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THE MINIMUM AREA OF CONVEX LATTICE n-GONS

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Let $A(n)$ be the minimum area of convex lattice n-gons. We prove that $\lim A(n)/n^3$ exists. Our computations suggest that the value of the limit is very close to $0.0185067\ldots$.

1. Introduction

What is the minimal area $A(n)$ a convex lattice polygon with n vertices can have? The first to answer this question was G.E. Andrews [1]. He proved that $A(n) \geq cn^3$ with some universal constant c. V.I. Arnol'd arrived to the same question from another direction [2], and proved the same estimate. Further proofs are due to W. Schmidt [10], Bárány–Pach [3]. The best lower bound comes form Rabinowitz [8] via an inequality of Rényi–Sulanke [9]

$$\frac{1}{8\pi^2} < \frac{A(n)}{n^3} \leq \frac{1}{54}(1+o(1)).$$

The upper bound follows from Remark 2 below.

Our main result is:

Theorem 1. *$\lim A(n)/n^3$ exists.*

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The value of the limit – as we are going to show – equals the minimum of finitely many explicit extremal problems. But the finitely many is about 10^{10}, too many to solve. Our computations show, however, that most likely

$$\lim \frac{A(n)}{n^3} = 0.0185067\dots.$$

We will also see that the convex lattice n-gon P with area $A(n)$ has elongated shape: after applying a suitable lattice preserving affine transformation, P has lattice width $c_1 n$ in direction $(0,1)$ and has width $c_2 n^2$ in direction $(1,0)$ where c_1, c_2 positive constants. Almost all the paper is devoted to the proof of Theorem 1.

Remark 1. Actually, Andrews [1] showed much more, namely the following (see also [10], [7]). If $P \subset \mathbf{R}^d$ is a convex lattice polytope with n vertices and volume $V > 0$, then

$$cn^{\frac{d+1}{d-1}} \leq V,$$

where c is a constant depending only on dimension.

2. Reduction

Define $\mathcal{P}_n$ as the set of all convex lattice n-gons in $\mathbf{R}^2$, then

$$A(n) = \min\{\text{Area}\, P : P \in \mathcal{P}_n\}.$$

In the next two claims, whose proof is given at the end of this section, we reduce the search for $A(n)$. As $A(n)$ is increasing it is enough to work with even n.

Claim 1. *For even n, there exists a centrally symmetric $P \in \mathcal{P}_n$ with $A(n) = \text{Area}\, P$.*

Fix a centrally symmetric $P \in \mathcal{P}_n$ with $A(n) = \text{Area}\, P$ ($n = 2k$ even). The edges are $z_1, z_2, \dots, z_k, -z_1, \dots, -z_k$ in this order. Clearly, each z_i is a primitive vector, i.e., its components are coprime. Write $\mathbf{P}$ for the set of all primitive vectors in $\mathbf{Z}^2$. Define

$$C = \text{conv}\{z_1, z_2, \dots, z_k, -z_1, \dots, -z_k\}.$$

Then P is the zonotope spanned by $\{z_1, \dots, z_k\}$, i.e., $P = \sum_{i=1}^k [0, z_i]$. As it is well-known and easy to check

$$\text{Area}\, P = \sum_{1 \leq i < j \leq k} |\det(z_i, z_j)|.$$

Write $\mathcal{C}$ for the set of 0-symmetric convex bodies in $\mathbf{R}^2$. So $C\in\mathcal{C}$ and define

$$A(C) = \frac{1}{8}\sum_{u\in C\cap\mathbf{P}}\sum_{v\in C\cap\mathbf{P}} |\det(u,v)|.$$

The following claim shows that $A(C)=\operatorname{Area} P(C)$.

Claim 2. *If $z\in C\cap\mathbf{P}$ then $z=z_i$, or $-z_i$ for some i.*

This means that the search for $A(n)$, or for minimal $P\in\mathcal{P}_n$ is reduced to the following minimization problem.

$$\operatorname{Min}(n) = \min\{A(C) : C\in\mathcal{C} \text{ with } |C\cap\mathbf{P}| = n\}.$$

Observe that the solution C to the problem $\operatorname{Min}(n)$ is invariant under lattice preserving linear transformation. Thus we may fix C in *standard position*. This means that the lattice width of C is $2b=2b(C)$ and is taken in direction $(0,1)$. Recall (from [6], say,) that the *width* of $K\subset\mathbf{R}^2$ in direction $z\in\mathbf{Z}^2$, $z\neq 0$ is

$$w(z,K) = \max\{zx - zy :\ x,y\in K\},$$

and the *lattice width* of K is, by definition,

$$w(K) = \min\{w(z,K) :\ z\in\mathbf{Z}^2, z\neq 0\}.$$

Now let $[-a,a]$ be the intersection of C with the x axis. We may further assume that the tangent line to C at $(a,0)$ has slope ≥ 1. A simple computation, (using the fact that the width of C in direction $(1,0)$ and $(1,-1)$ is at least $2b$) shows that $2a\geq b$. We fix C in this standard position. We record the following inequalities:

$$2a\geq b, \quad 2ab\leq \operatorname{Area} C\leq 4ab.$$

Remark 2. From now on we may assume $b\geq 2$ since for $b=1$, according to Claim 2, the minimal C is (with $n=2k$)

$$\operatorname{conv}\{\pm(0,1),\pm(1,1),\pm(2,1),\ldots,\pm(k-2,1),\pm(1,0)\}$$

which gives $\lim\frac{A(C)}{n^3}=\frac{1}{48}$, the example found in [8].

