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Authors
Antonio Peyrache and Tim Coelli

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School of Economics
University of Queensland
St. Lucia, Qld. 4072
Australia

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A Multiplicative Directional Distance Function

Antonio Peyrache*
and
Tim Coelli

School of Economics
University of Queensland, Brisbane, Australia.

Abstract

A Multiplicative Directional Distance Function (MDDF), which encompasses all commonly used multiplicative efficiency measures, is defined. We provide a discussion of its main properties (in particular scale invariance); a proof of a duality result involving the profitability ratio; and a set of equations that show how the MDDF and the existing (additive) Directional Distance Function (DDF) are mathematically related. The findings are then illustrated via a small numerical example.

Keywords: Hyperbolic Efficiency, Directional Distance Functions, Scale Invariance, Duality

* Corresponding author e-mail: a.peyrache@uq.edu.au

1. Introduction

Input, Output and Hyperbolic Distance Functions have been used for a number of decades (in efficiency analysis and production economics) to characterize technologies and estimate efficiency scores from input-output datasets. These efficiency measures involve a proportional (radial) contraction of inputs and/or expansion of outputs to the production frontier. More recently, Chambers et al. (1998) introduced the notion of a Directional Distance Function (DDF) where the projection onto the production frontier occurs on a pre-assigned (possibly non-radial) direction. Färe & Grosskopf (2000) showed that a number of the standard radial measures (Input, Output and Hyperbolic oriented) could be considered as special cases of the additive DDF (ADDF).

The ADDF is linear in nature and, as noted by Salnykov & Zelenyuk (2005), this fact leads to a structural lack of commensurability (i.e., scale invariance). As they emphasize, scale invariance is a very desirable property, since it avoids the danger that different researchers (using the same methodology and datasets) may obtain different results because they happen to define input and/or output measures using different units (e.g., specifying fertiliser in tonnes versus kilograms).

The duality of the ADDF to the profit function (see Chambers et al., 1998 and Färe & Grosskopf, 2000) gives rise to an additive decomposition of overall economic efficiency into technical and allocative components. Additive decompositions could be useful in application where the interest is in absolute profit; alternatively, when the focus is relative profit (i.e., profitability) the "return to the dollar" notion (the ratio of revenue over cost, see Färe et al., 2002) is a more suitable notion.

In this study we contribute to this literature by introducing the notion of a Multiplicative Directional Distance Function (MDDF) as the multiplicative counterpart to the ADDF. Some special cases of the MDDF include the Input, Output and Hyperbolic distance functions, as well as the modified hyperbolic distance function discussed in Cuesta et al. (2007) (which allows one to consider bad outputs in the production process). We show that the MDDF is multiplicative in nature, it satisfies commensurability (scale invariance) and that it is dual to the “return to the dollar”. The duality result permits a multiplicative decomposition of overall efficiency (into allocative and technical components) instead of the additive decomposition reached with ADDF.

The remainder of this paper is organized into sections. In section 2 we define the Multiplicative Directional Distance Function. Section 3 covers duality properties, while Section 4 is dedicated to the establishment of a formal connection between the ADDF and the MDDF. A numerical example is provided in section 5, with Section 6 containing some concluding remarks.

2. Definitions and Properties

A production process that produces M outputs ($\mathbf{y} \in R^M$) by means of N inputs ($\mathbf{x} \in R^N$) may be represented with the *production set* as

$$T = \left\{ (\mathbf{x}, \mathbf{y}) \in R^{N+M} : \mathbf{x} \; can \;\; produce \; \mathbf{y} \right\}. \tag{1}$$

We assume that the following standard regularity conditions apply to the production set:

(H1) *No free lunch*: $(\mathbf{x},\mathbf{0}) \in T$ and $(\mathbf{0},\mathbf{y}) \in T \;\; \Rightarrow \mathbf{y} = \mathbf{0}$;

(H2) the Production Set is Closed;

(H3) the Production Set is bounded;

(H4) *Strong disposability*: if $(\mathbf{x},\mathbf{y}) \in T$ then $(\mathbf{x}',\mathbf{y}') \in T$ for each $(\mathbf{x}' \geq \mathbf{x}) \cap (\mathbf{y}' \leq \mathbf{y})$;

(H5) *Convexity*: if $(\mathbf{x},\mathbf{y}),(\mathbf{x}',\mathbf{y}') \in T$ then $\{\alpha x + (1-\alpha)x', \alpha y' + (1-\alpha)y'\} \in T$.

