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The Primes Are a D-Complete Sequence

Paul S. Bruckman

Following Erdös and Lewin [1], we define a *d-complete sequence A* as an infinite increasing sequence of integers, such that no one element of A divides any other, and such that every sufficiently large integer is the sum of distinct elements of A. The purpose of this paper is to demonstrate that the sequence of primes is a *d*-complete sequence.

Clearly, no prime divides any other. Therefore, it suffices to prove that every sufficiently large integer is a sum of distinct primes. However, we shall prove a slightly stronger result than this

We denote the n-th prime as p_n (thus $p_1 = 2$, $p_2 = 3$, etc.), and let $S_n = 2+3+\ldots+p_n$. A few preliminary lemmas are required for our proof. Lemma 1 is actually a well-known theorem in number theory, commonly referred to (erroneously) as Bertrand's "Postulate".

Lemma 1: For all $n \geq 1$, $p_{n+1} < 2p_n$.

Lemma 2: If $n \geq 5$, $S_n - p_{n+1} \geq 15$.

Proof (by induction) : Let M denote the set of integers $n \geq 5$ such that the statement of the lemma is valid. Since $S_5 = 28$, $p_6 = 13$, we see that $5 \in M$. Suppose that $n \in M$. Then $S_{n+1} - p_{n+2} = S_n + p_{n+1} - p_{n+2} = S_n - p_{n+1} + 2p_{n+1} - p_{n+2} > 15$, by the inductive hypothesis and Lemma 1. Thus, $n \in M$ implies $(n+1) \in M$, completing the proof.

Thus, if $n \geq 5$, $S_n > p_{n+1} \geq 13$.

Now consider the following generating functions:

$$f_n(x) = \prod_{k=1}^{n} (1 + x^{p_k}) = \sum_{m=0}^{S_n} \theta(m,n)\, x^m, \tag{1}$$

$n = 1, 2, \ldots,$ given $|x| < 1$.

$$f(x) = \prod_{p} (1 + x^p) = \sum_{m=0}^{\infty} \theta(m)\, x^m. \tag{2}$$

Note that $\lim_{n\to\infty} \theta(m,n) = \theta(m)$. Also, for all $m \geq 0, n \geq 1$,

$$\theta(m) \geq \theta(m,n) \geq 0, \qquad (3)$$

and

$$\theta(m,n+1) \geq \theta(m,n). \qquad (4)$$

Theorem 1 : $\theta(m) \geq 1$ for all $m \geq 7$.

Proof : We first prove the auxiliary result :

If $n \geq 4$, $\theta(m,n) \geq 1$ for all m with $7 \leq m \leq S_n - 7$. (5)

Let N denote the set of integers $n \geq 4$ for which the statement of (5) is valid. Note that $S_4 = 17$; it is a trivial exercise to verify (by direct expansion) that (5) is valid for $n = 4$, and hence that $4 \in N$. Note that $f_{n+1}(x) = (1 + x^{p_{n+1}}) f_n(x)$. Comparison of coefficients yields the following relations, valid for all $n \geq 4$:

$$\theta(m, n+1) = \begin{cases} \theta(m,n) \text{ if } 0 \leq m < p_{n+1}; \\ \theta(m,n) + \theta(m - p_{n+1},n) \text{ if } p_{n+1} \leq m \leq S_n; \\ \theta(m - p_{n+1}, n) \text{ if } S_n < m \leq S_{n+1}. \end{cases} \qquad (6)$$

By the inductive hypothesis, if $n \geq 4$, $\theta(m - p_{n+1},n) \geq 1$ for all m with $7 \leq m - p_{n+1} \leq S_n - 7$, i.e., if $18 \leq 7 + p_{n+1} \leq m \leq S_{n+1} - 7$. Together with the other relations in (6), this implies that $\theta(m,n+1) \geq 1$ whenever $n \geq 4$ and $7 \leq m \leq S_{n+1} - 7$. Thus, $n \in N$ implies $(n+1) \in N$, which completes the proof by induction of (5). Now letting $n \to \infty$ yields the desired result.

We have shown that the sequence of primes is *d*-complete. However, a stronger result is actually true. If $m \geq 8$ is composite, it must be the sum of at least two distinct primes ; but can we make a similar claim if m is prime? The answer is found in the following :

Corollary: Every $m \geq 12$ is the sum of at least two distinct primes.

Proof: In light of Theorem 1, it is sufficient to prove that $\theta(p) \geq 2$ for all primes $p \geq 13$. We may set $m = p_{n+1}$ in (5), assuming that $n \geq 5$. Then it follows that $\theta(p_{n+1}, n) \geq 1$ if $n \geq 5$. However, from (6), it also follows that $\theta(p_{n+1}, n+1) = \theta(p_{n+1}, n) + 1$, which implies that $\theta(p_{n+1}, n+1) \geq 2$ if $n \geq 5$. Equivalently, $\theta(p_n, n) \geq 2$ for all $n \geq 6$. Then $\theta(p_n, N) \geq 2$ for all $N \geq n \geq 6$ (treating n as fixed). Letting $N \to \infty$ and replacing p_n by p ($p \geq 13$), it follows that $\theta(p) \geq 2$ for all $p \geq 13$. This establishes the Corollary.

Conclusion : It might appear at first glance that the result of the Corollary could have some application towards a resolution of the famous *Goldbach Conjecture*, which asserts that every even integer greater than or equal to 6 is a sum of two (possibly identical) primes. However, we have merely shown that every integer $n \geq 12$ is a sum of two *or more* distinct primes ; what is required to prove the Goldbach Conjecture is a demonstration that every even integer $n \geq 6$ is a sum of *exactly* two primes, and this is not implied by the Corollary. Nevertheless, a foundation may have been laid for further research into this question.

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The Case of the Missing Case: The Completion of a Proof by R.L. Graham

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1. Introduction: In his article *On primitive graphs and optimal vertex assignments*, Graham [3] defines the notions of primitive and completely decomposable graphs and proves several results regarding such graphs. We will focus on the proof of one of these results. To get started, a few definitions are in order. We assume throughout that G denotes a graph with vertex set $V(G)$ and edge set $E(G)$. Loops and multiple edges are prohibited.

A subset $C \subseteq E(G)$ is a *cutset* of G if the graph with vertex set $V(G)$ and edge set $E(G) - C$ is disconnected. Note that any disconnected graph has a *cutset*, namely the empty set. We say that C is a *simple cutset* if no two edges of C have a common vertex. If G has a simple cutset, we say that G is *decomposable*; otherwise, G is *indecomposable*. We will call G *completely decomposable* if every subgraph of G (including G itself) is decomposable.

The canonical example of a completely decomposable graph is the N-cube, the graph whose vertex set is the set of all binary N-tuples (or equivalently the set of integers $\{0, 1, ..., 2^N - 1\}$ expressed in base-two representation), and whose edge set consists of all unordered pairs of vertices that differ in exactly one coordinate position. If G is a connected subgraph of the N-cube with at least tow vertices, a simple cutset can be obtained as follows: If the edge ij appears in G, then the vertices i and j differ in exactly one coordinate position, say position n. The simple cutset C then consists of *all* edges joining vertices that differ in coordinate position n.

Graham goes on to define a *primitive* graph to be an indecomposable graph for which every proper subgraph is completely decomposable. We will not need this notion in the present work.

Another definition is required: Given a non-negative integer k, let $w(k)$ denote the sum of the digits in the binary (base-two) representation of k. Put another way, $w(k)$ is the number of ones in the binary representation of k. For $k \geq 1$ let

$$W(k) = w(0) + w(1) + \cdots + w(k - 1).$$

Since the numbers 0, 1, 2, 3, 4, and 5 have binary representations 0, 1, 10, 11, 100, and 101, respectively, we see for example that $W(6) = 0 + 1 + 1 + 2 + 1 + 2 = 7$. The theorem we wish to consider ties together this function W with the notion of complete decomposability[1].

Theorem 1. *(Graham) Let G be a completely decomposable graph with n vertices. Then* $|E(G)| \leq W(n)$. *This bound is the best possible in the sense that for every* $n > 0$ *there exists a completely decomposable graph with n vertices and exactly* $W(n)$ *edges.*

The proof of this theorem that appears in [3] omits an important case, and so is incomplete. We note that this theorem also appears in at least one other article [2] with the reader referred to the original paper for a proof. In this work we provide a complete proof of the above theorem. In so doing, we essentially duplicate many of the fine ideas that appear in Graham's proof, adding only a few of our own. We feel that presenting the complete proof provides a coherent means for understanding the theorem, as well as offering readers new to Graham's work a glimpse into one of the great mathematical minds of the century. With the exception of what we call Case D_2, all of what follows can be attributed to Ronald L. Graham.

2. A Few Facts Regarding The *w* Function. It is prudent to make a few simple observations now regarding Graham's w function. We first note that for $p > 0$,

$$2^p - 1 = \sum_{a=0}^{p-1} 2^a.$$

in other words, the number $2^p - 1$ has only ones in its binary expansion. From this it follows that for $0 \leq k < 2^p$, the binary expansions of k and $2^p - 1 - k$ do not agree in any digit. Hence we see that

$$w(k) + w(2^p - 1 - k) = w(2^p - 1) = p, \text{ for } 0 \leq k < 2^p.$$

All we will need is a slight generalization of this fact; we will restate it,

[1] In Graham's original paper [3], the function W was defined a little differently; it was given as the sum $W(k) = w(0) + w(1) + \cdots + w(k)$.

replacing 2^p by an arbitrary multiple of 2^p. Given a multiple of 2^p, express it as $2^q u$, where $q \geq p$ and u is odd. Being a multiple of 2^p, the last p digits in the binary expansion of $2^q u$ are all zeros, and hence the last p digits in the binary expansion of $2^q u - 1$ are all ones. It follows that for $0 \leq k < 2^p$, the binary expansions of k and $2^p u - 1 - k$ do not agree in any of the last p digits. Hence we see that

$$w(k) + w(2^p u - 1 - k) = w(2^p u - 1), \text{ for } 0 \leq k \leq 2^p. \qquad (1)$$

This fact will be used at the end of the proof of the theorem.

3. Proof of the Theorem. It is not difficult to see that the bound $W(n)$ can be attained for all n. Given $n > 0$, consider the graph G_n with

$V(G_n) = \{0, 1, ..., n - 1\}$, and

$E(G_n) = \{ij \mid$ the base-two representations of vertices i and j differ in exactly one position$\}$.

Choose N with $2^N > n$. Then G_n is easily seen to be a subgraph of the N-cube, so G_n is completely decomposable. Now since G_1 is a singleton, $|E(G_1)| = 0 = W(1)$. Noting that G_j is a subgraph of G_{j+1} for all $j > 0$, observe that G_{j+1} has exactly $w(j)$ more edges than does G_j. The additional edges are those incident with vertex j; there is an edge from vertex j to exactly those vertices whose base-two representations can be obtained from that of j's by replacing a 1 by a 0, and there are $w(j)$ ones in the base-two representation of j. Thus induction on n shows that

$$|E(G_n)| = w(0) + w(1) + \cdots + w(n - 1) = W(n).$$

The remainder of the proof is devoted to establishing the inequality $|E(G)| \leq W(n)$. The proof proceeds by induction of n. If $n = 1$ the result holds vacuously, and if $n = 2$ the result holds since G can have no more than one edge, and $W(2) = 1$. Now suppose $n > 2$ and the theorem holds for all completely decomposable graphs with fewer than n vertices. Let G be a completely decomposable graph with n vertices. Let C be a simple cutset of for G, and let G_1 and G_2 be two disjoint subgraphs of G satisfying

(i) $V(G_1) \neq \emptyset$, and $V(G_2) \neq \emptyset$,

(ii) $V(G_1) \cup V(G_2) = V(G)$,

(iii) $E(G_1) \cup E(G_2) = E(G) - C$, and

(iv) Every edge in C joins a vertex of G_1 with a vertex of G_2.

Essentially, G_1 and G_2 are two graphs on "either end" of the simple cutset C.

We write $n_1 = |V(G_1)|$ and $n_2 = |V(G_2)|$, so that $n = n_1 + n_2$. Without loss of generality, assume that $n_1 \geq n_2 > 0$. We invoke our inductive hypothesis on the graphs G_1 and G_2, and use the fact that C is a *simple* cutset to obtain

$$|E(G)| = |E(G_1)| + |E(G_2)| + |C|$$
$$\leq W(n_1) + W(n_2) + n_2.$$

The theorem will follow if we can establish for $n_1 \geq n_2 > 0$ with $n_1 + n_2 = n$, that

$$W(n_1) + W(n_2) + n_2 \leq W(n),$$

or equivalently

$$W(n_2) + n_2 \leq W(n) - W(n_1),$$

or equivalently

$$[w(0) + 1] + [w(1) + 1] + \cdots + [w(n_2 - 1) + 1] \leq w(n_1) + w(n_1 + 1) + \cdots + w(n - 1).$$

This follows immediately from part (ii) of the following lemma if one takes $s = n_1$ and $r = n_2 - 1$.

4. The Offending Lemma.

Lemma 1. Let r and s be integers greater than or equal to 0. For a 1 - 1 function

$$\varphi: \{0, 1, ..., r\} \to \{s, s + 1, \cdots, s + r\}$$

define $\delta(\varphi)$ by $\delta(\varphi) = \min\, [w(\varphi(k)) - w(k)]$, $0 \leq k \leq r$. Then,

(i) There exists a φ such that $\delta(\varphi) \geq 0$, and

(ii) If $s > r$, then there exists a φ such that $\delta(\varphi) \geq 1$.

Proof: The proof will be broken down into cases as follows:

Case A: The case when $s = 0$.

Case B: The case when $s > 0$ and $r = 0$.

Case C: The case when $0 < s \leq r$.

Case D: The case when $s > r > 0$.

<u>Case A</u>

If $s = 0$, then let φ: $\{0, 1, ..., r\} \to \{0, 1, ..., r\}$ be the identity function. Then for all k with $0 \leq k \leq r$, $w(\varphi(k)) - w(k) = w(k) - w(k) = 0$. Thus, (i) holds and (ii) holds vacuously since $s \not> r$.

Case B

If $r = 0$ and $s > 0$, φ: $\{0\} \to \{s\}$ is uniquely determined. Hence $\delta(\varphi) = w(\varphi(0)) - w(0) = w(s) - w(0) = w(s) - 0 = w(s) > 0$. Thus, $\delta(\varphi) \geq 1$. So, (ii) holds and this implies (i).

Thus, the lemma holds for $r = 0$ and all $s \geq 0$. The remaining cases proceed by induction on r. Having established the initial case $r = 0$, we assume $r > 0$.

Case C

If $0 < s \leq r$, then the two sets $\{0,\ldots,r\}$ and $\{s,\ldots,s+r\}$ overlap. We will partition these sets in the following manner:

$$\{0,\ldots,r\} = \{0,\ldots, s-1\} \cup \{s,\ldots,r\}, \text{ and}$$
$$\{s,\ldots,s+r\} = \{s,\ldots,r\} \cup \{1+r,\ldots,s+r\}.$$

Let φ_1: $\{s,\ldots,r\} \to \{s,\ldots,r\}$ be the identity map. Thus, $\delta(\varphi_1) = 0$. By the inductive hypothesis, there exists a map φ_2: $\{0,\ldots,s-1\} \to \{1+r,\ldots,s+r\}$ with $\delta(\varphi_2) \geq 1$. Define φ: $\{0, 1,\ldots,r\} \to \{s, s+1,\ldots,s+r\}$ so that φ agrees with φ_1 and φ_2 on their respective domains. Then $\delta(\varphi) = \min[\delta(\varphi_1), \delta(\varphi_2)] = 0$. Hence, (i) holds and (ii) holds vacuously since $s \not> r$.

Case D

If $s > r > 0$, then it suffices to establish (ii) since this implies (i). Let p satisfy $2^{p-1} \leq r < 2^p$. This implies that $2^p \in \{0, 1,\ldots,2r\}$. Furthermore, there is at least one multiple of 2^p in every set of $2r+1$ consecutive integers. In particular, there must be at least one multiple of 2^p in the set $\{s-r,\ldots,s,\ldots s+r\}$. Choose the largest of these to be expressed as 2^qu where $q \geq p$ and u is odd. Now, Case D can be broken down into two cases:

Case D_1: The case when $2^qu \in \{s+1,\ldots, s+r\}$.

Case D_2: The case when $2^qu \in \{s-r,\ldots,s\}$, and no multiple of 2^p lies in the set $\{s+1,\ldots,s+r\}$.

Graham's mistake in [3] was in not acknowledging the possibility of Case D_2. For example, if $r = 5$ and $s = 9$, then $2^p = 8$, and no multiple of 8 lies in the set $\{10, 11, 12, 13, 14\}$.

Case D_1.

Let $x = 2^qu - s$. Thus, $s = 2^qu - x$, and $1 \leq x \leq r$. Partition $\{0,\ldots,r\} = \{0,\ldots,r-x\} \cup \{r-(x-1),\ldots,r\}$ and

$$\{s,\ldots,s+r\} = \{2^qu - x,\ldots, 2^qu - 1\} \cup \{2^qu,\ldots, 2^qu + r - x\}.$$

Let φ_1: $\{0,\ldots,r-x\} \to \{2^qu,\ldots,2^qu+r-x\}$ be defined via $k \to 2^qu + k$ for each $k \in \{0,\ldots,r-x\}$. Then since $k \leq r - x < 2^p$, and since 2^qu is a multiple of 2^p, the binary expansions of k and $\varphi_1(k) = 2^qu + k$ agree in the last p digits. Hence $w(\varphi_1(k)) - w(k) = w(2^pu) \geq 1$ for all $k \in \{0,\ldots,r-x\}$, and we see that $\delta(\varphi_1) \geq 1$.

We will show that there exists a function φ_2: $\{r-(x-1),\ldots,r\} \to \{2^qu - x,\ldots,2^qu - 1\}$ with $\delta(\varphi_2) \geq 1$. Then φ can be defined to agree with φ_1 and φ_2 on their respective domains, and $\delta(\varphi) = \min[\delta(\varphi_1), \delta(\varphi_2)] \geq 1$, and Case D_1 is proved.

To establish the existence of φ_2 with $\delta(\varphi_2) \geq 1$, note that since $r < s = 2^qu - x$, we have $x - 1 < 2^qu - 1 - r$, and so by the inductive hypothesis there exists a function

φ_3: $\{0, 1,\ldots,x-1\} \to \{2^qu - 1 - r,\ldots,2^qu - 1 - r + x - 1\}$ with $\delta(\varphi_3) \geq 1$.

