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On forced and unforced triadic models of atmospheric flow

By R. LUPINI and C. PELLACANI, *Istituto di Fisica dell 'Università, Via Irnerio, 46, 40126 Bologna, Italy*

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ABSTRACT

The lowest-order spectral model for barotropic, non-divergent flow on the sphere is studied both in the forced and unforced case. It is found that the unforced, inviscid oscillations are typically non-periodic, the periodicity of the spectra notwithstanding. Such non-periodicity is characteristic of the spectral representation on the sphere, in the sense that typical solutions of the equivalent plane model are exactly periodic. By an analytic treatment, it is also found that for some triad configurations, the limit set of the corresponding non-conservative model with axisymmetric forcing may contain a limit cycle together with the stable, zonal-flow state, while only one attractor (limit cycle or zonal flow) is implied by the equivalent plane model. Numerical solutions are reported which confirm the above analysis.

1. Introduction

Severely truncated forms of the spectral, barotropic vorticity equation have been proposed by Lorenz (1960) in order to elucidate the climatologically relevant aspects of atmospheric flows at the planetary scale. In the inviscid case, the simplest truncated model allowing for non-linear energy exchange among spectral components is obtained by discarding all interactions except those involving two coupled waves and an arbitrary number of zonal-flow components. In spherical geometry, this model has been studied extensively by Platzman (1962) who showed that the time evolution of the variables describing the flow configuration can be expressed, in principle, in terms of elliptic functions. Galin (1974) has determined the stability conditions of the equilibrium configuration of the simple triadic system, while Dutton (1976a) has shown that most initial conditions produce periodic oscillations of the triad's spectrum, in the sense that aperiodic time-behaviour can only be exhibited by motions starting from points of a four-dimensional surface of the five-dimensional phase space of the triad. However, numerical computations reported by Baer (1970) suggested that the phases might be non-periodic, the periodicity of the spectrum notwithstanding. Non-periodic time evolution of the triad motions was also exhibited by numerical solutions reported by Galin and Kurbatkin (1975), but Dutton (1976a) remarked that these computations must be affected by numerical errors, as they violate the periodicity condition for the phase difference.

In this paper we reconsider the problem of the periodicity of the free oscillations of a rotating atmosphere, as described by Platzman's model, which all previous works have left open to question. Through a detailed mathematical analysis, we shall shown that such oscillations are typically non-periodic, although the spectra oscillate periodically.

Moreover, following more recent developments concerning the study of the long-range time evolution of atmospheric flows when forcing and dissipation are taken into account (Lorenz, 1963; Dutton, 1976b; Vickroy and Dutton, 1979; Mitchell and Dutton, 1981), we have analysed some properties of the limit set of the non-conservative, triadic model when forcing is applied to the zonal component. It is found that for most triad configurations, when the zonal equilibrium becomes unstable, the triadic flow tends to a state of stable, periodic oscillation. This result is the counterpart in spherical geometry of an analogous result obtained by Mitchell and Dutton (1981) for flows in the β-plane, except for a possible, nonsymmetric position of the couple of bifurcation points with respect to zero. However, for some triad configurations of large zonal wavelength but relatively small meridional wavelength, we find that a stable zonal flow configuration may coexist with a couple of states of periodic oscillation, one of which is unstable. As shown by a preliminary set of numerical computations, the unstable state exerts a significant effect on the pattern of the basins of attraction of the two stable configurations, and on the time-scales of approach to such asymptotic regimes.

2. Conservative flows: fundamental equations and periodicity condition

In spherical geometry, the truncated, spectral vorticity field of a two-dimensional flow, including a couple of waves and any number of zonal components can be written

$$\zeta = \zeta_1^0 P_1^0 + \sum_\alpha \zeta_\alpha P_{n_\alpha}^0 + \zeta_\beta Y_{n_\beta}^l + \zeta_\gamma Y_{n_\gamma}^l + \text{c.c.}, \tag{1}$$

where P_n^l denote Legendre functions, and Y_n^l denote spherical harmonics; the zonal components, ζ_1^0 and ζ_α^0 are real functions of time, while the wave components, ζ_β and ζ_γ are complex functions of time; c.c. signifies the complex conjugate of the (complex) wave field. The conservative, barotropic, vorticity equation for the truncated field of eq. (1) leads to the following set of ordinary differential equations (Platzman, 1962):

$$\dot{\zeta}_\alpha = K_\alpha M, \tag{1a}$$

$$\dot{\zeta}_\beta = i(g_\beta + h_\beta z)\zeta_\beta + i(a_\gamma + b_\gamma z)\zeta_\gamma, \tag{1b}$$

$$\dot{\zeta}_\gamma = i(g_\gamma + h_\gamma z)\zeta_\gamma + i(a_\beta + b_\beta z)\zeta_\beta, \tag{1c}$$

where

$$z = \frac{\zeta_\alpha - \zeta_\alpha^0}{K_\alpha},$$

$$g_\kappa = \sum_\alpha I_{\kappa\kappa\alpha}\zeta_\alpha^0 - l\omega_\kappa,$$

$$a_\kappa = \sum_\alpha (c_\kappa - c_\alpha) K_\alpha \zeta_\alpha^0, \tag{2}$$

$$b_\kappa = \sum_\alpha (c_\kappa - c_\alpha) K_\alpha^2,$$

$$h_\kappa = \sum_\alpha I_{\kappa\kappa\alpha} K_\alpha^2,$$

$$\zeta_\alpha^0 = \zeta_\alpha(0),$$

and the "structure" parameters c_κ, ω_κ, $I_{\kappa\kappa\alpha}$ are defined as

$$c_\kappa = \frac{1}{n_\kappa(n_\kappa + 1)},$$

$$K_\alpha = l\int_{-1}^{1} P_{n_\beta}^l P_{n_\gamma}^l \frac{\mathrm{d}}{\mathrm{d}x} P_{n_\alpha}^0(x)\,\mathrm{d}x,$$

