Alternating Permutations and Symmetric Functions

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Abstract

We use the theory of symmetric functions to enumerate various classes of alternating permutations w of $\{1, 2, \ldots, n\}$. These classes include the following: (1) both w and w^{-1} are alternating, (2) w has certain special shapes, such as $(m-1, m-2, \ldots, 1)$, under the RSK algorithm, (3) w has a specified cycle type, and (4) w has a specified number of fixed points. We also enumerate alternating permutations of a multiset. Most of our formulas are umbral expressions where after expanding the expression in powers of a variable E, E^k is interpreted as the Euler number E_k. As a small corollary, we obtain a combinatorial interpretation of the coefficients of an asymptotic expansion appearing in Ramanujan's Lost Notebook.

1 Introduction.

This paper can be regarded as a sequel to the classic paper [6] of H. O. Foulkes in which he relates the enumeration of alternating permutations to the representation theory of the symmetric group and the theory of symmetric functions. We assume familiarity with symmetric functions as presented in [17, Ch. 7]. Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $1, 2, \ldots, n$. A permutation $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ is *alternating* if $a_1 > a_2 < a_3 > a_4 < \cdots$. Equivalently, write $[m] = \{1, 2, \ldots, m\}$ and define the *descent set* $D(w)$ of $w \in \mathfrak{S}_n$ by

$$D(w) = \{i \in [n-1] \,:\, a_i > a_{i+1}\}.$$

Then w is alternating if $D(w) = \{1, 3, 5, \dots\} \cap [n-1]$. Similarly, define w to be *reverse alternating* if $a_1 < a_2 > a_3 < a_4 > \cdots$. Thus w is reverse alternating if $D(w) = \{2, 4, 6, \dots\} \cap [n-1]$. Also define the *descent composition* $\mathrm{co}(w)$ by

$$\mathrm{co}(w) = (\alpha_1, \alpha_2, \dots, \alpha_k), \tag{1}$$

where $D(w) = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \cdots + \alpha_{k-1}\}$ and $\sum \alpha_i = n$. Thus $\alpha \in \mathrm{Comp}(n)$, where $\mathrm{Comp}(n)$ denotes the set of compositions of n.

Let E_n denote the number of alternating permutations in $\mathfrak{S}_n$. Then E_n is called an *Euler number* and was shown by D. André [1] to satisfy

$$\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x. \tag{2}$$

(Sometimes one defines $\sum (-1)^n E_n x^{2n}/(2n)! = \sec x$, but we will adhere to (2).) Thus E_{2m} is also called a *secant number* and E_{2m+1} a *tangent number*. The bijection $w \mapsto w'$ on $\mathfrak{S}_n$ defined by $w'(i) = n + 1 - w(i)$ shows that E_n is also the number of reverse alternating permutations in $\mathfrak{S}_n$. However, for some of the classes of permutations considered below, alternating and reverse alternating permutations are not equinumerous.

Foulkes defines a certain (reducible) representation of $\mathfrak{S}_n$ whose dimension is E_n. He shows how this result can be used to compute E_n and other numbers related to alternating permutations, notably the number of $w \in \mathfrak{S}_n$ such that both w and w^{-1} are alternating. Foulkes' formulas do not give a "useful" computational method since they involve sums over partitions whose terms involve Littlewood-Richardson coefficients. We show how Foulkes' results can actually be converted into useful generating functions for computing such numbers as (a) the number of alternating permutations $w \in \mathfrak{S}_n$ with conditions on their cycle type (or conjugacy class). The special case of enumerating alternating involutions was first raised by Ehrenborg and Readdy and discussed further by Zeilberger [18]. Another special case is that of alternating permutations with a specified number of fixed points. Our proofs use, in addition to Foulkes' representation, a result of Gessel and Reutenauer [9] on permutations with given descent set and cycle type. (b) The number of $w \in \mathfrak{S}_n$ such that both w and w^{-1} are alternating, or such that w is alternating and w^{-1} is reverse alternating. (c) The number of alternating permutations of certain shapes (under the RSK algorithm). (d) The number of alternating permutations of a multiset of integers, under various interpretations of the term "alternating."