When $C\in\mathcal{C}$ is a circle with $|C\cap\mathbf{P}|=n$, its radius, and then $A(C)$ are estimated easily showing $\lim\frac{A(C)}{n^3}=\frac{1}{54}$. This is the estimate given in the introduction.

Proof of Claim 1. Let $Q\in\mathcal{P}_n$ with vertices $v_1,v_2,\dots,v_{2k}$ $(n=2k)$ in this order. The diagonal $[v_i,v_{i+k}]$ cuts Q into two parts. Reflecting the part with smaller (or equal) area to the point $(v_i+v_{i+k})/2$ produces a lattice polygon with area $\le \operatorname{Area} Q$. So it is enough to show that, for some $i\in\{1,\dots,k\}$, the reflected n-gon is convex. It is certainly convex if there are parallel tangent lines to Q at v_i and v_{i+k}.

If there are no such tangents then the lines of the edges incident to v_i intersect the ones incident to v_{i+k} on the same side of the line v_iv_{i+k}, on the left side, say. Then the lines of the edges, incident to v_{i+k} intersect the ones incident to v_{i+k+1} on the left side of the line $v_{i+1}v_{i+k+1}$, again. Starting with $i=1$ a contradiction is reached at $i=k+1$. ∎

We prove Claim 2 in stronger form:

Claim 2′. *Assume that $x_1,\dots,x_k\in\mathbf{R}^2$, and no two of them collinear. If $x\in\operatorname{conv}\{\pm x_1,\dots,\pm x_k\}$ and $x\ne\pm x_i$ $(\forall i)$, then there is a j such that replacing x_j by x gives a zonotope with smaller area.*

Proof. Assume first that x is on the boundary of $\operatorname{conv}\{\pm x_1,\dots,\pm x_k\}$. Then $x=(1-u)x_s+ux_t$ for some $0<u<1$. We may also assume that $\sum_{i=1}^k|\det(x_i,x_s)|\ge\sum_{i=1}^k|\det(x_i,x_t)|$. Let $y_s=x$ and $y_i=x_i$ if $i\ne s$. Then

$$\begin{aligned}
\sum_{i=1}^k|\det(y_i,y_s)| &= \sum_{i\ne s}|\det(x_i,x)| < \sum_{i=1}^k|\det(x_i,x)|\\
&= \sum_{i=1}^k|\det(x_i,(1-u)x_s+ux_t)|\\
&= \sum_{i=1}^k|(1-u)\det(x_i,x_s)+u\det(x_i,x_t)|\\
&\le (1-u)\sum_{i=1}^k|\det(x_i,x_s)|+u\sum_{i=1}^k|\det(x_i,x_t)|\\
&\le (1-u)\sum_{i=1}^k|\det(x_i,x_s)|+u\sum_{i=1}^k|\det(x_i,x_s)|\\
&= \sum_{i=1}^k|\det(x_i,x_s)|.
\end{aligned}$$

Thus, replacing x_s by x makes the area smaller.

If x is in the interior of $\operatorname{conv}\{\pm x_1,\dots,\pm x_k\}$, then λx is on the boundary of this set with a unique $\lambda>1$ (apart from the trivial case $x=0$). The previous argument shows that replacing x_s by λx makes the area smaller, and consequently, replacing x_s by x makes it smaller, too. ∎

In the next two sections we approximate $|C \cap \mathbf{P}|$ and $A(C)$ using that the density of $\mathbf{P}$ in $\mathbf{Z}^2$ is $6/\pi^2$ (cf. [5]). We need to measure approximation by a quantity invariant under lattice preserving linear transformations. This is going to be the lattice width $2b = 2b(C)$.

3. Approximating $|C \cap \mathbf{P}|$

Lemma 1.

$$\left| |C \cap \mathbf{P}| - \frac{6}{\pi^2} \operatorname{Area} C \right| \ll \operatorname{Area} C \cdot \frac{\log b}{b}.$$

Here and in what follows we use Vinogradov's $\ll$ notation. Thus $f(n) \ll g(n)$ means that $f(n) \le D g(n)$ with some universal constant D.