Hence

$$w(\varphi_3(k)) - w(k) \geq 1 \text{ for } 0 \leq k \leq x - 1. \tag{2}$$

We now define φ_2 as follows: For $k \in \{0,\ldots,x-1\}$,

$$\varphi_2: 2^qu - 1 - \varphi_3(k) \mapsto 2^qu - 1 - k.$$

To check that φ_2 satisfies $\delta(\varphi_2) \geq 1$, invoke equation (1) twice. For $0 \leq k < 2^p$, we have

$$w(k) + w(2^qu - 1 - k) = w(2^qu - 1),$$

and for $0 \leq 2^qu - 1 - \varphi_3(k) < 2^p$, we have

$$w(\varphi_3(k)) + w(2^qu - 1 - \varphi_3(k)) = w(2^qu - 1).$$

Since $x - 1 < r < 2^p$, and $2^qu - 1 - \varphi_3(k) \leq r < 2^p$, we can substitute each of these equations into inequality (2) to get for $0 \leq k \leq x - 1$,

$$\begin{aligned} 1 &\leq w(\varphi_3(k)) - w(k) \\ &= [w(2^qu - 1) - w(2^qu - 1 - \varphi_3(k))] - [w(2^qu - 1) - w(2^qu - 1 - k)] \\ &= w(2^qu - 1 - k) - w(2^qu - 1 - \varphi_3(k)). \end{aligned}$$

Hence for $j \in \{r-(x-1),\ldots,r\}$, we have $w(\varphi_2(j)) - w(j) \geq 1$, and thus $\delta(\varphi_2) \geq 1$.

Case D_2 (The missing case).

Let $x = s - 2^qu$, so $0 \leq x \leq r$. We will show the existence of φ_1:$\{0, \ldots, r\} \to \{x, \ldots, x+r\}$ and construct φ_2:$\{x,\ldots, x+r\} \to \{s, \ldots, s+r\}$. We will then define φ:$\{0,\ldots,r\} \to \{s,\ldots,s+r\}$ as the composition $\varphi = \varphi_2 \circ \varphi_1$.

Since $x \le r$, we may invoke the inductive hypothesis as in Case C to show there exists a 1 - 1 function φ_1: $\{0,...,r\} \to \{x,...,x+r\}$ with $\delta(\varphi_1) \ge 0$.

Define φ_2 via $k \mapsto 2^q u + k$ for all $k \in \{x, ..., x + r\}$. Since there is no multiple of 2^p in the set $\{2^q u + 1, ..., s, s + 1, ..., s + r\}$, and this set has cardinality $s + r - 2^q u = x + r$, we see that $x + r < 2^p$. Hence for each $k \in \{x, ..., x + r\}$, the binary expansions of k and $\varphi_2(k) = 2^q u + k$ agree in the last p digits. So $w(\varphi_2(k)) - w(k) = w(2^q u) \ge 1$ for all $k \in \{x, ..., x + r\}$, and we see that $\delta(\varphi_2) \ge 1$.

Therefore, $\delta(\varphi) = \delta(\varphi_2 \circ \varphi_1) \ge \delta(\varphi_2) + \delta(\varphi_1) \ge 1 + 0 = 1$. Hence, (ii) is shown, and the lemma is proved.

5. Remarks. The basic structure for the proof of this theorem was applied to prove a similar result in [1] in 1988. Here the authors, Graham among them, prove a somewhat weaker result in Lemma 4.1, which essentially states:

If G is a subgraph of the N-cube with n vertices, then $|E(G)| \le \frac{1}{2} n \log_2 n$.

Their proof works for the more general class of completely decomposable graphs, but the bound is not tight unless n is a power of 2. The advantage of this approach is that nothing along the lines of Lemma 1 is needed, so the proof is considerably shorter.

An immediate consequence of this result together with the tightness of the bound in Theorem 1 is that for all integers $n > 0$,

$$W(n) \le \frac{1}{2} n \log_2 n.$$

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Divisibility Tests - Making Order out of Chaos

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I. Introduction

If the last digit of a base ten numeral is even, then the number is divisible by 2. If the sum of its digits is divisible by 3, then the number is divisible by 3. If the sum obtained by alternately adding and subtracting its digits is divisible by 11, then the number is divisible by 11. A number is divisible by 7 if the number that results when you subtract twice its last digit from the number that remains is divisible by 7. It seems as if there is a distinctly different test to decide whether a given number is divisible by each different divisor. We shall make order from this chaos of tests by showing that all divisibility tests fall into just two categories and that one can readily find a test for any given divisor from at least one of these categories.

The First Divisibility Test

First, we introduce some helpful preliminary ideas, that is, theorems. We observe that, for a given divisor d, if two integers are divisible by d, then so also is their sum divisible by d. More generally, if integers a and b are both divisible by d and if m and n are any integers whatever, then $ma + nb$, that is, any *linear combination* of a and b, is divisible by d. Thus, since 26 and 39 are both divisible by 13, then we know that $26078 = 1000 \cdot 26 + 2 \cdot 39$ is divisible by 13. Also, $25961 = 1000 \cdot 26 + (-3) \cdot 13$ is divisible by 13. Finally, we note as a corollary that if $a + b = c$ for any integers a, b, and c, and if any two of these integers are divisible by a divisor d, then so also is the third one divisible by d. Hence, either none of a, b, and c, exactly one of them, or all three of them are divisible by d, but never just two of them.

An immediate result of these theorems is that a base ten numeral is divisible by 2 or by 5 if its units digit is so divisible. For example, $3517 = 3510 + 7$. Since 3510 is divisible by 10, it is divisible by 2 and by 5. If the units digit 7 is divisible by either 2 or 5, then the number 3517 must be so divisible. More importantly, since the difference between 3517 and 7 is a multiple of 2 and of 5, then when either number 3517 or 7 is divided by one of these two numbers, the remainder will be the same. That is, when you divide either 3517 or 7 by 5, the remainder is 2. This idea leads us to a definition.

The First Divisibility Test

Definition. We write $a \equiv b \pmod{n}$ and we say "*a is congruent to b, modulo n*" to denote that integers a and b have the same remainder when each is divided by the positive integer n, where $n \geq 2$. From the preceding paragraph we see that this congruence means that the difference between a and b is a multiple of n. That is, there is an integer k such that $a - b = kn$.

Hence, $3517 \equiv 7 \pmod{10}$ since the difference $3517 - 7 = 3510$ is divisible by 10. Of course, we also have $3517 \equiv 7 \pmod{5}$, $3517 \equiv 7 \pmod{2}$; and $3517 \equiv 7 \pmod{351}$, too. It is convenient to note some elementary properties of modular arithmetic which we shall use to justify our divisibility tests. We state these properties as a pair of theorems.

Theorem. Congruence modulo n is an equivalence relation. That is, the following three statements hold, where a, b, c, and n are integers and $n \geq 2$:

1) $a \equiv a \pmod{n}$,
2) If $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$, and
3) If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.

Proof. We illustrate the method of proof by establishing part (3) of this theorem. To that end we note that there are integers p and q such that $a - b = pn$ and $b - c = qn$ from the definition of congruence modulo n. Then

$$a - c = (a - b) + (b - c) = pn + qn = (p + q)n,$$

which shows that $a - c$ is a multiple of n and hence $a \equiv c \pmod{n}$. ⊛

Theorem. If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then

1) $a + c \equiv b + d \pmod{n}$,
2) $a - c \equiv b - d \pmod{n}$, and
3) $ac \equiv bd \pmod{n}$.

Proof. By the definition we have $a - b = pn$ and $c - d = qn$ for some integers

p and q. Parts (1) and (2) are easily established by adding and subtracting these two equations. For part (3), we rewrite them in the form $a = b + pn$ and $c = d + qn$ and then multiply them side for side, obtaining

$$ac = (b + pn)(d + qn) = bd + bqn + pnd + pqn^2$$

$$= bd + (bq + pd + pqn)n,$$

so $ac \equiv bd \pmod{n}$. ⊛

Recall that place value in a base ten numeral indicates that powers of ten are multiplied by the digits of the numeral. For example,

$$63405 = 6 \cdot 10^4 + 3 \cdot 10^3 + 4 \cdot 10^2 + 0 \cdot 10 + 5.$$

These powers of ten *weight* the digits of the number (just as one weights tests, so if a test t counts two quizzes q, then their average is $(2t + 1q)/3$ using the weights 2/3 and 1/3). Consider testing such a number for divisibility by a divisor n. If we replace each power of ten by a smaller number congruent to it modulo n, then the theorems stated above prove that the given number and the new *weighted digit sum* will have the same remainder when we divide them by n.

For example, since $10 \equiv 1 \pmod{3}$, $10^2 = 100 \equiv 1 \pmod{3}$, and in general, $10^k \equiv 1 \pmod{3}$ for every positive integer k, then

$$63405 \equiv 6 \cdot 1 + 3 \cdot 1 + 4 \cdot 1 + 0 \cdot 1 + 5 = 6 + 3 + 4 + 0 + 5 = 18 \pmod{3},$$

the sum of its digits. Since 18 divided by 3 leaves a remainder of 0, then so also does 63405 leave a remainder of 0. As another example, $10 \equiv -1 \pmod{11}$, $100 \equiv 1 \pmod{11}$, $1000 \equiv -1 \pmod{11}$, $10000 \equiv 1 \pmod{11}$, etc., so that

$$63405 \equiv 6 \cdot 1 + 3 \cdot (-1) + 4 \cdot 1 + 0 \cdot (-1) + 5 = 6 - 3 + 4 - 0 + 5 = 12 \pmod{11},$$

so 63405 leaves the remainder 1 when divided by 11 because 12 leaves the remainder 1. The weights 1, -1, 1, -1, ... by which we multiplied the digits are called *digit weights*. We make formal the ideas illustrated in this paragraph.

Definition. *Digit weights* are an ordered set of integers $\{b_0, b_1, \ldots, b_{k-1}, \ldots\}$

used to weight the digits of a base ten numeral $q = a_n a_{n-1} a_{n-2} \dots a_0$, whose digits are $a_n, a_{n-1}, a_{n-2}, \dots, a_0$, in the following manner to form the *weighted digit sum* (wds) w given by

$$w = b_0 a_0 + b_1 a_1 + \dots + b_{k-1} a_{k-1} + b_k a_k + \dots + b_n a_n.$$

A *divisibility set* (of digit weights) *for a given divisor n* is a set of digit weights where $b_0 = 1$ and for $k \geq 1$, $b_k \equiv 10^k \pmod{n}$.

Thus a divisibility set for the divisor 3 is $\{1, 1, 1, \dots\}$ and a divisibility set for 11 is $\{1, -1, 1, -1, \dots\}$, as we saw above. Note that the first digit weight b_0 is multiplied by the units digit, the next digit weight b_1 is multiplied by the tens digit, and so forth, working from left to right in the set of digit weights but multiplying by the digits of the number from right to left. To create such a divisibility set for a given divisor n, the first digit weight is always $b_0 = 1 = 10^0$. The next digit weights are taken to be the remainders when the successive powers of 10 are divided by n.

A divisibility set for 2 is thus $\{1, 0, 0, 0, \dots\}$ since the remainder is zero when any positive power of 10 is divided by 2. The wds is just the units digit of the number, and we arrive at the usual test for divisibility by 2. A divisibility set for 7 is $\{1, 3, 2, 6, 4, 5, 1, 3, 2, 6, 4, 5, \dots\}$ since

$$10 \equiv 3 \pmod{7},$$
$$100 = 10^2 \equiv 3^2 = 9 \equiv 2 \pmod{7},$$
$$1000 = 100 \cdot 10 \equiv 2 \cdot 3 = 6 \pmod{7},$$
$$10000 = 10 \cdot 1000 \equiv 3 \cdot 6 = 18 \equiv 4 \pmod{7},$$

and so forth. Similarly, a divisibility set for 6 is $\{1, 4, 4, 4, 4, \dots\}$. Thus

$$63405 \equiv 1 \cdot 5 + 4 \cdot 0 + 4 \cdot 4 + 4 \cdot 3 + 4 \cdot 6 = 57 \pmod{6}.$$

Since 57 leaves a remainder of 3 when divided by 6, then so also does 63405 leave a remainder of 3 when divided by 6. Of course, 57 can be further reduced because

$$57 \equiv 1 \cdot 7 + 4 \cdot 5 = 27 \pmod{6}.$$

The process of finding weighted digit sums can be applied as many times as one wishes to reduce a number to a smaller one.

Observe that every divisibility set we have seen repeats after a time. This repeating will always occur because each digit weight is a remainder when we divide by a divisor n. Since there are only n possible remainders $(0, 1, \dots, n - 1)$, it follows that when we get to the $(n + 1)$st digit weight, there must have been a repetition; a digit weight must have appeared a second time. As soon as a digit weight appears a second time, then all the digit weights from the first to the second appearance will repeat. The method by which we find divisibility sets shows this repetition.

For example, we have seen that $10^6 \equiv 1 \pmod{7}$. Then $10^7 = 10^6 \cdot 10 \equiv 1 \cdot 10 = 10 \equiv 3 \pmod{7}$, $10^8 = 10^6 \cdot 10^2 \equiv 1 \cdot 10^2 \equiv 2$, and so forth. Notice that the repetend may not start immediately, as in the divisibility set for 6. We simplify our notation because of our observation.

Definition. Henceforth we shall denote a set of digit weights as follows: $\{b_0, b_1, \dots, b_{k-1};; c_0, c_1, \dots, c_{m-1}\}$, where the weights $c_0, \dots, c_{m-1}$ form the repetend. If the repetend starts with the very first digit weight, then the b's and the double semicolon are omitted.

Thus we list the divisibility set for 2 as $\{1;;0\}$, that for 3 is $\{1\}$, that for 6 is $\{1;;4\}$, that for 7 is $\{1,3,2,6,4,5\}$, and the divisibility set for 11 is $\{1,-1\}$. Compare these lists with those we found earlier to be sure you understand the notation.

Of course, we could have used the divisibility set $\{1,10\}$ for 11, but it is easier to alternately add and subtract the digits of a number than to apply this set. The set $\{1,10\}$ applied to our earlier example would yield

$$63405 \equiv 1 \cdot 5 + 10 \cdot 0 + 1 \cdot 4 + 10 \cdot 3 + 1 \cdot 6 = 05 + 34 + 6 = 45,$$

which again has a remainder of 1 when divided by 11. Thus, a divisibility set is not unique. We may replace any digit weight by another that is congruent to it. Thus, for 4 we could use either the divisibility set $\{1,2;;0\}$ or more commonly $\{1,10;;0\}$. The test for divisibility by 4, using this latter divisibility set, says simply check whether the number formed by the last two digits of the given number is divisible by 4.

Table 1. Some Common Divisibility Sets

Divisor	Divisibility Set	Divisor	Divisibility Set
2	1;;0	13	1,-3,-4,-1,3,4
3	1	14	1;;-4,2,6,4,-2,-6
4	1,10;;0	15	1;;-5
5	1;;0	16	1,10,100,1000;;0
6	1;;-2	17	1,-7,-2,-3,4,6,-8,5, -1,7,2,3,-4,-6,8,-5
7	1,3,2,-1,-3,-2	18	1;;-8
8	1,10,100;;0	19	1,-9,5,-7,6,3,-8,-4,-2, -1,9,-5,7,-6,-3,8,4,2
9	1	20	1,10;;0
10	1;;0	25	1,10;;0
11	1,-1	1001	1,10,100,-1-10,-100
12	1,-2;;4		

Another example of a better divisibility set formed by changing the digit weights we use is the set for 7. Since 6 ≡ -1, 4 ≡ -3, and 5 ≡ -2 (mod 7), we write the divisibility set for 7 as {1,3,2,-1,-3,-2}, which is much easier to remember than our earlier set. In fact, it is easy to see that if the digit weight -1 ever occurs, then the rest of the divisibility set is just the negatives of the digits that have already occurred. Our method of calculating digit weights readily shows the truth of this statement.

We list some common divisibility sets for small divisors in Table 1. For convenience, they are listed without braces.

In this table we see many of the common tests for divisibility. For 2, 5, or 10, just look at the last digit of the number. For 4, 20, or 25, look at the number formed by the last two digits of the number. For 8, look at the number formed by the last three digits of the number. For 16, look at the number formed by the last four digits of the number. This pattern continues for higher powers of 2 such as 32 or 64. For 3 or for 9, look at the sum of the digits of the number. Since the divisibility set for 11 is either {1,-1} or {1,10}, we can use either test described above.

To check whether 65,221,806 is divisible by 7 and to find the remainder if it is not, let us perform the calculations as in the display below. Write the number to be tested, leaving space between its digits. In the line above, write the numbers of the divisibility set, starting with the units digit and repeating as necessary. Multiply down each column, writing the products in the third row. Finally, add the products along the third line to find the desired weighted digit sum. We have

$$\begin{array}{cccccccc} 3 & 1 & -2 & -3 & -1 & 2 & 3 & 1 \\ \underline{6} & \underline{5} & \underline{2} & \underline{2} & \underline{1} & \underline{8} & \underline{0} & \underline{6} \\ +18 & +5 & -4 & -6 & -1 & +16 & +0 & +6 \end{array} = 34.$$

The wds is 34, which leaves a remainder of 6 when divided by 7. Therefore, 65,221,806 is not divisible by 7 and in fact leaves a remainder of 6. This tableau is an easy way to apply the more complicated divisibility sets. Be sure to write the + and - signs in the product row so you are sure to add correctly, adding the 16 rather than 1 + 6 = 7, in the displayed example. The tableau also provides a format in which it is easy to check the computations.

The last, rather strange-looking, set of digit weights listed in Table 1 is most useful to us. Applying the 1001 test is easy: just separate the number into blocks of three digits each by the usual commas and attach alternating signs to the groups, taking the units group positive, the thousands group negative, and so forth. For example,

$$65{,}221{,}806 \equiv +806 - 221 + 65 = 650 \pmod{1001}.$$

The great importance of this test is that $1001 = 7\cdot 11\cdot 13$. Thus, the 1001 test is actually a convenient test for divisibility by 7, by 11, and by 13, all in one easy package. Thus, to find out whether 65,221,806 is divisible by 7, by 11, or by 13, one need only test the wds 650 for such divisibility. Since 650 is divisible by 13, but not by 7 or 11, then 65,221,806 is divisible only by 13 and not 7 or 11. Again, of course, the given number and the wds will both have the same remainder when divided by the divisor. Since 650 leaves a remainder of 6 when divided by 7, then so also does 65,221,806 leave a remainder of 6.

Thus we see that modular arithmetic, divisibility sets, and weighted digit sums are the basis for all these common divisibility tests. These ideas provide the unifying concept that ties all these seemingly distinct divisibility tests together in one neat mathematical package.

We observe that the number of digits in the divisibility set for a prime p, different from 2 and 5, is a divisor of p - 1. If a number is divisible by neither 2 nor 5, then its divisibility set is all repetend. That is, its repetend starts with the first digit. For numbers that are divisible by 2 or 5, the divisibility set will have at least one digit before the repetend.