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2 The work of Foulkes.

We now review the results of Foulkes that will be the basis for our work. Given a composition α of n, let B_α denote the corresponding border strip (or ribbon or skew hook) shape as defined e.g. in [5][17, p. 383]. Let s_{B_α} denote the skew Schur function of shape B_α. The following result of Foulkes [5, Thm. 6.2] also appears in [17, Cor. 7.23.8].

Theorem 2.1. *Let α and β be compositions of n. Then*

$$\langle s_{B_\alpha}, s_{B_\beta} \rangle = \#\{w \in \mathfrak{S}_n : \mathrm{co}(w) = \beta,\ \mathrm{co}(w^{-1}) = \alpha\}.$$

We let $\tau_n = B_\alpha$ where $\alpha = (1, 2, 2, \ldots, 2, j) \in \mathrm{Comp}(n)$, where $j = 1$ if n is even and $j = 2$ if n is odd. Thus if $'$ indicates conjugation (reflection of the shape about the main diagonal), then $\tau'_{2k+1} = \tau_{2k+1}$, while $\tau'_{2k} = (2, 2, \ldots, 2)$. We want to expand the skew Schur functions s_{τ_n} and $s_{\tau'_n}$ in terms of power sum symmetric functions. For any skew shape λ/μ with n squares, let $\chi^{\lambda/\mu}$ denote the character of $\mathfrak{S}_n$ satisfying $\mathrm{ch}(\chi^{\lambda/\mu}) = s_{\lambda/\mu}$. Thus by the definition [17, p. 351] of ch we have

$$s_{\lambda/\mu} = \sum_{\rho \vdash n} z_\rho^{-1} \chi^{\lambda/\mu}(\rho) p_\rho,$$

where $\chi^{\lambda/\mu}(\rho)$ denotes the value of $\chi^{\lambda/\rho}$ at any permutation $w \in \mathfrak{S}_n$ of cycle type ρ.

The main result [6, Thm. 6.1][17, Exer. 7.64] of Foulkes on the connection between alternating permutations and representation theory is the following.

Theorem 2.2. (a) *Let $\mu \vdash n$, where $n = 2k+1$. Then*

$$\chi^{\tau_n}(\mu) = \chi^{\tau'_n}(\mu) = \begin{cases} 0, & \text{if } \mu \text{ has an even part} \\ (-1)^{k+r}E_{2r+1}, & \text{if } \mu \text{ has } 2r+1 \text{ odd parts and} \\ & \text{no even parts.} \end{cases}$$

(b) *Let $\mu \vdash n$, where $n = 2k$. Suppose that μ has $2r$ odd parts and e even parts. Then*

$$\begin{aligned} \chi^{\tau_n}(\mu) &= (-1)^{k+r+e}E_{2r} \\ \chi^{\tau'_n}(\mu) &= (-1)^{k+r}E_{2r}. \end{aligned}$$

NOTE. Foulkes obtains his result from the Murnaghan-Nakayama rule. It can also be also be obtained from the formula

$$\sum_{n\geq 0} s_{\tau_n}t^n = \frac{1}{\sum_{n\geq 0}(-1)^n h_{2n}t^{2n}} + \frac{\sum_{n\geq 0}(-1)^n h_{2n+1}t^{2n+1}}{\sum_{n\geq 0}(-1)^m h_{2n}t^{2n}},$$

where s_{τ_n} denotes a skew Schur function. This formula is due to Carlitz [4] and is also stated at the bottom of page 520 of [17].

