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Some Linear Transformations on Symmetric Functions Arising From a Formula of Thiel and Williams

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Abstract: We consider a linear operator ψ_r from the ring $\Lambda_{\mathbb{Q}}$ of symmetric functions over $\mathbb{Q}$ to the polynomial ring $\mathbb{Q}[n]$ defined by $\psi_r m_\lambda = \left[\sum_{i=1}^{l}(\lambda_i)_r\right] m_\lambda(1^n)$, where m_λ is a monomial symmetric function, $(\lambda_i)_r$ denotes the falling factorial, and $m_\lambda(1^n)$ denotes m_λ evaluated at $x_1 = \cdots = x_n = 1$, $x_i = 0$ for $i > n$. We obtain formulas for many instances of $\psi_r b_\lambda$, where b_λ denotes one of the six standard bases for $\Lambda_{\mathbb{Q}}$. The formula for $\psi_2 s_\lambda$, where s_λ is a Schur function, is equivalent to a formula of M. Thiel and N. Williams on the expected square norm of the weight of an irreducible representation of the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$.

Keywords: Schur function; Symmetric function; Thiel-Williams formula
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1. Introduction

The motivation for this paper is a formula [5, Thm. 1.1] of M. Thiel and N. Williams, namely, for a complex simple Lie algebra $\mathfrak{g}$ with an irreducible representation V_λ of highest weight λ, the expected squared norm of a weight in V_λ is

$$\mathop{\mathbb{E}}_{\mu\in V_\lambda}(\langle \mu,\mu\rangle) := \frac{1}{\dim V_\lambda}\sum_{\mu\in V_\lambda}\dim(V_\lambda(\mu))\langle \mu,\mu\rangle = \frac{1}{h+1}\langle \lambda,\lambda+2\rho\rangle, \tag{1}$$

where $\dim V_\lambda(\mu)$ is the multiplicity of μ in V_λ, h is the Coxeter number of $\mathfrak{g}$, and ρ is the half-sum of the positive roots. (The sum over $\mu \in V_\lambda$ has only finitely many nonzero terms.)

In type A, that is, $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, equation (1) can be stated in terms of symmetric functions in the variables $x_1, \ldots, x_n$. Moreover, this restated formula stabilizes as $n \to \infty$, so we get a formula involving symmetric functions in infinitely many variables.

To state this formula, we will use standard notation and terminology from the theory of symmetric functions as found in [3, Ch. 7]. In particular, λ and μ now denote partitions (rather than weights). If λ is a partition of d, then we write $\lambda \vdash d$, $|\lambda| = d$, or $\lambda \in \mathrm{Par}(d)$. We also write $\lambda = \langle 1^{m_1}2^{m_2}\cdots d^{m_d}\rangle$ if λ has $m_i = m_i(\lambda)$ parts equal to i, so $\sum i m_i = |\lambda|$. The *length* $\ell(\lambda)$ is the total number of parts, so $\ell(\lambda) = \sum m_i$. Let λ_i' be the number of parts of λ that are greater than or equal to i. The partition $\lambda' = (\lambda_1', \lambda_2', \ldots, \lambda_k')$ is called the *conjugate* partition of λ. Thus $\lambda_1' = \ell(\lambda)$ and $\lambda_1 = \ell(\lambda')$.

Throughout this paper, $\mathbb{P}$ and $\mathbb{Q}$ respectively denote the sets of positive integers and rational numbers. Recall that the algebra $\Lambda_{\mathbb{Q}}(x)$ of symmetric functions has various bases that are indexed by the set Par of partitions, including $m_\lambda = m_\lambda(x)$ (monomial symmetric functions), p_λ (power sum symmetric functions), e_λ (elementary symmetric functions), h_λ (complete homogeneous symmetric functions), s_λ (Schur functions), and $\mathrm{fo}_\lambda = \omega m_\lambda$ (forgotten symmetric functions), where ω is the involution on $\Lambda_{\mathbb{Q}}$ defined by $\omega(h_\lambda) = e_\lambda$. For $f(x) \in \Lambda_{\mathbb{Q}}(x)$, let

$$f(1^n) = f(\overbrace{1,\ldots,1}^{n},0,0,\ldots).$$

For fixed f, the function $f(1^n)$ is always a polynomial in n.

Let $n, r \in \mathbb{P}$. For $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathrm{Par}$, define a $\mathbb{Q}$-linear transformation

$$\psi_r \colon \Lambda_{\mathbb{Q}} \to \mathbb{Q}[n]$$

by

$$\psi_r m_\lambda = \left[\sum_{i=1}^{l} (\lambda_i)_r\right] m_\lambda(1^n),$$

where $(a)_r = a(a-1)\cdots(a-r+1)$ and $l = \ell(\lambda)$.

We can now state (in an equivalent form) the result of Thiel and Williams [5] in the case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, namely, for $\lambda \vdash d$,

$$\psi_2 s_\lambda = \frac{2f^{\lambda/(2)}}{(d-2)!} \cdot \frac{\prod_{u\in\lambda}(n + c(u))}{n+1}, \tag{2}$$

where $f^{\lambda/(2)}$ is the number of standard tableaux of the skew shape $\lambda/(2)$ (interpreted to be 0 if $(2) \not\subseteq \lambda$, i.e., if $\lambda = \langle 1^d \rangle$), and where $c(u)$ is the content of the square u of (the Young diagram of) λ.