The Second Divisibility Test

There is another type of divisibility test we have not yet studied here, but did allude to in the opening paragraph. To test a number for divisibility by 7, cut the units digit off the number and subtract twice that digit from the number that remains. Repeat the process until the result is a two-digit number and then test it. For our 63405, we would find

$$6340 - 2\cdot 5 = 6330,\ 633 - 2\cdot 0 = 633, \text{ and } 63 - 2\cdot 3 = 57.$$

Since 57 is not divisible by 7, then neither is 63405. Observe that when 57 and 63405 are divided by 7, their remainders are different. This test does show whether a number is divisible by 7, but does not give a true remainder if the number is not divisible by 7.

Definition. Let $n = 10k + d$ be a positive integer, where d is the last digit of n and k is the number that remains when d is cut off from n. That is, $k = (n - d)/10$. For a given divisor p, if there is an integer m such that n and $k + md$ both are divisible by p or both are not divisible by p for all choices of k and d, then we say that "m is a *multiplier* for the *units digit multiplier divisibility test* for the divisor p."

We have stated that $m = -2$ is a multiplier for the divisor 7. We shall show that every positive integer greater than 1 and not divisible by 2 or 5 will have a multiplier. That is, any divisor greater than 1 that ends in 1, 3, 7, or 9 will have a multiplier for the units digit multiplier divisibility test. We first prove this fact for the divisor 7. If we wish to test a number n for divisibility by 7, and if mn is any multiple of n such that m is not divisible by 7, then n and mn are both divisible by 7 or are both not divisible by 7. Notice that $3\cdot 7 = 21 = 20 + 1$; that is, $3\cdot 7$ differs from a multiple of 10 by 1. Therefore, let us examine

$$2n = 20k + 2d = 21k - (k - 2d).$$

Since 7 divides 21, then 7 divides $21k$. Now, if 7 divides either one of the numbers n and $k - 2d$, then it must also divide the other number.

So we search for a multiple of the divisor that differs from a multiple of 10 by exactly 1. Any number that ends in 1 or in 9 already differs from a multiple of 10 by 1. If a number ends in 3 or in 7, then multiply it by 3 (as we did above with 7). Since $3\cdot 7 = 21$, which is 1 more than 2 tens, then $m = -2$. Since $13\cdot 3 = 39$, the test for the divisor 13 is found from the equation

$$4n = 40k + 4d = 39k + (k + 4d).$$

Since 39 is a multiple of 13, then $4n$ will be divisible by 13 if and only if $k + 4d$ is so divisible. Thus $m = +4$ for the divisor 13. In general, m is in absolute value equal to the multiplier of ten to which the divisor or three times the divisor, whichever applies, is closest. The sign on m is + if the divisor or three times it is less than its nearest multiple of 10, that is, when the divisor ends in 3 or 9, and the sign is - when the divisor ends in 1 or 7. It is easy to see which sign applies by writing the equation corresponding to the last two displayed equations.

To illustrate the units digit multiplier test again, let us check 63405 for divisibility by 13. Since $m = +4$, we have

$$6340 + 4\cdot 5 = 6360,\ 636 + 4\cdot 0 = 636,\ 63 + 4\cdot 6 = 87, \text{ and } 8 + 4\cdot 7 = 36.$$

Since 36 is not divisible by 13, then 63405 is not divisible by 13. Remember that the remainders may differ.

Table 2 lists the values of m for some common small divisors. Remember that the original number $n = 10k + d$ and $k + md$ both are or both are not divisible by the stated divisor. The remainders, however, may not be the same. The units digit multiplier divisibility test can be used for any divisor that ends in 1, 3, 7, or 9 and is larger than 1.

Table 2. Values of *m*, the Units Digit Multiplier

Divisor	m	Divisor	m
3	+1	23	+7
7	-2	27	-8
9	+1	29	+3
11	-1	31	-3
13	+4	33	+9
17	-5	37	-11
19	+2	39	+4
21	-2	41	-4

Notice the patterns in the table. The divisors 9 and 11 have m values of +1 and -1, 19 and 21 have +2 and -2, and so forth. The values of m for numbers ending in 3 increase by 3 each time, those ending in 7 decrease by 3. Hence, it is quite easy to extend the table as far as one might wish.

It is clear that the first divisibility test is based on arithmetic modulo the divisor. The second divisibility test also leans heavily on that arithmetic. For instance, $n \equiv 0 \pmod 7$ if and only if $2n \equiv 0 \pmod 7$. Since

$$2n = 20k + 2d = 21k - (k - 2d), \text{ then } 2n \equiv -(k - 2d) \pmod 7.$$

Of course, $-(k - 2d) \equiv 0 \pmod 7$ if and only if $k - 2d \equiv 0 \pmod 7$. We have shown that

$$n \equiv 0 \pmod 7 \text{ if and only if } k - 2d \equiv 0 \pmod 7.$$

That is, 7 divides n if and only if 7 divides $k - 2n$. Similarly, our work above shows that, when $n = 10k + d$, then

$$n \equiv 0 \pmod{13} \text{ if and only if } k + 4d \equiv 0 \pmod{13}.$$

Thus, modular arithmetic is the unifying concept behind both divisibility tests.

Conclusion

Certainly, for divisors such as 2, 3, 4, and 5, it is easy to use the tests based on digit weights. For 7, it is easier to use the units digit multiplier test than to remember the digit weights for 7. Testing for divisibility by 11 is easy either way. The 1001 test is an easy way to test for 7 or 11 or 13. Definitely, the units digit multiplier test is best for 19. Thus, to test 65,221,806 for divisibility by 19, we calculate

$$6522180 + 2\cdot 6 = 6522192,\ 652219 + 2\cdot 2 = 652223,\ 65222 + 2\cdot 3 = 65228,$$

$$6522 + 2\cdot 8 = 6538,\ 653 + 2\cdot 8 = 669,\ 66 + 2\cdot 9 = 84, \text{ and } 8 + 2\cdot 4 = 16,$$

which is not divisible by 19. The process is a bit tedious when the number being tested is large, but it does not require remembering a long divisibility set. Of course, if we need to know the remainder, we can use digit weights, provided a divisibility set is readily available, as it is here in Table 1. We have

$$\begin{array}{rrrrrrrr} -4 & -8 & 3 & 6 & -7 & 5 & -9 & 1 \\ \underline{6} & \underline{5} & \underline{2} & \underline{2} & \underline{1} & \underline{8} & \underline{0} & \underline{6} \\ -24 & -40 & +6 & +12 & -7 & +40 & +0 & +6 \end{array} = -7.$$

To calculate the remainder when -7 is divided by 19, find the smallest positive integer congruent to -7 modulo 19 by adding 19 or a multiple of 19 to the -7. Thus we get $-7 + 19 = 12$. One can check that 65,221,806 does indeed leave a remainder of 12 when divided by 19.

So both tests are useful, depending upon the divisor and the number being tested. Any divisibility test I have seen falls into one or both of the two categories listed here. For instance, one modification of the 1001 test is to cut off the last three digits of the given number and subtract the number formed by these three digits from the remaining number. The process can be repeated as desired. The resulting difference and the given number will both be or will both not be divisible by 7, by 11, or by 13. Remainders, however, are not preserved in such a test modification.

Perhaps divisibility tests are not very important now that calculators are so readily available to do the actual division. On the other hand, it is vital to understand the structure of the number system in our technical world and it is a waste of time to dig out a calculator to decide whether a number is divisible by 2 or by 5. In any given situation, if one understands divisibility tests, then he or she can apply whichever test seems appropriate for the situation.

Here we have seen that divisibility tests are not just a conglomeration of unrelated special devices, but are interconnected by a mathematical structure. Hence the reader is now able to construct tests for divisibility by any divisor whatever.

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A New Insight Into The Goldbach Conjecture

Paul S. Bruckman

In 1742, Goldbach wrote a letter to Leonhard Euler, in which he conjectured that every integer $n > 5$ is the sum of three primes. Euler replied that this is easily seen to be equivalent to the conjecture that every even integer $n \geq 4$ is the sum of two primes. In turn, since $4 = 2+2$, it is easily seen that we may state the conjecture in the following form :

<u>Goldbach Conjecture (GC)</u> : Every even integer $2n \geq 6$ is the sum of two (necessarily odd) primes .

Despite the best efforts of mathematicians for the past 250 years, this famous conjecture has resisted all attempts at a proof, although various results *very nearly* proving it have been obtained. The numerical evidence supporting GC is overwhelming, yet it continues to be a source of frustration that such a simple statement, so easily understood by the average layperson, has not been proved.

Perhaps the closest result that approaches a proof of GC is a result due to J.R. Chen [1], which states that every sufficiently large even integer may be written as $2n = p+m$, where p is prime and m is a product of two (not necessarily distinct) primes.

In a classic paper [2], Hardy and Littlewood made the following now-famous conjecture regarding $r_2(2n)$, the number of partitions of the even integer $2n$ into a sum of two primes :

$$r_2(2n) \sim 2C_2\, n\,/(\log 2n)^2 \prod_{p|n,\, p>2} (p-1)/(p-2) \quad (\text{as } n \to \infty), \tag{1}$$

$$\text{where } C_2 = \prod_{p>2} p(p-2)/(p-1)^2 \ . \tag{2}$$

Throughout the remainder of this paper, the letters "p" (or "q") and "n" stand for *odd* primes and for natural numbers, respectively.

The constant C_2 also occurs in the literature in connection with another famous conjecture, namely the "twin primes" conjecture, and is consequently known as the "twin primes" constant. Its value is approximately 0.66016 .

The aim of this paper is to furnish some apparently new insights into this problem . Although these insights do not yield a proof of GC, the author attributes this to his own shortcomings. It is hoped that some enterprising researcher, using the tools presented here (or some suitable technique involving them), may reach the desired goal of proving GC.

We first introduce some notation . Let P denote the set of odd primes, and let S denote any subset of P. Given any such S , let

$$F_S(x) = \sum_{p\in S} x^p, \tag{3}$$

where $|x|<1$.

$$\text{Consider } G_S(x) \equiv \{F_S(x)\}^2 = \sum_{n=3}^{\infty} \theta_S(2n)\, x^{2n} \ . \tag{4}$$

Note that $\theta_S(2n)$ is the number of ways to write $2n$ as a sum of two (not necessarily distinct) primes taken from the set S, taking order into consideration. For example, $\theta_P(28) = 4$, since $28 = 5+23 = 11+17 = 17+11 = 23+5$. We observe that $\theta_P(2n)$ is odd iff n is prime (in which case, $\theta_P(2n)$ counts the single representation $2n = n + n$). We also note that $\theta_P(2n) = 2r_2(2n) - \delta_P(n)$ (using Hardy and Littlewood's notation), where $\delta_P(n)$ is the characteristic function of the odd primes.

Next, given S, we say that S has the *Goldbach Property (GP)* if $\theta_S(2n) \geq 1$ for all $n \geq 3$. Let $\mathscr{U}$ denote the set of sets S having the GP. Then we may restate GC in the following form:

$$P \in \mathscr{U}. \tag{5}$$

If (5) is *true*, we would expect that there exists some proper subset S of P such that $S \in \mathscr{U}$. This, in turn, would lead us to postulate the existence of a *minimal* S (say S_0) satisfying GP and with the following additional properties:

(a) If $S_0 = \{p_1, p_2, p_3, \ldots\}$ and $S = \{q_1, q_2, q_3, \ldots\}$,

with $S_0 \in \mathscr{U}$, $S \in \mathscr{U}$, then $p_i \geq q_i$, $i = 1,2,\ldots$ for all such S ;

(b) $\{\{S_0\}-p_i\} \notin \mathscr{U}$, $i = 1,2,3,\ldots$.

Property (a) states that if S_0 is the minimal set, it must consist of prime exponents that are at least as large as those of any other set with the Goldbach Property. Property (b) states that we cannot eliminate any element of S_0; all its elements are needed to satisfy the Goldbach Property.

It should be reemphasized that we have not proven the existence of $\mathscr{U}$, and therefore of any minimal set. This will be established only after GC is proven. If we *assume* that GC is true, we may at least obtain the first few terms of the postulated minimal set S_0. We indicate below how we might proceed in the construction of such set.

Note that $6 = 3+3$ is the only possible partition of 6; hence $3 \in S_0$. Likewise, $8 = 3+5$ is the only possible partition of 8; hence $5 \in S_0$. Since $10 = 3+7 = 5+5$ are the only possible partitions of 10, this does not preclude the elimination of 7 from S_0. However, $12 = 5+7$ is the only possible partition of 12, which shows that we must have $7 \in S_0$. Now $14 = 3+11 = 7+7$ are the only possible partitions of 14, which does not preclude the elimination of 11 from S_0. Also, $16 = 3+13 = 5+11$ only, which shows that either 11 or 13 must be in S_0. Since $18 = 5+13 = 7+11$ only, this leads to the same conclusion. Next, $20 = 3+17 = 7+13$, which shows that either 13 or 17 must be in S_0. Also, $22 = 3+19 = 5+17 = 11+11$ only, which requires 17 or 19 to be in S_0, *if* we eliminate 11 from S_0, but leads to no conclusion if $11 \in S_0$. At this point, we make the decision that $11 \notin S_0$, keeping in mind Property (a) . It appears that we may wait until we reach $2n = 2p$ before we can decide whether we should eliminate p from S_0, if we have not already been able to do so previously.

Returning to our analysis of 16 (or 18), we see that $13 \in S_0$; thus the analyses for 20 and 22 are inconclusive regarding the inclusion of 17 and/or 19. Continuing, $24 = 5+19 = 7+17$ only ($11+13$ is not allowed, since we have already determined that $11 \notin S_0$); this also is inconclusive regarding the inclusion of 17 and/or 19. Next, $26 = 3+23 = 7+19 = 13+13$ only, yielding no new information. However, $28 = 5+23$ only (excluding $11+17$), which shows that $23 \in S_0$. Since $30 = 7+23$ only, this leads to no new information. Next, $32 = 3+29 = 13+19$ only , which shows that either $19 \in S_0$ or $29 \in S_0$. Also, $34 = 3+31 = 5+29 = 17+17$ only , which shows that either $29 \in S_0$ or $31 \in S_0$, *if* $17 \notin S_0$. At this point, we decide that $17 \notin S_0$.

We may continue in this fashion. If $2n = p + q$, where p and q have previously been determined to be elements of S_0, we reach no conclusion about any other summands of $2n$. If $2n = p + q$, where we have already determined that $p \notin S_0$, we cannot consider this as a possible representation of $2n$. If p may be excluded as a summand of $2n$ for $n = 3,4,\ldots,p$, we deem that p may be excluded from S_0. In this fashion, we arrive at the following first few elements of the hypothetical set S_0:

$$S_0 = \{3, 5, 7, 13, 19, 23, 31, 37, 43, 47, 53, 61, 79, 83, 89, 97, 101, \cdots\}.$$

It is left for the reader to verify that

$$G_{S_0}(x) = \{x^3 + x^5 + x^7 + x^{13} + x^{19} + x^{23} + x^{31} + x^{37} + x^{43} + x^{47} + x^{53} + x^{61} + x^{79} + x^{83} + x^{89} + x^{97} + \ldots\}^2$$
$$= x^6 + 2x^8 + 3x^{10} + 2x^{12} + x^{14} + \ldots$$
$$= \sum_{n=3}^{\infty} \theta_{S_0}(2n)x^{2n},$$

where it appears that if sufficient terms are taken in the expansion, then $\theta_{S_0}(2n) \geq 1$. To be more precise, let

$$F_{S_0}(x,N) = \sum_{p \notin S_0, p \leq N} x^p, \qquad G_{S_0}(x,N) = \{F_{S_0}(x,N)\}^2$$
$$= \sum_{n=3}^{N} \theta_{S_0}(2n,N)x^{2n}.$$

for any given N, we will find that $\theta_{S_0}(2n,N) = 0$ for some (relatively few) values of $n \leq N$. We interpret this to mean that N must be increased to eliminate the zero values. Stating this in another way, we hypothesize that $0 \leq \theta_{S_0}(2n,N) \leq \theta \leq \theta_{S_0}(2n,N+1) \leq \theta_{S_0}(2n)$, and moreover that $\theta_{S_0}(2n) \geq 1$, for all n, N with $3 \leq n \leq N$. The latter condition is simply the requirement for $S_0 \in \mathscr{U}$.

In conclusion, nothing has been proved, but it is hoped that a new perspective has been offered by which to attack this thorny old chestnut of a problem. The functions $F_P(x)$ and $G_P(x)$, in particular, seem to be the natural generating functions to use in any analytical approach.

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Determinants of Δ_m –Matrices Using The Method of Generating Functions

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1. Introduction

Many discrete stochastic processes encountered in applications (queues, inventories, and dams) have transition matrices which are special cases of a quasi-triangular matrix ("Δ_m- matrix"). These matrices were first introduced and investigated by Abolnikov [1], where a class of Markov stochastic processes with a Δ_m-transition matrix was analyzed. In this paper we will find the determinants of some special cases of Δ_m-matrices using the method of generating functions.

2. Δ_m-Matrices

Stochastic systems described by Markov chains with a Δ_m-transition matrix are quite common in queueing theory problems and problems on the control of resources. Because these matrices come up so commonly in these topics, it seems important to study the properties of these transition matrices. In this paper we find the determinants of some special forms of Δ_m-matrices using generating functions. For further analysis of Δ_m- and more general $\Delta_{m,n}$-matrices, the reader is directed to the original research of Abolnikov [1], Abolnikov and Dukhovny [2][3].

We now give the formal definition of a Δ_m-matrix.

Definition 2.1 We shall refer to a finite matrix $M = \{a_{i,j}\}$ as a quasi-triangular matrix, or simply a Δ_m-matrix, if $a_{i,j} = 0$ for $i - j > m$ for $m = 0, 1, 2, \ldots, n - 1$. This matrix has the following form:

$$M = \begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} & \cdots & a_{0,k-1} & a_{0,k} & a_{0,k+1} & \cdots & a_{0,n-1} \\ a_{1,0} & a_{1,1} & a_{1,2} & \cdots & a_{1,k-1} & a_{1,k} & a_{1,k+1} & \cdots & a_{1,n-1} \\ a_{2,0} & a_{2,1} & a_{2,2} & \cdots & a_{2,k-1} & a_{2,k} & a_{2,k+1} & \cdots & a_{2,n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ a_{m,0} & a_{m,1} & a_{m,2} & \cdots & a_{m,k-1} & a_{m,k} & a_{m,k+1} & \cdots & a_{m,n-1} \\ 0 & a_{m+1,1} & a_{m+1,2} & \cdots & a_{m+1,k-1} & a_{m,k} & a_{m+1,k+1} & \cdots & a_{m+1,n-1} \\ 0 & 0 & a_{m+2,2} & \cdots & a_{m+2,k-1} & a_{m+2,k} & a_{m+2,k+1} & \cdots & a_{m+2,n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{m+k,k} & a_{m+k,k+1} & \cdots & a_{n-1,n-1} \end{bmatrix} \quad (1)$$

3. Determinants of Δ_m-Matrices

In many applications one deals with the determinant of a Δ_m-matrix. Unfortunately, for a general Δ_m-matrix it is difficult to find a closed form for its determinant. Because of this we focus our attention on some special forms of Δ_m-matrices. A closed form for the determinants of these matrices can be found by using the method of generating functions.