Proof. The proof is standard and uses the Möbius function $\mu(d)$ see [5]. Set

$$C^+ = C \cap \{(x,y) \in \mathbf{R}^2 : y > 0\}.$$

Clearly, $|C \cap \mathbf{P}| = 2 + 2|C^+ \cap \mathbf{P}|$ and

$$|C^+ \cap \mathbf{P}| = \sum_{(u,v) \in C^+ \cap \mathbf{Z}^2} \sum_{\substack{d|u \\ d|v}} \mu(d) = \sum_{d=1}^{\infty} \mu(d) \sum_{\substack{(u,v) \in C^+ \cap \mathbf{Z}^2 \\ d|u, d|v}} 1 = \sum_{d=1}^{b} \mu(d) \left| \frac{1}{d} C^+ \cap \mathbf{Z}^2 \right|.$$

Claim 3.

$$\left| |\frac{1}{d} C^+ \cap \mathbf{Z}^2| - \frac{1}{d^2} \operatorname{Area} C^+ \right| \le \frac{12a}{d}.$$

Proof. It suffices to show this for $d = 1$. Let $Q(z)$ denote the unit square centered at $z \in \mathbf{Z}^2$. Call $z \in \mathbf{Z}^2$ *inside* if $Q(z) \subset C^+$, *boundary* if $z \in C^+$ but $Q(z) \not\subset C^+$, and *outside* if $z \notin C^+$ and $Q(z) \cap \operatorname{int} C^+ \ne \emptyset$. (Note that we will use the same inside, boundary, and outside squares $Q(z)$ in the next section.) Clearly

$$\begin{aligned} |C^+ \cap \mathbf{Z}^2| &= |\{z \in \mathbf{Z}^2 : \text{inside}\}| + |\{z \in \mathbf{Z}^2 : \text{boundary}\}| \\ &= \operatorname{Area} C^+ + \sum_{z \text{ boundary}} \operatorname{Area}(Q(z) \setminus C^+) - \sum_{z \text{ outside}} \operatorname{Area}(Q(z) \cap C^+), \end{aligned}$$

and the number of boundary and outside $z \in \mathbf{Z}^2$ is at most the perimeter of the smallest aligned box containing C^+, which is $2b + 2(2a+b) \le 12a$. ∎

With Claim 3 we have

$$\left|\sum_{d=1}^{b}\mu(d)\left(\left|\frac{1}{d}C^+\cap\mathbf{Z}^2\right|-\frac{1}{d^2}\mathrm{Area}\,C^+\right)\right|\le\sum_{d=1}^{b}\left|\left|\frac{1}{d}C^+\cap\mathbf{Z}^2\right|-\frac{1}{d^2}\mathrm{Area}\,C^+\right|$$
$$\le\sum_{d=1}^{b}\frac{12a}{d}\le 12a(1+\log b).$$

Thus

$$\left||C^+\cap\mathbf{P}|-\sum_{d=1}^{b}\frac{\mu(d)}{d^2}\mathrm{Area}\,C^+\right|\le 12a(1+\log b).$$

Note that $|\sum_{d=1}^{b}\frac{\mu(d)}{d^2}-\frac{6}{\pi^2}|\le\sum_{d=b+1}^{\infty}\frac{1}{d^2}<\frac{1}{b}$. Then

$$\left||C\cap\mathbf{P}|-\frac{6}{\pi^2}\mathrm{Area}\,C\right|\le 2+\frac{1}{b}\mathrm{Area}\,C+24a(1+\log b)$$
$$\le\mathrm{Area}\,C\left(\frac{2}{2ab}+\frac{1}{b}+\frac{12(1+\log b)}{b}\right)$$
$$\ll\mathrm{Area}\,C\cdot\frac{\log b}{b}.$$

This completes the proof of Lemma 1. ∎

4. Approximating $A(C)$

Lemma 2.

$$\left|\sum_{u\in C^+\cap\mathbf{P}}\ \sum_{v\in C^+\cap\mathbf{P}}|\det(u,v)|-\left(\frac{6}{\pi^2}\right)^2\int_{C^+}\int_{C^+}|\det(x,y)|\,dx\,dy\right|$$
$$\ll(\mathrm{Area}\,C^+)^3\frac{\log b}{b}.$$

Proof. It is very simple to see that

$$\det(u,v)=\int_{x\in Q(u)}\int_{y\in Q(v)}\det(x,y)\,dx\,dy.$$

Then this holds for $|\det(u,v)|$ with $|\det(x,y)|$ as the integrand if $\det(x,y)$ has constant sign on $Q(u)\times Q(v)$. This happens if $Q(u)$ and $Q(v)$ are separated by a line going through the origin. In case they are not separated, there are $\xi,\eta\in\mathbf{R}^2$ with $\|\xi\|_{\max},\|\eta\|_{\max}\le\frac{1}{2}$ such that

$$0=\det(u+\xi,v+\eta)=(u_1+\xi_1)(v_2+\eta_2)-(u_2+\xi_2)(v_1+\eta_1).$$

This shows that

$$\det(u,v) = u_1v_2 - u_2v_1 = -u_1\eta_2 - \xi_1 v_2 - \xi_1\eta_2 + u_2\eta_1 + v_1\xi_2 + \xi_2\eta_1$$

implying $|\det(u,v)| \le \frac{1}{2}(|u_1|+|u_2|+|v_1|+|v_2|+1) \le \frac{1}{2}(2a+2b+1) \le 4a$ if $u,v \in \mathbf{Z}^2 - \{0\}$.