The elegant formula (2) suggests that it might be interesting to apply ψ_2 to other symmetric function bases and to generalize from ψ_2 to ψ_r.

In the next section (Section 2) we prove that for any symmetric function f,

$$[z^r] f(z+1, \overbrace{1, \ldots, 1}^{n-1}, 0, 0, \ldots) = \frac{1}{n \cdot r!} \psi_r f,$$

where $[z^r]g$ denotes the coefficient of z^r in g (when expanded as a power series in z). This representation allows us to compute $\psi_r b_\lambda$ for various bases of $\Lambda_{\mathbb{Q}}$. In particular, if $\lambda \vdash d$ then

$$\psi_r s_\lambda = C_{\lambda r} \cdot \frac{\prod_{u\in\lambda}(n + c(u))}{(n+1)(n+2)\cdots(n+r-1)}.$$

Here $c(u)$ is the content of the square u of the (diagram of) λ and

$$C_{\lambda r} = \begin{cases} \frac{r!}{(d-r)!} f^{\lambda/(r)} & \text{if } \lambda_1 \geq r \\ 0 & \text{otherwise,} \end{cases}$$

where $f^{\lambda/(r)}$ is the number of standard Young tableaux of the skew shape $\lambda/(r)$.

Remark 1.1. *The actual formula of Thiel and Williams mentioned above dealt (essentially) with the operator* $\hat{\psi}_2 \colon \Lambda_{\mathbb{Q}} \to \mathbb{Q}[n]$ *defined by*

$$\hat{\psi}_2 m_\lambda = \left(\sum_{i=1}^{l} \lambda_i^2\right) m_\lambda(1^n).$$

Since for $\lambda \vdash d$ *we have*

$$\begin{aligned} \hat{\psi}_2 m_\lambda &= \psi_2 m_\lambda + \left(\sum_{i=1}^{l} \lambda_i\right) m_\lambda(1^n) \\ &= \psi_2 m_\lambda + d m_\lambda(1^n), \end{aligned}$$

it follows that for any homogeneous symmetric function f *of degree* d,

$$\hat{\psi}_2 f = \psi_2 f + d f(1^n).$$

More generally, we can define a linear transformation $\hat{\psi}_r$ *for* $r \geq 2$ *by*

$$\hat{\psi}_r m_\lambda = \left(\sum_{i=1}^{l} \lambda_i^r\right) m_\lambda(1^n).$$

Since in general (e.g., [2, (1.96)])

$$a^r = \sum_{k=1}^{r} S(r,k)(a)_k,$$

where $S(r,k)$ *is a Stirling number of the second kind, our formulas for* $\psi_r f$ *yield formulas for* $\hat{\psi}_r f$.

2. A formula for $\psi_r f$

Let $n \in \mathbb{P}$ and z be an indeterminate. For any symmetric function $f \in \Lambda_{\mathbb{Q}}$ write

$$\vartheta f = f(z+1, \overbrace{1, \ldots, 1}^{n-1}, 0, 0, \ldots).$$

It is clear from the definition that $\vartheta(m_\lambda)$ is a polynomial in z (with coefficients in $\mathbb{Q}[n]$) of degree at most λ_1. Hence ϑ is an algebra homomorphism $\Lambda_{\mathbb{Q}} \to \mathbb{Q}[n, z]$. For instance,

$$\vartheta s_{21} = (n-1)z^2 + (n-1)(n+1)z + \frac{1}{3}(n-1)n(n+1).$$

Theorem 2.1. *For any $r \in \mathbb{P}$ and $f \in \Lambda_{\mathbb{Q}}$, we have*

$$[z^r]\vartheta f = \frac{1}{n \cdot r!}\psi_r f.$$

Proof. By linearity it suffices to show that the theorem is true for $f = m_\lambda$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) = \langle 1^{m_1} 2^{m_2} \cdots d^{m_d} \rangle \vdash d$. Set

