+1}, y_{n+2}, \dots$ is purely periodic with period at most $\ell$. This plainly implies that $y_1, y_2, y_3, \dots$ is periodic, contrary to Lemma 1. □

The key difference between Lemma 2 and a respective result in [8] is that we claim that the sequence is not periodic without assumption that the sequence of integer parts contains only finitely many composite numbers. Note that the sequence can be periodic for integer $a$. For instance, we can take $\xi = 1/2$ and $a = 5$. Then $s_1 = s_2 = s_3 = \dots = 2$.

Let $\mathcal{P}$ be a finite set of prime numbers, and let $a = b/c$, where $b > c \geqslant 1$ are relatively prime. Assume that the numbers $x_n = [\xi a^n + \nu]$ are not divisible by a prime $p \in \mathcal{P}$ for every sufficiently large $n$. We know that $s_n \in S(a, \nu)$ for every $n \in \mathbb{N}$. If $p|bc$, then $p|b$ or $p|c$. Such $p$ cannot divide $s_n$ for $n$ sufficiently large, since otherwise $x_{n+1}$ or $x_n$ is divisible by $p$. We will thus be able to exclude all numbers divisible by such primes from the set $S(a, \nu)$. Furthermore, the prime 2 lying in all sets $\mathcal{P}(a)$, by an easy parity argument, allows to exclude all odd numbers from $S(a, \nu)$ in case $b$ and $c$ are both odd.

Throughout we will use the following notation. Instead of $s_1, s_2, s_3, \ldots$ we will consider the sequence of operations denoted by $A, B, \ldots$ corresponding to every $s \in S(a, \nu)$. If, say $A$ corresponds to $s$, this means that $A$ maps the integer $x$ to the integer $(bx + s)/c$ (which corresponds to $[\xi a^n + \nu] \to [\xi a^{n+1} + \nu]$) and the fractional part $y$ (of $\xi a^n + \nu$) to $(by - s + (c - b)\nu)/c$ (which is the fractional part of $\xi a^{n+1} + \nu$).

All this gives certain restrictions on the sequence of operations containing $A$'s, $B$'s etc. In particular, for every fixed prime $p$, every operation is a permutation of residues modulo $p$. (We will only use the primes 3, 5, 7, 11, 13 in all our arguments below.) For instance, if say we seek for a contradiction modulo 7, then $2A1$ means that $A$ maps the number of the form $7v + 2$, $v \in \mathbb{N}$, to the number of the form $7v' + 1$, $v' \in \mathbb{N}$, in the corresponding sequence of integer parts. In a more compact form we will write this in, say, the form $A = (12)(643|(5)$. This means that the only possible transformations modulo 7 are $1A2$, $2A1$, $6A4$, $4A3$ and $5A5$, whereas 3 maps to 0, i.e. the next integer part is divisible by 7, a contradiction. If, e.g., $B = (123)(4|(56)$, these two successive operations will 'multiply' as two permutations, namely, $AB = (413|(2)(56|$. Also, $AA = A^2 = (1)(2)(63|(4|(5)$. A pattern $AB \ldots A$ is said to be impossible if $AB \ldots A = (1|(2|(3|(4|(5|(6|$; this means that one of the corresponding integer parts, but not necessarily the last one, is divisible by 7. Because of this notation, it is convenient to write the residues 10 modulo 11 and 10,11,12 modulo 13 as a single digit numbers. Throughout we will use the notation $10 = \alpha$, $11 = \beta$ and $12 = \gamma$. Note that $A^\infty$ means the periodic sequence $AAAAA\ldots$. (Similarly, $(AB)^\infty$ means $ABABAB\ldots$.) Of course, periodic sequences for noninteger $a$ cannot occur by Lemma 2.

## 3. Theorem 1: easy proofs

For $a = 2$, the binary expansion of the number $\xi$ contains infinitely many zeros. So the set $[\xi 2^n]$, $n \in \mathbb{N}$, contains infinitely many even numbers.

For $a = 3$, we have $S(3, 0) = \{0, 1, 2\}$. Thus $x_{n+1} = 3x_n + s_n$ with $s_n \in \{0, 1, 2\}$. Since 3 and 1 are both odd, the sequence $x_n$, $n = 1, 2, \ldots$, contains infinitely many even numbers or infinitely many numbers divisible by 3, unless $s_n = 2$ for all large $n$. Assume that it is so. But then $y_{n+1} = 3y_n - 2$ for all sufficiently large $n$. This implies that $y_n \to -\infty$ as $n \to \infty$, a contradiction.

For $a = 4$, $S(4, 0) = \{0, 1, 2, 3\}$. We either have infinitely many even integer parts or there are just two possibilities $x_{n+1} = 4x_n + 1$ and $x_{n+1} = 4x_n + 3$ starting with certain $n$. Assume that $m$ is so large that $x_m, x_{m+1}, \ldots$ are not divisible by 3. Then, by a simple argument modulo 3, we see that the possibility $x_{n+1} = 4x_n + 1$ cannot occur more than once. It follows that $x_{n+1} = 4x_n + 3$ for all sufficiently large $n$. Then $y_{n+1} = 4y_n - 3$ for all large $n$, so $y_n \to -\infty$ as $n \to \infty$, a contradiction.