$$I_{\kappa\kappa\alpha} = l\frac{(c_\kappa - c_\alpha)}{2}\int_{-1}^{1} P_{n_\kappa}^{l\,2} \frac{\mathrm{d}}{\mathrm{d}x} P_{n_\alpha}^0(x)\,\mathrm{d}x, \tag{3}$$

$$\omega_\kappa = \omega(1 - 2c_\kappa),$$

$$\omega = \tfrac{1}{2}\sqrt{3}\,\zeta_1^0.$$

Clearly, the index κ is equal to β or γ in eqs. (2) and (3). By the selection rules, the interaction coefficients K_α are non-zero for

$$\alpha \in \{n_\gamma = n_\beta + j : j = 1, 3, \ldots, 2n_\beta - 1\},$$

while the advection coefficients $I_{\kappa\kappa\alpha}$ are non-zero for

$$\alpha \in \{2n_\kappa - j : j = 1, 3, \ldots, 2n_\kappa - 1\}$$

so that, if we assume $n_\beta < n_\gamma$, the dimension N of the phase space for the truncated field in eq. (1) is $4 + n_\gamma$, although there are only $n_\beta + 4$ non-trivial differential equations. ζ_1^0, which is proportional to the total angular momentum, is constant, and ω is the corresponding angular velocity of solid rotation (Platzman, 1962). It has been shown by Platzman (1962) that $z(t)$ satisfies the second-order differential equation

$$\ddot{z} = q_0 + q_1 z + q_2 z^2 + q_3 z^3 \tag{4}$$

with initial conditions $z(0) = 0$, $\dot{z}(0) = M(0)$, where

$$q_0 = gL(0) - a_\beta P_\beta(0) - a_\gamma P_\gamma(0),$$

$$q_1 = hL(0) - g^2 - 4a_\beta a_\gamma - b_\beta P_\beta(0) - b_\gamma P_\gamma(0),$$

$$q_2 = -\tfrac{3}{2}gh - 3(a_\beta b_\gamma + a_\gamma b_\beta),$$

$$q_3 = -\tfrac{1}{2}h^2 - 2b_\beta b_\gamma, \tag{5}$$

and

$$g = g_\beta - g_\gamma,$$

$$h = h_\beta - h_\gamma,$$

$$P_\kappa = 2(c_\beta - c_\gamma)|\zeta_\gamma|^2, \ \phi\ \kappa = \beta, \gamma,$$

$$L = 2(c_\beta - c_\gamma)\mathrm{Re}(\zeta_\beta \zeta_\gamma^*) = L(0) - gz - \tfrac{1}{2}hz^2, \tag{6}$$

$$M = 2(c_\beta - c_\gamma)\mathrm{Im}(\zeta_\beta \zeta_\gamma^*).$$

Eq. (4) can be integrated analytically in terms of elliptic functions, and it can be shown that z is periodic for all initial conditions, except for an $(N-1)$-dimensional surface of the phase space, where points initiate aperiodic z-motions (Dutton, 1976a). From the equations

$$P_\beta = P_\beta(0) + 2a_\gamma z + b_\gamma z^2,$$
$$P_\gamma = P_\gamma(0) + 2a_\beta z + b_\beta z^2, \quad (7)$$

it follows that if z is periodic, then the spectrum is also a periodic function of time. But, the periodicity of the spectrum notwithstanding, the total wave field $\zeta_\beta Y^l_{n_\beta} + \zeta_\gamma Y^l_{n_\gamma}$ may be non-periodic, unless the phases also turn out to be periodic (mod. 2π). Now, by the definitions given in eqs. (2) and (6) we find

$$\mathrm{tg}\, \Delta\phi = \frac{M}{L} = \frac{\dot{z}}{L(0) - gz - \frac{1}{2}hz^2}, \quad (8)$$

where $\Delta\phi = \phi_\beta - \phi_\gamma$ is the phase difference. Accordingly, denoting by T the period of z,

$$\Delta\phi]_0^T = 2\pi j,$$

where $j \in \{0,1,2\}$ is the number of zeros of $L(z)$ in the range of the z-oscillation. Thus, $\Delta\phi$ is periodic (mod. 2π), and the periodicity condition for the wave-field (given the periodicity of z) is equivalent to the periodicity (mod. 2π) for the sum of the phases, $\delta\phi = \phi_\beta + \phi_\gamma$, that is

$$\delta\phi]_0^T = 2\pi r, \quad (9)$$

where r is any rational number*. The differential equation for $\delta\phi$ can be deduced from eqs. (1a, b) and reads

$$\dot{\delta\phi} = (g_\beta + g_\gamma) + (h_\beta + h_\gamma)z + \frac{L}{L^2 + M^2}[(a_\gamma P_\gamma(0) - a_\beta P_\beta(0) + z(b_\gamma P_\gamma(0) - b_\beta P_\beta(0))]. \quad (10)$$

From eq. (10), the periodicity condition (eq. (9)) can be written

$$2\pi r = (g_\beta + g_\gamma)T + (h_\beta + h_\gamma)\int_0^T z\, dt' + \int_0^T \frac{L}{L^2 + M^2}[(a_\gamma P_\gamma(0) - a_\beta P_\beta(0)) + z(b_\gamma P_\gamma(0) - b_\beta P_\beta(0))]dt', \quad (11)$$

where r is any rational number and T is the period of z. It is clear that, as the right-hand side of eq. (11) is a continuous function of the initial conditions (except for the surface of aperiodic motions), for generic values of the structure parameters, eq. (11) can only be satisfied by points that belong to a sequence of $(N-1)$-dimensional surfaces of the phase space. In the Appendix it is in fact shown that the vanishing of the advection parameters ω_κ, $I_{\kappa\kappa\alpha}$ is a necessary (and sufficient) condition for the fulfilment of eq. (11) in the whole phase space, apart from the surface of aperiodic motions.