$$b_\lambda(n) := \binom{n-1}{l} \cdot \binom{l}{m_1, m_2, \ldots, m_d}.$$

Then we have

$$\begin{aligned}
\vartheta m_\lambda &= \left[\sum_{i=1}^{d} (z+1)^i \cdot \binom{n-1}{l-1} \cdot \binom{l-1}{m_1, \ldots, m_{i-1}, m_i - 1, m_{i+1}, \ldots, m_d}\right] + b_\lambda(n) \\
&= \left[\sum_{i=1}^{d} m_i (z+1)^i \cdot \binom{n-1}{l-1} \cdot \binom{l-1}{m_1, m_2, \ldots, m_d}\right] + b_\lambda(n) \\
&= \left[\sum_{i=1}^{l} (z+1)^{\lambda_i}\right] \cdot \binom{n-1}{l-1} \cdot \binom{l-1}{m_1, m_2, \ldots, m_d} + b_\lambda(n) \\
&= \left[\sum_{i=1}^{l} \sum_{r=0}^{\lambda_i} \binom{\lambda_i}{r} z^r\right] \cdot \frac{1}{n}\binom{n}{l}\binom{l}{m_1, m_2, \ldots, m_d} + b_\lambda(n) \\
&= \frac{1}{n}\binom{n}{l}\binom{l}{m_1, m_2, \ldots, m_d} \cdot \left[\sum_{r=0}^{\lambda_1} \frac{1}{r!}\left(\sum_{i=0}^{l} (\lambda_i)_r\right) z^r\right] + b_\lambda(n) \\
&= \frac{1}{n} m_\lambda(1^n) \cdot \left[\sum_{r=0}^{\lambda_1} \frac{1}{r!}\left(\sum_{i=0}^{l} (\lambda_i)_r\right) z^r\right] + b_\lambda(n),
\end{aligned}$$

so the proof follows. ☐

We can also prove Theorem 2.1 by applying ϑ (acting on x variables only, so y variables are regarded as scalars) to both sides of the following identity

$$\sum_\lambda m_\lambda(x) h_\lambda(y) = \prod_{i,j} \frac{1}{1 - x_i y_j} = \exp\left(\sum_{i \geq 1} \frac{1}{i} p_i(x) p_i(y)\right)$$

to get

$$\begin{aligned}
\sum_\lambda \vartheta m_\lambda(x) \cdot h_\lambda(y) &= \exp\left(\sum_{i \geq 1} \frac{1}{i} \vartheta p_i(x) \cdot p_i(y)\right) \\
&= \exp\left\{\sum_{i \geq 1} \frac{1}{i}\left[(n-1) + (z+1)^i\right] p_i(y)\right\}
\end{aligned}$$

$$= \left[\exp\left(\sum_{i\geq 1}\frac{1}{i}p_i(y)\right)\right]^{n-1}\cdot\left[\exp\left(\sum_{i\geq 1}\frac{1}{i}(z+1)^i p_i(y)\right)\right]$$

$$= \left[\sum_{k=0}^{\infty}h_k(y)\right]^{n-1}\cdot\left[\sum_{j=0}^{\infty}(z+1)^j h_j(y)\right].$$

Then we can complete the proof by comparing the coefficient of $h_\lambda(y)$; we omit the details.

3. Schur functions

Theorem 3.1. *For $\lambda \vdash d$ and $r \in \mathbb{P}$, we have*

$$\psi_r s_\lambda = C_{\lambda r}\cdot\frac{\prod_{u\in\lambda}(n+c(u))}{(n+1)(n+2)\cdots(n+r-1)}$$

where

$$C_{\lambda r} = \begin{cases} \frac{r!}{(d-r)!}f^{\lambda/(r)}, & \text{if } \lambda_1 \geq r,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. Firstly, we claim that

$$s_\lambda(x_1+1, x_2+1, \ldots, x_n+1) = \sum_{\mu\subseteq\lambda}\frac{f^{\lambda/\mu}}{|\lambda/\mu|}\left(\prod_{u\in\lambda/\mu}(n+c(u))\right)s_\mu(x_1, x_2, \ldots, x_n). \tag{3}$$

Indeed, using standard notation from [3, §7.15] we have

$$s_\lambda(x_1+1,\ldots,x_n+1) = \frac{a_{\lambda+\delta}(x_1+1,\ldots,x_n+1)}{a_\delta(x_1+1,\ldots,x_n+1)}$$
$$= \frac{a_{\lambda+\delta}(x_1+1,\ldots,x_n+1)}{a_\delta(x_1,\ldots,x_n)}.$$

We can expand the entries of $a_{\lambda+\delta}(x_1+1,\ldots,x_n+1)$ and use the multilinearity of the determinant to get (see [1, Example I.3.10, p. 47])

$$s_\lambda(x_1+1,\ldots,x_n+1) = \sum_{\mu\subseteq\lambda}d_{\lambda\mu}s_\mu,$$

where

$$d_{\lambda\mu} = \det\left(\binom{\lambda_i+n-i}{\mu_j+n-j}\right)_{1\leq i,j\leq n}.$$

We can factor out factorials from the numerators of the row entries and denominators of the column entries of the above determinant. These factorials altogether yield $\prod_{u\in\lambda/\mu}(n+c(u))$. What remains is exactly the determinant for $f^{\lambda/\mu}/|\lambda/\mu|!$ given by Corollary 7.16.3 in [3]. This completes the proof of equation (3). Set $x_1 = z$ and $x_2 = x_3 = \cdots = x_n = 0$ in (3). Then we have

$$\vartheta s_\lambda = \sum_{\mu\subseteq\lambda}\frac{f^{\lambda/\mu}}{|\lambda/\mu|!}\left(\prod_{u\in\lambda/\mu}(n+c(u))\right)s_\mu(z,0,0,\cdots,0).$$

Note that

$$s_\mu(z,0,0,\cdots,0) = \begin{cases} z^r & \text{if } \mu=(r),\\ 0 & \text{otherwise.}\end{cases}$$