Definition 3.1 If M is an n x n matrix such that $a_{i,j} = a_{k,l}$, when $i - j = k - l$, then M is called a Toeplitz matrix [4]. In other words, in Topelitz matrices all entries belonging to the same diagonal are equal to each other.

3.1 Determinants of Δ_m-Toeplitz Matrices

We now consider the determinants of two different forms of Δ_m-Toeplitz matrices.

1. Let M be an n x n Δ_1-Toeplitz matrix of the following form:

$$M = \begin{bmatrix} a_1 & a_2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_0 & a_1 & a_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & a_0 & a_1 & a_2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & a_0 & a_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_1 & a_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_0 & a_1 \end{bmatrix} \qquad (2)$$

To find the determinant of M, first we will find a recurrence relation for the determinant of M by expanding the determinant along the first column of M. We will then solve this recurrence relation using generating functions.

Let D_n be the determinant of M. When we delete the first row and first column then the remaining $(n - 1) \times (n - 1)$ matrix is of the same form as the original. We denote this as D_{n-1}. There is a similar result if we delete the first row and second column. At first we are left with a matrix that is not at all like the original, but the only choice we have left is to delete its first row and first column. This last deletion leaves us with an $(n - 2) \times (n - 2)$ matrix of the same form as the original, which we denote as D_{n-2}. This process can be expressed in the following recurrence relation,

$$D_n = a_1 D_{n-1} - a_0 a_2 D_{n-2},\ n \geq 2. \qquad (3)$$

where

$$D_0 = 1,\ D_1 = a_1,\ D_2 = a_1^2 - a_0 a_2$$

The solution to this recurrence relation can be found using the method of generating functions. Once this solution is found, we will have a closed formula for the determinant of M. If $D(x) = \sum_{n=0}^{\infty} D_n x^n$, where $D(x)$ converges in some interval $(-r, r)$, then:

$$D_n = a_1 D_{n-1} - a_0 a_2 D_{n-2},\ n \geq 2.$$

$$\sum_{n=2}^{\infty} D_n x^n = \sum_{n=2}^{\infty} a_1 D_{n-1} x^n - \sum_{n=2}^{\infty} a_0 a_2 D_{n-2} x^n$$

$$D_0 + D_1 x + \sum_{n=2}^{\infty} D_n x^n = a_1 \sum_{n=2}^{\infty} D_{n-1} x^n - a_0 a_2 \sum_{n=2}^{\infty} D_{n-2} x^n + D_0 + D_1 x$$

$$D(x) = a_1 x \sum_{n=2}^{\infty} D_{n-1} x^{n-1} - a_0 a_2 x^2 \sum_{n=2}^{\infty} D_{n-2} x^{n-2} + D_0 + D_1 x$$

$$D(x) = a_1 x \sum_{n=2}^{\infty} D_{n-1} x^{n-1} + a_1 x D_0 - a_0 a_2 x^2 \sum_{n=2}^{\infty} D_{n-2} x^{n-2} + D_0 + D_1 x - a_1 x D_0$$

$$D(x) = a_1 x D(x) - a_0 a_2 x^2 D(x) + D_0 + D_1 x - a_1 x D_0$$

$$D(x)[1 - a_1 x + a_0 a_2 x^2] = D_0 + D_1 x - a_1 x D_0$$

$$D(x) = \frac{D_0 + D_1 x - a_1 x D_0}{1 - a_1 x + a_0 a_2 x^2}$$

Replacing D_0 and D_1 with their values from (3), we have:

$$D(x) = \frac{1}{1 - a_1 x + a_0 a_2 x^2}. \qquad (4)$$

Since $D(x) = \sum_{n=0}^{\infty} D_n x^n = \frac{1}{1 - a_1 x + a_0 a_2 x^2}$, all that is needed now is to find the coefficient of x^n in $D(x)$. This coefficient is D_n, the determinant of the $n \times n$ matrix M.

Factoring $1 - a_1 x + a_0 a_2 x^2$, we have:

$$D(x) = \frac{1}{1 - a_1 x + a_0 a_2 x^2} = \frac{1}{a_0 a_2 (x - \beta_1)(x - \beta_2)}$$

where

$$\beta_1 = \frac{a_1 + \sqrt{a_1^2 - 4a_0 a_2}}{2a_0 a_2},\ \beta_2 = \frac{a_1 - \sqrt{a_1^2 - 4a_0 a_2}}{2a_0 a_2}. \qquad (5)$$

Note: We are assuming that $a_0a_2 \neq 0$.

Using partial fractions,

$$D(x) = \frac{1}{a_0a_2}\left[\frac{\frac{1}{\beta_1-\beta_2}}{(x-\beta_1)} + \frac{\frac{1}{\beta_2-\beta_1}}{(x-\beta_2)}\right] = C\left[\frac{1}{(x-\beta_1)} - \frac{1}{(x-\beta_2)}\right] \qquad (6)$$

where

$$C = \frac{\frac{1}{\beta_1-\beta_2}}{a_0a_2}.$$

$$D(x) = -\frac{C}{\beta_1}\left[\frac{1}{1-\frac{x}{\beta_1}}\right] + \frac{C}{\beta_2}\left[\frac{1}{1-\frac{x}{\beta_2}}\right]$$

$$D(x) = -\frac{C}{\beta_1}\left[1+\frac{x}{\beta_1}+\left(\frac{x}{\beta_1}\right)^2+\left(\frac{x}{\beta_1}\right)^3+\cdots\right]+\frac{C}{\beta_2}\left[1+\frac{x}{\beta_2}+\left(\frac{x}{\beta_2}\right)^2+\left(\frac{x}{\beta_2}\right)^3+\cdots\right].$$

The coefficient of x^n is easily found now. It is

$$D_n = -\frac{C}{\beta_1^{n+1}} + \frac{C}{\beta_2^{n+1}} = \det(M),$$

where β_1, β_2 are defined in (5) and C is defined in (6).

A similar same type of analysis can be used to find the determinant of other special forms of Δ_m-Toeplitz matrices.

2. Let M be an $n \times n$ Δ_2-Toeplitz matrix of the following form:

$$M = \begin{bmatrix} a_2 & a_3 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & a_2 & a_3 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & \cdots & 0 & 0 \\ 0 & 0 & 0 & a_0 & a_1 & a_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_2 & a_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_1 & a_2 \end{bmatrix} \qquad (7)$$

The determinant of M can be expressed as a recurrence relation in the same fashion as (3). Let D_n be the determinant of M, then

$$D_n = a_2D_{n-1} - a_1a_3D_{n-2} + a_0a_3^2D_{n-3}, n \geq 3 \qquad (8)$$

where

$$D_0 = 1,\ D_1 = a_2,\ D_2 = a_2^2 - a_1a_3 .$$

We will now find the generating function for the determinant of this matrix. If

$D(x) = \sum_{n=0}^{\infty} D_nx^n$, where $D(x)$ converges in some interval $(-r, r)$, then:

$$D_n = a_2D_{n-1} - a_1a_3D_{n-2} + a_0a_3^2D_{n-3},\ n \geq 3.$$

$$\sum_{n=3}^{\infty} D_nx^n = a_2\sum_{n=3}^{\infty} D_{n-1}x^n - a_1a_3\sum_{n=3}^{\infty} D_{n-2}x^n + a_0a_3^2\sum_{n=3}^{\infty} D_{n-3}x^n$$

$$D(x) = a_2x\sum_{n=3}^{\infty} D_{n-1}x^{n-1} - a_1a_3x^2\sum_{n=3}^{\infty} D_{n-2}x^{n-2} + a_0a_3^2x^3\sum_{n=3}^{\infty} D_{n-3}x^{n-3} + D_0 + D_1x + D_2x^2$$

$$D(x) = a_2xD(x) - a_1a_3x^2\sum_{n=3}^{\infty} D_{n-2}x^{n-2} + a_0a_3^2x^3\sum_{n=3}^{\infty} D_{n-3}x^{n-3} + D_0 + D_1x + D_2x^2 - a_2D_0x - a_2D_1x^2$$

$$D(x) = a_2xD(x) - a_1a_3x^2D(x) + a_0a_3^2x^3D(x) + D_0 + D_1x + D_2x^2 - a_2D_0x - a_2D_1x^2 + a_1a_3D_0x^2$$

$$D(x)\left[1 - a_2x + a_1a_3x^2 - a_0a_3^2x^3\right] = D_0 + D_1x + D_2x^2 - a_2D_0x - a_2D_1x^2 + a_1a_3D_0x^2$$

$$D(x) = \frac{D_0 + D_1x + D_2x^2 - a_2D_0x - a_2D_1x^2 + a_1a_3D_0x^2}{1 - a_2x + a_1a_3x^2 - a_0a_3^2x^3}.$$

Replacing D_0, D_1, and D_2 with their values from (8), we have:

$$D(x) = \frac{1}{1 - a_2x + a_1a_3x^2 - a_0a_3^2x^3}.$$

We now have a generating function for finding the determinant of M. The determinant of M can be found in a way similar to (4).

This paper is the result of my independent studies at Loyola Marymount University, under the guidance of Dr. Lev Abolnikov. I would like to extend my thanks to Dr. Abolnikov for his time and patience.

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A Proof of the Pythagorean Theorem Using A Circle

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In book [1], Dr. Elisha Loomis has collected 370 proofs of the Pythagorean Theorem. They are mainly divided into two categories: algebraic proofs using ratio and geometric proofs using area. Among algebraic proofs, there are about 35 using circles. Here a new proof using a circle is provided. It is interesting because it is really algebraic.

Assume that ABC is a right triangle with angle $ABC = 90°$, $BC = a$, $AB = b$ and $AC = c$. From vertex B draw altitude BD. If $BD = r$, then $rc = ab = 2$ (area of triangle ABC). Hence $r = ab/c$. Using B as center and r as radius, draw a circle $O(B, r)$ to intersect AB and BC at E and F respectively. Then AD is a tangent line to circle $O(B, r)$. Let $CD = x$, $AD = y$. Then $c = x + y$. By the Tangent and Secant Segment Theorem [1, p157], the following equalities are immediate.

$$x^2 = (a-r)(a+r) = a^2 - (ab/c)^2,$$
$$y^2 = (b-r)(b+r) = b^2 - (ab/c)^2.$$

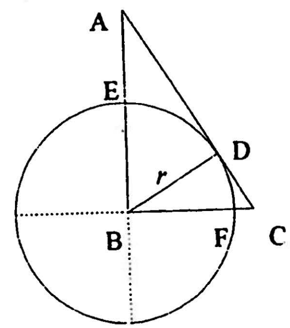

Figure 1

Therefore

$$c = x+y = \sqrt{a^2-(ab/c)^2} + \sqrt{b^2-(ab/c)^2},$$

or

$$c^2 = a\sqrt{c^2-b^2} + b\sqrt{c^2-a^2}.$$

Now we simplify the above identity,

$$(a\sqrt{c^2-b^2})^2 = (c^2-b\sqrt{c^2-a^2})^2,$$
$$a^2c^2-a^2b^2 = c^4-2bc^2\sqrt{c^2-a^2}+b^2c^2-a^2b^2,$$
$$2b\sqrt{c^2-a^2} = c^2+b^2-a^2,$$
$$4b^2c^2-4a^2b^2 = c^4+b^4+a^4+2b^2c^2-2a^2c^2-2a^2b^2,$$
$$c^4+b^4+a^4-2b^2c^2+2a^2b^2-2a^2c^2 = 0,$$
$$(c^2-b^2-a^2)^2 = 0,$$
$$c^2 = a^2+b^2.$$

The proof is completed.

From the discussion above, we have a pure algebraic problem as follows:

Problem. Let a, b, c be three positive real numbers with $c>a$, $c>b$. Then $c^2 = a^2 + b^2$ if and only if

$$c^2 = a\sqrt{c^2-b^2}+b\sqrt{c^2-a^2}.$$

Reference

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Pythagorean Theorem

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Two new proofs of the Pythagorean theorem are given in this article. They belong to the category of geometric proof in book [1], where 370 proofs are collected. My proofs are simpler than most of the geometric proofs in that book.

Proof 1

Consider the following drawing, where BDC and BAG are two congruent right triangles with BD perpendicular to AB. Let $BD = BA = a$, $DC = AG = b$ and $BC = BG = c$. Then $CF = a+b$, $FG = b-a$. Obviously the following identity is true:

area (ABDF) + area (BDC) + area (GFC) = area (GBC) + area (BAG).

Therefore

$$a^2+\frac{ab}{2}+\frac{(b-a)(b+a)}{2} = \frac{c^2}{2}+\frac{ab}{2},$$

$$2a^2+(b^2-a^2) = c^2,$$

$$a^2+b^2 = c^2.$$

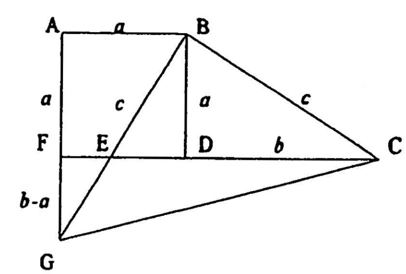

Figure 1

Proof 2

Assume that ABC is a right triangle such that $AB = a$, $BC = b$ and $AC = c$. Draw the squares outwardly on the sides of AB, BC and AC. For convenience, call them Ha, Hb and Hc. Draw the altitude BD to AC. Let $BD = d$. It is easily seen that triangles ABC, BDC and ADB are similar. Therefore

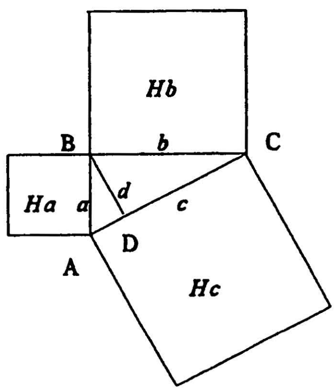

Figure 2

$$\frac{a}{c} = \frac{d}{b};\ \frac{b}{c} = \frac{CD}{b}.$$
$$\frac{a}{c} = \frac{AD}{a};\ \frac{b}{c} = \frac{d}{a}.$$

Now we have

$$\frac{ab}{c^2} = \frac{(d)(CD)}{b^2};$$
$$\frac{ab}{c^2} = \frac{(AD)(d)}{a^2}.$$

The above equalities can be written as

$$\frac{2area(ABC)}{area(Hc)} = \frac{2area(DBC)}{area(Hb)} = \frac{2area(DAB)}{area(Ha)}.$$

As shown below, there is a nonzero constant k such that

$$\frac{area(ABC)}{c^2} = \frac{area(DBC)}{b^2} = \frac{area(DAB)}{a^2} = k.$$

Since

$$area\ (ABC) = area\ (DBC) + area\ (DAB),$$

we have

$$kc^2 = kb^2 + ka^2,$$
$$c^2 = b^2 + a^2.$$

The second proof of the Pythagorean theorem is completed. The readers could ask the question what this nonzero constant k is. It is easily seen that

$$k = \frac{ab}{2c^2} = \frac{\sin C\ \sin A}{2} = \frac{\cos A\ \sin A}{2} = \frac{\sin(2A)}{4}.$$

The constant k is completely determined by the shape of the triangle ABC.

Reference

1. Elisha Loomis, *The Pythagorean Proposition*, NCTM, 1968.

The Cantor Shadow Problem: Using Geometry to Compute Sums of Cantor Sets

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Given two sets of real numbers A and B, their sum is $A + B = \{ a + b \mid a \in A, b \in B \}$. Sets of the form $A + B$ play a significant role in mathematics. For example, Fermat's Last Theorem says that, for $A_k = \{ n^k \mid n=1, 2, 3, \ldots \}$, $k > 2$ we have $(A_k + A_k) \cap A_k = \varnothing$. As another example, if P is the set of prime numbers, then the Goldbach Conjecture claims that $P + P$ contains all of the even integers greater than two.

Let C denote the Cantor ternary set (or simply the Cantor set). To fix notation, we shall quickly outline its construction. C is a subset of the real numbers. Start with the interval $C_0 = [0,1]$. Remove the open middle third of the interval, i.e. $(1/3,2/3)$ to obtain $C_1 = [0,1/3] \cup [2/3,1]$. Construct C_2 by removing the middle third of each subinterval, giving $C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$. In general, C_n is obtained from C_{n-1} by removing the middle third from each interval in C_{n-1}. C_n consists of 2^n intervals, each of length $1/3^n$. The Cantor set is defined as

$$C = \bigcap_{n=0}^{\infty} C_n.$$

Alternatively, notice that the creation of C_n has the effect of removing all interior points from C_{n-1} whose ternary (base 3) expansion has a 1 in the n^{th} place; hence we could also define C to be the set of all real numbers in the unit interval whose ternary expansions contain no 1's. (If a number has two ternary expansions, then the number is at an endpoint of a C_n, and one of the expansions will contain no 1's. For example, 1/3 has a ternary expansion 0.1 as well as 0.0222... .)

It has been known since early this century (see [6], for example) that $C + C = [0,2]$: if $y \in [0,2]$ we take $x = (1/2)\, y$, which has a ternary

expansion $x = x_1 x_2 x_3 \ldots$. Each x_n can be written as the sum $a_n + b_n$; where $a_n = 0$ if $x_n = 0$ and $a_n = 1$ otherwise, $b_n = 0$ if $x_n = 0$ and $b_n = 1$ otherwise. Let $a = a_1 a_2 a_3 \ldots$ and $b = b_1 b_2 b_3 \ldots$ (ternary expansions). Then $a + b = x$, hence $2a + 2b = y$. As $2a$, $2b \in C$ we get $C + C = [0,2]$.

The objective here is to calculate, for any fixed real numbers a and b, the set $aC + bC = \{ ac_1 + bc_2 \mid c_1, c_2 \in C \}$ Sums of Cantor sets, some more general than the one described above, have been studied in great detail recently (for example [1], [3], [4], [5]); in fact the answer to our problem can be found using some very technical tools. This paper will provide a much simpler argument: the only tools needed are basic geometry and induction. The geometric version of this problem is called the Cantor Shadow Problem. First posed by Edward Thomas in 1993, it involves looking at linear projections of the Cartesian product $C \times C$ ont the $x-$ axis; or in other words to look at the "shadow" cast when exposed to a light source emitting parallel rays at a given angle θ. Linear projections of this Cartesian product were also used in [5]; however in that paper the angle was always the same, and only the results for $C + C$ could be recovered, By allowing the angle to vary, the shadow we compute will give us information about $aC + bC$. It turns out that $aC + bC$ is either a single interval or a union of disjoint intervals (for nonzero a and b) which resembles one of the partial Cantor sets C_n.

1. The Cantor Shadow

The Cartesian product $C \times C$ can be viewed as ordered pairs (c_1,c_2), $c_1 \in C$, $c_2 \in C$. It is often helpful to view the Cartesian product $C \times C$ as the intersection of the product of the partial Cantor sets $C_n \times C_n$. Consider an imaginary light source in the plane such that all of the rays of light are parallel, as shown in Figure 1.