For $a = 6$, $S(6, 0) = \{0, 1, 2, 3, 4, 5\}$. Now, either we have infinitely many integer parts divisible by 2 or 3 or there are just two possibilities $x_{n+1} = 6x_n + 1$ and

$x_{n+1} = 6x_n + 5$ starting with certain $n$. Assume that $m$ is so large that $x_m, x_{m+1}, \dots$ are not divisible by 5. Then, by a simple argument modulo 5, we see that the possibility $x_{n+1} = 6x_n + 1$ cannot occur more than three times. It follows that $x_{n+1} = 6x_n + 5$ for all sufficiently large $n$. Then $y_{n+1} = 6y_n - 5$ for all large $n$, so $y_n \to -\infty$ as $n \to \infty$, a contradiction.

For $a = 4/3$, $S(4/3, 0) = \{-2, -1, 0, 1, 2, 3\}$. Again, either there are infinitely many integer parts divisible by 2 or 3 or only two possibilities can occur $3x_{n+1} = 4x_n - 1$ (type $A$) and $3x_{n+1} = 4x_n + 1$ (type $B$) starting with certain $n$. Assume that there are only finitely many integer parts divisible by 5. Of course, $A = (1)(324|$ and $B = (231|(4)$ modulo 5. Since the sequence of $A$'s and $B$'s is not periodic, the pattern $AB$ occurs infinitely often. More precisely, since $AB = (1|(3)(24|$, this pattern can only be $3AB3$ or $2AB4$. The second case is impossible, because we then must have $B^\infty$, which is periodic, contrary to Lemma 2. Similarly, $3AB^2 1$ leads to $A^\infty$, a contradiction again. So we can only have $3ABA2$. In order to avoid 1 and 4 in the sequence of residues modulo 5, we must have $(AB)^\infty$ which is also periodic, a contradiction.

For $a = 3/2$, $S(3/2) = \{-1, 0, 1, 2\}$. Now, either there are infinitely many even integer parts or there are two possibilities $2x_{n+1} = 3x_n - 1$ (type $A$) and $2x_{n+1} = 3x_n + 1$ (type $B$) starting with certain $n$. The arguments modulo 5, 7, 11 of Forman and Shapiro [8] show however that this sequence must be periodic, unless there are infinitely many integer parts divisible by 5, 7 or 11. This proves the result, by Lemma 2. The fact that they only consider the case $\xi = 1$ is not essential. We will not repeat their argument, although it can also be made much shorter than in [8].

## 4. Integer parts of powers of 5/4

Consider the case $a = 5/4$. Then $S(5/4, 0) = \{-3, -2, -1, 0, 1, 2, 3, 4\}$. At the expense of the prime 2, we can exclude $-2, 0, 2, 4$. The four remaining cases correspond to the recurrences $4x_{n+1} = 5x_n + s_n$ and $y_{n+1} = (5y_n - s_n)/4$. They are $s_n = -1$ (type $A$), $s_n = 3$ (type $B$), $s_n = 1$ (type $C$) and $s_n = -3$ (type $D$). Note that the operation $A$ can only occur if $y_n \in [0, 3/5)$. On applying it, the fractional part $y_{n+1}$ will be in the interval $[1/4, 1)$. So $A$ acts on fractional parts as $[0, 3/5) \to [1/4, 1)$. Similarly, $B : [3/5, 1) \to [0, 1/2)$, $C : [1/5, 1) \to [0, 1)$ and $D : [0, 1/5) \to [3/4, 1)$. We will write this in the form as below, where $x, y$ refer to integer and fractional parts, respectively:

$$\begin{aligned}
A &: x \to (5x-1)/4, \quad y \to (5y+1)/4, \quad [0, 3/5) \to [1/4, 1);\\
B &: x \to (5x+3)/4, \quad y \to (5y-3)/4, \quad [3/5, 1) \to [0, 1/2);\\
C &: x \to (5x+1)/4, \quad y \to (5y-1)/4, \quad [1/5, 1) \to [0, 1);\\
D &: x \to (5x-3)/4, \quad y \to (5y+3)/4, \quad [0, 1/5) \to [3/4, 1).
\end{aligned}$$

In order to avoid confusion we must say that the composition of operations in the proof of Lemma 3 below is very unusual for a reader with an algebraic background. The composition of operations, say, $BC$ is read from left to right giving

$BC : y \to (25y-19)/16$. This of course contradicts to the usual rule of composition from right to left. However, in all our arguments following Lemma 3 we always use $A, B, C$ and $D$, firstly, as elements of an infinite sequence and, secondly, as a kind of permutations. In both cases, it is much more convenient to write (and read), say, $BC$ from left to right.