It can be concluded that the inviscid, unforced flows of a barotropic rotating atmosphere, as described by the truncated set of eqs. (1a, b, c) are, typically, non-periodic, although the spectra oscillate periodically.

3. Conservative flows: an analytic example

As an example illustrating the effect of absolute rotation on the periodicity of the free oscillations in spherical geometry, we shall consider the case of triadic truncation with n_α even, $\Delta\phi(0) = \frac{1}{2}\pi$, $\zeta_\alpha(0) = 0$, $|\zeta_\beta(0)| = |\zeta_\gamma(0)|$. By the selection rules, the advection coefficients $I_{\kappa\kappa\alpha}$ are zero and eq. (4) can be easily solved analytically. We find

$$z = z_* \, sn\,(\nu t | m),$$

where sn is the elliptic sine function,

$$\nu = \frac{w}{z_*}, \quad (12)$$

$$m = -1 + \frac{g^2 + w(c_\beta - c_\gamma)}{\nu^2}, \quad (13)$$

$$z_*^2 = \frac{g^2 + (b_\beta + b_\gamma)w - \{[g^2 + (b_\beta - b_\gamma)w]^2 + 4b_\beta b_\gamma w^2\}^{\frac{1}{2}}}{-2b_\beta b_\gamma}, \quad (14)$$

and $w = P_\beta(0)$. The equation for the phase difference,

$$\Delta\phi = \mathrm{tg}^{-1}\left(-\frac{\dot{z}}{gz}\right), \quad (15)$$

implies that $\Delta\phi$ is an increasing function of time,

* When $r = m/n$ is rational, repetition of the motion occurs after n periods of z-oscillations.

which is periodic (mod. 2π), as $\Delta\phi(T) = \Delta\phi(0) + 2\pi$.

The differential equations for the phases

$$\dot{\phi}_\beta = g_\gamma + g \frac{1}{1 + b_\gamma z^2/w}, \tag{16a}$$

$$\dot{\phi}_\gamma = g_\beta - g \frac{1}{1 - b_\beta z^2/w}, \tag{16b}$$

can be integrated analytically in terms of elliptic integrals of the third kind (Abramovitz and Stegun, 1972). For the case $b_\beta \cdot b_\gamma < 0$ (that is $n_\beta > n_\alpha > n_\gamma$) we find

$$\phi_\beta|_0^T = 4g_\beta \frac{K(m)}{\nu} + 2\pi|1 - \Lambda_0(\theta_\beta|m)|, \tag{17}$$

$$\phi_\gamma|_0^T = 4g_\gamma \frac{K(m)}{\nu} - 2\pi|1 - \Lambda_0(\theta_\gamma|m)|, \tag{18}$$

where

$$\sin\theta_{\beta,\gamma} = \left(\frac{1 - m_{\beta,\gamma}}{1 - m}\right)^{1/2} \tag{19}$$

$$m_{\beta,\gamma} = -\frac{b_{\beta,\gamma}}{w} z_*^2, \tag{20}$$

and Λ_0 is the Heuman's Lambda function. By the definitions of ν, m, z_* (eqs. (12), (13) and (14)), it is easily verified that m, $m_{\beta,\gamma}$, ν^{-1} $g_{\beta,\gamma}$ are monotone functions of the parameter $\varepsilon = lw/|w|^{\frac{1}{2}}$, which is a measure of the ratio of the angular momentum of solid rotation to the angular momentum of the wave-field. As K and Λ_0 are monotone functions of their arguments, it follows that the necessary (and sufficient) condition for periodicity of the flow, that is $\phi_\beta|_0^T = 2\pi r$, where r is any rational number, can only be satisfied by a countable set of values of ε, as long as $\omega = 0$. The asymptotic limits of eqs. (16a, b) as ε tends to zero, in the case $b_\beta + b_\gamma > 0$ (the case $b_\beta + b_\gamma < 0$ leads to the same results apart from an interchange of the indices β,γ), are given by

$$\phi_\beta|_0^T = f(\varepsilon), \tag{20a}$$

$$\phi_\gamma|_0^T = -2\pi + f(\varepsilon), \tag{20b}$$

where $f(\varepsilon)$ is of order ε as ε tends to zero. Eqs. (20) show that the periodic, standing oscillations performed by the triad in the non-rotating case ($\varepsilon = 0$) (Lorenz, 1960; Galin, 1974) are "destabilized" by the introduction of a small field of absolute rotation ($0 < \varepsilon \ll 1$), in the sense that one of the waves acquires a finite zonal phase speed while the other one remains nearly standing; moreover the entire flow pattern becomes almost-periodic for most initial conditions.

4. Non-conservative triadic model: equilibria and periodic trajectories

The inviscid model discussed in the previous chapters describes the time evolution of unforced planetary components of flow whose dissipation time scales are much larger than their intrinsic periods. In the present section, we shall study the asymptotic properties of the motions of the simple triadic model when an axisymmetric field of vorticity generation is taken into account, together with dissipation of the energy. Introducing the same formal representation of the forcing and dissipation terms as discussed by Mitchell and Dutton (1981), the basic set of equations can be written

$$\dot{\zeta}_\alpha = 2K_\alpha(c_\beta - c_\gamma)\,\mathrm{Im}\,(\zeta_\beta\zeta_\gamma^*) - \nu_\alpha\zeta_\alpha + f_\alpha,$$
$$\dot{\zeta}_\beta = [i(I_{\beta\beta\alpha}\zeta_\alpha + g_\beta) - \nu_\beta]\zeta_\beta + i(c_\gamma - c_\alpha)K_\alpha\zeta_\alpha\zeta_\gamma, \tag{21}$$
$$\dot{\zeta}_\gamma = i(c_\beta - c_\alpha)K_\alpha\zeta_\alpha\zeta_\beta + [i(I_{\gamma\gamma\alpha}\zeta_\alpha + g_\gamma) - \nu_\gamma]\zeta_\gamma,$$