Therefore we have

$$\vartheta s_\lambda = \sum_{r=0}^{\lambda_1}\frac{f^{\lambda/(r)}}{(d-r)!}\left(\prod_{u\in\lambda/(r)}(n+c(u))\right)z^r$$

$$= \sum_{r=0}^{\lambda_1} \frac{f^{\lambda/(r)}}{(d-r)!} \cdot \frac{\prod_{u\in\lambda}(n+c(u))}{n(n+1)\cdots(n+r-1)} z^r.$$

Then it follows from Theorem 2.1 that, for any $1 \leq r \leq \lambda_1$,

$$\begin{aligned}\psi_r s_\lambda &= n \cdot r! \cdot [z^r](\vartheta s_\lambda) \\ &= \frac{r!}{(d-r)!} f^{\lambda/(r)} \cdot \frac{\prod_{u\in\lambda}(n+c(u))}{(n+1)(n+2)\cdots(n+r-1)}.\end{aligned}$$

□

4. Formulas for $\psi_2(p_\lambda)$, $\psi_2(e_\lambda)$ and $\psi_2(h_\lambda)$

For any $f \in \Lambda_{\mathbb{Q}}$, by Theorem 2.1 and by the definition of ϑ, we have

$$\vartheta f = f(1^n) + \frac{1}{n} \sum_{r\geq 1} \frac{\psi_r f}{r!} z^r. \tag{4}$$

This implies that ϑf can be regarded as the generating function for $\psi_r f$. Then it is natural to consider $\psi_r b_\lambda$ for other bases $\{b_\lambda\}$. We shall show that, for general r, $\psi_r e_\lambda$ also has a nice formula that can be written as the product of linear factors. Although for general r, $\psi_r p_\lambda$, $\psi_r h_\lambda$ and $\psi_r \mathrm{fo}_\lambda$ do not have such nice formulas, the case for $r = 2$ turns out to be simple.

In this section, we will exploit the relation (4) to get formulas for $\psi_r e_\lambda$, $\psi_2 p_\lambda$ and $\psi_2 h_\lambda$. For the forgotten symmetric function fo_λ, this method seems to be not very effective. We will use another tool in the next section to derive the formula for $\psi_2 \mathrm{fo}_\lambda$.

Theorem 4.1. *For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash d$, we have*

(1) $\psi_r e_\lambda = \widetilde{C}_{\lambda r} \cdot n^{l-r+1} \cdot \prod_{i\geq 2}(n-i+1)^{\lambda'_i}$, *where*

$$\widetilde{C}_{\lambda r} = \begin{cases} \frac{r!}{\prod_{i=1}^{l} \lambda_i!} \left(\sum_{i_1<i_2<\cdots<i_r} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_r} \right) & \text{if } r \leq l, \\ 0 & \text{otherwise.} \end{cases}$$

(2) $\psi_2 p_\lambda = n^{l-1} \cdot \left[\left(\sum_{i=1}^{l} \lambda_i^2 - d \right) n + d^2 - \sum_{i=1}^{l} \lambda_i^2 \right].$

(3) $\psi_2 h_\lambda = \frac{\left[2(\sum_{i=1}^{l}\lambda_i^2) - 2d + 2\sum_{i<j}\lambda_i\lambda_j\right]\cdot n + 2\sum_{i<j}\lambda_i\lambda_j}{n(n+1)\prod_{i=1}^{l}\lambda_i!} \cdot \prod_{i\geq 1}(n+i-1)^{\lambda'_i}.$

Proof. (1) By equation (4), we have

$$\begin{aligned}\psi_r e_\lambda &= r!\, n \cdot [z^r](\vartheta e_\lambda) \\ &= r!\, n \cdot [z^r] \left(\vartheta e_{\lambda_1} \cdot \vartheta e_{\lambda_2} \cdots \vartheta e_{\lambda_l} \right) \\ &= r!\, n \cdot [z^r] \left(\vartheta s_{1^{\lambda_1}} \cdot \vartheta s_{1^{\lambda_2}} \cdots \vartheta s_{1^{\lambda_l}} \right).\end{aligned}$$

By the proof of Theorem 3.1, we get

$$\vartheta s_{1^k} = \frac{(n-1)(n-2)\cdots(n-k+1)}{k!}(n+kz).$$

Then we obtain that

$$\begin{aligned}\psi_r e_\lambda &= r!\, n \cdot [z^r] \left[\prod_{i=1}^{l} \frac{(n-1)(n-2)\cdots(n-\lambda_i+1)}{\lambda_i!}(n+\lambda_i z) \right] \\ &= r!\, n \cdot \frac{\prod_{i\geq 2}(n-i+1)^{\lambda'_i}}{\prod_{i=1}^{l}\lambda_i!} \cdot [z^r] \left[\prod_{i=1}^{l}(n+\lambda_i z) \right]\end{aligned}$$

$$= \begin{cases} r!\, n \cdot \dfrac{\prod_{i\geq 2}(n-i+1)^{\lambda_i'}}{\prod_{i=1}^{l}\lambda_i!}\left(\displaystyle\sum_{i_1<i_2<\cdots<i_r}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_r}\right)\cdot n^{l-r}, & \text{if } r\leq l \\ 0, & \text{otherwise} \end{cases}$$

$$= \widetilde{C}_{\lambda r}\cdot n^{l-r+1}\cdot\prod_{i\geq 2}(n-i+1)^{\lambda_i'}$$

where

$$\widetilde{C}_{\lambda r} = \begin{cases} \dfrac{r!}{\prod_{i=1}^{l}\lambda_i!}\left(\displaystyle\sum_{i_1<i_2<\cdots<i_r}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_r}\right) & \text{if } r\leq l, \\ 0 & \text{otherwise.} \end{cases}$$