**Lemma 3.** *The patterns $AD$, $DA$, $D^2$, $B^2$, $BC^kB$, where $k \in \mathbb{N}$, $BC^4$, $(DB)^2D$, $BDCBDC^uBD$, $u \in \{0,1,2,3\}$, $(BD)^3$, $(DB)^2CD$, $(DB)^2C^2DBC^vD$, $v \in \{0,1\}$, cannot occur.*

*Proof.* The result is evident for $AD$, $DA$, $D^2 = DD$ and $B^2 = BB$ from fractional parts. Also, $BC : \ y \to (25y-19)/16$ which is smaller than $3/8$. Since $C$ maps every $y$ to a smaller number, we deduce that $BC^k$ maps $y$ to a number smaller than $3/8$ for every $k \in \mathbb{N}$. Hence $BC^kB$ cannot occur. Similarly, $C^4 : \ y \to (625y-369)/256$, so $C^4$ can only be applied to $y \geqslant 369/625 > 1/2$. Hence $BC^4$ cannot occur. Also, $(DB)^2 : \ y \to (625y+123)/256$ which is greater than $1/5$, so $(DB)^2D$ cannot occur.

Since $BD : \ y \to (25y-3)/16$, it can only be applied if $3/5 \leqslant y < 19/25$. But $BDCBD : \ y \to (3125y-967)/1024$ which is greater than or equal to $227/256 = 0.88671\ldots$. On applying $C$ at most three times to this number we will get a number greater than $0.77 > 0.76 = 19/25$, so the pattern $BDCBDC^uBD$, where $0 \leqslant u \leqslant 3$, cannot occur. Note that $(BD)^2 : \ y \to (625y-123)/256$. For $y \geqslant 3/5$, this is greater than or equal to $63/64 > 19/25$, so $(BD)^2$ cannot be followed by one more $BD$, i.e. $(BD)^3$ cannot occur.

Similarly, $(DB)^2C : \ y \to (3125y+359)/1024$ which is greater than $0.35 > 1/5$, so $(DB)^2CD$ is impossible. Furthermore, this implies that $(DB)^2C^2$ maps every $y$ to a number $y' > 3/16$. But $DB$ maps $y'$ to $(25y'+3)/16$ which is greater than 0.48, since $y' > 3/16$. Thus $(DB)^2C^2DBC^v$, $v \in \{0,1\}$, is greater than 0.35 and cannot be followed by $D$. □

**Lemma 4.** *Suppose that the set of integer parts contains only finitely many elements divisible by 2 and 3. Then there are only finitely many A's. Furthermore, starting from some place, the sequence of operations is either $BDC^{k_1}BDC^{k_2}\ldots$ or $DBC^{k_1}DBC^{k_2}\ldots$, where $k_1, k_2, \cdots \geqslant 0$.*

*Proof.* Note first that the patterns $AC$, $CA$, $ABA$, $ABDC$, $CBC$, $CDC$ cannot occur. Indeed, modulo 3 we have $A = (1)(2|$, $B = (12)$, $C = (1|(2)$, $D = (12)$. This implies the above claim. We will frequently use it without referring to it. Sometimes we will combine it with Lemma 3 which gives other restrictions on patterns.

Assume that there are infinitely many $A$'s. Then, since the sequence is not periodic and since the patterns $AC$, $CA$, $AD$ and $DA$ cannot occur, every pattern $A^k$, $k \in \mathbb{N}$, can only occur between two $B$'s. Similarly, since $BC^kB$ is impossible, every pattern $C^k$ can only occur between $B$ and $D$, giving $BC^kD$, $D$ and $B$, giving $DC^kB$, or $D$ and $D$, giving $DC^kD$. Fix a fragment $BA^kB$. Let's forget for a moment about $A$'s and $C$'s and consider the remaining subsequence of $B$'s and $D$'s (to the right of the fixed fragment $BA^kB$ which can only be $2BA^kB2$). Next operation in this subsequence should be $D$, because neither $B^2$ nor $ABA$ can occur, so the next operation cannot be $B$. Furthermore, this $D$ should be of the form $2D1$. Now, the next operation in the subsequence should be $B$. Indeed, assume that it is $D$. Since the pattern $D^2$ is impossible, these two $D$'s should be separated by $C^k$, $k \in \mathbb{N}$. Since $CDC$ cannot occur, we must have the pattern $ABDC^kD$, but its subpattern $ABDC$ cannot occur, a contradiction. Furthermore, this $B$ must be of the form $1B2$. Since $BC^kB$, where $k \geqslant 0$, is impossible, after $D$, $B$ it should be $2D1$ again, etc. We thus deduce that the subsequence is $D, B, D, B, D, B, \dots$. These can only be separated by $C^k$, so there are no more $A$'s, as claimed.

