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SHORT CONTRIBUTION

Note on cellular convection with nonisotropic eddies

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1. Introduction

"According to Krueger and Fritz (1961), one of the outstanding features calling for explanation in the cellular cloud patterns revealed by satellites is that the diameter-to-depth ratio is around 30, instead of about 3 as predicted by classical theory and verified by laboratory experiments." This call for explanation in the introductory sentence of a short communication by Priestley (1962) is as urgent today as 21 years ago. He advocated and Ray (1965) calculated the influence of anisotropic eddy diffusion as a possible solution to the problem of cell flatness. Using the classical Boussinesq approximation to describe Rayleigh–Bénard convection and introducing turbulent horizontal heat and momentum diffusion coefficients of about 100 times the vertical value, results in cell aspect ratios of the required magnitude. This was confirmed recently by Sheu et al. (1980) making eddy anisotropy a major candidate for explanation of the difference between observation and theory (also Agee and Mitchell, 1977).

2. Influence of eddy anisotropy

Applying the classical Rayleigh–Bénard equations to the atmosphere, suggests the replacement of molecular conductivity and viscosity by their turbulent counterparts. These are not necessarily isotropic, especially when different horizontal and vertical scales are involved. In the following, the nomenclature of Ray is used where eddy anisotropies m and n are defined as the ratio of turbulent horizontal to vertical conductivity and viscosity, respectively.

Their introduction leads in the linear analysis to the following differential equation for the vertical velocity perturbation $W(\zeta)$ (eq. (1.13) in the paper by Ray):

$$((D^2 - ma^2)(D^2 - na^2)(D^2 - a^2) + Ra^2)W(\zeta) = 0, \tag{1}$$

where $D^2 = \partial^2/\partial\zeta^2$, ζ the dimensionless vertical coordinate, a the dimensionless wave number, and R the Rayleigh number. The solution for the free slip case is given by

$$W(\zeta) \propto \cos \pi\zeta \tag{2}$$

which yields the condition

$$R = (\pi^2 + ma^2)(\pi^2 + na^2)\left(1 + \frac{\pi^2}{a^2}\right). \tag{3}$$

The critical Rayleigh number R_c is found as the minimum of this expression when varied with respect to a. Table 1 shows critical Rayleigh and wave numbers for various combinations of m and

Table 1. *Critical wave number a_c, aspect ratio $2\pi/a_c$ and critical Rayleigh number R_c for different values of m and n in the case of free boundaries*

m or n	m or n	a_c	$2\pi/a_c$	R_c
0	0	∞	0	97.4
0	1	3.14	0.64	389.6
1	1	2.22	2.8	657.5
1	100	0.821	7.6	12,744.6
10	100	0.537	11.7	17,388.6
10	1000	0.305	20.6	119,010.3
100	100	0.311	20.2	39,349.5
100	1000	0.176	35.7	169,288.4

n. The aspect ratio at the onset of convection, and also the critical Rayleigh number grow monotonously with increasing eddy anisotropy m or/and n, R_c and the critical aspect ratio being smallest at $m = n = 0$. Values of several hundred for m and n result in the required aspect ratio of about 30. In the nonlinear study by Sheu et al. (1980), eddy anisotropy was introduced in the same way as a parameter yielding similar results concerning the cell aspect ratio at the onset of instability.

Unfortunately, there exists no simple and straightforward answer to the question, which value of m and n is typical for the atmosphere in the case of cellular convection. Turbulent exchange coefficients as a parameterization of small-scale turbulent transport, depend on the length scales considered. The functional form of this dependence is not quite clear. While the simplest combination of dimensions needs a length times a velocity scale to produce the dimension of the exchange coefficient (length × length/time), mixing length theory suggests the square of the length scale times a velocity gradient. Assuming that either the typical velocity or its gradient are independent of the length scale would give $K \propto l$ or $K \propto l^2$, respectively. Following Sutton (1953), similarity considerations in good agreement with observations suggest a dependence $K \propto l^{4/3}$. Therefore, a general form of

$$K \propto l^{\alpha} \tag{4}$$

will be used in the following where $0 \leq \alpha \leq 2$. The typical horizontal dimension for cellular convection is the diameter of convection cells L. Thus, the horizontal exchange coefficients are given by

$$K_{Mh} = f_{Mh} L^{\alpha_{Mh}}, K_{Hh} = f_{Hh} L^{\alpha_{Hh}}.$$

The indices M and H refer to momentum and heat transport, respectively, and h indicates horizontal exchange. With the convection layer depth d as the typical vertical dimension, analogous expressions for the vertical exchange (index v) are

