Reading 02: Vacuous Truth

Recall that we defined the implication $p \to q$ with this truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

In this reading, we're going to discuss **why** implications are defined this way.

1. Intuition for Implication

Remember our intuition for implication was a promise: $p \to q$. A verification that p is true gives you a guarantee that q is true.[1] That intuition gave us the first two lines of the truth table. Both propositions p and q being true is consistent with the implication itself being true; having p evaluate to true but q evaluate to false means the promise was broken, so $p \to q$ must be false.

2. An Example

Now let's figure out those last two lines. The easiest way to see the reason for the last two lines is an example. Let's think about something at least somewhat familiar. You learned a theorem like this one in your calculus courses – it says that for a continuous function f, if it goes from below the x-axis to above the x-axis, it must have been exactly 0 between those points.[2]

Theorem 1. *If f is continuous over $[a, b]$, $f(a) < 0$, and $f(b) > 0$, then there exists a point z such that $a < z < b$ and $f(z) = 0$.*

This is a good promise – whenever $f, a,$ and b meet the requirements in the hypothesis, we can guarantee the existence of a z. Let's try to translate this theorem into symbolic (predicate) logic.

The heart of the statement is an implication, we have a hypothesis with three parts (f being continuous and the signs of $f(a)$ and $f(b)$) and our consequence is an existential statement (there is a z between a and b where f evaluates to 0). So far that would be something like

$$\Big(\texttt{ContinuousOnRange}(f, a, b) \wedge \texttt{LessThan}(f(a), 0) \wedge \texttt{GreaterThan}(f(b), 0)\Big)$$
$$\to \Big(\exists z \big(\texttt{LessThan}(a, z) \wedge \texttt{LessThan}(z, b) \wedge \texttt{Equal}(f(z), 0)\big)\Big)$$

There's one extra piece we need to add, remember when we have unquantified variables (here f, a, b) we interpret them as universally quantified, so our final statement is:

$$\forall f, a, b. \Big[\Big(\texttt{ContinuousOnRange}(f, a, b) \wedge \texttt{LessThan}(f(a), 0) \wedge \texttt{GreaterThan}(f(b), 0)\Big)$$
$$\to \Big(\exists z \big(\texttt{LessThan}(a, z) \wedge \texttt{LessThan}(z, b) \wedge \texttt{Equal}(f(z), 0)\big)\Big)\Big]$$

[1] Provided that the implication $p \to q$ itself is a true implication.

[2] Our theorem is a version of the Intermediate Value Theorem. This version has a more restricted hypothesis so we can have a slightly shorter description in symbolic logic. (This version only handles going from below the x-axis to above it, not from above to below.)

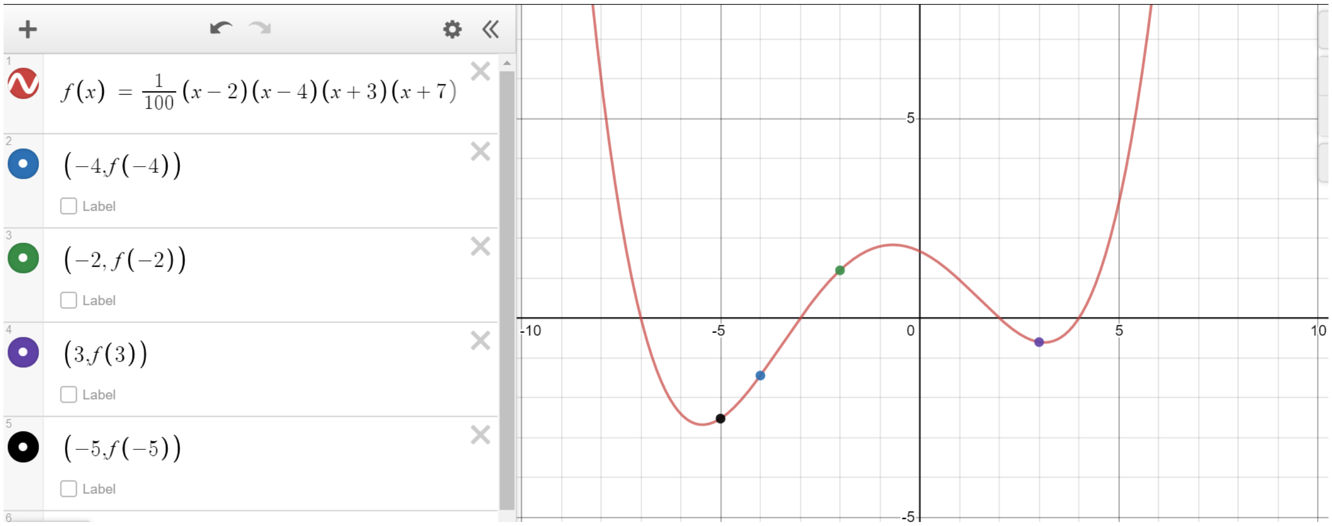

Figure 1: Our function f and some points we'll plug into Theorem 1.

We said at the beginning of this section that this implication is a good promise. That predicate formula should be true. So when we evaluate the implication for any particular f, a, b we have to get true. Let's look at a few values of f, a, b and see what we get:

Figure 1 shows a sample function f, and some points we'll try plugging into our theorem.

If we plug in $a = -4$ and $b = -2$, we'll be using the theorem as a promise. The hypothesis is met, which guarantees the existence of a z ($z = -3$ works).

What happens if we plug in other points...let's say $a = -4$ and $b = 3$. The implication should still be true, we're just not applying it well. We don't meet the hypothesis: $f(3) < 0$. Nonetheless, the conclusion still holds ($z = 2$ is still between a and b and $f(2) = 0$). So the implication evaluates to $\mathtt{F} \rightarrow \mathtt{T}$. We still want this implication to evaluate to true though, this doesn't make our promise invalid. And we really want formula * to be true! Which means we need $\mathtt{F} \rightarrow \mathtt{T}$ to evaluate to true.

Similarly, we can plug in $a = -5$, $b = -4$. The hypothesis doesn't hold ($f(-4) < 0$) and neither does the consequence f is negative over $[-5, -4]$. The implication evaluates to $\mathtt{F} \rightarrow \mathtt{F}$. Again, we need to evaluate $\mathtt{F} \rightarrow \mathtt{F}$ to true for * to be true.

Notice that no matter how hard we try, we can't find a, b to plug in where the hypothesis will be true, but the consequence will be false. That's what the implication means – it's a guarantee that we can't find a counter-example.

3. Vacuous Truth

Now we can see why "vacuous truth" exists. The promise made by the theorem is true. But the promise can be "misapplied." For an implication to be useful, it should apply universally (promises that are sometimes true aren't great promises). But if the implication applies universally, you'll probably be able to find values where the promise is meaningless, like our last two paragraphs.

This decision to make $\mathtt{F} \rightarrow \mathtt{T}$ and $\mathtt{F} \rightarrow \mathtt{F}$ both evaluate to true doesn't just apply when we're looking at good promises. For example:
Theorem 2. *If I am the King of England, then I can prove $P \neq NP$.*

Neither the hypothesis nor the conclusion is true. And there's no causal connection between being the King of England and being able to solve computer science problems. Nonetheless, the implication is still true. "Untestable" promises – where the hypothesis cannot be met – have to be true. We have to stay consistent on what $p \rightarrow q$ means, and making sure we can universally quantify real promises is worth this strange side-effect.