Similarly, if $Q(u)$ and $Q(v)$ are not separated, then for all $(x,y) \in Q(u) \times Q(v)$ (with $u,v \in \mathbf{Z}^2 - \{0\}$ again)

$$|\det(x,y)| \le 8a.$$

We start estimating $\sum\sum|\det(u,v)|$ via

$$\begin{aligned}
&\sum_{u\in C^+\cap \mathbf{P}} \sum_{v\in C^+\cap \mathbf{P}} |\det(u,v)| \\
&\qquad = \sum_{u\in C^+\cap \mathbf{Z}^2} \sum_{v\in C^+\cap \mathbf{Z}^2} |\det(u,v)| \sum_{s|u_1, s|u_2} \mu(s) \sum_{t|v_1, t|v_2} \mu(t) \\
&\qquad = \sum_{s=1}^{\infty}\sum_{t=1}^{\infty} \mu(s)\mu(t) \sum_{\substack{u\in C^+\cap \mathbf{Z}^2\\ s|u_1,\, s|u_2}} \sum_{\substack{v\in C^+\cap \mathbf{Z}^2\\ t|v_1,\, t|v_2}} |\det(u,v)| \\
&\qquad = \sum_{s=1}^{b}\sum_{t=1}^{b} s\mu(s)t\mu(t) \sum_{u\in \frac{1}{s}C^+\cap \mathbf{Z}^2} \sum_{v\in \frac{1}{t}C^+\cap \mathbf{Z}^2} |\det(u,v)| \\
&\qquad = \sum(1) - \sum(2) + \sum(3)
\end{aligned}$$

where

$$\sum(1) = \sum_s\sum_t s\mu(s)t\mu(t) \sum_{u\in \frac{1}{s}C^+\cap \mathbf{Z}^2} \sum_{v\in \frac{1}{t}C^+\cap \mathbf{Z}^2} \int_{Q(u)}\int_{Q(v)} |\det(x,y)|\,dx\,dy,$$

$\sum(2)$ is the same as $\sum(1)$ but for non-separated $Q(u), Q(v)$, and

$$\sum(3) = \sum_s\sum_t s\mu(s)t\mu(t) \sum_{u\in \frac{1}{s}C^+\cap \mathbf{Z}^2} \sum_{v\in \frac{1}{t}C^+\cap \mathbf{Z}^2} |\det(u,v)|\,dx\,dy,$$

again for non-separated $Q(u), Q(v)$. For fixed s and t, the number, $N(s,t)$, of non-separated pairs $Q(u), Q(v)$ with $u \in \frac{1}{s}C^+ \cap \mathbf{Z}^2$ and $v \in \frac{1}{t}C^+ \cap \mathbf{Z}^2$ can be estimated generously via Claim 3:

$$\begin{aligned}
N(s,t) &\le \left(\operatorname{Area}\frac{1}{s}C^+ + \frac{10a}{s}\right)\left(\operatorname{Area}\frac{1}{t}C^+ + \frac{10a}{t}\right) \\
&\ll (\operatorname{Area} C^+)^2 \frac{1}{s^2t^2}.
\end{aligned}$$

In $\sum(3)$, $|\det(u,v)| \ll \frac{a}{s} + \frac{a}{t}$, and in $\sum(2)$ the integrand $|\det(x,y)| \ll \frac{a}{s} + \frac{a}{t}$ as well. Consequently

$$\left|\sum(2)\right|, \left|\sum(3)\right| \ll \sum_{s=1}^{b}\sum_{t=1}^{b} st\,(\operatorname{Area} C^+)^2 \frac{1}{s^2t^2}\left(\frac{1}{s}+\frac{1}{t}\right)a$$
$$\ll (\operatorname{Area} C^+)^3 \frac{\log b}{b}.$$

Now we turn to $\sum(1)$. Define $R(s) = \bigcup_{u\,\text{boundary}}(Q(u) \setminus \frac{1}{s}C^+)$ and $T(s) = \bigcup_{u\,\text{outside}}(Q(u) \cap \frac{1}{s}C^+)$. We have to integrate over

$$\left[\frac{1}{s}C^+ \cup R(s) \setminus T(s)\right] \times \left[\frac{1}{t}C^+ \cup R(t) \setminus T(t)\right].$$

The main term comes from integrating over $\frac{1}{s}C^+ \times \frac{1}{t}C^+$. We are going to estimate the remaining 8 integrals.