If there are only finitely many $C$'s in the original sequence, we have $(DB)^\infty$, a contradiction. Assume that the first occurrence of $C$'s is between $D$ and $B$, i.e. there is a pattern $DC^kB$ with $k > 0$. Modulo 3 we must have $1DC^kB1$. This cannot be followed by $C$, so it must be followed by $D$ and gives $1DC^kBD2$. This is clearly followed by $C^{k_1}B$, where $k_1 \geqslant 0$, and gives $1DC^kBDC^{k_1}B1$. Further, we must have $D$, then $C^{k_2}BD$, etc. Hence the sequence from certain place is $BDC^{k_1}BDC^{k_2}\dots$, $k_1, k_2, \dots \geqslant 0$. The argument when the first occurrence of $C$'s is between $B$ and $D$ is precisely the same and gives the sequence $DBC^{k_1}DBC^{k_2}\dots$, where $k_1, k_2, \dots \geqslant 0$. □

**Lemma 5.** *Suppose that the set of integer parts contains only finitely many elements divisible by* 2, 3, 7, 11 *and* 13. *Then the sequence* $BDC^{k_1}BDC^{k_2}\dots$ *is impossible.*

*Proof.* Modulo 7 we have $B = (63125|(4)$, $C = (21534|(6)$, $D = (14652|(3)$. Hence $BD = (1|(2)(346)(5|$.

We claim that there are infinitely many patterns $4BD6$. There are four possible cases $3BD4$, $6BD3$, $2BD2$ and $4BD6$. In the first case, we cannot have $C$ next, so we must have $4BD6$ straight after $3BD4$. In the second case, $6BD3$, we have next either $3BD4$ (i.e. we are back to the first case), or $3C4$ which can only be followed by $4BD6$. Finally, in the third case, $2BD2$, since the sequence is not $(BD)^\infty$, we must have some $C$'s later on. So next there is $2(BD)^u2C^v$, where $u \geqslant 0$ and $v \in \{1, 2, 3, 4\}$, which ends with 1, 5, 3, 4, respectively, and then $BD$ again. But $BD$ cannot begin with 1 or 5. So we either immediately get $4BD6$ or we are back to the case $3BD4$, which is already considered. This proves our claim.

Each pattern $4BD6$ is either followed immediately by $6BD3$ or it is followed by $6C^kBD3$. We thus have $4BDC^kBD3$, where $k \geqslant 0$. If next we would have $3BD4$ then this should be followed again by $BD$. But by Lemma 3 $(BD)^3$ cannot occur, so $BD$ cannot be repeated more than twice, a contradiction. Thus we have the pattern $4BDC^kBDC4$ which must be followed by $4BD6$, etc. Hence our sequence is formed by the patterns $BDCBD$ which are separated by $C^{u_i}$, where $u_i \geqslant 0$. So the sequence is

$$BDCBDC^{u_1}BDCBDC^{u_2}\dots.$$

Furthermore, Lemma 3 implies that each $u_i$ is greater than or equal to 4.

The number 10 modulo 11 occurs as one of the residues. Recall that throughout 10 will be denoted by $\alpha$. We have $B : x \to 4x - 2$, $C : x \to 4x + 3$ and $D : x \to 4x + 2$ modulo 11, hence $B = (9126|(3\alpha574)(8)$, $C = (17965)(3482|(\alpha)$,

$D = (16478)(2\alpha 95|(3)$. Consequently, $BDCBDC^4 = (25|(79|(81|(3|(4|(6|(\alpha|$. Thus $BDCBDC^{u_i}$ can only end with one of the numbers $1, 7, 9, 6, 5$. However the next block $BDCBDC^{u_{i+1}}$ can only begin with 7. Thus each of the blocks $BDCBDC^{u_i}$ is of the form $7BDCBDC^{u_i}7$. This only happens if each $u_i$ is of the form $8 + 5v_i$, where $v_i$ is a nonnegative integer. Furthermore, there exist positive $v_i$, for otherwise we have $(BDCBDC^8)^\infty$. Hence $u_i \geqslant 13$ for some $i$.

We have no other choice modulo 13, but to continue with our curious notation $10 = \alpha$, $11 = \beta$ and $12 = \gamma$. Now, $B : x \to -2x + 4$, $C : x \to -2x + 10$, $D : x \to -2x + 9$. Hence $B = (49\gamma 6573\beta 812|(\alpha)$, $C = (\alpha 3426\beta 18795|(\gamma)$, $D = (941786\alpha 25\gamma\beta|(3)$. Thus $BDCBD = (1|(2|(953|(8\beta 6\alpha\gamma|(74|$. It follows that the block $BDCBDC^{u_i}$ with $u_i \geqslant 13$ can only be of the form $\alpha BDCBDC^{u_i}\gamma$, because $\gamma C^{u_i}\gamma$ is the only possibility if $u_i \geqslant 11$. However, the next block $BDCBDC^{u_{i+1}}$ cannot begin with $\gamma$, since the pattern $BDCBD$ cannot begin with $\gamma$, a contradiction. □

**Lemma 6.** *Suppose that the set of integer parts contains only finitely many elements divisible by* 2, 3, 7, 11 *and* 13. *Then the sequence* $DBC^{k_1}DBC^{k_2}\ldots$ *is impossible.*

*Proof.* We will first argue modulo 7 and claim that there are infinitely many patterns $3DB1$. Since $DB = (143)(5)(2|(6|$, other possibilities are $4DB3$, $1DB4$ and $5DB5$. Recall that $C = (21534|(6)$. The first possibility, $4DB3$, leads to $3DB1$ next or we must have $4DBC4$. So the only alternative to $4DB3$ to occur is $(DBC)^\infty$, a contradiction with Lemma 2. The pattern $1DB4$ leads to $4DB3$, so to the case which we just considered. After repeating $5DB5$ at most twice (Lemma 3), we should apply either $C$ or $C^2$ and then $DB$ again (modulo 7). This gives, respectively, $3DB1$ (as required) or $4DB3$ (which is the first possibility). The claim is proved.