Foulkes' result leads immediately to our main tool in what follows. Throughout this paper we will use *umbral notation* [15] for Euler numbers. In other words, any polynomial in E is to be expanded in terms of powers of E, and then E^k is replaced by E_k. The replacement of E^k by E_k is always the *last* step in the evaluation of an umbral expression. For instance,

$$(E^2-1)^2 = E^4 - 2E^2 + 1 = E_4 - 2E_2 + 1 = 5 - 2\cdot 1 + 1 = 4.$$

Similarly,

$$
\begin{aligned}
(1+t)^E &= 1+Et+\binom{E}{2}t^2+\binom{E}{3}t^3+\cdots \\
&= 1+Et+\frac{1}{2}(E^2-E)t^2+\frac{1}{6}(E^3-3E^2+2E)t^3+\cdots \\
&= 1+Et+\frac{1}{2}(E_2-E_1)t^2+\frac{1}{6}(E_3-3E_2+2E_1)t^3+\cdots \\
&= 1+1\cdot t+\frac{1}{2}(1-1)t^2+\frac{1}{6}(2-3\cdot 1+2\cdot 1)t^3+\cdots \\
&= 1+t+\frac{1}{6}t^3+\cdots .
\end{aligned}
$$

If $f = f(x_1, x_2, \dots)$ is a symmetric function then we use the notation $f[p_1, p_2, \dots]$ for f regarded as a polynomial in the power sums. For instance, if $f = e_2 = \sum_{i<j} x_i x_j = \frac{1}{2}(p_1^2 - p_2)$ then

$$e_2[E, -E, \dots] = \frac{1}{2}(E^2+E) = 1.$$

Theorem 2.3. *Let f be a homogenous symmetric function of degree n. If n is odd then*

$$\langle f, s_{\tau_n}\rangle = \langle f, s_{\tau_n'}\rangle = f[E,0,-E,0,E,0,-E,\dots] \tag{3}$$

If n is even then

$$
\begin{aligned}
\langle f, s_{\tau_n}\rangle &= f[E,-1,-E,1,E,-1,-E,1,\dots] \\
\langle f, s_{\tau_n'}\rangle &= f[E,1,-E,-1,E,1,-E,-1,\dots].
\end{aligned}
$$

Proof. Suppose that $n = 2k+1$. Let $\mathrm{OP}(n)$ denote the set of all partitions of n into odd parts. If $\mu \in \mathrm{OP}(n)$ and μ has $\ell(\mu) = 2r+1$ (odd) parts, then write $r = r(\mu)$. Let $f = \sum_{\lambda \vdash n} c_\lambda p_\lambda$. Then by Theorem 2.2 we have

$$
\begin{aligned}
\langle f, s_{\tau_n}\rangle &= \left\langle \sum_\lambda c_\lambda p_\lambda, \sum_{\mu \in \mathrm{OP}(n)} z_\mu^{-1}(-1)^{k+r(\mu)} E_{\ell(\mu)} p_\mu \right\rangle \\
&= \sum_{\mu \in \mathrm{OP}(n)} c_\mu (-1)^{k+r(\mu)} E_{\ell(\mu)}.
\end{aligned}
$$

If $\mu \in \mathrm{OP}(n)$ and we substitute $(-1)^j E$ for p_{2j+1} in p_μ then we obtain

$$\begin{aligned}\prod_{i=1}^{\ell(\mu)} (-1)^{\frac{1}{2}(\mu_i-1)} E &= (-1)^{\frac{1}{2}(2k+1-(2r(\mu)+1))} E^{\ell(\mu)} \\ &= (-1)^{k+r(\mu)} E^{\ell(\mu)},\end{aligned}$$

and equation (3) follows. The case of n even is analogous. $\square$

3 Inverses of alternating permutations.

In this section we derive generating functions for the number of alternating permutations in $\mathfrak{S}_n$ whose inverses are alternating or reverse alternating. This problem was considered by Foulkes [6, §5], but his answer does not lend itself to easy computation. Such "doubly alternating" permutations were also considered by Ouchterlony [14] in the setting of pattern avoidance. A special class of doubly alternating permutations, viz., those that are Baxter permutations, were enumerated by Guibert and Linusson [10].