(2) Again, by equation (4), we have

$$\begin{aligned} \psi_2 p_\lambda &= 2n\cdot[z^2](\vartheta p_\lambda) \\ &= 2n\cdot[z^2]\left(\vartheta p_{\lambda_1}\cdot\vartheta p_{\lambda_2}\cdots\vartheta p_{\lambda_l}\right) \\ &= 2n\cdot[z^2]\left\{\prod_{i=1}^{l}\left[n-1+(z+1)^{\lambda_i}\right]\right\} \\ &= 2n\cdot\left[n^{l-2}\left(\sum_{i<j}\lambda_i\lambda_j\right)+n^{l-1}\left(\sum_{i=1}^{l}\binom{\lambda_i}{2}\right)\right] \\ &= n^{l-1}\cdot\left[2\sum_{i<j}\lambda_i\lambda_j+n\left(\sum_{i=1}^{l}\lambda_i^2-d\right)\right] \\ &= n^{l-1}\cdot\left[\left(\sum_{i=1}^{l}\lambda_i^2-d\right)n+d^2-\sum_{i=1}^{l}\lambda_i^2\right]. \end{aligned}$$

This completes the proof of the formula for $\psi_2 p_\lambda$.

(3) The formula of $\psi_2 h_\lambda$ can be proved similarly, by using the fact that h_λ is a multiplicative basis of $\Lambda_{\mathbb{Q}}$ and by Theorem 2.1. We omit the details here.

□

Remark 4.1. *For general r, the formulas of $\psi_r p_\lambda$ and $\psi_r h_\lambda$ do not necessarily have such nice decompositions. For instance,*

$$\psi_3 h_{321} = n(n+1)(19n^2+35n+6)$$

and

$$\psi_3 p_{3211} = 6n^2(n^2+17n+17).$$

5. Forgotten symmetric functions

To prove the formula for $\psi_2\mathrm{fo}_\lambda$, we need the following reformulation of the $\mathbb{Q}$-linear transformation ψ_2 in terms of a differential operator.

Since we have

$$\frac{\partial^2}{\partial x_i^2}(x_1^{\alpha_1}x_2^{\alpha_2}\cdots) = \alpha_i(\alpha_i-1)x_1^{\alpha_1}\cdots x_i^{\alpha_i-2}\cdots,$$

then it is easy to see that

$$\left[\left(\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\right)m_\lambda\right]_{\substack{x_1=\cdots=x_n=1\\ x_{n+1}=\cdots=0}} = \left(\sum_{i=1}^{l}\lambda_i(\lambda_i-1)\right)m_\lambda(1^n). \tag{5}$$

For simplicity of notation we define a $\mathbb{Q}-$linear transformation $D_n^2 : \Lambda_{\mathbb{Q}}(x) \to \mathbb{Q}[n]$ by

$$D_n^2 f = \left[\left(\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}\right) f\right]_{\substack{x_1=\cdots=x_n=1\\ x_{n+1}=\cdots=0}}.$$

By equation (5), we have $D_n^2 f = \psi_2 f$ for any $f \in \Lambda_{\mathbb{Q}}$.

Theorem 5.1. *For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash d$, we have*

$$\psi_2 \mathrm{fo}_\lambda = \frac{\varepsilon_\lambda \prod_{i=1}^l (n+i-1)}{(n+1)\prod m_i(\lambda)!} \left[\left(\sum_{i=1}^l \lambda_i^2 - d\right) n + 2d^2 - d - \sum_{i=1}^l \lambda_i^2\right],$$

where $\varepsilon_\lambda = (-1)^{|\lambda|+\ell(\lambda)}$.

Proof. By regarding variables y as scalars and applying ω to the identity

$$H(x,y) = \prod_{i,j=1}^\infty (1 + x_i y_j) = \sum_\lambda m_\lambda(x) e_\lambda(y),$$

we then obtain

$$C(x,y) = \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda \mathrm{fo}_\lambda(x) \cdot e_\lambda(y). \tag{6}$$

By applying D_n^2 to the left hand side of (6) and ψ_2 to the right hand side, we deduce that

$$nC(1^n, y)\left[\left(\sum_{m=1}^\infty p_m(y)\right)^2 + \sum_{m=1}^\infty (m-1) p_m(y)\right] = \sum_\lambda \psi_2 \mathrm{fo}_\lambda \cdot e_\lambda(y).$$