If $3DB1$ is followed by $DB$ again, we cannot have further $DB$, by Lemma 3, so it must be followed by $C$. But $(DB)^2C$ cannot begin with 3, a contradiction. Hence $3DB1$ must be followed by $C$, i.e. we have infinitely many patterns $3DBC5$. What can happen between two successive $3DBC5$? Assume that the next operation after the first $3DBC5$ is $C$. Then either $3DBC^23$ is followed by $3DBC5$ or we have $3DBC^3DB3$ and then further $(CDB)^k$ until the second $3DBC5$. Both cases can be written as $DBC^2(CDB)^u$, where $u \geqslant 0$. Alternatively, assume that after the first $3DBC5$ is $DB$. The whole pattern is $3DBCDB5$. We can have at most one $DB$ until the next $C$, so the pattern can be written as $3DBC(DB)^vC3$, where $v \in \{1, 2\}$. We will show that the case $v = 2$ is impossible. Indeed, by Lemma 3, $(DB)^2C$ should be followed by $C$ which gives $3DBC(DB)^2C^24$. This must be followed by $4DB3$ (modulo 7), so we get $(DB)^2C^2DB3$. Further, by Lemma 3, this cannot be followed by $DB$ or by $CDB$. So it must be followed by $C^2$, which is impossible modulo 7. Hence $v = 1$, i.e. we have $3(DBC)^23$. The second $3DBC5$ begins either immediately or after inserting $(CDB)^k$, $k \in \mathbb{N}$. We conclude that the whole sequence consists of just two type blocks $DBC^2(CDB)^u$ and $(DBC)^2(CDB)^k$, where $k, u \geqslant 0$.

We now show that $k$ and $u$ can only take two values 0 and 1. Set $E = (DBC)^2$, $F = (DBC)^2(CDB)^k$, $G = DBC^2$, $H = DBC^2(CDB)^u$, where $k, u \in \mathbb{N}$. Note that the same letter $F$ (and $H$) can denote different patterns.

Modulo 11, we have $E = (371|(68|(\alpha 2|(9)(4|(5|$, $G = (427|(3\alpha 968|(1|(5|$, $CDB = (\alpha 185|(9342|(7)(6|$. (Here, we use the expressions for $B, C, D$ from Lemma 5.) Assume that there is $F$ with $k \geqslant 2$. By Lemma 3, $(DB)^2CD$ is impossible, so $F$ cannot be followed by another $F$ (which can be different from the first $F$) or $E$. Thus it must be followed by $G$ or $H$. Note that $F$ with $k \geqslant 2$ can end up only with the residues $7, 5, 4, 2$. So $FH$ can end up only with 7. $FG$ can end up only with 2 or 7. In case it ends up with 2, $FG$ must be followed by $G$ or $H$ which ends by 7. With 7 can only begin $E$ or $F$; this ends with $1, 8$ or 5. But neither of $E, F, G, H$ begins with $1, 8$ or 5, a contradiction. Similarly, assume that there is an $H$ with $u \geqslant 2$. By Lemma 3, the patterns $HE$ and $HF$ cannot occur. $H$ with $u \geqslant 2$ can end with $7, 8, 5, 4, 2$. This shows that $HH$, where both $H$ can be different, can only end with 7, whereas $HG$ can end with 2 or 7. In case it is 2, $HG$ should be followed by $G$ or $H$. This ends up with 7 and we get a contradiction as above. Hence $k = u = 1$ and $F, H$ are uniquely determined. Thus the sequence can contain only four possible patterns $E = (DBC)^2$, $F = (DBC)^2CDB$, $G = DBC^2$ and $H = DBC^2CDB = DBC^3DB$.