Theorem 3.1. *Let $f(n)$ denote the number of permutations $w \in \mathfrak{S}_n$ such that both w and w^{-1} are alternating, and let $f^*(n)$ denote the number of $w \in \mathfrak{S}_n$ such that w is alternating and w^{-1} is reverse alternating. Let*

$$\begin{aligned}L(t) &= \frac{1}{2}\log\frac{1+t}{1-t} \\ &= t + \frac{t^3}{3} + \frac{t^5}{5} + \cdots .\end{aligned}$$

Then

$$\sum_{k\geq 0} f(2k+1)t^{2k+1} = \sum_{r\geq 0} E_{2r+1}^2 \frac{L(t)^{2r+1}}{(2r+1)!} \tag{4}$$

$$f^*(2k+1) = f(2k+1) \tag{5}$$

$$\sum_{k\geq 0} f(2k)t^{2k} = \frac{1}{\sqrt{1-t^2}} \sum_{r\geq 0} E_{2r}^2 \frac{L(t)^{2r}}{(2r)!} \tag{6}$$

$$f^*(2k) = f(2k) - f(2k-2). \tag{7}$$

Proof. By Theorem 2.1 we have $f(n) = \langle s_{\tau_n}, s_{\tau_n}\rangle$. Let $n = 2k+1$. Then it follows from Theorems 2.2 and 2.3 that (writing $r = r(\mu)$)

$$\begin{aligned} f(n) &= \sum_{\mu\in \mathrm{OP}(n)} z_\mu^{-1}(-1)^{k+r}E_{2r+1}(-1)^{k+r}E^{2r+1} \\ &= \sum_{\mu\in \mathrm{OP}(n)} z_\mu^{-1}E_{2r+1}^2. \end{aligned} \tag{8}$$

Now by standard properties of exponential generating functions [17, §5.1] or by specializing the basic identity

$$\sum_\lambda z_\lambda^{-1}p_\lambda = \exp \sum_{n\geq 1} \frac{1}{n} p_n,$$

we have

$$\begin{aligned} \sum_{k\geq 0}\sum_{\mu\in \mathrm{OP}(2k+1)} z_\mu^{-1}y^{\ell(\mu)}t^{2k+1} &= \exp\left(y\left(t+\frac{t^3}{3}+\frac{t^5}{5}+\cdots\right)\right) \\ &= \exp(yL(t)). \end{aligned}$$

The coefficient of y^{2r+1} in the above generating function is therefore $L(t)^{2r+1}/(2r+1)!$, and the proof of (4) follows.

Since $\tau_n = \tau_n'$ for n odd we have

$$f^*(n) = \langle s_{\tau_n}, s_{\tau_n'}\rangle = \langle s_{\tau_n}, s_{\tau_n}\rangle = f(n),$$

so (5) follows.

The argument for $n = 2k$ is similar. For $\mu \vdash n$ let $e = e(\mu)$ denote the number of even parts of μ and $2r = 2r(\mu)$ the number of odd parts. Now the relevant formulas for computing $f(n)$ are

$$\begin{aligned} f(n) &= \sum_{\mu \vdash n} z_\mu^{-1}(-1)^{k+r+e} E_{2r}(-1)^{k+r+e} E^{2r} \\ &= \sum_{\mu \vdash n} z_\mu^{-1} E_{2r}^2 \end{aligned}$$

and

$$\begin{aligned} \sum_{k \geq 0} \sum_{\mu \vdash n} z_\mu^{-1} y^{2r(\mu)} t^n &= \exp\left(y\left(t + \frac{t^3}{3} + \frac{t^5}{5} + \cdots \right) + \left(\frac{t^2}{2} + \frac{t^4}{4} + \cdots \right) \right) \\ &= (1 - t^2)^{-1/2} (\exp(yL(t)), \end{aligned}$$

from which (6) follows.