It is readily seen that for $x, y \in C^+$ $|\det(x,y)| \le \operatorname{Area} C^+$. We need a slight strengthening of this (whose simple proof is omitted).

Claim 4. *When $x \in R(s) \cup T(s)$ and $y \in R(t) \cup T(t)$ and $s, t \le b$, then*

$$|\det(x,y)| \ll \frac{\operatorname{Area} C^+}{st}.$$

Using Claim 4

$$\int_{\frac{1}{s}C^+}\int_{R(t)} |\det(x,y)|\,dx\,dy \ll \frac{\operatorname{Area} C^+}{st}\int_{\frac{1}{s}C^+} dx \int_{R(t)} dy$$
$$\ll \frac{\operatorname{Area} C^+}{st}\frac{1}{s^2}\operatorname{Area} C^+ \frac{a}{t} \ll \frac{(\operatorname{Area} C^+)^3}{b}\frac{1}{s^3t^2}.$$

So the sum of these terms multiplied by st is

$$\ll \sum_{s=1}^{b}\sum_{t=1}^{b} st\,\frac{(\operatorname{Area} C^+)^3}{b}\frac{1}{s^3t^2} \ll (\operatorname{Area} C^+)^3\frac{\log b}{b}.$$

The same applies to the integral over $\frac{1}{s}C^+ \times T(t)$ and when t and s are interchanged. Similarly

$$\int_{T(s)}\int_{T(t)} |\det(x,y)|\,dx\,dy \ll \frac{\operatorname{Area} C^+}{st}\frac{a}{s}\frac{a}{t} \ll \frac{(\operatorname{Area} C^+)^3}{b^2}\frac{1}{s^2}\frac{1}{t^2},$$

and the same works for the remaining three integrals. Thus we have

$$\left| \sum_{u\in C^+\cap \mathbf{P}} \sum_{v\in C^+\cap \mathbf{P}} |\det(u,v)| - \sum_{s=1}^{b}\sum_{t=1}^{b} s\mu(s)t\mu(t) \int_{\frac{1}{s}C^+}\int_{\frac{1}{t}C^+} |\det(x,y)|\,dx\,dy \right| \ll (\operatorname{Area} C^+)^3 \left(\frac{\log b}{b} + \frac{\log^2 b}{b^2} \right) \ll (\operatorname{Area} C^+)^3 \frac{\log b}{b}.$$

Here

$$\begin{aligned} &\sum_{s=1}^{b}\sum_{t=1}^{b} s\mu(s)t\mu(t) \int_{\frac{1}{s}C^+}\int_{\frac{1}{t}C^+} |\det(x,y)|\,dx\,dy \\ &= \sum_{s=1}^{b}\sum_{t=1}^{b} \frac{\mu(s)}{s^2}\frac{\mu(t)}{t^2} \int_{C^+}\int_{C^+} |\det(x,y)|\,dx\,dy \\ &= \left(\frac{6}{\pi^2} - \sum_{s=b+1}^{\infty} \frac{\mu(s)}{s^2} \right) \left(\frac{6}{\pi^2} - \sum_{t=b+1}^{\infty} \frac{\mu(t)}{t^2} \right) \int_{C^+}\int_{C^+} |\det(x,y)|\,dx\,dy. \end{aligned}$$

By Claim 4 $\int_{C^+}\int_{C^+} |\det(x,y)|\,dx\,dy \ll (\operatorname{Area} C^+)^3$. Thus

$$\left| \sum_{s=1}^{b}\sum_{t=1}^{b} s\mu(s)t\mu(t) \int_{\frac{1}{s}C^+}\int_{\frac{1}{t}C^+} |\det(x,y)|\,dx\,dy - \left(\frac{6}{\pi^2}\right)^2 \int_{C^+}\int_{C^+} |\det(x,y)|\,dx\,dy \right| \ll (\operatorname{Area} C^+)^3 \frac{1}{b}$$

finishing the proof of Lemma 2. ∎

5. Symmetrization

Theorem 2. *Assume $K \in \mathcal{C}$, and $B \in \mathcal{C}$ is a disk with $\operatorname{Area} K = \operatorname{Area} B$. Then*

$$\int_K\int_K |\det(x,y)|\,dx\,dy \geq \int_B\int_B |\det(x,y)|\,dx\,dy.$$

Equality holds iff K is an ellipsoid.