$$K_{Mv} = f_{Mv} d^{\alpha_{Mv}}, K_{Hv} = f_{Hv} d^{\alpha_{Hv}}.$$

Assuming $\alpha_{Mh} = \alpha_{Mv} = \alpha_m$ and $\alpha_{Hh} = \alpha_{Hv} = \alpha_n$, and defining $f_m = f_{Mh}/f_{Mv}$, $f_n = f_{Hh}/f_{Hv}$ yields for the eddy anisotropies:

$$m = f_m a^{-\alpha_m} \propto \left(\frac{L}{d}\right)^{\alpha_m}, \tag{5}$$

$$n = f_n a^{-\alpha_n} \propto \left(\frac{L}{d}\right)^{\alpha_n}. \tag{6}$$

As the aspect ratio L/d is related to the wavenumber of the cells by $a = 2\pi d/L$, the eddy anisotropies are inversely proportional to the wavenumber. A further simplification is introduced by assuming the anisotropy of heat and momentum exchange to be identical ($\alpha = \alpha_m = \alpha_n$, $f = f_m = f_n$), which yields for the Rayleigh number

$$R = (\pi^2 + f a^{2-\alpha})^2 \left(1 + \frac{\pi^2}{a^2}\right). \tag{7}$$

f is set to $(2\pi)^{\alpha}$ which corresponds to isotropic turbulence ($m = n = 1$) when the horizontal and vertical scales are equal. Resulting critical parameter values for $\alpha = 0, 1, 2$ are given in Table 2. Obviously, the introduction of any $\alpha > 0$ results in increasing critical wave number and decreasing aspect ratio. At $\alpha \geq 2$, the smallest critical Rayleigh number is reached with $a_c = \infty$ and vanishing aspect ratio.

Table 2. *Critical wave number, aspect ratio and Rayleigh number of different values of α in the case of free boundaries*

α	a_c	$2\pi/a_c$	R_c
0	2.22	2.83	657.5
1	2.49	2.52	1687.3
2	∞	0	2435.2
>2	∞	0	97.4

3. Conclusions

The introduction of a constant eddy anisotropy greater than 1 leads to an aspect ratio of the starting convection that is larger than in the classical case of an isotropic exchange coefficient. Not only the aspect ratio, but also the critical Rayleigh number grow with increasing eddy anisotropy. Eddy anisotropies smaller than 1 result in smaller aspect ratios and critical Rayleigh numbers. The minimum value of R_c would be found for $a_c = \infty$ or vanishing aspect ratio if no scale dependence of the eddy exchange coefficient were introduced. And even with such an introduction, the minimum R_c appears at an aspect ratio that is smaller than in the case of isotropic turbulent exchange. Admittedly, the adapted functional form of the scale dependence of the exchange coefficients may be wrong in detail, but the above results hold qualitatively for any functional dependence as long as the eddy anisotropy increases with the cell aspect ratio.

Usually it is assumed that a layer heated from below will become convectively unstable at the smallest possible, the critical Rayleigh number. Interpreting observed atmospheric convection patterns with use of the results of such a linear analysis implies the further assumption that the fully developed convection pattern is still strongly related to the pattern at the onset of convection. Assuming all this as usual, the above analysis predicts the preference of eddy anisotropies smaller than one, that are related to cell aspect ratios even smaller than those of classical analysis.

These results strongly suggest that the eddy anisotropy cannot be the reason for the observed cell flatness. Therefore, Priestley's introductory call for explanation is still urgent.

REFERENCES

Agee, E. M. and Mitchell, D. L. 1977. The theoretical investigation of atmospheric convective modes as a function of Rayleigh number, Prandtl number and eddy anisotropy. *J. Meteorol. Soc. Japan 55*, 341–363.

Krueger, A. F. and Fritz, S. 1961. Cellular cloud patterns revealed by Tiros I. *Tellus 13*, 1–7.

Priestley, C. M. B. 1962. Width-height ratio of large convection cells. *Tellus 14*, 123–124.

Ray, D. 1965. Cellular convection with nonisotropic eddys. *Tellus 17*, 434–439.

Sheu, Pi-J., Agee, E. M. and Tribbia, J. J. 1980. A numerical study of physical processes affecting convective cellular geometry. *J. Meteorol. Soc. Japan 58*, 489–498.

Sutton, O. G. 1953. *Micrometeorology*. McGraw-Hill Publ. Co., London.