For the case $f^*(n)$ when n is even we have

$$\begin{aligned} f^*(n) &= \sum_{\mu \vdash n} z_\mu^{-1}(-1)^{k+r+e} E_{2r}(-1)^{k+r} E^{2r} \\ &= \sum_{\mu \vdash n} z_\mu^{-1}(-1)^e E_{2r}^2 \end{aligned}$$

and

$$\begin{aligned} \sum_{k \geq 0} \sum_{\mu \vdash 2k} z_\mu^{-1}(-1)^{e(\mu)} y^{2r(\mu)} t^{2k} &= \exp\left(y\left(t + \frac{t^3}{3} + \frac{t^5}{5} + \cdots \right) \right. \\ &\qquad \left. - \left(\frac{t^2}{2} + \frac{t^4}{4} + \cdots \right) \right) \\ &= \sqrt{1 - t^2} \exp yL(t). \end{aligned}$$

Hence

$$\sum_{k \geq 0} f^*(2k) t^{2k} = (1 - t^2) \sum_{k \geq 0} f(2k) t^{2k},$$

from which (7) follows. $\square$

Whenever we have explicit formulas or generating functions for combinatorial objects we can ask for combinatorial proofs of them. Bruce Sagan has pointed out that equation (5) follows from reversing the permutation, i.e., changing $a_1a_2\cdots a_n$ to $a_n\cdots a_2a_1$. We do not know combinatorial proofs of equations (4), (6) and (7). To prove equations (4) and (6) combinatorially, we probably need to interpret them as exponential generating functions, e.g., write the left-hand side of (4) as $\sum_{k\geq 0}(2k+1)!f(2k+1)t^{2k+1}/(2k+1)!$. Let us also note that if $g(n)$ denotes the number of reverse alternating $w\in\mathfrak{S}_n$ such that w^{-1} is also reverse alternating, then $f(n)=g(n)$ for all n. This fact can be easily shown using the proof method above, and it is also a consequence of the RSK algorithm. For suppose that w and w^{-1} are alternating, $w\overset{\mathrm{rsk}}{\to}(P,Q)$ and $w'\overset{\mathrm{rsk}}{\to}(P^t,Q^t)$ (where ^{t} denotes transpose). Then by [17, Lemma 7.23.1] the map $w\mapsto w'$ is a bijection between permutations $w\in\mathfrak{S}_n$ such that both w and w^{-1} are alternating, and permutations $w'\in\mathfrak{S}_n$ such that both w and $(w')^{-1}$ are reverse alternating. Is there a simpler proof that $f(n)=g(n)$ avoiding RSK?

4 Alternating tableaux of fixed shape.

Let T be a standard Young tableau (SYT). The *descent set* $D(T)$ is defined by [17, p. 351]

$$D(T)=\{i\,:\,i+1 \text{ is in a lower row than } i\}.$$

For instance, if

$$T=\begin{array}{lll}1&2&5\\3&4\\6,\end{array}$$

then $D(T)=\{2,5\}$. We also define the *descent composition* $\mathrm{co}(T)$ in analogy with equation (1). A basic property of the RSK algorithm asserts that $D(w)=D(Q)$ if $w\overset{\mathrm{rsk}}{\to}(P,Q)$. An SYT T of size n is called *alternating* if $D(T)=\{1,3,5,\dots\}\cap[n-1]$ and *reverse alternating* if $D(T)=\{2,4,6,\dots\}\cap[n-1]$. The following result is an immediate consequence of Theorem 7.19.7 and Corollary 7.23.6 of [17].

Theorem 4.1. *Let $\lambda \vdash n$ and $\alpha \in \mathrm{Comp}(n)$. Then $\langle s_\lambda, s_{B_\alpha}\rangle$ is equal to the number of SYT of shape λ and descent composition α.*

Let $\mathrm{alt}(\lambda)$ (respectively, $\mathrm{ralt}(\lambda)$) denote the number of alternating (respectively, reverse alternating) SYT of shape λ. The following result then follows from Theorems 2.3 and 4.1.