This theorem is known as Busemann's random simplex inequality [4]. The proof goes by standard symmetrization (see e.g. [4] or [11]), so we only give a sketch. Let K^* be the symmetral of K with respect to a line l passing through O. We may assume, without loss of generality, that l is the x axis. To prove the theorem, it suffices to show the following:

Claim 5.

$$\int_K \int_K |\det(x,y)|\,dx\,dy \geq \int_{K^*} \int_{K^*} |\det(x,y)|\,dx\,dy.$$

Proof. Write $x=(x_1,x_2)$ and $y=(y_1,y_2)$. Fix $x_1 \in \mathbf{R}$ and define $I(x_1)=\{x_2 \in \mathbf{R} : (x_1,x_2) \in K\}$ which is clearly an interval, $I(x_1)=[A,B]$, say. Then $\{x_2 \in \mathbf{R} : (x_1,x_2) \in K^*\} = [-(B-A)/2,(B-A)/2]$ is an interval again which we denote by $I^*(x_1)$. Similarly, for fixed y_1, $\{y_2 : (y_1,y_2) \in K\} = I(y_1)$ is an interval. Now the part of the integral $\int_K \int_K |\det(x,y)|\,dx\,dy$ with x_1 and y_1 fixed and the same part of $\int_{K^*}\int_{K^*} |\det(x,y)|\,dx\,dy$ can be compared easily. First, $\det(x,y)=x_1y_2-x_2y_1$, and

$$\int_{I(x_1)} \int_{I(y_1)} |x_1y_2 - x_2y_1|\,dx_2\,dy_2 \leq \int_{I^*(x_1)} \int_{I^*(y_1)} |x_1y_2 - x_2y_1|\,dx_2\,dy_2.$$

This is true since the integrand is the absolute value of a linear function on the rectangle $(x_2,y_2) \in I(x_1) \times I(y_1)$ which is clearly the smallest when the linear function is 0 at the center of the rectangle. This is the case exactly for the symmetral. ∎

If B is a disk centered at the origin, then

$$\int_B \int_B |\det(x,y)|\,dx\,dy = \frac{8}{9\pi^2} (\mathrm{Area}\, B)^3.$$

Thus Lemmas 1 and 2, and Theorem 2 give

Corollary 1. *If $C \in \mathcal{C}$ with $|C \cap \mathbf{P}| = n$, then*

$$A(C) \geq \left(\frac{1}{54} - D\,\frac{\log b}{b}\right) n^3$$

where D is a universal constant.

Remark 3. It turns out that one can take $D=5000$ here when working with explicit constants instead of $\ll$.

6. The value of $\lim A(n)/n^3$

Set

$$f(z_1,\ldots,z_b) = \\ = \frac{2}{3}\sum_{i=1}^{b} \left(2\frac{\phi^2(i)}{i} + \frac{\phi(i)}{i^2}\left(\sum_{j=1}^{i-1} j\phi(j)\right)\right) z_i^3 + 2\sum_{i=1}^{b} \phi(i) z_i \left(\sum_{j=1}^{i-1} \frac{\phi(j)}{j} z_j^2\right).$$

We say that $z_1,\dots,z_b$ is *special* if the z_i are decreasing, $z_b \ge 0$ and they are convex, i.e., they satisfy $2z_i \ge z_{i-1}+z_{i+1}$ for $i=2,\dots,b-1$. Define the following minimization problem:

$$\text{minimize } f(z_1,\dots,z_b) \text{ subject to } 4\sum_{i=1}^{b}\frac{\phi(i)}{i}z_i = 1, z_1,\dots,z_b \text{ is special.}$$

The minimum, which clearly exists, will be denoted by $M(b)$.

Theorem 3. $\lim A(n)/n^3 = \min_{b\le 10^{10}} M(b)$.

Before the proof we give the following construction. Assume $b>1$, and $x_1,\dots,x_b$ is special with $x_1>0$. Define

$$K = \operatorname{conv}\{(\pm x_i, \pm i)\in \mathbf{R}^2 : i=1,\dots,b\}.$$

K is a convex set which is symmetric with respect to both axes. Define the 2 by 2 diagonal matrix H_n, with diagonal elements λ_n and 1, that satisfies $|\mathbf{P}\cap H_nK| = n$. (There might be a little ambiguity in this definition since several, but at most $4b$, elements may appear on the boundary of H_nK. Resolve it by considering some of these points as belonging, while some others as not belonging, to H_nK.) Setting $K_n = H_nK$ we see that $|\mathbf{P}\cap K_n| = n$.

Define $I(e,f;i) = \{(x,i)\in \mathbf{Z}^2 : e\le x<f\}$. It is clear that the density of $\mathbf{P}$ on the line $y=i$ is $\phi(i)/i$, and there are exactly $\phi(i)$ primitive points on an interval of the form $I(si,(s+1)i;i)$. Write $I(i,s)$ for this interval. Now

$$\left|n - 4\sum_{i=1}^{b}\frac{\phi(i)}{i}\lambda_n x_i\right| \le 4\sum_{1}^{b} i \le 4b^2,$$

since, for each $i>0$, the error term comes from the two subintervals $I_{left}(i)$ and $I_{right}(i)$ of $I(-\lambda_n x_i, \lambda_n x_i; i)$ that remain after deleting all $I(i,s)$ contained in it. A similar argument gives the following claim.