where ν_α, ν_β and ν_γ are (positive) dissipation coefficients, $\sum_m f_m P_m^0$ is the (real) field of vorticity generation, $g_\kappa = \sum_{m'} I_{\kappa\kappa m'} f_{m'}/\nu_{m'} - l\omega_\kappa$, and $\zeta_{m'} = f_{m'}/\nu_{m'}$ are the amplitudes of the non-interactive zonal components that have already reached their steady-state, forced regime. Non-linear interactions only occur within the triad (α, β, γ).

The equations for the mean square vorticity and energy, respectively, can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}\zeta_\alpha^2 + |\zeta_\beta|^2 + |\zeta_\gamma|^2)$$
$$= -2(\nu_\alpha\tfrac{1}{2}\zeta_\alpha^2 + \nu_\beta|\zeta_\beta|^2 + \nu_\gamma|\zeta_\gamma|^2) + f_\alpha\zeta_\alpha, \tag{22a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(c_\alpha\tfrac{1}{2}\zeta_\alpha^2 + c_\beta|\zeta_\beta|^2 + c_\gamma|\zeta_\gamma|^2)$$
$$= -2(\nu_\alpha c_\alpha\tfrac{1}{2}\zeta_\alpha^2 + \nu_\beta c_\beta|\zeta_\beta|^2 + \nu_\gamma c_\gamma|\zeta_\gamma|^2) + c_\alpha f_\alpha\zeta_\alpha \tag{22b}$$

By eqs. (22) we find

$$\frac{\mathrm{d}}{\mathrm{d}t}[(c_\beta - c_\alpha)|\zeta_\beta|^2 + (c_\gamma - c_\alpha)|\zeta_\gamma|^2]$$
$$= -2[\nu_\beta(c_\beta - c_\alpha)|\zeta_\beta|^2 + \nu_\gamma(c_\gamma - c_\alpha)|\zeta_\gamma|^2]. \tag{23}$$

Eq. (22a) implies that all triad trajectories are uniformly bounded, while eq. (23) implies that for $n_\alpha \neq (n_\beta, n_\gamma)$, the wave-energy decays exponentially to zero, and purely zonal flow is an asymptotically stable state of equilibrium. This is in agreement with a general result reported in the above-cited paper by Mitchell and Dutton, which we shall quote as MD in the following. We shall then consider the case of intermediate forcing, that is $n_\beta < n_\alpha < n_\gamma$. Looking for exponential behaviour of the couple of waves, that is letting

$$\frac{\mathrm{d}}{\mathrm{d}t}(\zeta_\beta, \zeta_\gamma) = \lambda(\zeta_\beta, \zeta_\gamma),$$

we find

$$\begin{aligned}\lambda = i(\bar{I}\zeta_\alpha + \bar{g}) - \bar{\nu} \pm \tfrac{1}{2}[-(\Delta I\zeta_\alpha + \Delta g)^2 \\ + 4(c_\beta - c_\alpha)(c_\alpha - c_\gamma)K_\alpha^2\zeta_\alpha^2 + \Delta\gamma^2 \\ + 2i\Delta\gamma(\Delta I\zeta_\alpha - \Delta g)]^{1/2},\end{aligned} \quad (24)$$

where the overbar signifies average, and Δ the difference of the quantities to which they apply. By eq. (24), the local instability condition for forced zonal flow (FZF henceforth) is given by

$$2\bar{\nu} < |\,\mathrm{Re}\,(\text{square root term of eq. (24)})|. \quad (25)$$

A considerable simplification of the mathematical treatment is obtained by assuming $\nu_\beta = \nu_\gamma = \bar{\nu}$. By this assumption, in fact, eq. (25) reduces to

$$\begin{aligned}4\bar{\nu}^2 + (\Delta I\zeta_\alpha + \Delta g)^2 - 4(c_\alpha - c_\gamma) \\ \times (c_\beta - c_\alpha)K_\alpha^2\zeta_\alpha^2 < 0,\end{aligned} \quad (26)$$

where we have to take $\zeta_\alpha = f_\alpha/\nu_\alpha$. Now, let $\zeta_\alpha^{(1)}$ and $\zeta_\alpha^{(2)}$ be the roots of the polynomial of eq. (26) and assume $\zeta_\alpha^{(1)} < \zeta_\alpha^{(2)}$ (in the real case). Clearly, the following three cases may occur:

(a) $S = \Delta I^2 + 4(c_\beta - c_\alpha)(c_\gamma - c_\alpha)K_\alpha^2 < 0$; the roots are real, $\zeta_\alpha^{(1)} \cdot \zeta_\alpha^{(2)} < 0$ and the instability range for FZF is the exterior of the interval $(\zeta_\alpha^{(1)}, \zeta_\alpha^{(2)})$. In particular, the roots are given by

$$\zeta_\alpha^{(1,2)} = \frac{\left\{\begin{aligned}-\Delta g\Delta I \pm 2[(\Delta g^2 + 4\bar{\nu}^2)(c_\alpha - c_\gamma) \\ \times (c_\beta - c_\alpha)K_\alpha^2 - \bar{\nu}^2\Delta I^2]^{1/2}\end{aligned}\right\}}{S} \quad (27)$$

(b) $S > 0$ and $\bar{\nu}^2 < \bar{\nu}_0^2 = \Delta g^2(c_\alpha - c_\gamma)(c_\beta - c_\alpha)K_\alpha^2/S$; the roots are real, $\zeta_\alpha^{(1)} \cdot \zeta_\alpha^{(2)} > 0$ and the instability range for FZF is the interior of the interval $(\zeta_\alpha^{(1)}, \zeta_\alpha^{(2)})$. We distinguish the subcase (b_1), when $\zeta_\alpha^{(1)} > 0$, and the subcase (b_2), when $\zeta_\alpha^{(2)} < 0$.