Theorem 4.2. *Let $\lambda \vdash n$ and $\alpha \in \mathrm{Comp}(n)$. If n is odd, then*

$$\mathrm{alt}(\lambda) = \mathrm{ralt}(\lambda) = s_\lambda[E, 0, -E, 0, E, 0, -E, \dots].$$

If n is even then

$$\begin{aligned} \mathrm{alt}(\lambda) &= s_\lambda[E, -1, -E, 1, E, -1, -E, 1, \dots] \\ \mathrm{ralt}(\lambda) &= s_\lambda[E, 1, -E, -1, E, 1, -E, -1, \dots]. \end{aligned}$$

Theorem 4.2 "determines" the number of alternating SYT of any shape λ, but the formula is not very enlightening. We can ask whether there are special cases for which the formula can be made more explicit. The simplest such case occurs when λ is the "staircase" $\delta_m = (m-1, m-2, \dots, 1)$. For any partition λ write H_λ for the product of the hook lengths of λ [17, p. 373]. For instance,

$$H_{\delta_m} = 1^{m-1}\,3^{m-2}\,5^{m-3}\cdots(2m-3).$$

Theorem 4.3. *If $m = 2k$ then*

$$\mathrm{alt}(\delta_m) = \mathrm{ralt}(\delta_m) = E^k \prod_{j=1}^{m-2} (E^2 + j^2)^{k - \lceil j/2 \rceil}.$$

If $m = 2k+1$ then

$$\mathrm{alt}(\delta_m) = \mathrm{ralt}(\delta_m) = E^k \prod_{j=1}^{m-2} (E^2 + j^2)^{k - \lfloor j/2 \rfloor}.$$

Proof. By the Murnaghan-Nakayama rule, s_{δ_m} is a polynomial in the odd power sums $p_1, p_3, \ldots$ [17, Prop. 7.17.7]. Assume that m is odd. Then by the hook-content formula [17, Cor. 7.21.4] we have

$$\begin{aligned} s_{\delta_m}[E, 0, E, 0, \ldots] &= s_{\delta_m}[E, E, E, \ldots] \\ &= \frac{E^k \prod_{j=1}^{m-2}(E^2 - j^2)^{k-\lfloor j/2 \rfloor}}{H_{\delta_m}}. \end{aligned} \tag{9}$$

Let $n = \binom{m}{2}$, and suppose that n is odd, say $n = 2r+1$. Let $\lambda \in \mathrm{OP}_n$ and $2j + 1 = \ell(\lambda)$. Thus

$$p_\lambda[E, 0, E, 0, E, 0, \ldots] = E^{2j+1}.$$

A simple parity argument shows that

$$p_\lambda[E, 0, -E, 0, E, 0, -E, 0, \ldots] = (-1)^{r-j} E^{2j+1}.$$

It follows that we obtain $s_{\delta_m}[E, 0, -E, 0, E, 0, -E, 0, \ldots]$ from the polynomial expansion of $s_{\delta_m}[E, 0, E, 0, E, 0, \ldots]$ by replacing each power E^{2j+1} with $(-1)^{r-j} E^{2j+1}$. The proof for m odd and n odd now follows from equation (9).

The argument for the remaining cases, viz., (a) m odd, n even, (b) m even, n odd, and (c) m even, n even, is completely analogous. $\square$

There are some additional partitions λ for which $\mathrm{alt}(\lambda)$ and $\mathrm{ralt}(\lambda)$ factor nicely as polynomials in E. One such case is the following.