Then it follows that $\psi_2 \mathrm{fo}_\lambda$ is the coefficient of $e_\lambda(y)$ in

$$\left[n \sum_\mu \mathrm{fo}_\mu(1^n) \cdot e_\mu(y)\right] \cdot \left[\left(\sum_{m=1}^\infty p_m(y)\right)^2 + \sum_{m=1}^\infty (m-1) p_m(y)\right].$$

By Newton's identities, we have

$$\begin{aligned} p_m(y) &= \sum_{\substack{(r_1, r_2, \ldots, r_m) \in \mathbb{N}^m \\ r_1 + 2r_2 + \cdots + m r_m = m}} (-1)^m \frac{m(r_1 + r_2 + \cdots + r_m - 1)!}{r_1! \cdots r_m!} \prod_{i=1}^m (-e_i(y))^{r_i} \\ &= \sum_{\nu \vdash m} (-1)^{|\nu| + \ell(\nu)} \frac{|\nu|(\ell(\nu)-1)!}{\prod m_i(\nu)!} e_\nu(y) \\ &= \sum_{\nu \vdash m} \varepsilon_\nu |\nu| \frac{(\ell(\nu)-1)!}{\prod m_i(\nu)!} e_\nu(y) \end{aligned}$$

where $\varepsilon_\nu = (-1)^{|\nu|+\ell(\nu)}$. Note that for a partition $\mu \in \mathrm{Par}$, we have

$$\begin{aligned} \mathrm{fo}_\mu(1^n) &= (-1)^{|\mu|} \binom{-n}{\ell(\mu)} \binom{\ell(\mu)}{m_1(\mu), m_2(\mu), \ldots} \\ &= \frac{\varepsilon_\mu (n + \ell(\mu) - 1)!}{(\prod m_i(\mu)!)(n-1)!}. \end{aligned}$$

Write Par^* for $\mathrm{Par} \setminus \emptyset$, the set of all partitions excluding the partition $\emptyset$ of 0. We then deduce that for any $\lambda \vdash d$ with $\ell(\lambda) \geq 2$,

$$\begin{aligned} &\psi_2 \mathrm{fo}_\lambda \\ &= n \sum_{\substack{(\mu,\nu,\rho) \in \mathrm{Par} \times \mathrm{Par}^* \times \mathrm{Par}^*, \\ \mu \cup \nu \cup \rho = \lambda \text{ as multisets}}} \frac{\varepsilon_\mu (n + \ell(\mu) - 1)!\, \varepsilon_\nu |\nu| (\ell(\nu)-1)!\, \varepsilon_\rho |\rho| (\ell(\rho)-1)!}{(\prod m_i(\mu)!)(n-1)!\, (\prod m_i(\nu)!)(\prod m_i(\rho)!)} \end{aligned}$$

$$+ n \sum_{\substack{(\mu,\nu)\in \mathrm{Par}\times\mathrm{Par}^*,\\ \mu\cup\nu=\lambda \text{ as multisets}}} \frac{\varepsilon_\mu (n+\ell(\mu)-1)!\, \varepsilon_\nu |\nu|(|\nu|-1)(\ell(\nu)-1)!}{(\prod m_i(\mu)!)(n-1)!\,(\prod m_i(\nu)!)}$$

$$= \frac{n\varepsilon_\lambda}{\prod m_i(\lambda)!} \sum_{(\mu,\nu,\rho)} (n)^{\overline{\ell(\mu)}} (1)^{\overline{\ell(\nu)-1}} (1)^{\overline{\ell(\rho)-1}} \frac{|\nu||\rho| \prod m_i(\lambda)!}{\prod_i (m_i(\mu)!\, m_i(\nu)!\, m_i(\rho)!)} \tag{7}$$

$$+ \frac{n\varepsilon_\lambda}{\prod m_i(\lambda)!} \sum_{(\mu,\nu)} (n)^{\overline{\ell(\mu)}} (1)^{\overline{\ell(\nu)-1}} \frac{|\nu|(|\nu|-1)\prod m_i(\lambda)!}{\prod_i (m_i(\mu)!\, m_i(\nu)!)}, \tag{8}$$

where $(x)^{\overline{k}}$ denotes the rising factorial. Now we simplify the summands (7) and (8) respectively. Note that the summand (7) is equal to

$$\frac{n\varepsilon_\lambda}{\prod m_i(\lambda)!} \sum_{\substack{(l_1,l_2,l_3)\\ l_1\geq 0, l_2,l_3\geq 1\\ l_1+l_2+l_3=\ell(\lambda)}} (n)^{\overline{l_1}} (1)^{\overline{l_2-1}} (1)^{\overline{l_3-1}} \sum_{\substack{(\mu,\nu,\rho)\\ \ell(\mu)=l_1\\ \ell(\nu)=l_2\\ \ell(\rho)=l_3\\ \mu\cup\nu\cup\rho=\lambda}} \frac{|\nu||\rho| \prod m_i(\lambda)!}{\prod_i (m_i(\mu)!\, m_i(\nu)!\, m_i(\rho)!)}.$$