The Multiplicative Directional Distance Function (MDDF) is defined as

$$H(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{q}) = \inf\left\{ \theta > 0 : \left(\theta^{\mathbf{Z}}\mathbf{x}, \theta^{-\mathbf{Q}}\mathbf{y} \right) \in T \right\} \tag{2}$$

where $\mathbf{z} \in R^N$ and $\mathbf{q} \in R^M$ are two exogenously given vectors that we call the *orientation vector* and $\mathbf{Z} = diag(\mathbf{z})$, $\mathbf{Q} = diag(\mathbf{q})$ are the diagonal matrices built with **z**, **q**.

Some properties of the MDDF are as follows.

P1. The MDDF characterizes the technology: $0 < H(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{q}) \leq 1$ if and only if $(\mathbf{x},\mathbf{y}) \in T$.

P2. The MDDF is equal to 1 for efficient points.

P3. The MDDF is *almost homogeneous* of degree $(-\mathbf{z},\mathbf{q},1)$ in $(\mathbf{x},\mathbf{y})$:

$$H\left(\mu^{-\mathbf{Z}}\mathbf{x},\mu^{\mathbf{Q}}\mathbf{y},\mathbf{z},\mathbf{q}\right) = \mu H(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{q}).$$

<u>*Proof:*</u>

Following Cuesta & Zofio (2005) and Cuesta et al. (2007), we define a function $F(x,y)$ as almost homogeneous of degree (k_1,k_2,k_3) if

$$F\left(\mu^{k_1}x,\mu^{k_2}y\right) = \mu^{k_3}F(x,y),$$

where the parameters (k_1,k_2,k_3) are given. Using this, we have

$$H\left(\mu^{-\mathbf{Z}}\mathbf{x},\mu^{\mathbf{Q}}\mathbf{y},\mathbf{z},\mathbf{q}\right) = \inf\left\{\theta > 0 : \left(\theta^{\mathbf{Z}}\mu^{-\mathbf{Z}}\mathbf{x},\theta^{-\mathbf{Q}}\mu^{\mathbf{Q}}\mathbf{y}\right) \in T\right\} =$$
$$= \mu \inf\left\{\frac{\theta}{\mu} > 0 : \left(\left(\frac{\theta}{\mu}\right)^{\mathbf{Z}}\mathbf{x},\left(\frac{\theta}{\mu}\right)^{-\mathbf{Q}}\mathbf{y}\right) \in T\right\} = \mu H(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{q}).$$

□

P4. The MDDF is *homogeneous* of degree –1 in $(\mathbf{z},\mathbf{q})$:

$$H(\mathbf{x},\mathbf{y},\lambda\mathbf{z},\lambda\mathbf{q}) = H(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{q})^{1/\lambda}, \ \lambda > 0;$$

<u>*Proof:*</u>

$$H(\mathbf{x},\mathbf{y},\lambda\mathbf{z},\lambda\mathbf{q}) = \inf\{\theta > 0 : (\theta^{\lambda\mathbf{Z}}\mathbf{x}, \theta^{-\lambda\mathbf{Q}}\mathbf{y}) \in T\} =$$

$$[\inf\{\theta^{\lambda} > 0 : (\theta^{\lambda\mathbf{Z}}\mathbf{x}, \theta^{-\lambda\mathbf{Q}}\mathbf{y}) \in T\}]^{1/\lambda} = H(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{q})^{1/\lambda}.$$

□

P5. MDDF is scale invariant (commensurable) [1]: $H(\mathbf{\Omega}_x\mathbf{x}, \mathbf{\Omega}_y\mathbf{y}, \mathbf{z}, \mathbf{q}) = H(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{q})$.