(c) $S > 0$ and $\bar{\nu} \geqslant \bar{\nu}_0$, the roots are complex and the instability range for FZF is void.

From eq. (24) we can also deduce that the basic system (21) allows for solutions in the form of forced, periodic oscillations (FPO). In fact, if ζ_α is one of the (real) marginal values $\zeta_\alpha^{(1,2)}$, then λ is imaginary and the following is a solution of eqs. (21):

$$(\zeta_\beta, \zeta_\gamma) = (\zeta_\beta(0), \zeta_\gamma(0))\exp(i\sigma t), \quad (28a)$$

$$\zeta_\gamma(0) = \frac{I_{\gamma\gamma\alpha}\zeta_\alpha + g_\gamma - i\bar{\nu}}{(c_\gamma - c_\alpha)K_\alpha\zeta_\alpha}\zeta_\beta(0), \quad (28b)$$

$$|\zeta_\gamma(0)|^2 = \left(\frac{f_\alpha}{\nu_\alpha} - \zeta_\alpha\right)\left(\frac{c_\alpha - c_\gamma}{c_\beta - c_\gamma}\right)\frac{\nu_\alpha}{\bar{\nu}}\zeta_\alpha, \quad (28c)$$

$$\sigma = \bar{I}\zeta_\alpha + \bar{g}. \quad (28d)$$

Eq. (28d) implies that, denoting by $\mathrm{FPO}_{1,2}$ the periodic solutions corresponding to $\zeta_\alpha^{(1,2)}$, respectively, FPO_2 (FPO_1) is a meaningful solution if $f_\alpha/\nu_\alpha > \zeta_\alpha^{(2)}$ ($< \zeta_\alpha^{(1)}$), for case (a); for case (b) both FPO are meaningful when $f_\alpha/\nu_\alpha > \zeta_\alpha^{(2)}$ (in subcase (b_1)) or $f_\alpha/\nu_\alpha < \zeta_\alpha^{(1)}$ (in subcase (b_2)), but only FPO_1 (FPO_2) is meaningful when $\zeta_\alpha^{(1)} < f_\alpha/\nu_\alpha < \zeta_\alpha^{(2)}$. For case (c) there is no FPO, and FZF is unconditionally stable. We note that in eqs. (28), the phase of one of the two waves is arbitrary, so that the FPO's cover a two-dimensional circular torus of the 5-dimensional phase space of the triad. We can conclude this section by saying that, besides purely zonal flow, the limit set of the basic system (21) contains one or two tori composed of periodic orbits, depending on the value of the bifurcation parameter f_α/ν_α and of the dissipation coefficient $\bar{\nu}$.

4.1. *Stability of periodic oscillations*

In order to study the stability of the FPO's, we have to determine the asymptotic behaviour of triad motions starting in a neighbourhood of such periodic trajectories. Fortunately, this problem can be reduced to one of stability for critical points, which is much easier to solve. In terms of the variables A and B, defined by $A + iB = \zeta_\beta\zeta_\gamma^* = |\zeta_\beta||\zeta_\gamma|\exp(i\Delta\phi)$, the system of eqs. (21) can be written

$$\begin{aligned}\dot{\zeta}_\alpha &= 2K_\alpha(c_\beta - c_\gamma)B - \nu_\alpha\zeta_\alpha + f_\alpha, \\ \dot{A} &= -(\Delta I\zeta_\alpha + \Delta g)B - 2\bar{\nu}A, \\ \dot{B} &= (\Delta I\zeta_\alpha + \Delta g)A - 2\bar{\nu}B - [(c_\alpha - c_\gamma)|\zeta_\gamma|^2 \\ &\quad + (c_\beta - c_\alpha)|\zeta_\beta|^2]K_\alpha\zeta_\alpha.\end{aligned} \quad (29)$$

The assumption $\nu_\beta = \nu_\gamma = \bar{\nu}$ allows for direct integration of eq. (23), which gives

$$W = W(0) \exp(-2\bar{\nu}t),$$

where $W = (c_\beta - c_\alpha)|\zeta_\beta|^2 - (c_\alpha - c_\gamma)|\zeta_\gamma|^2$. It follows that, asymptotically, $W = 0$ and the triad motion satisfies (approximately) the following third-order system of equations

$$\dot{\zeta}_\alpha = 2K_\alpha(c_\beta - c_\gamma)B - \nu_\alpha \zeta_\alpha + f_\alpha,$$

$$\dot{A} = -(\Delta l \zeta_\alpha + \Delta g)B - 2\bar{\nu}A, \tag{30}$$

$$\dot{B} = (\Delta l \zeta_\alpha + \Delta g)A - 2\bar{\nu}B - 2K_\alpha|(c_\alpha - c_\gamma)(c_\beta - c_\alpha)|^{1/2}\zeta_\alpha(A^2 + B^2)^{1/2}.$$

The mean square vorticity equation can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{1}{2}\zeta_\alpha^2 + (A^2 + B^2)^{1/2}\right] = -2\bar{\nu}(A^2 + B^2)^{1/2} - \nu_\alpha \zeta_\alpha^2 + f_\alpha \zeta_\alpha$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{c_\alpha - c_\gamma}{c_\beta - c_\gamma}\tfrac{1}{2}\zeta_\alpha^2 + |\zeta_\beta|^2\right] = -2\bar{\nu}|\zeta_\beta|^2 + (f_\alpha - \nu_\alpha \zeta_\alpha)\zeta_\alpha \frac{c_\alpha - c_\gamma}{c_\beta - c_\gamma}. \tag{31}$$