Claim 6.

$$A(K_n) = \lambda_n^3 f(x_1,\dots,x_b) + O(\lambda_n^2),$$

where the implied constant depends only on b.

Proof. We only give a sketch. The basic observation is that $\sum|\det(u,v)|$ over $(u,v)\in (I(i,s)\cap\mathbf{P})\times(I(j,t)\cap\mathbf{P})$ is the same as $\phi(i)\phi(j)/(ij)$ times the same sum over $(u,v)\in I(i,s)\times I(j,t)$, provided $\det(u,v)$ does not change sign on the box $I(i,s)\times I(j,t)$. The error terms come from two sources: First, from boxes where sign change occurs, but there are few of those, and there $|\det(u,v)|\le ij$. Secondly, when $u\in I_{left}(i)$ or $I_{right}(i)$, and similarly for v.

But these intervals are short, with $|\det(u,v)|$ at most $4b\lambda_n$ (cf. Claim 4). The statement follows by summing $\phi(i)\phi(j)|\det(u,v)|/(ij)$ over all $(u,v)\in(K_n\cap\mathbf{Z}^2)\times(K_n\cap\mathbf{Z}^2)$. ∎

So this construction satisfies $|\mathbf{P}\cap K_n|=n$ and

$$\lim\frac{A(K_n)}{n^3}=f(x_1,\dots,x_b)\left(4\sum_{i=1}^{b}\frac{\phi(i)}{i}x_i\right)^{-3}.$$

Proof of Theorem 1. Let C_n be the solution of the extremal problem $\operatorname{Min}(n)$ from Section 2. Let n_j be a sequence along which $\operatorname{Min}(n_j)$ tends to $M=\liminf\operatorname{Min}(n)$. If $b(C_n)\to\infty$ along a subsequence of n_j, then, according to Corollary 1

$$\lim\frac{A(C_n)}{n^3}=\frac{1}{54}$$

along the sequence n_j. But then this is true along the sequence n as well since, for the disk B_n containing n primitive points, $\lim A(B_n)/n^3=1/54$.

Assume now that $b(C_n)$ is bounded along n_j. Then we can choose a subsequence of n_j along which $b(C_n)=b$ for some fixed $b>1$. To save writing, we denote this subsequence by n_j as well.

Let C_n^* denote the symmetral of C_n with respect to the y axis. The discrete analogue of the proof of Theorem 2 shows (we omit the straightforward details) that, along the sequence n_j,

$$\lim\frac{A(C_n)}{n^3}=\lim\frac{A(C_n^*)}{n^3}.$$

For $n=n_j$ and $i=0,1,\dots,b$ define $x_i(n)\geq 0$ by

$$[-x_i(n),x_i(n)]=C_n^*\cap\{(x,i):x\in\mathbf{R}\}.$$

Then with our previous notation $x_0(n)=a(C_n^*)=a(C_n)$, and $a(C_n)\to\infty$ (along n_j) since $\operatorname{Area}C_n\leq 4a(C_n)b$. Choose now a subsequence of n_j along which $x_i(n)/a(C_n)$ is convergent, with limit x_i, for $i=1,\dots,b$. As the sequence $x_1,\dots,x_b$ is special and $x_1>0$, the above construction works and gives the sequence K_n. It is obvious that, along the last subsequence of n_j, $\lim A(K_n)/n^3=\lim A(C_n^*)/n^3=M$. Then

$$\lim_{n\to\infty}\frac{A(K_n)}{n^3}=M$$

as well. ∎

Proof of Theorem 3. Given a special $x_1,\dots,x_b$ with $4\sum_1^b \frac{\phi(i)}{i} x_i = 1$, we constructed a sequence of bodies K_n with $|K_n \cap \mathbf{P}| = n$ and $\lim A(K_n)/n^3 = f(x_1,\dots,x_b)$. So the value of the limit in Theorem 1 is less than $1/54$ if we find a single special sequence on which f is smaller than $1/54$. Here is such a sequence with $b = 15$:

$$\begin{aligned}
x_1 &= 0.03352589244, & x_2 &= 0.03335447314, & x_3 &= 0.03300806459,\\
x_4 &= 0.03251038169, & x_5 &= 0.03186074614, & x_6 &= 0.03104245531,\\
x_7 &= 0.03004944126, & x_8 &= 0.02886386937, & x_9 &= 0.02745127878,\\
x_{10} &= 0.02577867736, & x_{11} &= 0.02380388582, & x_{12} &= 0.02143223895,\\
x_{13} &= 0.01851220227, & x_{14} &= 0.01470243266, & x_{15} &= 0.008861427136,
\end{aligned}$$

giving

$$f(x_1,\dots,x_{15}) = 0.0185067386955\dots$$

which is smaller than $1/54 = 0.0185185185\dots$ by about 10^{-5}.