<u>*Proof:*</u> Let $\tilde{\mathbf{x}} = \mathbf{\Omega}_x\mathbf{x}$ and $\tilde{\mathbf{y}} = \mathbf{\Omega}_y\mathbf{y}$ be a transformation of the input-output vector, where $\mathbf{\Omega}_x, \mathbf{\Omega}_y$ are two diagonal matrices with strictly positive constants on the diagonal (these matrices are a re-scaling of the original input-output space). The transformed technology can be defined as

$$\tilde{T} = \{(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) : (\mathbf{x},\mathbf{y}) \in T\} = \{(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) : (\mathbf{\Omega}_x^{-1}\tilde{\mathbf{x}}, \mathbf{\Omega}_y^{-1}\tilde{\mathbf{y}}) \in T\}.$$

Using definition (2) and the definition of the transformed technology we can write

$$H(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \mathbf{z}, \mathbf{q}) = \inf\{\theta > 0 : (\theta^{\mathbf{Z}}\tilde{\mathbf{x}}, \theta^{-\mathbf{Q}}\tilde{\mathbf{y}}) \in \tilde{T}\} =$$

$$= \inf\{\theta > 0 : (\theta^{\mathbf{Z}}\mathbf{\Omega}_x^{-1}\tilde{\mathbf{x}}, \theta^{-\mathbf{Q}}\mathbf{\Omega}_y^{-1}\tilde{\mathbf{y}}) \in T\} =$$

$$= \inf\{\theta > 0 : (\theta^{\mathbf{Z}}\mathbf{x}, \theta^{-\mathbf{Q}}\mathbf{y}) \in T\} = H(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{q})$$

□

Special cases:

The traditional input and output distance functions can be obtained by placing restrictions on the orientation vector $(\mathbf{z},\mathbf{q})$. The input distance function is obtained using the vector $(\mathbf{z},\mathbf{q}) = (\mathbf{1},\mathbf{0})$ and the almost homogeneity condition (**P3**) becomes a homogeneity condition of degree -1 in inputs:

[1] This is an important property that the MDDF satisfies while the ADDF does not.

$$H(\mu\mathbf{x},\mathbf{y},\mathbf{z}=\mathbf{1},\mathbf{q}=\mathbf{0}) = \frac{1}{\mu}\cdot H(\mathbf{x},\mathbf{y},\mathbf{z}=\mathbf{1},\mathbf{q}=\mathbf{0})$$

The output distance function is obtained if we choose $(\mathbf{z},\mathbf{q})=(\mathbf{0},\mathbf{1})$. The almost homogeneity condition (**P3**) then becomes a homogeneity condition of degree 1 in outputs:

$$H(\mathbf{x},\mu\mathbf{y},\mathbf{z}=\mathbf{0},\mathbf{q}=\mathbf{1}) = \mu\cdot H(\mathbf{x},\mathbf{y},\mathbf{z}=\mathbf{0},\mathbf{q}=\mathbf{1}).$$

Finally, if we choose $(\mathbf{z},\mathbf{q})=(\mathbf{1},\mathbf{1})$ we obtain the Hyperbolic Distance Function or graph efficiency measure. This function is almost homogeneous of degree $(-\mathbf{1},\mathbf{1},1)$

$$H(\mu^{-1}\mathbf{x},\mu\mathbf{y},\mathbf{z}=\mathbf{1},\mathbf{q}=\mathbf{1}) = \mu\cdot H(\mathbf{x},\mathbf{y},\mathbf{z}=\mathbf{1},\mathbf{q}=\mathbf{1}).$$

3. Duality

Färe et al. (2002) and Färe & Grosskopf (2004) pointed out that the Graph Efficiency Measure (i.e., the hyperbolic distance function) is dual to the "Return to the Dollar" when the maximum feasible profit is zero. In efficiency measurement, duality results play an important role in permitting an economic interpretation of the efficiency indexes, therefore it is important to establish a dual theorem for the MDDF.