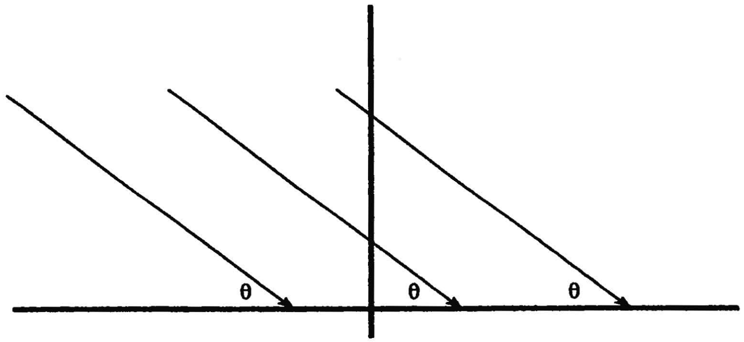

Figure 1

Let $0 < \theta < \pi$ be the angle these rays make, measured from the negative x - axis. For a given θ we shall denote the shadow of $C \times C$ on the x - axis by I_θ, which we call the Cantor shadow for the angle θ.

Given any point $(x,y) \in [0,1] \times [0,1]$ its shadow on the x - axis is $x + y\cot\theta$, as can be seen in Figure 2. In the case $\theta = \pi/4$ the image of (c_1,c_2) is $c_1 + c_2$ hence $I_{\pi/4} = C + C = [0,2]$. Notice a square with vertices (x_1,y_1), (x_2,y_1), (x_1,y_2), (x_2,y_2); $x_1 < x_2$, $y_1 < y_2$ has shadow

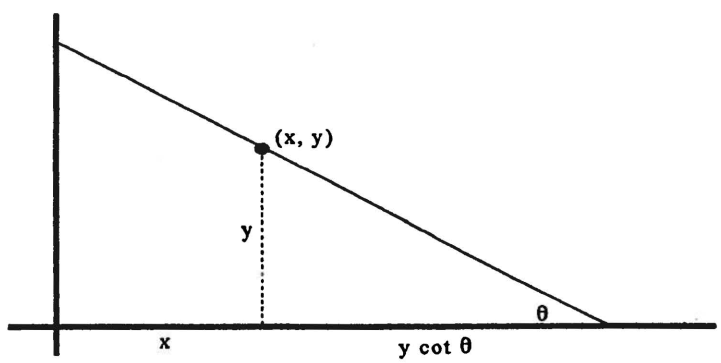

Figure 2

$[x_1 + y_1\cot\theta,\ x_2 + y_2\cot\theta]$.

2. The General Shadow Solution

The shadow is easy to describe when the angle is "near" $\pi/4$.

Proposition 2.1. For any θ with $1/3 \le \cot\theta \le 3$, $I_\theta = [0,\ 1 + \cot\theta]$.

Proof. Assume $\cot\theta \neq 1$, since we already know $I_{\pi/4} = [0,2]$.

We claim that the shadow cast by each $C_n \times C_n$ is $[0,\ 1 + \cot\theta]$. Once this is established, then for any $x \in [0,\ 1 + \cot\theta]$ we can find a square F_n in $C_n \times C_n$ so that its shadow includes x. Furthermore, these F_n's can be chosen so that $F_n \subset F_{n-1}$ for all n. The collection of these sets satisfies the finite intersection condition (i.e. every finite subcollection has a finite intersection), and hence by [2, Theorem 5.9] we get $F = \bigcap F_n \neq 0$. As the diameters of the F_n go to zero, F must consist of exactly one point, say (c_1,c_2). This will show that $I_\theta = [0,\ 1 + \cot\theta]$.

The claim is proved by induction. By the observations at the end of the previous section, the shadow cast by the square $C_0 \times C_0$ is precisely $[0,\ 1 + \cot\theta]$. Assume that the image of $C_n \times C_n$ is also $[0,\ 1 + \cot\theta]$. Consider any of the squares in $C_n \times C_n$. Let (x,y) be the coordinate of the lower left endpoint. The shadow cast by this square is $[x + y\cot\theta,\ (x + 1/3^n) + (y + 1/3^n)\cot\theta]$. It suffices to show that the four squares created by the next iteration cast the same shadow. The four shadows cast are

$$[x + y\cot\theta,\ (x + 1/3^{n+1}) + (y + 1/3^{n+1})\cot\theta]$$
$$[(x + 2/3^{n+1}) + y\cot\theta,\ (x + 1/3^n) + (y + 1/3^{n+1})\cot\theta]$$
$$[x + (y + 2/3^{n+1})\cot\theta,\ (x + 1/3^{n+1}) + (y + 1/3^n)\cot\theta]$$
$$[(x + 2/3^{n+1}) + (y + 2/3^{n+1})\cot\theta,\ (x + 1/3^n) + (y + 1/3^n)\cot\theta].$$

It is easy to check that, for $1/3 \le \cot\theta \le 3$, the union of these four intervals is precisely $[x + y\cot\theta,\ (x + 1/3^n) + (y + 1/3^n)\cot\theta]$. As the shadow cast by each square of $C_n \times C_n$ is preserved at the next iteration, the entire shadow must be preserved. Hence, by induction, the claim is proved, and $I_\theta = [0,\ 1 + \cot\theta]$. ■

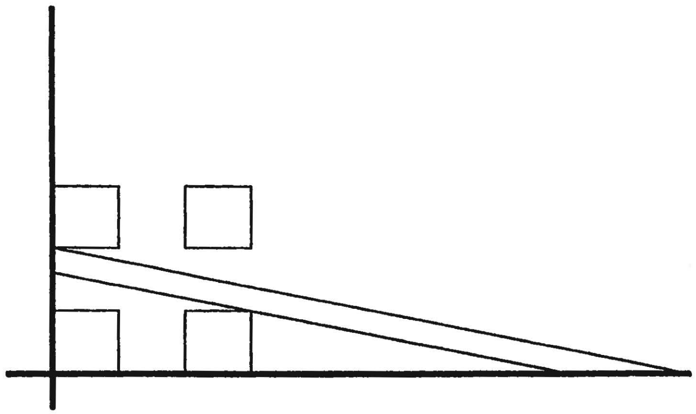

Figure 3

Now suppose that $3^k < \cot\theta \leq 3^{k+1}$ for some integer $k \geq 1$. Figure 3 shows that, after only the first iteration, the shadow is no longer connected. Instead, we get the following result.

Lemma 2.2. Let $\cot\theta > 3^k$, $k \geq 1$. Let x_n be the left endpoint of the n^{th} interval of C_k. Then the shadow cast by $C_k \times C_k$ is

$$\bigcup_{n=1}^{2^k} [x_n \cot\theta,\ x_n \cot\theta + 1 + 1/3^k \cot\theta].$$

Proof. Consider any row of squares in $C_k \times C_k$. The maximum distance between any two squares is 1/3. One can show that the shadows of any two adjacent squares overlap: clearly, we may assume that this row is on the x - axis. Figure 4 shows the case where $k = 2$:

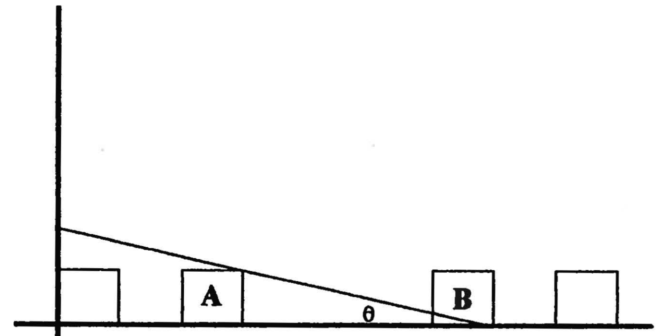

Figure 4

Let A and B be two adjacent squares, having height $1/3^k$. Square A casts a shadow with right-hand endpoint $1/3^k \cot\theta$ units to the right of the right-hand endpoint of the base of A. As $\cot\theta > 3^k$, $1/3^k \cot\theta > 1 > 1/3$, so all of the shadows overlap. Thus any shadow cast by the row of squares on the x - axis is $[0, 1 + 1/3^k \cot\theta]$.

Now consider any two adjacent rows of squares. We may assume they are the bottom two rows. Here the shadows created by the these rows do not overlap (see Figure 2). To show this, note that the distance between the two rows is $1/3^k$. The bottom row casts a shadow on $[0,\ 1 + 1/3^k \cot\theta]$. The left endpoint of the shadow cast by the upper row is $2/3^k \cot\theta$. Since $\cot\theta > 3^k$, these shadows cannot overlap as $1 + 1/3^k \cot\theta < 2/3^k \cot\theta$. Since there are 2^k intervals in C_k, the shadow related by $C_k \times C_k$ consists of 2^k disjoint closed intervals, and they are of length $1 + 1/3^k \cot\theta$. Since each row of squares has its distance to the x - axis given by x_n, it follows that each interval has left endpoint $x_n \cot\theta$. ■

Proposition 2.3. If $3^k < \cot\theta \leq 3^{k+1}$, $k \geq 1$, then the shadow cast by $C \times C$ is equal to the shadow cast by $C_k \times C_k$.

Proof. Using the method of Proposition 2.1, it can easily be shown that the shadow cast by $C_{k+1} \times C_{k+1}$ is identical to the shadow cast by $C_k \times C_k$. The proof starts with the same inductive argument, using $C_k \times C_k$ as the starting point. Then use the finite intersection condition to guarantee that an element of $C \times C$ will cast a shadow on any point in the shadow of $C_k \times C_k$. ■

Remark. Similar results hold for $3^{-k-1} \leq \cot\theta < 3^{-k}$. Reflections and translations will give us answers when $\pi/2 < \theta < \pi$, and the case $\theta = \pi/2$ is projection onto the first coordinate. The results are summarized below:

Theorem 2.4. Let I_θ denote the Cantor shadow for the angle θ. For each k, let $x_{k,n}$ denote the left-hand endpoint of the n^{th} interval of C_k.

a. If $\theta < \pi/4$, let k denote the unique nonnegative integer such that $3^k < \cot\theta \leq 3^{k+1}$. Then

$$I_\theta = \bigcup_{n=1}^{2^k} [x_{k,n}\cot\theta,\ x_{k,n}\cot\theta + 1 + 1/3^k\cot\theta].$$

b. If $\pi/4 < \theta < \pi/2$, let k denote the unique nonnegative integer such that $3^{-k-1} \leq \cot\theta < 3^{-k}$. Then

$$I_\theta = \bigcup_{n=1}^{2^k} [x_{k,n},\ x_{k,n} + 1/3^k + \cot\theta].$$

c. $I_{\pi/4} = [0,2]$, $I_{\pi/2} = C$; and if $\pi/2 < \theta < \pi$, then $I_\theta = 1 - I_{\pi-\theta}$. ■

3. The General Structure of $aC + bC$

The answer to our initial problem is now quite simple.

Corollary 3.1. If a and b are real numbers with $b \neq 0$, and if $\theta = \cot^{-1}(a/b)$, then $aC + bC = bI_\theta$.

Proof. Let $(c_1,c_2) \in C \times C$. The shadow cast by (c_1,c_2) with angle θ is $c_1 + c_2\cot\theta$, so $C + (\cot\theta)C = I_\theta$. Thus $C + (a/b)C = I_\theta$. Multiplying both sides of this equation by b gives us $aC + bC = bI_\theta$. ■

Remark. When $1/3 \leq a/b \leq 3$ we have $aC + bC = [0, a+b]$ when both a and b are positive, $aC + bC = [a+b, 0]$ when they are both negative. These are the only instances when a single interval is generated, and we can create an interval of any length we want by picking the constants appropriately.

Acknowledgements. The authors would like to thank Edward Thomas and Julian Fleron for the inspiration and motivation received during the preparation of this paper.

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Factoring $n^4 + 4^n$ with a Braille 'n Speak

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To better understand our story, we need to introduce ourselves. I am a professor of mathematics with all the usual quirks that this implies. Angie, when this story was written, was a junior mathematics major at Washington and Lee University from which she graduated in June 1997. Her chief interests are music and mathematics, and she actively pursues both. She is currently doing graduate work in mathematics at the University of Virginia, and she believes she would enjoy teaching at the college level. She would also like to work with blind students in science and mathematics since she has been totally blind since infancy.

Angie relies heavily upon Braille and recorded materials to complete assignments, but she has benefited greatly from access to the Internet. For Angie, email is much more than a high-tech way to procrastinate (although she certainly appreciates that aspect of the Information Superhighway). She uses a curious little device known as a Braille 'n Speak to communicate with her email account. The Braille 'n Speak has seven buttons (six "dots" and a space key) that input Braille characters. The space key can be pressed in combination with other keys to perform special functions; such as entering the machine's calculator mode, checking the time, or executing a macro. The "Speak" part is a voice synthesizer that attempts to speak whatever is entered from the keyboard or through the serial port and it is in this manner that Angie "reads" her email. When mathematics is entered, the result from the voice synthesizer is somewhat strange.

Using T_EX code, Angie types her notes and other mathematics directly into her Braille 'n Speak. Most of her mathematics professors send homework to her by email and she returns her proofs in the same manner. When she needs to present something in class, she first connects the Braille 'n Speak to a Braille embosser to produce a hard copy that she can read. She then emails the

professor her notes and the professor prints the TEXed version for her classmates.

Number Theory is not one of our regularly offered courses at Washington and Lee, but Angie needed a one credit course to supplement her one credit voice lesson and the one credit she received for being a University Chamber singer. Another student needed a three credit mathematics course and so I agreed to teach a Number Theory course. Angie then decided that having three other mathematics courses was not enough, so she dropped her voice lesson and took Number Theory for three credits. We decided to used the third edition of Kenneth H. Rosen's *Elementary Number Theory and Its Applications* since it was the most recently published book that Angie could get on tape.

Exercise 5.1.20 of the text asks for which positive integers n is $f(n) = n^4 + 4^n$ prime. After assigning this problem, I decided to see if I could do it and quickly discovered that it was a wee bit harder than I anticipated. Obviously, $f(1) = 5$ is prime and $f(2k)$ is not prime for any positive integer k. By using Fermat's Little Theorem, $n^4 \equiv 1 (\text{mod } 5)$ for $n \not\equiv 0 (\text{mod } 5)$. Now $4 \equiv -1 (\text{mod } 5)$ and so for n odd, $4^n \equiv (-1)^n \equiv -1 \equiv 4 (\text{mod } 5)$. Thus, $n^4 + 4^n \equiv 1 - 1 \equiv 0 (\text{mod } 5)$ and hence 5 divides $f(n)$.

The next step is to try $n = 5$. Now $f(5) = 17 \cdot 97$ and with great expectation that 17 divides $f(n)$ for n divisible by 5, I computed $15^4 + 4^{15} (\text{mod } 17)$. With great disappointment, I found that $15^4 + 4^{15} \equiv (-2)^4 + (4^2)^7 \cdot 4 \equiv 16 + (-1)^7 \cdot 4 \equiv 16 - 4 \equiv 12 (\text{mod } 17)$. Recognizing that I was in serious trouble, I asked the class for advice. As we were discussing this, Angie decided to write a macro on her Braille 'n Speak to factor $f(15) = 1{,}073{,}792{,}449$.

The macro that Angie wrote during class was simply trial division. She decided to divide $f(15)$ by each odd positive integer and store the quotient in a file. She later modified her algorithm so that the Braille 'n Speak would only divide by the odd integers that were not divisible by three. I suggested that she run the program while she was sleeping during her real analysis class (fortunately for me, her aim with her cane is not that good). During her choir rehearsal, however, she let the Braille 'n Speak divide to its heart's content.

Periodically, she would search the file of quotients for a period followed by a space; that would indicate a quotient with no fractional part.

Two days later, I received an email message from Angie with the subject line, "Brute force," and the factorization $f(15) = 29153 \cdot 36833$. Noting that $2^{15} = 32768$ lies almost midway between these two factors, it didn't take long to see that $29153 + 36833 = 2(2^{15} + 15^2)$. Returning to $f(5) = 17 \cdot 97$, we see that $17 + 97 = 2(2^5 + 5^2)$. Could it be true that $f(n) = ab$ with $a < b$ and $a + b = 2(2^n + n^2)$? Observing that $36833 - 29153 = 2^9 \cdot 15$ and $97 - 17 = 2^4 \cdot 5$, perhaps it is also true that $b - a = 2^m n$ for some integer m. With two equations and two unknowns, a and b, it is easy to see that

$$b = 2^n + n^2 + n2^{m-1},$$

and

$$a = 2^n + n^2 - n2^{m-1}.$$

We can solve for m by using $n = 5$ and checking with $n = 15$. Hence $m = (n + 3)/2$. It is now trivial to verify that $n^4 + 4^n = ab$ where $a = 2^n + n^2 - n2^{m-1}$ and $b = 2^n + n^2 + n2^{m-1}$ with $m = (n + 3)/2$.

Since we have factored $f(n)$ for $n > 1$ and answered the exercise, it is time to state the morals of our story. First, brute force is sometimes a useful technique. Second, use whatever technology is appropriate and available. Without the Braille 'n Speak, we would not have factored $f(15)$ and without this factorizátion, we would not have found the general factorization of $f(n)$. Our third and final moral: work the problem *before* you assign it!

Acknowledgment: We want to thank Paul Humke of St. Olaf College for his enthusiasm and encouragement. Without these, this paper would not have been written.

Curious Numbers

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The last four digits of 9376^2 are 9376. An n-digit nonnegative integer x is called a *curious number* if and only if $x^2 - x$ is divisible by 10^n. The purpose of this paper is to answer the following two questions.

1. Is there a closed-form expression for curious numbers?
2. Are there infinitely many curious numbers?

Closed-form expressions for the positive roots of $x^k - x = n$ (n a nonnegative integer) for $2 \le k \le 5$ have already been obtained by Elia and Filipponi [1, 2]. In the case of $k = 2$,

$$x = \frac{1 + \sqrt{4n + 1}}{2} \quad (n = 0, 1, 2, \ldots)$$

However there are difficulties when we try to apply the above formula and solve for curious numbers, because we do not know the value of n. We only know that n is a multiple of 10^m if x is an m-digit integer.

The following theorem is useful for constructing curious numbers.

Theorem 1: If an n-digit number $A_{n-1} A_{n-2} \cdots A_0$ is a curious number, so is the $(n - 1)$-digit number $A_{n-2} A_{n-3} \cdots A_0$ $(n > 1)$.