Our next claim is that only the patterns of the form $GH$ and $G^2E^kF$, where $k \geqslant 0$, can occur. We will still argue modulo 11. Recall that $E = (371|(68|(\alpha 2|(9)(4|(5|$, $G = (427|(3\alpha 968|(1|(5|$, $F = ECDB = (9378|(65|(1|(2|(4|(\alpha|$, $H = (27|(4|(\alpha 31|(9|(65|(8|$. None of the operations $E, F, G, H$ begins with $1, 5, 8$, so they cannot end with $1, 5, 8$. With 7 can begin only $E$ and $F$, but they end with 1 and 8, respectively. This is impossible, so no operation can begin or end with $1, 5, 7, 8$. All remaining possibilities are $\alpha E2$, $9E9$, $4G2$, $3G\alpha$, $\alpha G9$, $9G6$, $9F3$, $\alpha H3$. None of these begins with 2 or 6, so $\alpha E2$, $4G2$, $9G6$ cannot occur. Remaining are $9E9$, $3G\alpha$, $\alpha G9$, $9F3$, $\alpha H3$. It is easily seen that $F$ and $H$ must be followed by $3G\alpha$, so we have infinitely many $3G\alpha$, unless the sequence is $E^\infty$. What can happen between two consecutive $3G\alpha$'s? If $3G\alpha$ is followed by $\alpha H3$, we immediately get the fragment $GH$, because the next $3G\alpha$ should follow. Otherwise, we have $3G^29$. If the next is $9F3$, we have $3G^2F3$ and the fragment is finished. The alternative is that we have several $E$'s (which are all of the form $9E9$ inserted between $G^2$ and $F$). So another possible fragment is $G^2E^kF$, where $k \geqslant 0$.

We now derived that the sequence contains just two possible fragments $GH$ and $G^2E^kF$. A contradiction will be obtained modulo 13. By a simple computation using the expressions for $A, B, C, D$ from the previous lemma, we have modulo 13

$$\begin{aligned}
G &= DBC^2 = (125)(7)(64\beta|(38|(\gamma 9|(\alpha|,\\
H &= DBC^3DB = (\gamma 679|(431\alpha|(5)(2|(8|(\beta|,\\
E &= (DBC)^2 = (16)(34)(\gamma 8|(25|(9\beta|(7|(\alpha|,\\
F &= (DBC)^2CDB = (937|(\gamma 1|(65|(42|(8|(\alpha|(\beta|.
\end{aligned}$$

Clearly, $G^2 = (152)(7)(6\beta|(3|(4|(8|(9|(\alpha|(\beta|(\gamma|$. In case if the sequence is not $(GH)^\infty$, we must have infinitely many fragments of the form $G^2E^kF$ (with may be different $k \geqslant 0$). But $G^2$ can end only with $1, 2, 5, 7, \beta$, so $G^2E^k$ can end with $1, 2, 5, 7, \beta, 6$. Among these numbers, $F$ can only begin with 6 thus giving 5 at the end of each fragment $G^2E^kF$. If we have at least one fragment $GH$ after certain $G^2E^kF$, then it must be $5GH\alpha$. However $G$ cannot begin with $\alpha$, a contradiction. So we only have the fragments of the form $G^2E^kF$ with may be different $k$, but each ending (and so beginning) with 5. So we have $5G^22$. This cannot be followed neither by $F$ nor by $E^2$, so it must be followed by $E$ and then by $F$ which is impossible modulo 13, a contradiction. This completes the proof of Lemma 6 and the proof of Theorem 1.

## 5. Integer parts of $\xi 5^n$

For $a = 5$, we have $x_{n+1} = 5x_n + s_n$ with $s_n \in S(5, 0) = \{0, 1, 2, 3, 4\}$. Assume that the sequence of integer parts contains only finitely many elements divisible by 2 and 5. Then, starting with some $n$, there are two possibilities $x_{n+1} = 5x_n + 2$ (type $A$) and $x_{n+1} = 5x_n + 4$ (type $B$). Suppose that there are also only finitely many elements divisible by 3. But $A = (1)(2|$ and $B = (1|(2)$ modulo 3, so the patterns $AB$ and $BA$ cannot occur. Thus we have either $A^\infty$ or $B^\infty$. In the second case, $y_{n+1} = 5y_n - 4$, hence $y_n \to -\infty$ as $n \to \infty$, a contradiction. So we must have $A^\infty$, i.e. $x_{n+1} = 5x_n + 2$ and $y_{n+1} = 5y_n - 2$ for all sufficiently large $n$. In case if there is no $n$ for which $y_n = 1/2$, we obtain a simple contradiction using fractional parts as in Lemma 2 and getting $y_n \to \infty$ or $y_n \to -\infty$ as $n \to \infty$. So $y_u = 1/2$ for some $u \in \mathbb{N}$. Setting $q = x_u$, we deduce that $\xi 5^u = q + 1/2$. Hence either we have infinitely many integer parts $[\xi 5^n]$ divisible by at least one number of the set $\{2, 3, 5\}$ or $\xi = (q + 1/2)5^{-u}$.

Let us choose the smallest nonnegative integers $t$ and $r$ for which we can write $\xi = (2q + 1)/(2 \cdot 5^u) = (2t + 1)/(2 \cdot 5^r)$. Then

$$x_{n+r} = [(t + 1/2)5^n] = t5^n + (5^n - 1)/2.$$

If $t$ is even, then the numbers $t5^n + (5^n - 1)/2$, $n = 1, 2, 3, \dots$, are all even. Hence the sequence $[\xi 5^n]$, $n = 1, 2, \dots$, contains infinitely many even numbers, but this is already covered by the previous case, because $2 \in \{2, 3, 5\}$. So assume without loss of generality that $t$ is odd: $t = 2k+1$, where $k \geqslant 0$. Then $\xi = (4k+3)/(2\cdot 5^r)$, where $(4k + 3, 5^r) = 1$.