To see the bound $b \le 10^{10}$, we use Corollary 1: if $b > 10^{10}$, and $C \in \mathcal{C}$ with $|C \cap \mathbf{P}| = n$, then

$$\frac{A(C)}{n^3} \ge \frac{1}{54} - 5000\frac{\log 10^{10}}{10^{10}} > 0.018507. \qquad \blacksquare$$

Remark 4. The 10^{10} bound can be improved to about 10^7 by proving a stability version of Theorem 2: informally stated, this would say that if the left hand side of the inequality in Theorem 2 is smaller than $1+\varepsilon$ times its right hand side, then K can be sandwiched between two ellipsoids E and $(1+c\sqrt{\varepsilon})E$.

7. Remarks on computation

It seems hard to solve the minimization problems explicitly. We used the following heuristics. Let $x_1,\dots,x_b$ be a solution to the problem. What can we expect about $x_1,\dots,x_b$? According to Theorem 2, it is reasonable to assume that $(x_1,1),\dots,(x_b,b)$ are almost on the boundary of an ellipsoid. So let E_t be an ellipsoid whose half-axes are of length 1 and t with $b \le t \le b+1$, and define

$$w(i,t) = \sqrt{1-\left(\frac{i}{t}\right)^2} \quad (0 \le i \le t).$$

Set $W = 4\sum_{i=1}^b \frac{\phi(i)}{i} w(i,t)$ and $z_i = w(i,t)/W$ for $i = 1,\dots,b$. Then these $z_1,\dots,z_b$ form a good approximation for the solution of the minimization problem. In fact, this method gives, with $t = 15.56$ (then $b = 15$) the points

$z_1,\dots,z_{15}$, that already satisfy $f(z_1,\dots,z_{15})<1/54$. The even better solution $x_1,\dots,x_{15}$ giving $f(x_1,\dots,x_{15})=0.0185067\dots$ was found near the previous $z_1,\dots,z_{15}$ by solving the set of equations that constitute the necessary conditions for the extremum.

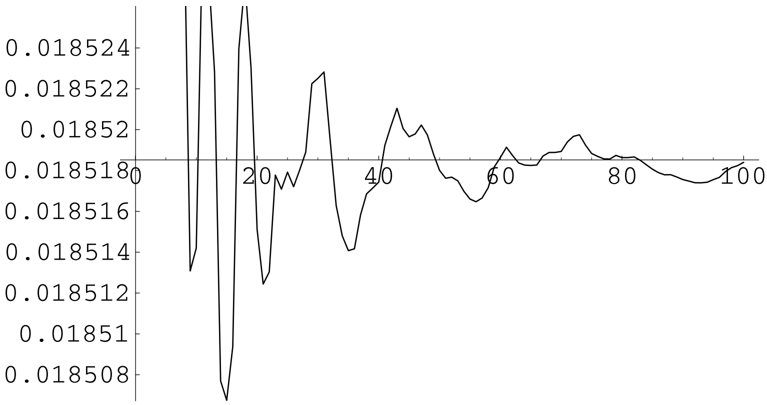

Figure 1.

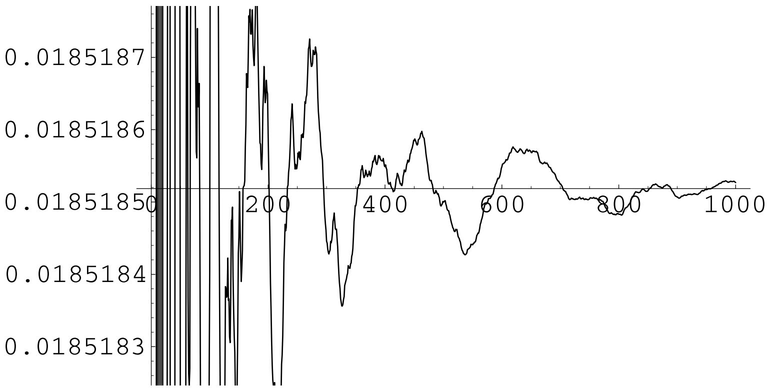

Figure 2.

Using this heuristics we have checked the value of f near the ellipsoid E_t for $b=1,\dots,100$ carefully (Figure 1) and for $b=101,\dots,1000$ roughly (Figure 2). The computation suggests that the true limit of $A(n)/n^3$ is very close to the above value $0.0185067\dots$. If this is the case then the minimizer P_n is very close, but not equal to, the ellipsoid with equation $x^2/A^2+y^2/B^2=1$ where $A=0.003573n^2$ and $B=1.656n$. But even if the minimum is different, the shape of the minimizer P_n is oblong: it is $c_1 n$ wide $c_2 n^2$ long in its lattice width direction.

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