Proof: Since $A_{n-1}A_{n-2}A_{n-3}\cdots A_0 = 10^{n-1} \cdot A_{n-1} + A_{n-2}A_{n-3}\cdots A_0$,

$$A_{n-1}A_{n-2}A_{n-3}\cdots A_0^2 = (10^{n-1} \cdot A_{n-1} + A_{n-2}A_{n-3}\cdots A_0)^2$$
$$= 10^{2n-2} \cdot A_{n-1}^2 + 2 \cdot 10^{n-1} \cdot A_{n-1} \cdot A_{n-2}A_{n-3} \cdots A_0 + A_{n-2}A_{n-3} \cdots A_0^2.$$

Since $A_{n-1}A_{n-2}A_{n-3}\cdots A_0$ is a curious number, the last $n - 1$ digits of its square are $A_{n-2}A_{n-3} \cdots A_0$. By examining the last equation, we see that the only term that contributes to the last $n - 1$ digits of $A_{n-1}A_{n-2}A_{n-3} \cdots A_0^2$ is $A_{n-2}A_{n-3}\cdots A_0^2$. Therefore the last $n - 1$ digits of $A_{n-2}A_{n-3}\cdots A_0^2$ are also $A_{n-2}A_{n-3} \cdots A_0$. By definition, $A_{n-2}A_{n-3}\cdots A_0$ is a curious number. **Q.E.D.**

This result provides an easy way of finding curious numbers recursively. Start with a known curious number, and then attach a single digit in front of it, square the newly formed number and see if this number repeats itself in its square. Be aware that using this method, 0 also counts as a leading digit. For example, if we choose curious number 6 to start with, we have the ten possible combinations 06, 15, 26, 36, 46, 56, 66, 76, 86 and 96 to square. Then

$$06^2 = 36$$
$$16^2 = 256$$
$$26^2 = 676$$
$$36^2 = 1296$$
$$46^2 = 2116$$
$$56^2 = 3136$$
$$66^2 = 4356$$
$$76^2 = 5776$$
$$86^2 = 7396$$
and
$$96^2 = 9216.$$

So we see that number 76 is another curious number. We can carry out the process again by now squaring 076, 176, 276, 376, 476, 576, 676, 776, 876 and 976. Then

$$076^2 = 5776$$
$$176^2 = 30976$$
$$276^2 = 76176$$
$$376^2 = 141376$$
$$476^2 = 226576$$
$$576^2 = 331776$$
$$676^2 = 456976$$
$$776^2 = 602176$$
$$876^2 = 767376$$
and
$$976^2 = 952576.$$

Now we see that number 376 is another curious number.

In order to find a closed form formula, we have to employ some ideas from the number theory. Assume x is an n-digit curious number. By definition,

$x^2 - x$ must be divisible by 10^n. Therefore,

$$\frac{x^2-x}{10^n} = \frac{x\cdot(x-1)}{2^n 5^n}$$

is an integer. Since x and $x - 1$ are relatively prime, there are two cases.

Case 1. x contains all the factors of 2 and $x - 1$ contains all the factors of 5.

$$x = 2^n \cdot p$$

$$x - 1 = 5^n \cdot q \ (p, q \in Z)$$

Therefore,

$$2^n \cdot p \equiv 1 (\text{mod } 5^n).$$

Case 2. x contains all the factors of 5 and $x - 1$ contains all the factors of 2.

$$x = 5^n \cdot r$$

$$x - 1 = 2^n \cdot s \ (r, s \in Z)$$

Therefore,

$$5^n \cdot r \equiv 1 (\text{mod } 2^n).$$

Both cases can be solved using Euler's Theorem [3]. Euler's Theorem says that if $(a, m) = 1$, then

$$a^{\varphi(m)} \equiv 1 (\text{mod } m)$$

where $\varphi(m)$ is Euler's totient function. $\varphi(m)$ equals the number of positive integers which are less than or equal to m and relatively prime to m. For example, $\varphi(2^n) = 2^{n-1}$ and $\varphi(5^n) = 4 \cdot 5^{n-1}$. By Euler's Theorem,

$$2^{\varphi(5^n)} \equiv 1 (\text{mod } 5^n)$$

and so

$$2^{4 \cdot 5^{n-1}} \equiv 1 (\text{mod } 5^n).$$

Therefore

$$p \equiv 2^{4\cdot 5^{n-1}-n} (\mathrm{mod}\, 5^n)$$

$$x = 2^n \cdot (2^{4\cdot 5^{n-1}-n}\ (\mathrm{mod}\, 5^n)) \quad (n \in \mathrm{N})\ (\text{Case 1}).$$

Similarly,

$$5^{\varphi(2^n)} \equiv 1 (\mathrm{mod}\, 2^n)$$

and so

$$5^{2^{n-1}} \equiv 1 (\mathrm{mod}\ 2^n).$$

Therefore

$$r \equiv 5^{2^{n-1}-n} (\mathrm{mod}\, 2^n)$$

$$x = 5^n \cdot (5^{2^{n-1}-n} (\mathrm{mod}\, 2^n)) \quad (n \in N)(\text{Case 2}).$$

Now we have considered both cases and derived two formulas. Each of them produces a sequence of curious numbers. Since n can be any positive integer, it is obvious that there is an infinite number of curious numbers.

Conclusion:

1. All curious numbers larger than one can be given by one of the following two formulas.

$$x = 2^n \cdot (2^{4\cdot 5^n - n} (\mathrm{mod}\, 5^n)) \qquad (n \in \mathrm{N})$$

$$x = 5^n \cdot (5^{2^{n-1}} (\mathrm{mod}\, 2^n)) \qquad (n \in \mathrm{N})$$

2. There are infinitely many curious numbers.

Lastly, it is possible to rewrite the two formulas using a common modulus, namely 10^n.

Case 1.

$$2^n \cdot (2^{4\cdot 5^{n-1}-n} (\mathrm{mod}\, 5^n)) = 2^n \cdot \left(\frac{2^{4\cdot 5^{n-1}}}{2^n}\ (\mathrm{mod}\, 5^n) \right)$$

$$= 2^n \cdot \left(\frac{2^{4\cdot 5^{n-1}}\ (\mathrm{mod}\, 10^n)}{2^n} \right)$$

$$= 2^{4\cdot 5^{n-1}} (\mathrm{mod}\, 10^n)$$

$$= 16^{5^{n-1}} (\mathrm{mod}\, 10^n).$$

Case 2.

$$5^n \cdot (5^{2^{n-1}-n} (\mathrm{mod}\, 2^n)) = 5^n \cdot \left(\frac{5^{2^{n-1}}}{5^n}\ (\mathrm{mod}\, 2^n) \right)$$

$$= 5^n \cdot \frac{5^{2^{n-1}} (\mathrm{mod}\, 10^n)}{5^n}$$

$$= 5^{2^{n-1}} (\mathrm{mod}\, 10^n).$$

Acknowledgments: The author would like to thank Professor Brigitte Servatius for her help with this paper and consistent support and guidance. Professor Joseph D. Fehribach deserves gratitude for providing insightful comments. The author also owes a thank you to the anonymous referee for careful reading and pointing out valuable references.

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BIOGRAPHICAL SKETCHES OF THE AUTHORS

Paul S. Bruckman was born in Florence, Italy, and received an M.S. degree in mathematics from the University of Illinois at Chicago in 1974. From 1960 through 1990, he was employed with private actuarial firms, most recently as a pension plan actuary. Mr. Bruckman has been and continues to be a frequent contributor to **The Fibonacci Quarterly** and the **Pi Mu Epsilon Journal** .

Julie C. Jones graduated with honors from Randolph-Macon College in May of 1997. The joint research that led to this paper constituted part of her departmental honors project. Julie is also a charter member of the Virginia Iota chapter of **Pi Mu Epsilon**. She is currently pursuing a Ph.D. in Mathematics from University of Southwestern Louisiana, near her hometown of New Orleans.

Clayton W. Dodge received his B.A. and M.A. degrees from the University of Maine in 1956 and 1959. He taught there for 41 years, retiring as Professor Emeritus of Mathematics in 1997. His interests include geometry, teacher education, and problems. He was an editor for the Elementary Problem Department of the *American Mathematical Monthly* and has served as Problems Editor for this journal since 1980.

Anthony Shaheen is an undergraduate Computer Science major/Mathematics minor at Loyola Marymount University in Los Angeles, California. His academic interests are in Theoretical Computer Science and Mathematics. His recent research has been in the area of caching and scheduling algorithms, which was conducted at DIMACS (Center for Disrete Mathematics and Theoretical Computer Science) in Rutgers University. His personal interests include basketball, weightlifting, and jazz.

Melisa Hicks is a senior, pursuing a degree in Math Education, at the University of North Florida. She enjoys tutoring and watching movies in her spare time. She thanks her family for all their love and support. **Beverly Collins** is a junior majoring in Mathematics Education. In grade six through twelve she was educated by parents at home. She sincerely thanks her parents for their love and support.

Tammy M. Muhs graduated in May, 1998, with a B.S. in mathematics. She will be pursuing a graduate degree in mathematical science beginning in the Fall semester. In addition to being a wife and mother of two children, she keeps busy with part time employment at University of North Florida, a variety of interests and volunteer work.

Alan Koch obtained his Ph.D. in 1995 from the State University of New York at Albany. He has taught at both Rensselaer Polytechnic Institute in Troy, NY and Hope College in Holland, MI. He is currently an Assistant Professor at St. Edward's University in Austin, TX. **James D. Panariello** received his Ph.D. in 1996 from the State University of New York at Albany. He spent a year as a Visiting Assistant Professor at Lafayette College in Easton, PA, and is currently employed by the software development company Keane, Inc.

Wayne M. Dymàček is a Professor of Mathematics at Washington and Lee University and had previously worked for the National Security Agency. He received both his B.S. and Ph.D. degrees from Virginia Tech in 1974 and 1978, respectively. His research interests are in graph theory and combinatorics with a special interest in Steinhaus graphs. He has published several papers and is the co-author of a textbook on discrete mathematics. Along with a computer science colleague, he has directed undergraduate students in research work, from which four papers have been published. For twelve years he has helped grade the AP calculus exams and perhaps this is why he prefers to teach discrete mathematics. He is currently on the Mathematical Association of America's Committee on Testing. **Angie Matney** is originally from Iaeger, West Virginia. She received her B.S. in mathematics from Washington and Lee University in historic Lexington, Virginia in 1997. While at W & L, she served as co-president of the Virginia Theta chapter of **Pi Mu Epsilon**. Ms. Matney is now working towards her Ph.D. in mathematics at the University of Virginia. She is an active member of the Blue Ridge Chapter of the National Federation of the Blind of Virginia. Her non-mathematical interests include science fiction and music. She also enjoys spending time with her guinea pig Molly.

Xiaolong Ron Yu is currently a fourth year undergraduate student studying at Worcester Polytechnic Institute in Massachusetts. He is double majoring in Mathematics and Electrical Engineering. After graduation, he is planning to attend graduate school. Presently, his mathematical interests include number theory, combinatorics, differential equations and optimization problems. In his spare time, Xiaolong loves to play with logic and mathematical puzzles and enjoys sports, including tennis, table tennis, basketball and swimming.

THE RICHARD V. ANDREE AWARDS

The Richard V. Andree Awards are given annually to the authors of the three papers written by students that have been judged by the officers and councilors of **Pi Mu Epsilon** to be the best that have appeared in the **Pi Mu Epsilon Journal** in the past year.

Until his death in 1987, Richard V. Andree was Professor Emeritus of Mathematics at the University of Oklahoma. He has served **Pi Mu Epsilon** for many years and in a variety of capacities: as President, as Secretary-Treasurer, and as Editor of the **Journal**.

Listed alphabetically, the three winners for 1998 are:

1. **Johanna Miller** and **C. Ryan Vinroot** for their paper "Mauldin-Williams Graphs with Unique Dimension", this **Journal** 10(1994-99)#8, 620-628.

2. **Loi Nguyen** and **Tu Tran** for their paper "Seeing Is Not Always Believing", this **Journal** 10(1994-99)#9, 689-691.

3. **Tricia Stone (Hovorka)** for her paper "Where Do My Sequences Lead?", this **Journal** 10(1994-99)#8, 629-633.

The officers and councilors of the Society congratulate the winners on their achievements and wish them well for their futures.

1998 NATIONAL PI MU EPSILON MEETING

The Annual Meeting of the Pi Mu Epsilon National Honorary Mathematics Society was held at Toronto from July 15 through July 18. As in the past, the meeting was held in conjunction with the national meeting of the Mathematical Association of America's Student Sections.

The J. Sutherland Frame Lecturer was **Joseph A. Gallian**, University of Minnesota - Duluth. His presentation was on "Breaking Drivers' License Codes."

The following thirty-three student papers were presented at the meeting.

Program-Student Paper Pi Mu Epsilon Sessions

Cryptography: The Rivest-Shamir-Adleman (RSA) Encryption System — **Andrew D. Barlow**, Steven F. Austin State University, Texas Delta

Algorithm For Classifying Orientable Surfaces — **Stephen Bochanski**, St. Joseph's University, Pennsylvania Xi

What Does A Part Time Job Tell Us About High School Students? — **Joyce Cannone**, Youngstown State University, Ohio Xi

Determining Winners Of Round-Robin Tournaments — **Mary Elizabeth Cassells**, Lafayette College, Pennsylvania Tau

Using Polya's Theorem To Count Figures — **Michael DeCoster**, Miami University, Ohio Delta

Possible Orders Of Graphs For A Given Degree Set — **Melissa Desjarlais**, Alma College, Michigan Theta

Fractals, Geometry And Their Dimensions — **Joe Ferguson**, Youngstown State University, Ohio Xi

Molecular Computation And Graph Theory — **Nathan L. Gibson**, **Angela Komorowski**, Worcester Polytechnic Institute, Massachusetts Alpha

When Bad Things Happen To Good Trees — **William Gravemann**, Lafayette College, Pennsylvania Tau

Maximum Degree Growth Of The Iterated Line Graph — **Stephen Hartke**, University of Dayton, Ohio Zeta

Don't Let Topspin Put You In A Tailspin — **Erin Huebner**, St. Norbert College, Wisconsin Delta

From X-rays To CAT-Scans — **Tina Huss**, St. Norbert College, Wisconsin Delta

Investigating The Relationship Between The Shape Of A Coffee Cup And The Shape Of The Caustic Inside It — **Asif Iqbal**, University of Minnensota, Minnesota Eta

We Have To Use What? — **Ben Jantson**, Youngstown State University, Ohio Xi

The *p*-Colorability Of The Rational Knot — **Michael Kern**, Kent State University, Ohio Epsilon

Lottery Games In Ohio — **Rob King**, Youngstown State University, Ohio Xi

Just About Right Scales — **Mary Beth Lake**, Elmhurst College, Illinois Iota

The Golden Proportion: An Advertising Venture — **Laura Lemke**, St. Norbert College, Wisconsin Delta

Picking Digital Locks With Elliptic Curves — **Vincent Lucarelli**, Youngstown State University, Ohio Xi

Isomorphisms Of Circulant Graphs — **Kimball Martin**, University of Maryland Baltimore County, Mayland Gamma

A Long Line Of Dead Men — **Jodie Matulja**, Youngstown State University, Ohio Xi

Need For Speed — **Adam Messner**, **Vincent Lucarelli**, Youngstown State University, Ohio Xi

The Isoperimetric Problem On Surfaces — **Ting Fai Ng**, University of Pennsylvania, Pennsylvania Alpha

Baysian Statistics On The TI-83 Calculator — **Cathy O'Bryant**, Villanova University, Pennsylvania Iota

Quantification Of Shape Difference Using SL(2, R) — **Joseph Maxwell Oppong, Jr.**, University of Richmond, Virginia Alpha

The Economics Of Game Theory — **Erica Pagel**, St. Norbert College, Wisconsin Delta

An Iterative Algorithm For Deletion From AVL-Balanced Binary Trees — **Ben Pfaff**, Michigan State University, Michigan Alpha

Mathematical Models And Potato Crops — **Kate Rendall**, St. Norbert College, Wisconsin Delta

Carving The Great Pumpkins — **John Slanina**, Youngstown State University, Ohio Xi

An Algorithm For Classifying Non-Orientable Surfaces — **Harry Smith**, St. Joseph's University, Pennsylvania Xi

Height Ridges And Medial Loci For Image Analysis — **James Tripp**, University of Richmond, Virginia Alpha

Fractal Image Compression

Jason Walter
Mount Union College
Ohio Omicron

Let's Make An Ideal

Emilie B. Wiesner
Washington and Lee University
Virginia Theta

For the tenth consecutive year, the American Mathematical Society has given **Pi Mu Epsilon** a grant to be used as prize money for excellent student presentations. This year six prizes of $150.00 each and two prizes of $75.00 each were awarded. The winning speakers were:

Stephen Bochanski, St. Joseph's University,
Algorithm for Classifying Oreintable Surfaces

Joe Ferguson, Youngstown State University,
Fractals, Geometry and Their Dimensions

Nathan L. Gibson, Worcester Polytechnic Institute,
Molecular Computation and Graph Theory

Stephen Hartke, University of Dayton
Maximum Degree Growth of the Interated Line Graph

Kimball Martin, University of Maryland Baltimore County,
Isomorphisms of Circulant Graphs

Ting Fai Ng, University of Pennsylvania,
The Isoperimetric Problem on Surfaces

John Slanina, Youngstown State University,
Carving the Great Pumpkin

Harry Smith, St. Joseph's University,
An Algorithm for Classifying Non-Orientable Surfaces

Miscellany

Dane W. Wu made the following observation regarding the differentiation of $f(x)^{g(x)}$.

In calculus, there are various techniques to show that

$$\frac{d}{dx}[f(x)^{g(x)}] = f(x)^{g(x)}\ln[f(x)]\frac{dg(x)}{dx} + g(x)f(x)^{g(x)-1}\frac{df(x)}{dx} \qquad (1)$$

for nonzero, differentiable functions $f(x)$ and $g(x)$.

If f is a nonzero constant function, then

$$\frac{d}{dx}\left[f^{g(x)}\right] = f^{g(x)}\ln(f)\frac{dg(x)}{dx}. \qquad (2)$$

If g is a nonzero constant function, then

$$\frac{d}{dx}[f(x)^{g}] = gf(x)^{g-1}\frac{df(x)}{dx}. \qquad (3)$$

It is interesting to note that the sum of the right hand sides of (2) and (3) formally agrees with the right hand side of (1). This interesting discovery offers a useful way for students to remember how to differentiate $f(x)^{g(x)}$.

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, 5752 Neville/Math, University of Maine, Orono, ME 04469-5752. E-mail: dodge@gauss.umemat.maine.edu. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed to arrive by December 1, 1999.

Problems for Solution

953. *Proposed by Mike Pinter, Belmont University, Nashville, Tennessee.*

Since we want to enjoy our cake with a minimum amount of guilt, find the solution to the following base 10 alphametic that yields the minimum value for *ICING*.

$$ICING + CAKE = YUMMY.$$

954. *Proposed by Florian Luca, Syracuse University, Syracuse, New York.*

For any real number y let $[y]$ be the largest integer less than or equal to y. Suppose M is a set of positive integers with the following property: if $x > 1$ is an element of M, then both $[x \ln x]$ and $[\sqrt{x}]$ are elements of M. Show that, if M contains a positive integer greater than 3, then M contains all positive integers.