We need to show that the sequence of integer parts $(2k + 1)5^n + (5^n - 1)/2$, $n = 0, 1, 2, \dots$, contains infinitely many elements divisible by $10k + 7$. Let us take $n$ of the form $1 + \varphi(10k + 7)\ell$, where $\varphi$ is Euler's function and $\ell \in \mathbb{N}$. Then

$$(2k + 1)5^n + (5^n - 1)/2 = (10k + 7)5^{\varphi(10k+7)\ell} + (5^{\varphi(10k+7)\ell} - 1)/2$$

is divisible by $10k + 7$, by Euler's theorem, because $(5, 10k + 7) = 1$. This completes the proof of Theorem 2. □

**Proposition.** *Let a be a positive integer of the form* $4k+1$, *where* $k \in \mathbb{N}$, *and let* $\mathcal{P}$ *be an arbitrary finite set of prime numbers. Then there exists* $\xi > 0$ *such that every integer part* $[\xi a^n]$, $n = 1, 2, \dots$, *is relatively prime with every prime number of the set* $\mathcal{P}$.

*Proof.* Let $P$ be the product of all odd primes of $\mathcal{P}$, and let $\delta = 1$ if $P$ is of the form $4v + 3$, $v \geqslant 0$, and $\delta = 3$ if $P$ is of the form $4v + 1$, $v \geqslant 0$. Put $\xi = \delta P/2$. Then $[\xi a^n] = (\delta P a^n - 1)/2$ is odd, so there are no numbers among integer parts divisible by 2. Also, if $p$ is an odd prime which belongs to $\mathcal{P}$, then $(\delta P a^n - 1)/2$ is not divisible by $p$. □

## 6. Shifted integer parts

*Proof of Theorem 3.* For $x_n = [\xi(5/2)^n - 1]$, $S(5/2, -1) = \{2, 3, 4, 5, 6, 7\}$. So we either have infinitely many shifted integer parts $x_n$ divisible by 2 or 5 or two types of linear recurrences $2x_{n+1} = 5x_n + 3$ (type $A$) and $2x_{n+1} = 5x_n + 7$ (type $B$). Modulo 3, we have $A = (1)(2)$ and $B = (21|$. Hence, if only finitely many elements are divisible by 3, we cannot have more than one operation $B$ which must be followed by $A^\infty$, a contradiction with Lemma 2. (Note that the same proof applies without change to every set of the form $[\xi(5/2)^n] - 1 + 30k$, where $k$ is a fixed integer and where $n$ runs over every positive integer.)

Similarly, for $x_n = [\xi(6/5)^n - 1]$, $S(6/5, -1) = \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$. All numbers in this set, except for $-1$ and 1, are divisible by $2, 3$ or 5. This, assuming that there are only finitely many shifted integer parts divisible by 2, 3, 5, leaves just two possibilities. The corresponding formulas for fractional parts are $y_{n+1} = (6y_n + 2)/5$ and $y_{n+1} = 6y_n/5$. For $n$ sufficiently large, $y_n > 0$, since we must have at least once the first operation. (Otherwise this contradicts to Lemma 2.) But, as in both cases $y_{n+1} \geqslant 6y_n/5$, we deduce that $y_n \to \infty$, a contradiction. □

*Proof of Theorem 4.* For 7, we have $S(7, 1/2) = \{-3, -2, -1, 0, 1, 2, 3\}$. Assume that we have only finitely many shifted integer parts $x_n$ divisible by 2 and 7. Then, starting with certain $n$, we must have $x_{n+1} = 7x_n - 2$ (operation $A$) or $x_{n+1} = 7x_n + 2$ (operation $B$). Modulo 3 we have $A = (12|$ and $B = (21|$, so either there are infinitely many integer parts divisible by 3 or, starting with some place, we have $(AB)^\infty$. (There is no contradiction with Lemma 2, because for $a = 7$ it cannot be applied.) However, this means that there is an infinite subsequence of primes defined by the recurrent formula $x_{n+2} = 7(7x_n - 2) + 2 = 49x_n - 12$. Take one of these $x_n = p > 7$. Take $q$ such that $4q + 1$ is divisible by $p$. Then, since $-12 \equiv 48q \pmod{p}$, we get $x_{m+2} + q \equiv 49(x_m + q) \pmod{p}$ for every $m = n, n+2, n+4, \ldots$. Choosing $e \in \mathbb{N}$ such that $p|(49^e - 1)$ and multiplying the first $e$ congruences we get $x_{n+2e} + q \equiv (x_n + q) \pmod{p}$, hence $x_{n+2e} - x_n = x_{n+2e} - p$ is divisible by $p$. So $x_{n+2e} > p$ is divisible by $p$ and thus cannot be prime, a contradiction. This proves part $(i)$.