Considering the inner sum of the above equation, we have

$$\begin{aligned}
\sum_{\substack{(\mu,\nu,\rho)\\ \ell(\mu)=l_1\\ \ell(\nu)=l_2\\ \ell(\rho)=l_3\\ \mu\cup\nu\cup\rho=\lambda}} \frac{|\nu||\rho| \prod m_i(\lambda)!}{\prod_i (m_i(\mu)!\, m_i(\nu)!\, m_i(\rho)!)} &= \sum_{\substack{(S,T)\\ S,T\subseteq[l], S\cap T=\emptyset\\ |S|=l_2, |T|=l_3}} \left(\sum_{i\in S}\lambda_i\right)\left(\sum_{j\in T}\lambda_j\right)\\
&= \sum_{i\neq j} \lambda_i\lambda_j \cdot \binom{l-2}{l_2-1, l_3-1, l_1}\\
&= 2\left(\sum_{i<j}\lambda_i\lambda_j\right)\binom{l-2}{l_2-1, l_3-1, l_1},
\end{aligned}$$

since for each pair (i,j) with $i\neq j$, there are exactly $\binom{l-2}{l_2-1,l_3-1,l_1}$ pairs (S,T) such that $i\in S$ and $j\in T$. Therefore the summand (7) can be simplified to be

$$\frac{2n\varepsilon_\lambda}{\prod m_i(\lambda)!}\left(\sum_{i<j}\lambda_i\lambda_j\right) \sum_{\substack{(l_1,l_2,l_3)\\ l_1\geq 0, l_2,l_3\geq 1\\ l_1+l_2+l_3=\ell(\lambda)}} (n)^{\overline{l_1}} (1)^{\overline{l_2-1}} (1)^{\overline{l_3-1}} \binom{l-2}{l_2-1, l_3-1, l_1}$$

$$= \frac{2n\varepsilon_\lambda}{\prod m_i(\lambda)!}\left(\sum_{i<j}\lambda_i\lambda_j\right)(n+2)^{\overline{l-2}},$$

where we use the fact that rising factorials are Sheffer sequences of binomial type, namely, we use the following relation

$$(a+b+c)^{\overline{n}} = \sum_{\substack{(i,j,k)\\ i+j+k=n}} \binom{n}{i,j,k} (a)^{\overline{i}} (b)^{\overline{j}} (c)^{\overline{k}}.$$

Similarly, the summand (8) can be represented as

$$\frac{n\varepsilon_\lambda}{\prod m_i(\lambda)!} \sum_{\substack{(l_1,l_2)\\ l_1\geq 0, l_2\geq 1\\ l_1+l_2=l}} (n)^{\overline{l_1}} (1)^{\overline{l_2-1}} \sum_{\substack{(\mu,\nu)\\ \ell(\mu)=l_1\\ \ell(\nu)=l_2\\ \mu\cup\nu=\lambda}} \frac{\prod m_i(\lambda)! |\nu|(|\nu|-1)}{\prod_i (m_i(\mu)!\, m_i(\nu)!)}.$$

And the inner sum of the above equation can be simplified as follows.

$$\sum_{\substack{S\subseteq[l]\\ |S|=l_2}} \left(\sum_{i\in S}\lambda_i\right)\left(\sum_{j\in S}\lambda_j - 1\right)$$

$$= \sum_{i=1}^{l} \lambda_i^2 \binom{l-1}{l_2-1} + 2\sum_{i<j} \lambda_i\lambda_j \binom{l-2}{l_2-2} - \sum_{i=1}^{l} \lambda_i \binom{l-1}{l_2-1}.$$

Therefore, we can simplify the summand (8) as follows:

$$\frac{n\varepsilon_\lambda}{\prod m_i(\lambda)!}\left[\left(\sum_{i=1}^{l}\lambda_i^2 - d\right)(n+1)^{\overline{l-1}} + 2\sum_{i<j}\lambda_i\lambda_j \sum_{\substack{l_1\geq 0, l_2\geq 1\\ l_1+l_2=l}} (n)^{\overline{l_1}}(1)^{\overline{l_2-1}}\binom{l-2}{l_2-2}\right]$$

$$= \frac{n\varepsilon_\lambda}{\prod m_i(\lambda)!}\left[\left(\sum_{i=1}^{l}\lambda_i^2 - d\right)(n+1)^{\overline{l-1}} + 2\sum_{i<j}\lambda_i\lambda_j \sum_{\substack{l_1\geq 0, l_2\geq 2\\ l_1+l_2=l}} (n)^{\overline{l_1}}(2)^{\overline{l_2-2}}\binom{l-2}{l_2-2}\right]$$

$$= \frac{n\varepsilon_\lambda}{\prod m_i(\lambda)!}\left[\left(\sum_{i=1}^{l}\lambda_i^2 - d\right)(n+1)^{\overline{l-1}} + 2\sum_{i<j}\lambda_i\lambda_j (n+2)^{\overline{l-2}}\right]$$

$$= \frac{n\varepsilon_\lambda}{\prod m_i(\lambda)!}(n+2)^{\overline{l-2}}\left[\left(\sum_{i=1}^{l}\lambda_i^2 - d\right)n + d^2 - d\right].$$