955. *Proposed by Peter A. Lindstrom, Batavia, New York.*

Let G be a finite geometric series whose terms are all positive integers. If G has a sum that is a prime number, then prove that the first term is 1 and the number of terms of the series is a prime.

956. *Proposed by Charles Ashbacher, Decisionmark, Cedar Rapids, Iowa.*

For any positive integer n, the value of the *Smarandache function* $S(n)$ is the smallest positive integer m such that n divides $m!$. Thus, for example, $S(1) = 1$, $S(2) = 2$, $S(6) = 3$, and $S(8) = 4$. Let p be an odd prime. Prove that the following summation diverges:

$$\sum_{k=1}^{\infty} \frac{1}{S(p^k)}.$$

957. *Proposed by the late Jack Garfunkel, Flushing, New York.*

Triangle ABC is inscribed in a circle. The angle bisectors of ABC are drawn and extended to the circle to points A', B', C'. Triangle $A'B'C'$ is drawn. Prove that $s/r \geq s'/r'$ where s, s', r, r' are respectively the semiperimeters and inradii of triangles ABC and $A'B'C'$.

958. *Proposed by George Tsapakidis, Agrinio, Greece.*

In a triangle ABC the length of the bisector AD is equal to the length of the median AM, both drawn from the same vertex A. Prove that triangle ABC is isosceles.

959. *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

Find in closed form the sum

$$\sum_{k=1}^{n} k\binom{n}{k}.$$

960. *Proposed by Timothy Sipka, Alma College, Alma, Michigan.*

A *triangular number* is any number of the form $n(n+1)/2$, where n is a positive integer. Prove that the units digit of any triangular number is 0, 1, 3, 5, 6, or 8.

961. *Proposed by Charles Ashbacher, Charles Ashbacher Technologies, Hiawatha, Iowa.*

Given any positive integer n, the value of the *Pseudo-Smarandache function* $Z(n)$ is the smallest positive integer m such that n exactly divides

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}.$$

Thus $Z(1) = 1$, $Z(2) = 3$, $Z(3) = 2$, $Z(4) = 7$, etc.

a) Prove there is an infinite family of integers n such that $3 \cdot Z(n) = n$.

b) Prove that there are an infinite number of pairs (m, n) such that $m \cdot Z(n) = n \cdot Z(m)$.

962. *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

Shoelace clock. You are given a shoelace, some matches, and a pair of scissors. The shoelace burns like a fuse when lit at either end and takes exactly 60 minutes to burn. The burn rate may vary from one point on the shoelace to another, but it has a symmetry property in that the burn rate a distance x from the left end is the same as the burn rate the same distance x from the right end.

a) Find the shortest time interval you can measure.

b) Find the shortest time interval you can measure if you have two such laces that are identical.

c) Repeat part b if the two laces, which still burn for 60 minutes each, are not identical and not symmetric.

963. *Proposed by Peter A. Lindstrom, Batavia, New York.*

Consider the functions

$$f(x) = \sin(\cos x)) + \cos x \quad \text{and} \quad g(x) = \sin(\cos x) - \cos x,$$

on the interval $0 \le x \le \pi$. Without using the calculus,

a) show that their graphs are each symmetric about the point $(\pi/2, 0)$.

b) show that f is always decreasing, so that $f(\pi) \le f(x) \le f(0)$.

c) show that g is always increasing, so that $g(0) \le g(x) \le g(\pi)$.

***964.** *Proposed by Ice B. Risteski, Skopje, Macedonia.*

There are n_k balls of color k for $k = 1, 2, \ldots, r$. The total number of balls is $n_1 + n_2 + \ldots + n_r = 2m$, where m is a positive integer.

a) In how many ways can these balls be separated into unordered color pairs?

b) Find the probability of selecting a particular color pair.

965. *Proposed by David Iny, Baltimore, Maryland.*

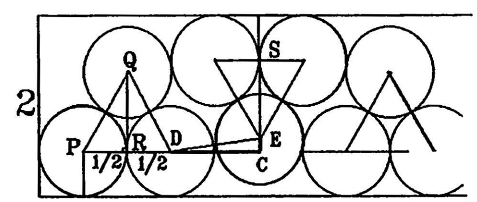
Evaluate the integral

$$\int_0^\infty \frac{e^{-x}}{1+x} dx.$$

Solutions

860. [Spring 1995, Spring 1996, Spring 1997] *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

This problem originally appeared in a column by the Japanese problems columnist Nob Yoshigahara. Find the minimal positive integer n so that $2n + 1$ circles of unit diameter can be packed inside a 2 by n rectangle.

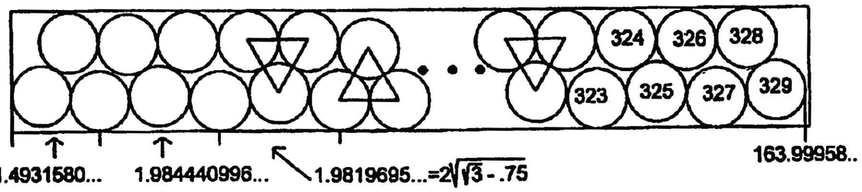


Problem 860

III. *Solution by Nick Baxter, Hillsborough, California.*

The circles need to be packed as shown in the figure on the next page. For $n = 164$ there is just enough room for 329 circles. There are 7 circles at each end with 105 "triangles" of three circles between them, occupying 163.99958... units of length.

Comment by the editor. Although he gave no figure or explanation, Baxter commented that the smallest rectangle found so far containing 329 circles has 13 circles on each end and total length 163.9973967.... It is interesting to speculate whether this is the best possible solution. The editor welcomes the opportunity to further test his computational and geometric skills by checking other solutions.



Problem 860 - Baxter's solution

927. [Spring 1998] *Proposed by Mike Pinter, Belmont University, Nashville, Tennessee.*

In the following base ten alphametic,

a) find the maximum value for *FINAL*,

b) find the minimum value for *FINAL*, and

c) can you find a solution yielding any other value for *FINAL*?

$$PASS + THE = FINAL.$$

Solution to parts (a) and (b) by Karl Bittenger, student, Austin Peay State University, Clarksville, Tennessee.

a) Since $F \neq 0$, then $P = 9$, $F = 1$, and $I = 0$. Since the hundreds column must sum to more than 10, the largest value for N occurs if $\{A,T\} = \{7,8\}$ and N is 5 or 6. For $N = 6$ the tens column must sum to more than 10, which is not possible with the remaining digits. Hence we take $N = 5$.

Only 2, 3, 4, and 6 remain unused. Next we maximize A. If $A = 8$ and $T = 7$, then $\{S,H\} = \{2,6\}$ and it is not possible to assign values to E and L. So let $A = 7$ and $T = 8$. Then $\{S,H\} = \{3,4\}$ and $\{E,L\} = \{2,6\}$. We find that $S = 4$, $H = 3$, $E = 2$, and $L = 6$ is a solution. Then we obtain

$$9744 + 832 = 10576$$

and 10576 is the maximum value for *FINAL*.

b) Again $P = 9$, $F = 1$, and $I = 0$. To minimize *FINAL* we try $N = 2$. Then none of the ten possibilities for A and T leads to a solution. So we try $N = 3$. The least value permissible for A is 4 with $T = 8$. We obtain the solution

$$9477 + 865 = 10342$$

and 10342 is the minimum value for *FINAL*.

Editorial note. Two other solutions exist: 9855 + 632 = 10487 and 9588 + 764 = 10352.

Also solved by Charles D. Ashbacher, *Charles Ashbacher Technologies, Hiawatha, IA,* Karen Bernard, *Arkansas Governor's School, Conway,* Paul S. Bruckman, *Edmonds, WA,* William Chau, *A T & T Laboratories, Middletown, NJ,* Wilson Davis, *Arkansas Governor's School, Conway,* Mark Evans, *Louisville, KY,* Stephen I. Gendler, *Clarion University of Pennsylvania,* Richard I. Hess, *Rancho Palos Verdes, CA,* Eric Jones, *Arkansas Governor's School, Conway,* Carl Libis, *Rust College, Holly Springs, MS,* Mimi Liu, *Arkansas Governor's School, Conway,* Yoshinobu Murayoshi, *Okinawa, Japan,* Jawad Sadek *and* Russell Euler, *Northwest Missouri State University, Maryville,* Rex H. Wu, *Brooklyn, NY,* Kenneth L. Yokom, *South Dakota State University, Brookings, and the* Proposer.

928. [Spring 1998] *Proposed by the late J. L. Brenner, Palo Alto, California.*

Is it true that, as n increases through the integers, the number of primes in the open interval $(n, 2n)$ can stay the same, increase by one, or decrease by one, but never change by two or more? Student solutions are especially invited.

Solution by Stephen I. Gendler, Clarion University of Pennsylvania, Clarion, Pennsylvania.

When we change from $(n, 2n)$ to $(n+1, 2n+1)$, we lose $n+1$ and add $2n$ and $2n+1$. If $n+1$ is a prime, then we lose a prime. If $2n+1$ is a prime, we add a prime. Since $2n$ is never a prime, we may lose one prime, gain one prime, or stay the same if both $n+1$ and $2n+1$ are prime or if neither is a prime.

Also solved by Paul S. Bruckman, *Edmonds, WA,* William Chau, *A T & T Laboratories, Middletown, NJ,* Richard I. Hess, *Rancho Palos Verdes, CA,* Carl Libis, *Rust College, Holly Springs, MS,* Yoshinobu Murayoshi, *Okinawa, Japan,* Jawad Sadek *and* Russell Euler, *Northwest Missouri State University, Maryville,* H.-J. Seiffert, *Berlin, Germany,* Rex H. Wu, *Brooklyn, NY, and the* Proposer. *Curiously, there were no student solutions to this problem.*

929. [Spring 1998] *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

On the ground floor of a building there are on the wall three light switches of the usual kind that show whether they are on or off. One of them controls a lamp with an ordinary incandescent 100-watt bulb located on the third floor. The other two switches are not connected to anything, although you have no way of telling which switch is the live one. You are allowed to toggle the switches at will before climbing to the third floor, which you can do just once. From the ground floor you cannot tell whether the lamp is on or off, but you have full access to the lamp when you are on the third floor.

a) Tell how to determine which switch controls the lamp.

b) Solve the problem if the switches are not marked with on and off positions, but you know that the lamp is initially off.

c) Solve part (b) if you do not know if the lamp is initially on or off.

I. *Solution by Rex H. Wu, Brooklyn, New York.*

a) We know the lamp is cold and off to begin with. First, turn the first switch on, wait for a while to let the lamp warm up and then turn it off. Turn the second switch on and immediately run up to the third floor (assuming the problem solver can do so). If the lamp is on, then it is the second switch. If it is off and warm, it is the first switch. Otherwise, it is the third.

b) It does not matter whether or not the switches are marked on and off. Just flip the switches as in part (a). The result is the same.

c) Using only on/off and cold/warm is not sufficient in this case. So we assume one can distinguish hot, warm, and cold temperatures. Also, if the bulb is on originally, it is assumed to be hot; if off, then cold. First, flip the first switch and wait long enough for the bulb to reach warm temperature if this switch controls the bulb. Then flip the first switch back, flip the second switch, and run upstairs. If the first switch is connected to the lamp, then the lamp will be warm, since it has had a chance to cool from hot or warm from cold. If it is the second switch, the lamp will be either hot and off or cold and on. If it is the third switch, the light will be on and hot or off and cold. The table below indicates which switch controls the lamp, according to whether the lamp was initially on or off.

Although this is my original solution, I do not like it. I am waiting for a better solution.

Connected Switch	Lamp originally on	Lamp originally off
1	on/warm	off/warm
2	off/hot	on/cold
3	on/hot	off/cold

II. *Solution to part (c) by the Proposer.*

c) By direct observation it takes a 100 watt bulb 30-45 seconds to heat up or cool down to a point where it can be comfortably held and twice as long to get fully hot or to cool. It takes the bulb socket 3-5 minutes to get fully warm or to cool. It takes about 20 seconds to climb to the third floor. Change switch 1. Wait 2 minutes. Then change it back, change switch 2, and hurry upstairs. If the [light, bulb, socket] is [off, hot, cool] or [on, cool, warm], then it is switch 1. If it is [off, hot, warm] or [on, cool, cool], then it is switch 2. If it is [off, cool, cool] or [on, hot, warm], then it is switch 3.

Also solved by the Proposer.

Editorial note. In early June of 1998 part (a) of this problem was presented on the National Public Radio program Car Talk, posed by Jim Gardner, son of the problemist Martin Gardner. Apparently one of the car talk brothers had stated earlier that Martin Gardner had died. So Jim called in to say that his father disagreed. He then posed the light switch problem as Car Talk's puzzler of the week. In a subsequent letter to this editor Martin Gardner credited Hess as the originator of this proposal.

930. [Spring 1998] *Proposed by the late J. L. Brenner, Palo Alto, California.*

By direct calculation, that is, without using published theorems, show that the permutation group generated by (127) and (135)(246) contains all of the following types (*shapes*): 3^1, 2^2, 4^12^1, 3^2, 3^12^2, 7^1, and 5^1, where (127) is of type 3^1 and (135)(246) is of type 3^2.

Solution by Rex H. Wu, Brooklyn, New York.

Let $\alpha = (127)$ and $\beta = (135)(246)$. Then we have, by direct computation,
type 2^2: $(\alpha\beta^2\alpha\beta)(\alpha\beta^2\alpha^2\beta) = (17)(25)$,
type 4^12^1: $(\alpha\beta^2\alpha\beta)(\alpha\beta^2\alpha^2\beta)(\alpha\beta\alpha^2) = (1746)(23)$,
type 3^12^2: $\alpha\beta\alpha^2\beta = (15)(263)(47)$,
type 7^1: $\alpha\beta = (1352467)$, and
type 5^1: $\alpha\beta\alpha\beta^2 = (12734)$.

Also solved by Stephen I. Gendler, *Clarion University of Pennsylvania, and the* Proposer.

931. [Spring 1998] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Determine the maximum value of

$$\frac{1}{S_u^r}\{S_{nur_1} + S_{nur_2} + \cdots + S_{nur_m}\},$$

where the x_i, r_i, and u all are positive and

$$S_t = x_1^t + x_2^t + \cdots + x_n^t \quad \text{and} \quad r = r_1 + r_2 + \cdots + r_m \geq 1.$$

Solution by Paul S. Bruckman, Edmonds, Washington.

We assume that the intent is to allow the x_j's to vary, while all other parameters are constant. Let

$$K = (S_u)^{-r}\sum_{i=1}^{m} S_{nur_i}.$$

Then

$$(S_u)^r K = \sum_{i=1}^{m}\sum_{j=1}^{n}(x_j)^{nur_i} = \sum_{j=1}^{n}\sum_{i=1}^{m}(x_j)^{nur_i}.$$

If all the x_j's are equal (say to x), this last expression becomes

$$(S_u)^r K = n\sum_{i=1}^{m} x^{nur_i} \leq mnx^{nur}.$$

Also, $(S_u)^r = n^r x^{ur}$. Then $K \leq mn^{1-r}x^{(n-1)ur}$ in this case.

More generally, let $y = \min(x_j)$ and $Y = \max(x_j)$ for $j = 1, 2, \ldots, n$. By the AM-GM inequality, $S_u \geq n(\Pi_{j=1}^{n} x_j)^{u/n}$; thus $(S_u)^r \geq n^r y^{ur}$. Also, using Hölder's inequality,

$$\left(\frac{S_{nur_i}}{n}\right)^{1/nur_i} \leq \left(\frac{S_{nur}}{n}\right)^{1/nur},$$

which implies

$$S_{nur_i} \leq n^{1-r_i/r}(S_{nur})^{r_i/r} \leq n^{1-\rho/r}S_{nur},$$

where $\rho = \min(r_i)$ for $i = 1, 2, \ldots, m$. Then

$$\sum_{i=1}^{m} S_{nur_i} \leq mn^{1-\rho/r}S_{nur} \leq mn^{2-\rho/r}Y^{nur},$$

and hence

$$K \leq mn^{(2-r-\rho/r)}y^{-ur}Y^{nur} \leq mn^{1-r}y^{-ur}Y^{nur}.$$

Equality is attained if and only if all $x_j = x$ and $m = 1$, in which case $\rho = r$, $y = Y$ and $K = n^{1-r}x^{(n-1)ur}$.

Also solved by the Proposer.

932. [Spring 1998] *Proposed by David Iny, Baltimore, Maryland.*

a) For $0 < \mu < \varepsilon \leq 1$, define a sequence recursively by $x_0 = \varepsilon$, and $x_{n+1} = \sin x_n$ for $n \geq 0$. Thus $\{x_n\}$ is a monotone decreasing sequence of positive numbers. Estimate the smallest value of m such that $x_m \leq \mu$.

b) Repeat Part (a) using the recursion formula $x_{n+1} = \ln(1 + x_n)$.

Solution by Mark Evans, Louisville, Kentucky.

a) By Taylor's approximation, $x_{n+1} = \sin x_n \approx x_n - x_n^3/6$, so then

$$\Delta x = x_n - x_{n+1} \approx \frac{x_n^3}{6}.$$

Thus, to cover the interval dx requires $(x^3/6)^{-1}\,dx$ iterations. This leads to the value

$$m = \int_\mu^\epsilon 6x^{-3}dx = 3\left(\frac{1}{\mu^2} - \frac{1}{\epsilon^2}\right).$$

b) Similarly, since $\ln(1 + x) \approx x - x^2/2$, we have

$$\Delta x = x_n - x_{n+1} = x_n - \ln(1 + x_n) \approx \frac{x_n^2}{2}.$$

Again, to cover the interval dx requires $(x^2/2)^{-1}\,dx$ iterations. This leads to the value

$$m = \int_\mu^\epsilon 2x^{-2}dx = 2\left(\frac{1}{\mu} - \frac{1}{\epsilon}\right).$$

I tested these formulae by computer and they are either within 1 iteration of being correct or the error is a fraction of a percent.

Also solved by Paul S. Bruckman, *Edmonds, WA,* Richard I. Hess, *Rancho Palos Verdes, CA, and the* Proposer.