It is clear that the amplitude stationary motions described by eqs. (28) become critical points of the approximate system (30), which can be written as

$$(\zeta_\alpha, A, B) = \left(\zeta_\alpha, -\frac{\Delta l \zeta_\alpha + \Delta g}{2\bar{\nu}}\,\frac{\nu_\alpha \zeta_\alpha - f_\alpha}{2K_\alpha(c_\beta - c_\gamma)}, \frac{\nu_\alpha \zeta_\alpha - f_\alpha}{2K_\alpha(c_\beta - c_\gamma)}\right), \tag{32}$$

where ζ_α is at the marginal stability boundary of the equilibrium zonal flow. Thus, the stability analysis of the FPO's reduces to the stability analysis of the critical points of eqs. (32) for the reduced system (30). After some algebraic manipulation, we find that the linear variational system (Cronin, 1980) of system (30) relative to the equilibria of eq. (32) is stable if

$$\left(\frac{f_\alpha}{\nu_\alpha} - \zeta_\alpha\right)(S\zeta_\alpha + \Delta l\,\Delta g) < 0. \tag{33}$$

For triads of class (a) ($S < 0$) it is immediately seen, via eq. (27), that eq. (33) always holds; for such triads, as the bifurcation parameter f_α/ν_α crosses one of the bifurcation points $\zeta_\alpha^{(1,2)}$ from the interior of the interval $(\zeta_\alpha^{(1)}, \zeta_\alpha^{(2)})$ a new stable periodic solution appears, while the zonal flow becomes unstable (supercritical Hopf bifurcation). The corresponding bifurcation diagram is depicted in Fig. 1a.

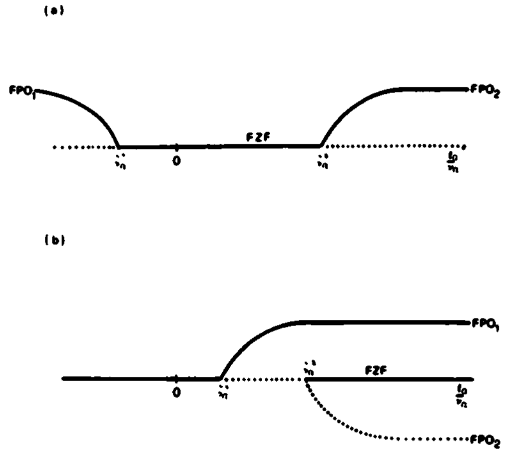

Fig. 1. (a) Bifurcation diagram for triads of class (a). (b) Bifurcation diagram for triads of class (b). FPO = forced periodic oscillation; FZF = forced zonal flow. Dotted lines denote unstable equilibria; continuous lines denote stable equilibria.

For triads of class (b) we find that eq. (33) holds for FPO_1 (FPO_2) when $f_\alpha/\nu_\alpha > \zeta_\alpha^{(1)}$ ($< \zeta_\alpha^{(2)}$), but it is not satisfied by FPO_2 (FPO_1) when $f_\alpha/\nu_\alpha > \zeta_\alpha^{(2)}$ ($< \zeta_\alpha^{(1)}$); thus, in subcase (b_1), the original system (21) undergoes two Hopf bifurcations, a supercritical one at $\zeta_\alpha^{(1)}$, and a subcritical one at $\zeta_\alpha^{(2)}$. The corresponding bifurcation diagram is depicted in Fig. 1b, which also represents class (b_2) after interchange of the indices 1 and 2, and reflection with respect to the origin.

We notice that triads with η_α even belong to class (a), the advection parameters being all zero in this case. The vanishing of advection in truncated models for flows in plane geometry implies that only case (a) occurs in such a geometry, as long as we maintain the zonality of forcing. Our case (a) corresponds in fact to triads which are called of type I in MD.

Numerical simulations confirming the above analysis are reported in the next section. We conclude this section with a few considerations which will be helpful for the interpretation of the qualitative properties of the trajectories to be reported later.

If we consider the set $\{(\zeta_\alpha, |\zeta_\beta|, \Delta\phi): 0 \leqslant \Delta\phi < 2\pi\}$ as the phase space of the reduced system (30), it is clear from eq. (31) that the equilibrium states belong to the elliptic cylinder, say C_0, of equations $G = 0, 0 \leqslant \Delta\phi < 2\pi$, where G is the function on the left-hand side of eq. (31). The triad trajectories enter the region bounded by the elliptic cylinder, say C_1, which is a tangent on the exterior to C_0; subsequently, they proceed towards the origin from points inside C_0 and recede from the origin, from points outside C_1, while crossing the surfaces of the equation

$$V = |\zeta_\beta|^2 + \frac{c_\alpha - c_\gamma}{c_\beta - c_\gamma}\frac{\zeta_\alpha^2}{2} = \text{cte.}$$

Typical projections on the $(\zeta_\alpha, |\zeta_\beta|)$ plane are sketched in Fig. 2a, for the case $\nu_\alpha \leq \bar{\nu}$, and in Fig. 2b for the case $\nu_\alpha > \bar{\nu}$. It is clear that the way a trajectory approaches one of the equilibrium points on C_0 depends on the transversality of the family of surfaces V = cte. with respect to the surface $G = 0$. We expect very low approach velocities for trajectories crossing a neighbourhood of the point where C_1 touches C_0, as confirmed by numerical solutions to be reported below.

4.2. *Examples of triad trajectories*

As an example of a triad configuration belonging to class (a), we have chosen $n_\alpha = 3$, $n_\beta = 2$, $n_\gamma = 4$, $l = 2$. The angular velocity of solid rotation, ω, has been taken to be 1, and the dissipation coefficients $\nu_\alpha = \bar{\nu} = 0.05$, which corresponds to a dissipation time scale of about 20 days.