Let $\mathbf{w}\in R_+^N$ be the vector of input prices and $\mathbf{p}\in R_+^M$ the vector of output prices. The profit function is

$$\Pi(\mathbf{p},\mathbf{w}) = \sup_{(\mathbf{x},\mathbf{y})}\{\mathbf{py}-\mathbf{wx} : (\mathbf{x},\mathbf{y})\in T\} \tag{3}$$

Under CRS, the maximum feasible profit is zero, therefore the maximum "return to the dollar" $\frac{\mathbf{py}}{\mathbf{wx}}$ is one. Given that the profit function cannot be smaller than the profit associated with any feasible production plan in the production set, it is also true that

$$\Pi(\mathbf{p},\mathbf{w}) \geq \mathbf{p}H^{-\mathbf{Q}}\mathbf{y} - \mathbf{w}H^{\mathbf{Z}}\mathbf{x}. \tag{4}$$

Furthermore, given that under CRS the maximum feasible profit is zero, we have $0 \geq \mathbf{p}H^{-\mathbf{Q}}\mathbf{y} - \mathbf{w}H^{\mathbf{Z}}\mathbf{x}$. Then rearranging terms and multiplying by the "return to the dollar" one obtains:

$$\frac{\mathbf{py}}{\mathbf{wx}} \leq \frac{\mathbf{py}}{\mathbf{wx}} \cdot \frac{\mathbf{w}H^{\mathbf{Z}}\mathbf{x}}{\mathbf{p}H^{-\mathbf{Q}}\mathbf{y}}. \tag{5}$$

We note that the right-hand side is the actual loss in profitability due to technical inefficiency. Defining the following exponential factor

$$\Delta = \frac{1}{\ln H}\left[\ln\left(\mathbf{w}H^{\mathbf{Z}}\mathbf{x}\right) - \ln(\mathbf{wx}) + \ln(\mathbf{py}) - \ln\left(\mathbf{p}H^{-\mathbf{Q}}\mathbf{y}\right)\right] \tag{6}$$

it is easy to show that $[H(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{q})]^{\Delta} = \frac{\mathbf{py}}{\mathbf{wx}} \cdot \frac{\mathbf{w}H^{\mathbf{Z}}\mathbf{x}}{\mathbf{p}H^{-\mathbf{Q}}\mathbf{y}}$ and the duality theorem for the MDDF can be expressed as

$$\frac{\mathbf{py}}{\mathbf{wx}} \leq [H(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{q})]^{\Delta}. \tag{7}$$

The duality of the MDDF to the "return to the dollar", supports the interpretation of the MDDF as a "loss in profitability measure" (once the adjustment exponential factor Δ has been taken into account). In the case of the hyperbolic distance function ($\mathbf{z} = \mathbf{1}, \mathbf{q} = \mathbf{1}$) the exponential factor becomes independent of prices: $\Delta = 2$ (see Färe et al., 2002). If price information is available, using the exponential factor it is possible to have a measure of technical efficiency that is independent from any rescaling of the orientation vector (see **P4**). In fact, using **P4**, we see that:

$$[H(\mathbf{x},\mathbf{y},\lambda\mathbf{z},\lambda\mathbf{q})]^{\Delta} = \frac{\mathbf{py}}{\mathbf{wx}} \cdot \frac{\mathbf{w}H(\mathbf{x},\mathbf{y},\lambda\mathbf{z},\lambda\mathbf{q})^{\lambda\mathbf{Z}}\mathbf{x}}{\mathbf{p}H(\mathbf{x},\mathbf{y},\lambda\mathbf{z},\lambda\mathbf{q})^{-\lambda\mathbf{Q}}\mathbf{y}} =$$
$$= \frac{\mathbf{py}}{\mathbf{wx}} \cdot \frac{\mathbf{w}H(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{q})^{\frac{1}{\lambda}\lambda\mathbf{Z}}\mathbf{x}}{\mathbf{p}H(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{q})^{-\frac{1}{\lambda}\lambda\mathbf{Q}}\mathbf{y}} = \frac{\mathbf{py}}{\mathbf{wx}} \cdot \frac{\mathbf{w}H(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{q})^{\mathbf{Z}}\mathbf{x}}{\mathbf{p}H(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{q})^{-\mathbf{Q}}\mathbf{y}} = [H(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{q})]^{\Delta}$$

This finding is not surprising since, when price information is available, the efficiency measure $[H(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{q})]^{\Delta}$ is the actual loss in profitability due to technical inefficiency. It follows that market prices give us the opportunity to choose a very natural normalization for the MDDF.

Using this duality result it is possible to (multiplicatively) decompose overall efficiency, $OE = \frac{\mathbf{py}}{\mathbf{wx}}$, into a technical efficiency component, $TE = [H(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{q})]^{\Delta}$, and a residual allocative efficiency component, $AE = \frac{OE}{TE}$:

$$OE = AE \cdot TE . \tag{8}$$

As emphasized by Färe et al. (2002) the "return to the dollar" is bounded between zero and one only under CRS. Deviations from CRS imply that the allocative component can take any value, with a value of one signalling allocative efficiency. Moreover, if CRS doesn't hold, the production plan that maximizes the "return to the dollar" could be different from the one that maximizes absolute profit (the profit function).

4. Connections between the ADDF and the MDDF

Following Chambers et al (1998) and Färe & Grosskopf (2000, 2004), we define the ADDF as

$$D(\mathbf{x},\mathbf{y},\mathbf{g}_x,\mathbf{g}_y) = \sup\{\beta : (\mathbf{x} - \beta\mathbf{g}_x, \mathbf{y} + \beta\mathbf{g}_y) \in T\}, \tag{9}$$

where the $(N+M)\times 1$ exogenously given *direction vector* $(\mathbf{g}_x, \mathbf{g}_y)$ specifies the direction in which the input-output vector is projected onto the production frontier. The slack vector[2] associated to optimization problem (9) is $(\mathbf{s}_x, \mathbf{s}_y) = \beta \cdot (\mathbf{g}_x, \mathbf{g}_y)$, or equivalently:

[2] The slack vector describes the absolute change in the input-output vector for it to reach the frontier.

$$\begin{cases} s_{xi} = Dg_{xi}\ , & i = 1,\dots,N \\ s_{yj} = Dg_{yj}\ , & j = 1,\dots,M \end{cases} \tag{10}$$

For the case of the MDDF, we use definition (2) to define the slack vector as

$$\begin{cases} s_{xi} = \left(1 - H^{z_i}\right)x_i\ , & i = 1,\dots,N \\ s_{yj} = \left(H^{-q_j} - 1\right)y_j\ , & j = 1,\dots,M. \end{cases} \tag{11}$$

Combining equations (10) and (11) we obtain

$$\begin{cases} g_{xi} = \dfrac{\left(1 - H^{z_i}\right)x_i}{D}, & i = 1,\dots,N \\ g_{yj} = \dfrac{\left(H^{-q_j} - 1\right)y_j}{D}, & j = 1,\dots,M, \end{cases} \tag{12}$$

which after rearrangement, becomes

$$\begin{cases} z_i = \ln\left(\dfrac{x_i}{Hg_{xi}}\right), & i = 1,\dots,N \\ q_j = \ln\left(\dfrac{Hg_{yj}}{y_j}\right), & j = 1,\dots,M. \end{cases} \tag{13}$$