For 5/3, we have $S(5/3, 1/2) = \{-3, -2, -1, 0, 1, 2, 3\}$. Assume that there are only finitely many shifted integer parts divisible by 2 and 3. This leaves us two options $-2$ and 2 with two respective operations for fractional parts $A : y \to (5y + 1)/3$ (which maps $[0, 2/5)$ to $[1/3, 1)$) and $B : y \to (5y - 3)/3$ (which maps $[3/5, 1)$ to $[0, 2/3)$). If the sequence is not $A^\infty$, $B^\infty$ or $(AB)^\infty$ ($(BA)^\infty$ is the same), then we must have either $A^2$ or $B^2$. Since $A^2 : y \to (25y + 8)/9$ which is greater than or equal to $8/9$, $A^2$ must be followed by $B^2$. Similarly,

$B^2: y \to (25y-24)/9$ which is smaller than $1/9$, so $B^2$ should be followed by $A^2$. We thus have $(A^2B^2)^\infty$, a contradiction with Lemma 2, which completes the proof of $(ii)$.

Finally, $S(7/5, 1/2) = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$. At the expense of prime numbers $2, 5$ we can exclude all numbers from $S(7/5, 1/2)$ except for $-4, -2, 2, 4$. The four remaining possibilities are

$$\begin{aligned}
A:&\ x \to (7x-4)/5,\ \ y \to (7y+3)/5,\ \ [0, 2/7) \to [3/5, 1);\\
B:&\ x \to (7x-2)/5,\ \ y \to (7y+1)/5,\ \ [0, 4/7) \to [1/5, 1);\\
C:&\ x \to (7x+2)/5,\ \ y \to (7y-3)/5,\ \ [3/7, 1) \to [0, 4/5);\\
D:&\ x \to (7x+4)/5,\ \ y \to (7y-5)/5,\ \ [5/7, 1) \to [0, 2/5).
\end{aligned}$$

Modulo 3 we have $A = C = (1|(2)$ and $B = D = (1)(2|$. The sequence thus contains either the operations $A$ and $C$ only or the operations $B$ and $D$ only. (Otherwise there are infinitely many shifted integer parts divisible by 3.) We will consider the $A, C$ case. Assume without loss of generality that the sequence is not $C^\infty$. We then have infinite number of $A$'s. Each $A$ should be followed by a pattern of $C$'s. But $C^4: y \to (2401y - 2664)/625$, so it cannot occur. We thus can have the patterns $AC$, $AC^2$ and $AC^3$ only. Note that $AC: y \to (49y+6)/25$, $AC^2: y \to f(y) = (343y-33)/125$, $AC^3: y \to g(y) = (2401y-606)/625$. Since the functions $(49y+6)/25$ and $f(y)$ at $6/25$ are greater than $2/7$, each $AC$ should be followed by $AC^3$. By Lemma 2, the sequence is not $(AC^2)^\infty$, so this implies that there are infinitely many patterns $AC^3$. We claim that only the patterns $AC^3AC$ and $AC^3AC^2AC$ can occur. Indeed, since $g(2/7) = 16/125$ and $g(16/125) < 0$, $AC^3$ cannot be followed by $AC^3$, so it must be followed either by $AC$ or by $AC^2$. In the first case we have $AC^3$ next after the pattern $AC^3AC$. In the second case, $AC^3AC^2$, since $f(16/125) < 0.09 < 16/125$ we cannot have $AC^3$ next. Also, since $f(0.09) < 0$, we cannot have $AC^2$ next, so $AC^3AC^2$ must be followed by $AC$ which is always followed by $AC^3$. This proves that only the patterns $U = AC^3AC^2AC$ and $V = AC^3AC$ can occur.

We will now seek for a contradiction modulo 11. Assume that there are only finitely many elements divisible by 11. $A$ acts as $x \to 8 - 3x$ and $C$ acts as $x \to 7 - 3x$. This gives $A = (86154793\alpha|(2)$ and $C = (785392146|(\alpha)$. Thus $U = (18|(24|(76|(9)(3|(5|(\alpha|$ and $V = (1)(24\alpha|(93|(78|(5|(6|$. By Lemma 2, the sequence is not $U^\infty$ or $V^\infty$, so there are infinitely many patterns $VU$. But $VU$ can only be of the form $1VU8$. This leads to a contradiction, because $U$ and $V$ cannot begin with 8, so neither $VU^2$ nor $VUV$ can occur.

Finally, note that on replacing each pair $x_n, y_n$ by $-x_n, 1-y_n$, $D$ becomes $A$ and $B$ becomes $C$. The endpoints of the intervals will be the only difference: e.g., instead of $[0, 2/7)$ the respective interval will be $(0, 2/7]$. This makes no difference in our argument, so we do not need to repeat it in the case when $B$ and $D$ are the only operations which occur. The proof of Theorem 4 is now completed. □

In connection with shifted integer parts considered in Theorems 3 and 4 we conclude with the following open question.

**Open question.** Prove that for any pair of real numbers $\xi > 0$ and $\nu$ the set $[\xi 2^n + \nu]$, $n \in \mathbb{N}$, contains infinitely many composite numbers.

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