The computed values of the structure parameters are $I_{223} = -0.189$, $I_{443} = -0{\cdot}268$ and $K_3 = 6.546$. All passive zonal components have been assumed to be zero, so that $g_\kappa = -l\omega_\kappa$; the computed values of the stability margins of FZF are $\zeta_\alpha^{(1)} = -0.78$ and $\zeta_\alpha^{(2)} = 0.62$. For $f_\alpha/\nu_\alpha < -0.78$, the corresponding FZF is unstable and FPO$_1$ is stable. The value of $|\zeta_\beta|$ corresponding to FPO$_1$ is 0.28.

The projections on the $(\zeta_\alpha, |\zeta_\beta|)$ plane of two trajectories starting at $\zeta_\alpha = 0.01$, $|\zeta_\beta| = 0.025$, for two values of the bifurcation parameter ($f_\alpha/\nu_\alpha = -0.43$ and $f_\alpha/\nu_\alpha = -1.48$) are reported in Fig. 3. Also shown in the same figure is the projection of the cylinder C_0 for the second value of f_α/ν_α. Apparently, the first trajectory tends, aperiodically, to FZF, while the latter spirals towards FPO$_1$, following the qualitative behaviour predicted in Section 3. Other computations not reported here suggest that for triads of class (a), the stable equilibrium state (FZF or FPO, depending on the value of f_α/ν_α) is globally attracting.

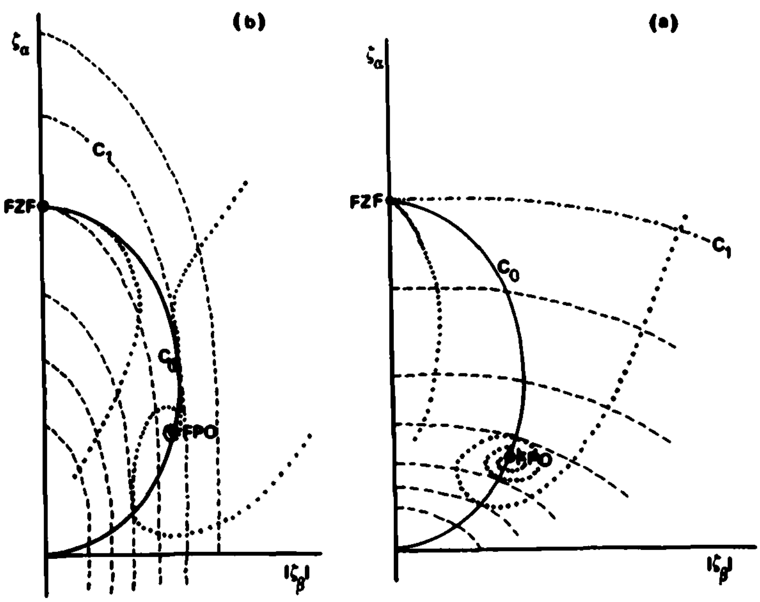

Fig. 2. Sketch of the family of ellipses V = cte (---), of the ellipses C_1 (-·-) of the ellipses C_0 (——) and of the projection of sample trajectories on the $(\zeta_\alpha, |\zeta_\beta|)$ plane (•••), for two different geometrical configurations ((a), (b)).

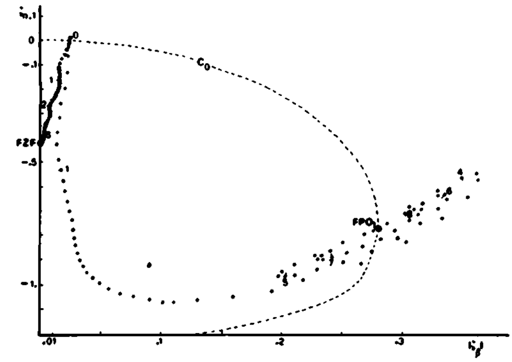

Fig. 3. Projection of the computed trajectories for the triad $n_\alpha = 3$, $n_\beta = 2$, $n_\gamma = 4$, $l = 2$. Parameter values are $\nu_\alpha = \bar{\nu} = 0.05$, $\omega = 1$, $f_\alpha/\nu_\alpha = -0.43$ (•••) and $f_\alpha/\nu_\alpha = -1.48$ (•••). (●) denotes stable equilibria. Time unit is 10 days.

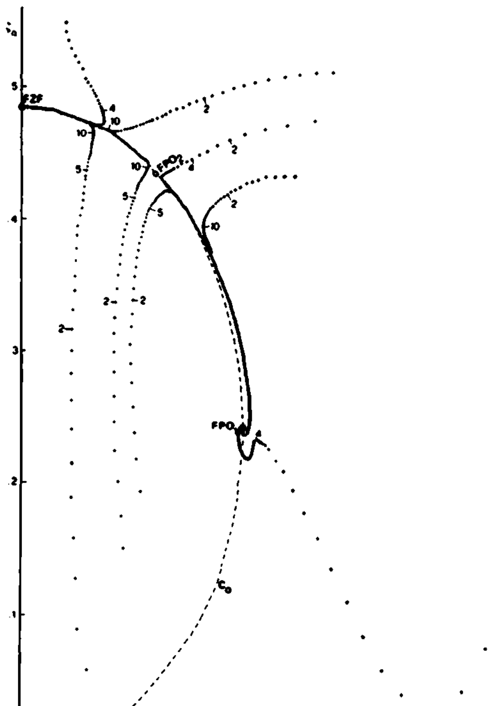

Fig. 4. Projection of sample trajectories (•••) for the triad $n_\alpha = 5$, $n_\beta = 4$, $n_\gamma = 8$, $l = 1$. Parameter values are $\omega = 1$, $\nu_\alpha = 0.05$, $\bar{\nu} = 0.03$, $f_\alpha/\nu_\alpha = 0.484$. (○) denotes unstable equilibrium. (– – –) denotes C_0. Time unit is 10 days.