Here we see that with $\left(\mathbf{g}_x, \mathbf{g}_y\right)$ fixed, the values of the vector $\left(\mathbf{z}, \mathbf{q}\right)$ depend both on the value of the input-output vector and the measure of distance. By considering $\left(\mathbf{g}_x, \mathbf{g}_y\right)$ as fixed we are imposing a translation invariance property upon the MDDF, meaning that we cannot then choose a fixed orientation vector $\left(\mathbf{z}, \mathbf{q}\right)$. Conversely, from equation (12) we note that when the orientation vector $\left(\mathbf{z}, \mathbf{q}\right)$ is fixed, the directional vector $\left(\mathbf{g}_x, \mathbf{g}_y\right)$ depends both on the value of the input-output vector and the distance from the frontier. With the fixity of $(\mathbf{z},\mathbf{q})$ we impose an *almost homogeneity* property on the MDDF measure, meaning that the ADDF requires a non-fixed directional vector $(\mathbf{g}_x,\mathbf{g}_y)$.

5. Numerical Example

In this numerical example we consider (for ease of exposition) only those projections which are orthogonal to the output set ($\mathbf{g}_y = \mathbf{0}$; $\mathbf{q} = \mathbf{0}$), and a constant returns to scale technology. It follows that we can focus our attention on the production coefficients ($a_1 = {}^{x_1}\!/_{y}$; $a_2 = {}^{x_2}\!/_{y}$) when considering a simple two-input, one-output example. Our example is presented in Table 1 and Figure 1, where points $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ are efficient (they define the convex production frontier), while point $\mathbf{a}_0$ is inefficient.

To begin, let us consider the standard input oriented distance function. To represent this function using the MDDF we impose $\mathbf{z} = \mathbf{1}$, while for the ADDF the elements of the corresponding directional vector are $g_{xi} = \frac{x_i}{\sqrt{\sum_i x_i^2}}$. With radial projection, points $\mathbf{a}_2$ and $\mathbf{a}_3$ are the peers of the inefficient observation. The efficiency target for $\mathbf{a}_0$ is $\mathbf{a}_0^E = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}$, the associated slack vector is $\mathbf{s} = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}$, the MDDF is 0.5 while the ADDF is $\frac{5\sqrt{2}}{2}$. The efficiency target can be found by solving optimization problem (2) or alternatively problem (9).

Now, let us choose a different orientation vector, such as $\mathbf{z} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The peers for the inefficient observation are now $\mathbf{a}_3$ and $\mathbf{a}_4$, and the new efficiency target now has to lie on the convex linear combination

$$\mathbf{a}_0^{E*} = \alpha \mathbf{a}_3 + (1-\alpha)\mathbf{a}_4 = \begin{bmatrix} 5-2\alpha \\ 1+\alpha \end{bmatrix}$$

Moreover we know that $\mathbf{a}_0^{E*} = H^Z \mathbf{a}_0$ where H is unknown. Thus we can write the following system of equations:

$$\begin{cases} 5-2\alpha = 5H \\ 1+\alpha = 5H^2 \end{cases}$$

Solving this system we obtain the MDDF measure, the efficiency target and hence the slacks: $H \cong 0.623$, $\alpha \cong 0.942$, $s_1 \cong 1.884$, $s_2 \cong 3.058$. The ADDF information can then be found using the slacks: $D = 3.592$, $g_1 = 0.525$, $g_2 = 0.851$. If we now use this directional vector $g_x = (0.525, 0.851)$ we can obtain the same efficiency target by using optimization problem (9).

6. Conclusions

In this study we provide a formal definition of the Multiplicative Directional Distance Function (MDDF) and its properties. We show that the MDDF satisfies commensurability (scale invariance). Moreover, we show that the MDDF is dual to the "return to the dollar" and this duality allows one to decompose (multiplicatively) overall efficiency into a technical efficiency component and a residual allocative efficiency component. In addition, we derive a formal connection between the MDDF and the ADDF. A numerical example is used to illustrate the findings of the paper.