933. [Spring 1998] *Proposed by David Iny, Baltimore, Maryland.*
Define for nonnegative integers k and n the sums

$$J_{kn} = \frac{1}{1+k}\binom{n}{0} + \frac{1}{2+k}\binom{n}{1} + \cdots + \frac{1}{n+1+k}\binom{n}{n}.$$

a) Find closed form expressions for J_{kn} for $k = 0, 1, 2, \ldots$

b) Let p be any nonnegative real number and $[x]$ the greatest integer less than or equal to x. Evaluate

$$\lim_{n\to\infty} n2^{-n}(J_{[pn]n}).$$

I. *Solution to part (a) by H.-J. Seiffert, Berlin, Germany.*

a) A closed form expression for J_{kn} is not known. We can, however, give some integral representations and other sum formulas. For example, we have

$$J_{kn} = \sum_{i=0}^{n}\frac{1}{i+1+k}\binom{n}{i} = \sum_{i=0}^{n}\binom{n}{i}\int_0^1 x^{i+k}dx = \int_0^1\left(\sum_{i=0}^{n}\binom{n}{i}x^{i+k}\right)dx,$$

or, by the binomial theorem,

$$J_{kn} = \int_0^1 x^k(x+1)^n dx, \quad k, n \in N_0.$$

II. *Solution to part (b) by Richard I. Hess, Rancho Palos Verdes, California.*

By taking the derivative of $x^{k+1}(1 + x)^n/(n + k + 1)$, we find that

$$\int x^k(1+x)^n dx = \frac{x^{k+1}(1+x)^n}{n+k+1} + \frac{n}{n+k+1}\int x^k(1+x)^{n-1}dx.$$

It follows that

$$J_{kn} = \frac{1}{n+k+1}\left(2^n + \frac{n2^{n-1}}{n+k} + \frac{n(n-1)2^{n-2}}{(n+k)(n+k-1)} + \cdots + \frac{n!}{(n+k)(n+k-1)\cdots(1+k)}\right).$$

Now we have, for p a positive integer,

$$J_{pn,n} = \frac{1}{n+pn+1}\left(2^n + \frac{2^{n-1}}{1+p} + \frac{1(1-\frac{1}{n})2^{n-2}}{(1+p)(1+p-\frac{1}{n})} + \cdots + \frac{(1-\frac{1}{n})(1-\frac{2}{n})\cdots(\frac{1}{n})}{(1+p)(1+p-\frac{1}{n})\cdots(\frac{1}{n}+p)}\right)$$

$$\approx \frac{1}{n(1+p)+1}\left(2^n + \frac{2^{n-1}}{1+p} + \frac{2^{n-2}}{(1+p)^2} + \frac{2^{n-3}}{(1+p)^3} + \cdots\right) = \frac{2^n}{n(1+p)+1}\cdot\frac{2(p+1)}{2p+1}.$$

Therefore,

$$\lim_{n\to\infty}\frac{n}{2^n}J_{[pn]n} = \lim_{n\to\infty}\frac{2(p+1)}{([1+p]+\frac{1}{n})(2p+1)} = \frac{2}{2p+1}.$$

Also solved by Paul S. Bruckman, *Edmonds, WA,* Richard I. Hess, *Rancho Palos Verdes, CA,* Carl Libis, *Rust College, Holly Springs, MS,* Cecil Rousseau, *The University of Memphis, TN,* H.-J. Seiffert, *Berlin, Germany, and the* Proposer. *Not all solvers agreed with the above solution.*

934. [Spring 1998] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Evaluate the integral

$$I = \int_{-\infty}^{\infty} \sin\left(\pi^4 x^2 + \frac{1}{x^2}\right) dx.$$

Solution by Roger Zarnowski and Charles Diminnie, Angelo State University, San Angelo, Texas.

Our solution uses the well-known Fresnel integrals

$$\int_0^\infty \sin x^2 dx = \int_0^\infty \cos x^2 dx = \sqrt{\frac{\pi}{8}}.$$

Since the integrand is an even function, we get

$$I = 2\int_0^\infty \sin\left(\pi^4 x^2 + \frac{1}{x^2}\right) dx$$

$$= 2\left[\int_0^{1/\pi} \sin\left(\pi^4 x^2 + \frac{1}{x^2}\right) dx + \int_{1/\pi}^\infty \sin\left(\pi^4 x^2 + \frac{1}{x^2}\right) dx\right],$$

provided both integrals exist. Now let $u = 1/(\pi^2 x^2)$ in the left integral to get

$$\int_0^{1/\pi} \sin\left(\pi^4 x^2 + \frac{1}{x^2}\right) dx = \int_\infty^{1/\pi} \sin\left(\frac{1}{u^2} + \pi^4 u^2\right)\left(\frac{-1}{\pi^2 u^2}\right) dx$$

$$= \int_{1/\pi}^\infty \sin\left(\pi^4 x^2 + \frac{1}{\pi^2 x^2}\right)\left(\frac{1}{\pi^2 x^2}\right) dx,$$

and hence,

$$I = 2\int_{1/\pi}^\infty \sin\left(\pi^4 x^2 + \frac{1}{x^2}\right)\left(1 + \frac{1}{\pi^2 x^2}\right) dx$$

$$= 2\int_{1/\pi}^\infty \sin\left[\left(\pi^2 x - \frac{1}{x}\right)^2 + 2\pi^2\right]\left(1 + \frac{1}{\pi^2 x^2}\right) dx.$$

Finally, letting $z = \pi^2 x - 1/x$, we obtain

$$I = \frac{2}{\pi^2}\int_0^\infty \sin(z^2 + 2\pi^2) dz$$

$$= \frac{2}{\pi^2}\left[(\cos 2\pi^2)\int_0^\infty \sin z^2 dz + (\sin 2\pi^2)\int_0^\infty \cos z^2 dz\right]$$

$$= \frac{2}{\pi^2}\left[(\cos 2\pi^2)\sqrt{\frac{\pi}{8}} + (\sin 2\pi^2)\sqrt{\frac{\pi}{8}}\right]$$

$$= \frac{1}{\sqrt{2\pi^3}}(\cos 2\pi^2 + \sin 2\pi^2).$$

It is interesting to note that a slight change in the original problem would give the interesting evaluation

$$\int_{-\infty}^\infty \sin\left(\pi^2 x^2 + \frac{1}{x^2}\right) dx = \frac{1}{\sqrt{2\pi}}.$$

Also solved by Paul S. Bruckman, *Edmonds, WA,* Bob Prielipp, *University of Wisconsin-Oshkosh,* Cecil Rousseau, *The University of Memphis, TN,* H.-J. Seiffert, *Berlin, Germany, and the* Proposer.

Seiffert noted the interesting evaluation in the Zarnowski-Diminnie solution and also

$$\int_{-\infty}^\infty \sin\left(\pi x^2 + \frac{\pi}{x^2}\right) dx = \frac{1}{\sqrt{2}}.$$

***935.** [Spring 1998] *Proposed by M. A. Khan, Lucknow, India.*

From a deck of n cards numbered 1, 2, ..., n, select m cards ($3 \le m \le n$) at random. Show that the probability p that the numbers on the selected cards are in arithmetic progression is given by

$$p = \frac{(q + 1)(R + n + 1 - m)}{2\binom{n}{m}},$$

where q is the integral quotient and R the remainder when $n-m+1$ is divided by $m-1$.

Solution by Paul S. Bruckman, Edmonds, Washington.

Let N_k denote the number of ways in which the m cards can be drawn in such a way that the card numbers are in arithmetic progression with common difference k (irrespective of the order in which the numbers are drawn). Also, let $N = \Sigma N_k$, summed over all acceptable values of k. Then $p = N/{}_nC_m$. Now N_1 counts the following possible draws: $\{1,2,...,m\}$, $\{2,3,...,m+1\}$, ..., $\{n-m+1, n-m+2, ..., n\}$, irrespective of the order in which the numbers are drawn. Thus $N_1 = n - m + 1$.

Similarly, N_2 counts the sets $\{1,3,5,,,2m-1\}$, $\{2,4,6,...,2m\}$, ..., $\{n-2m+2, n-2m+4, ..., n\}$, so $N_2 = n - 2m + 2$. In general, we see that $N_k = n - k(m - 1)$, subject to restrictions on k.

By definition, $n - m + 1 = q(m - 1) + R$, so $n = (q + 1)(m - 1) + R$, with $0 \le R \le m - 1$. Thus, we must restrict k so that $N_k \ge 0$, which requires that $1 \le k \le q + 1$. Then

$$N = \sum_{k=1}^{q+1} [n - k(m - 1)] = n(q + 1) - (m - 1)(q + 1)(q + 2)/2$$

$$= \frac{1}{2}(q + 1)[2n - (m - 1)(q + 2)] = \frac{1}{2}(q + 1)(R + n + 1 - m),$$

which leads to the indicated expression for p.

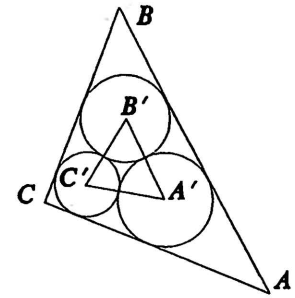
Also solved by Mark Evans, *Louisville, KY, and the* Proposer.

936. [Spring 1998] *Proposed by the late Jack Garfunkel, Flushing, New York.*

Given the Malfatti configuration, where three mutually external, mutually tangent circles with centers A', B', C' are inscribed in a triangle ABC so that circle (A') is tangent to the two sides of angle A, circle (B') is tangent to the sides of angle B, and (C') to the sides of C. See the figure. If $\angle A \le \angle B \le \angle C$ and $\angle A < \angle C$, then prove that we have $\angle C' - \angle A' < \angle C - \angle A$.

Solution by Rex H. Wu, Brooklyn, New York.

Let $r_{A'}$, $r_{B'}$, and $r_{C'}$ be the radii of circles A', B', and C' respectively. Observe that when $\angle A \le \angle B \le \angle C$, then $r_{A'} \ge r_{B'} \ge r_{C'}$ and $\angle C' \ge \angle B' \ge \angle A'$. Furthermore, because $r_{A'} \ge r_{C'}$ and $r_{B'} \ge r_{C'}$, then $\angle C' \le \angle C$. This can be seen by drawing lines through C' parallel to CA and to CB. Similarly, $\angle A' \ge \angle A$, so that $\angle C - \angle A \ge \angle C' - \angle A'$. Also, we have equality only when $\angle A = \angle B = \angle C$. Hence, if $\angle A < \angle C$, then $\angle C - \angle A > \angle C' - \angle A'$.



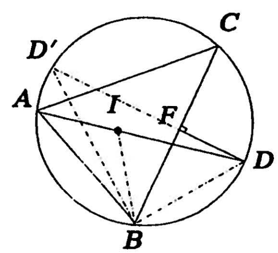
Problem 936

Also solved by Paul S. Bruckman, *Edmonds, WA*, Richard I. Hess, *Rancho Palos Verdes, CA, and the* Proposer.

Other solvers showed the inequality $\angle C' \ge \angle B' \ge \angle A'$ by examining trigonometrically the angles between lines CA and $C'A'$, etc.

937. [Spring 1998] *Proposed by R. S. Luthar, Janesville, Wisconsin.*

Let I be the incenter of triangle ABC, let AI cut the triangle's circumcircle (again) at point D, and let F be the foot of the perpendicular dropped from D to side BC, as shown in the figure. Prove that $DI^2 = 2R \cdot DF$, where R is the circumradius of triangle ABC.



Problem 937

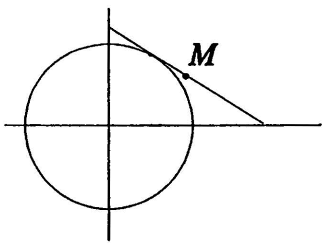
Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

Extend DF to cut the circle again at D', forming diameter DD', which subtends a right angle at B, as seen in the figure. Now BF is the altitude to the hypotenuse of right triangle DBD'. By standard mean proportion,

$$BD^2 = DF \cdot DD' = DF \cdot 2R.$$

Now $\angle BAD = \angle BAI = \frac{1}{2}\angle A$ and $\angle ABI = \frac{1}{2}\angle B$, so we have $\angle BID = \frac{1}{2}(\angle A + \angle B)$. Since $\angle IDB = \angle ADB = \angle ACB = \angle C$, then we also have that $\angle DBI = \frac{1}{2}(\angle A + \angle B)$. Therefore, $BD = DI$ and the desired result follows.

Also solved by Paul S. Bruckman, *Edmonds, WA,* Yoshinobu Murayoshi, *Okinawa, Japan,* William H. Peirce, *Delray Beach, FL,* Rex H. Wu, *Brooklyn, NY, and the* Proposer.



Problem 938

938. [Spring 1998] *Proposed by R. S. Luthar, Janesville, Wisconsin.*

Find the locus of the midpoints M of the line segments in the first quadrant lying between the two axes and tangent to the unit circle centered at the origin. See the figure.

Solution by Keith Maceli, student, Loyola College, Baltimore, Maryland.

We consider the equation $f(x) = \sqrt{1 - x^2}$ for $0 \le x \le 1$ of the circle in the first quadrant and we find an equation for the tangent line at $x = a$, where $0 < a < 1$. Since $f'(a) = -a/\sqrt{1 - a^2}$, we have

$$y - \sqrt{1 - a^2} = \frac{-a}{\sqrt{1 - a^2}}(x - a).$$

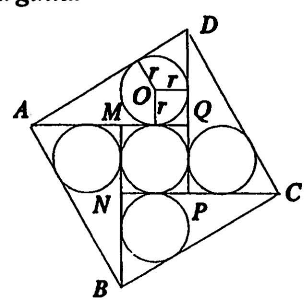
By setting $x = 0$ and $y = 0$ we obtain respectively the y and x intercepts $(0, 1/\sqrt{1 - a^2})$ and $(1/a, 0)$. Therefore, the midpoint M of the segment is at $(1/(2a), 1/(2\sqrt{1 - a^2}))$. Now let $x = 1/(2a)$, so that $a = 1/(2x)$. Substitute into the equation $y = 1/(2\sqrt{1 - a^2})$ to get the equation of the locus of midpoints M

$$y = \frac{1}{2\sqrt{1 - \left(\frac{1}{2x}\right)^2}} = \frac{x}{\sqrt{4x^2 - 1}} \quad \text{for } x > 1/2.$$

Thus, the graph, though not a hyperbola, has asymptotes $x = 1/2$ and $y = 1/2$.

Also solved by Cheril Lin Abeel-Wescoat, *Alma College, MI,* Miguel Amengual Covas, *Cala Figuera, Mallorca, Spain,* Ayoub B. Ayoub, *Pennsylvania State University, Abington,* Paul S. Bruckman, *Edmonds, WA,* Rob Downes, *Plainfield, NJ,* Russell Euler *and* Jawad Sadek, *Northwest Missouri State University, Maryville,* Mark Evans, *Louisville, KY,* Jayanthi Ganapathy, *University of Wisconsin-Oshkosh,* Robert C. Gebhardt, *Hopatcong, NJ,* Stephen I. Gendler, *Clarion University of Pennsylvania,* Richard I. Hess, *Rancho Palos Verdes, CA,* Joe Howard, *New Mexico Highlands University, Las Vegas,* Peter A. Lindstrom, *Batavia, NY,* H.-J. Seiffert, *Berlin, Germany,* Skidmore College Problem Group, *Saratoga Springs, NY,* Kenneth L. Yokom, *South Dakota State University, Brookings,* Monte J. Zerger, *Adams State College, Alamosa, CO, and the* Proposer.

939. [Spring 1998] *Proposed by Khiem Viet Ngo, Virginia Polytechnic Institute, Blacksburg, Virginia.*



Problem 939

In the accompanying figure both quadrilaterals $ABCD$ and $MNPQ$ are squares, each side of square $ABCD$ has length 1, and the five inscribed circles are all congruent to one another. Find their common radius.

Solution by Abhiram Shandilya, student, Angelo State University, San Angelo, Texas.

Let O denote the center of the incircle of triangle AQD and draw radii from O to the three sides of that triangle. Let $AQ = x$ and $DQ = y$. Since the area of triangle AQD is the sum of the areas of triangles OAD, OAQ, and OQD, and since $AD = 1$, we have that

$$\frac{1}{2}xy = \frac{1}{2}r + \frac{1}{2}rx + \frac{1}{2}ry, \text{ so } xy = r(1 + x + y).$$

Multiply both sides of this last equation by $1 - (x + y)$ and use the fact that $x^2 + y^2 = 1$ to get

$$1 - x - y = -2r.$$

Since $AQ = AM + MQ$ and triangles AQD and BMA are congruent, we have $x = y + 2r$, which, when taken with the last displayed equation and the equation $x^2 + y^2 = 1$ yields $y = 1/2$ (so triangle AQD is a 30°-60° right triangle) and $8r^2 + 4r - 1 = 0$. Thus

$$r = \frac{\sqrt{3} - 1}{4}.$$

Also solved by Miguel Amengual Covas, *Cala Figuera, Mallorca, Spain,* Ayoub B. Ayoub, *Pennsylvania State University, Abington,* Scott H. Brown, *Auburn University, AL,* Paul S. Bruckman, *Edmonds, WA,* Russell Euler *and* Jawad Sadek, *Northwest Missouri State University, Maryville,* Mark Evans, *Louisville, KY,* Robert C. Gebhardt, *Hopatcong, NJ,* Richard I. Hess, *Rancho Palos Verdes, CA,* Joe Howard, *New Mexico Highlands University, Las Vegas,* Yoshinobu Murayoshi, *Okinawa, Japan,* William H. Peirce, *Delray Beach, FL,* Carter Price, *Arkansas Governor's School, Conway,* Shiva K. Saksena, *University of North Carolina at Wilmington,* Nicholas Seward, *Arkansas Governor's School, Conway,* Rex H. Wu, *Brooklyn, NY,* Kenneth L. Yokom, *South Dakota State University, Brookings,* Monte J. Zerger, *Adams State College, Alamosa, CO, and the* Proposer.

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Find out by attending the 1999 Meeting of the Pi Mu Epsilon National Honorary Mathematics Society, which will be held in Providence, RI, on July 31 and August 1. The meeting will be held in conjunction with the MAA Mathfest, which will run from July 31-August 2. Pi Mu Epsilon will again coordinate its national meeting with that of the MAA student chapters.

The Pi Mu Epsilon meeting will begin with a reception on the evening of Friday, July 30. On Saturday morning, July 31, the Pi Mu Epsilon Council will have its annual summer meeting. The student presentations will begin later that same day. The presentations will continue on Sunday, August 1. The Pi Mu Epsilon Banquet will take place that evening, followed by the J. Sutherland Frame Lecture. This year's Frame lecture will be given by Fred Rickey, of Bowling Green State University. Pi Mu Epsilon members are encouraged to participate in the MAA Student Chapter Workshop and Student Lecture, both of which will take place on Monday, August 2.

Pi Mu Epsilon will provide travel support for student speakers at the national meeting. The first speaker is eligible for 31 cents per mile, up to a maximum of $600. If a student chooses to use public transportation, PME will reimburse for the actual cost of transportation, up to a maximum of $600. In case this request exceeds 31 cents per mile, receipts should be presented. The first four additional speakers from a given chapter are eligible for 20% of whatever amount the first speaker receives. In the case of more than one speaker from one chapter, the speakers may share the support in any way that they see fit. If a chapter is not represented by a student speaker, Pi Mu Epsilon will provide one-half support for a student delegate. **Every Pi Mu Epsilon student member is encouraged to give a presentation at this summer meeting!** For further information about attending the meeting, preparing a talk to present, and receiving travel support:

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