Triad configurations of class (b) are more rare than those of class (a) and most of them correspond to the smallest values of the zonal index ($l = 1,2$). An example is offered by $n_\alpha = 5$, $n_\beta = 4$, $n_\gamma = 8$, $l = 1$, provided that $\bar{\nu}$ is small enough to make the roots of eq. (26) real. Choosing $\nu_\alpha = 0.05$, $\bar{\nu} = 0.03$, $\omega = 1$, and zero values for all passive zonal components, the instability range for FZF is (0.242, 0.433); accordingly, if $0.433 < f_\alpha/\nu_\alpha$, the limit set contains the stable FZF, the unstable FPO and the unstable FPO. A sample of the computed trajectories in the case $f_\alpha/\nu_\alpha = 0.484$ is reported in Fig. 4, together with the projection of C_0 on the (ζ_α, $|\zeta_\beta|$) plane. It is evident that the stability properties of the three critical points agree with the analysis of Section 3; moreover, as predicted there, the form of the basins of attraction is largely influenced by the presence of the unstable critical point, which is in fact attractive for distant states and weakly repulsive locally. It is also clear that the upper part of the ellipsis C_0, is nearly parallel locally to members of the family $V =$ cte, so that many trajectories are attracted by it; as they cannot cross this curve, nor recede from it, these trajectories spend a long time in a very small neighbourhood of C_0 before reaching one of the two attractors.

5. Conclusions

It has been shown that the free oscillations of a rotating, barotropic atmosphere, described by the lowest-order truncated model in spherical geometry which allows for triadic interaction, are typically non-periodic, although the spectra oscillate periodically. It is found that such non-periodicity is due to advection of the waves in the field of the zonal flow components and to absolute rotation; as this effect is absent in the analogous model written in plane geometry, we can say that flows in spherical geometry show time evolutions with qualitative features not shared by plane flows, given the same level of truncation.

Some properties of the limit set of the forced, dissipative triadic model have been studied in the case of axisymmetric forcing. Asymptotic regimes in the form of periodic oscillation have been found and their stability properties analysed. For most triad configurations, the zonal flow is the only state of equilibrium for small forcing values, but it becomes unstable at suitably large forcing and a periodic attractor enters into play. Numerical simulations suggest that such asymptotic regimes are globally attracting. For this class of triad configurations, we find in fact the same qualitative properties shown by the analogous triadic model in plane geometry (MD).

However, for other triad configurations, characterized by low zonal indices and somewhat large meridional indices, stable zonal flow may coexist with a periodic attractor, in a convenient range of forcing values. Numerical simulations confirm the above analysis and suggest the existence of surfaces of the phase space which can act as "weak" attractors for certain open sets of states. Generalization of the results obtained in this work, both in the inviscid case and in the forced case, to higher-order models is in progress.

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7. Appendix

Let us denote by A the surface of the phase space whose points initiate aperiodic z-motions and let P be the complementary set with respect to A (see text). The periodicity condition (eq. (11) of the text) is identically satisfied at P (an open set) if and only if the continuous function on the right-hand side of eq. (11), say ψ, is constant at P. We show below that, unless all the advection parameters ω_κ, $I_{\kappa\kappa\alpha}$, $\kappa = \beta, \gamma$ are zero, this function is not bounded in P, as it tends to infinity when its argument approaches the boundary surface A. In fact, as the initial condition tends to A, the period T of the z-oscillation tends to infinity and the motion becomes asymptotic to an amplitude stationary oscillation (Dutton, 1976a). Accordingly, $T^{-1}\, S\phi]_0^T$ (see text) is asymptotic to a constant value, say $\overline{S\phi}$, given by

$$\overline{S\phi} = \bar{g}_\beta + \bar{g}_\gamma + \frac{\bar{a}_\gamma \bar{P}_\gamma - \bar{a}_\beta \bar{P}_\beta}{\bar{L}}, \tag{A1}$$

where the overbar signifies the value taken at an amplitude stationary motion. On the other hand, the equations defining the subset of initial conditions giving rise to an amplitude stationary oscillation, which can be derived from eq. (4) of the text by letting $\dot{z}(0) = \ddot{z}(0) = 0$, are

$$(\bar{g}_\beta - \bar{g}_\gamma)\bar{L} - (\bar{a}_\beta \bar{P}_\beta + \bar{a}_\gamma \bar{P}_\gamma) = 0, \tag{A2a}$$

$$\bar{M} = 0. \tag{A2b}$$

From eq. (A2a), eq. (A1) becomes

$$\bar{S\phi} = 2\frac{\bar{g}_\beta \bar{a}_\gamma \bar{P}_\gamma + \bar{g}_\gamma \bar{a}_\beta \bar{P}_\beta}{\bar{a}_\beta \bar{P}_\beta + \bar{a}_\gamma \bar{P}_\gamma}. \tag{A3}$$

It is now clear that, unless $\bar{g}_\beta = \bar{g}_\gamma = 0$, $\overline{S\phi} \neq 0$ at some point, which implies that $S\phi$ is not bounded. But the equations

$$\bar{g}_\beta = \sum_\alpha I_{\beta\beta\alpha}\zeta_\alpha^0 - l\omega_\beta = 0,$$

$$\bar{g}_\gamma = \sum_\alpha I_{\gamma\gamma\alpha}\zeta_\alpha^0 - l\omega_\gamma = 0,$$

are identically satisfied only when all the coefficients $I_{\beta\beta\alpha}$, $I_{\gamma\gamma\alpha}$, ω_β, ω_γ, are zero.

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