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Table 1: Production coefficients of five observations

	a_1	a_2
$\mathbf{a}_0$	5	5
$\mathbf{a}_1$	1	5
$\mathbf{a}_2$	2	3
$\mathbf{a}_3$	3	2
$\mathbf{a}_4$	5	1

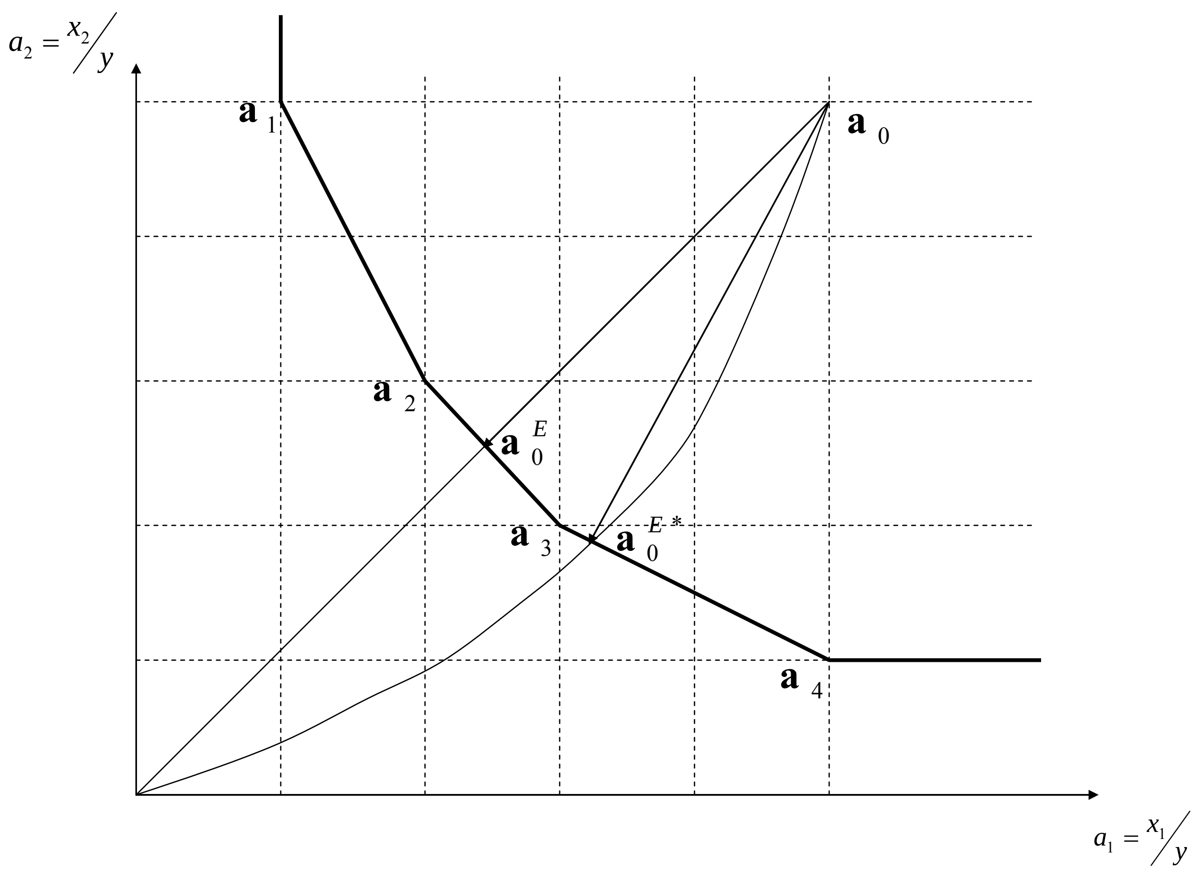

Figure 1: Numerical example