Hence, for $\lambda \vdash d$ with $\ell(\lambda) \geq 2$, we have

$$\begin{aligned}
&\psi_2 \mathrm{fo}_\lambda \\
&= \frac{n\varepsilon_\lambda}{\prod m_i(\lambda)!}(n+2)^{\overline{l-2}}\left[\left(\sum_{i=1}^{l}\lambda_i^2 - d\right)n + d^2 - d + 2\sum_{i<j}\lambda_i\lambda_j\right] \\
&= \frac{n\varepsilon_\lambda}{\prod m_i(\lambda)!}(n+2)^{\overline{l-2}}\left[\left(\sum_{i=1}^{l}\lambda_i^2 - d\right)n + 2d^2 - d - \sum_{i=1}^{l}\lambda_i^2\right] \\
&= \frac{\varepsilon_\lambda}{\prod m_i(\lambda)!}\cdot\frac{\prod_{i=1}^{l}(n+i-1)}{n+1}\left[\left(\sum_{i=1}^{l}\lambda_i^2 - d\right)n + 2d^2 - d - \sum_{i=1}^{l}\lambda_i^2\right].
\end{aligned}$$

When $\ell(\lambda) = 1$, i.e., $\lambda = (d)$, it is easy to show that the above formula for $\psi_2\mathrm{fo}_\lambda$ still holds. □

Remark 5.1. *We can also use the differential operator D_n^2 to deduce formulas for $\psi_2 p_\lambda$, $\psi_2 e_\lambda$ and $\psi_2 h_\lambda$. The computation will be simpler than the case for* fo_λ*; we leave the proof to the reader.*

6. Final remarks

Based on Theorem 2.1 and the operator D_n^2, we derive nice formulas for $\psi_2(b_\lambda)$ when $b_\lambda \in \{s_\lambda, p_\lambda, e_\lambda, h_\lambda, \mathrm{fo}_\lambda\}$. It would be of interest if some nice formulas can still be obtained when applying ψ_2 to other symmetric functions. We will conclude this paper with a nice formula for $\psi_r(G_k^{(a,b,c)})$, where $G_k^{(a,b,c)}$ denotes a generalization of the (r,k)-parking symmetric functions introduced by Stanley and Wang [4].

Theorem 6.1. *Let a, b, r, k be positive integers, and let c be an indeterminate. Let*

$$H(t) = \sum_{n\geq 0} h_n t^n = \frac{1}{(1-x_1t)(1-x_2t)\cdots}$$

$$F_k^{(a,b)} = \frac{b}{ak+b}[t^k](H(t))^{ak+b}$$

and

$$G_k^{(a,b,c)} = [y^k]\left(\sum_{j=0}^{\infty} F_j^{(a,b)} y^j\right)^c.$$

Then we have

$$\psi_r(G_k^{(a,b,c)}) = (r-1)!bcn\binom{ak+bc+r-1}{r-1}\binom{(ak+bc)n+k-1}{k-r}. \tag{9}$$

Proof. It suffices to prove the theorem for positive integer c, since both sides of equation (9) are polynomials in c. Now let c be an positive integer, by the relation in [4, Theorem 3.1], we deduce that $G_k^{(a,b,c)} = F_k^{(a,bc)}$. So we only need to verify that

$$\psi_r(F_k^{(a,b)}) = (r-1)!bn\binom{ak+b+r-1}{r-1}\binom{(ak+b)n+k-1}{k-r}. \tag{10}$$

The remainder of the proof is just routine computation as follows.

$$\begin{aligned}
\psi_r(F_k^{(a,b)}) &= \frac{b}{ak+b}[t^k]\psi_r(H(t))^{ak+b} \\
&= \frac{r!bn}{ak+b}[t^k z^r]\vartheta(H(t))^{ak+b} \\
&= \frac{r!bn}{ak+b}[t^k z^r]\frac{1}{(1-(z+1)t)^{ak+b}(1-t)^{(ak+b)(n-1)}} \\
&= \frac{r!bn}{ak+b}\sum_{m=0}^{k-r}\binom{-(ak+b)(n-1)}{m}(-1)^m\binom{-(ak+b)}{k-m}(-1)^{k-m}\binom{k-m}{r} \\
&= \frac{r!bn}{ak+b}\sum_{m=0}^{k-r}\binom{-(ak+b)(n-1)}{m}(-1)^m\binom{ak+b+k-m-1}{k-m-r}\binom{ak+b+r-1}{r} \\
&= \binom{ak+b+r-1}{r}\frac{r!bn}{ak+b}\sum_{m=0}^{k-r}\binom{-(ak+b)(n-1)}{m}(-1)^m\binom{-(ak+b)-r}{k-m-r}(-1)^{k-m-r} \\
&= \binom{ak+b+r-1}{r}\frac{r!bn}{ak+b}(-1)^{k-r}\binom{-(ak+b)(n-1)-(ak+b+r)}{k-r} \\
&= \frac{r!bn}{ak+b}\binom{ak+b+r-1}{r}\binom{(ak+b)n+k-1}{k-r} \\
&= (r-1)!bn\binom{ak+b+r-1}{r-1}\binom{(ak+b)n+k-1}{k-r}.
\end{aligned}$$

□

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