Theorem 4.4. *Let p be odd, and let $p \times p$ denote the partition of p^2 whose shape is a $p \times p$ square. Then*

$$\begin{aligned} \mathrm{alt}(p \times p) &= \mathrm{ralt}(p \times p) \\ &= \frac{E^p (E^2 + 2^2)^{p-1} (E^2 + 4^2)^{p-2} \cdots (E^2 + (2(p-1))^2)}{H_{p \times p}}. \end{aligned}$$

Proof (sketch). Let h_n denote the complete symmetric function of degree n. From the identity

$$\sum_{n \geq 0} h_n t^n = \exp \sum_{n \geq 1} \frac{p_n t^n}{n}$$

we obtain

$$\begin{aligned}\sum_{n\geq 0} h_n[E,0,E,0,E,0,\ldots]t^n &= \exp\sum_{n\,\mathrm{odd}}\frac{Et^n}{n}\\ &= \left(\frac{1+t}{1-t}\right)^{E/2}.\end{aligned}$$

Write

$$\left(\frac{1+t}{1-t}\right)^{E/2} = \sum_{n\geq 0} a_n(E)t^n.$$

The Jacobi-Trudi identity [17, §7.16] implies that $s_{p\times p} = \det(h_{p-i+j})_{i,j=1}^p$. Hence

$$s_{p\times p}[E,0,E,0,E,0,\ldots] = \det(a_{p-i+j}(E))_{i,j=1}^p. \tag{10}$$

I am grateful to Christian Krattenthaler and Dennis Stanton for evaluating the above determinant. Krattenthaler's argument is as follows. Write

$$\left(\frac{1+t}{1-t}\right)^{E/2} = \left(1+\frac{2t}{1-t}\right)^{E/2} = 1+\sum_{n\geq 1} t^n \sum_{k=1}^{n}\binom{n-1}{k-1}\binom{E/2}{k}2^k.$$

After substituting $k+1$ for k, we see that we want to compute the Hankel determinant

$$\det_{0\leq i,j\leq n}\left(\sum_{k=0}^{i+j}\binom{i+j}{k}\binom{E/2}{k+1}2^{k+1}\right).$$

Now by a folklore result [13, Lemma 15] we conclude that this determinant is the same as

$$\det_{0\leq i,j\leq n}\left(\binom{E/2}{i+j+1}2^{i+j+1}\right).$$

When this determinant is expanded all powers of 2 are the same, so we are left with evaluating

$$\det_{0\leq i,j\leq n}\left(\binom{E/2}{i+j+1}\right).$$

This last determinant is well-known; see e.g. [13, (3.12)]. $\square$

Stanton has pointed out that the determinant of (10) is a special case of a Hankel determinant of Meixner polynomials $M_n(x; b, c)$, viz., $a_p(E) = 2EM_{p-1}(E-1; 2, -1)$. Since the Meixner polynomials are moments of a Jacobi polynomial measure [12, Thm. 524] the determinant will explicitly factor.

Neither of these two proofs of factorization is very enlightening. Is there a more conceptual proof based on the theory of symmetric functions?

NOTE. Permutations whose shape is a $p \times p$ square have an alternative description as a consequence of a basic property of the RSK algorithm [17, Cor. 7.23.11, Thm. 7.23.17], viz., they are the permutations in $\mathfrak{S}_{p^2}$ whose longest increasing subsequence and longest decreasing subsequence both have length p.

There are some other "special factorizations" of $\mathrm{alt}(\lambda)$ and $\mathrm{ralt}(\lambda)$ that appear to hold, which undoubtedly can be proved in a manner similar to the proof of Theorem 4.4. Some of these cases are the following, together with those arising from the identity $\mathrm{alt}(\lambda) = \mathrm{alt}(\lambda')$ when $|\lambda|$ is odd, and $\mathrm{alt}(\lambda) = \mathrm{ralt}(\lambda')$ when $|\lambda|$ is even. We write $\lambda = \langle 1^{m_1} 2^{m_2} \cdots \rangle$ to indicate that λ has m_i parts equal to i.

- $\mathrm{ralt}(\langle p^{p-1} \rangle)$
- $\mathrm{alt}(\langle 1, p^p \rangle)$, p odd
- certain values of $\mathrm{alt}(b, b-1, b-2, \dots, a)$ or $\mathrm{ralt}(b, b-1, b-2, \dots, a)$.

There are numerous other values of λ for which $\mathrm{alt}(\lambda)$ or $\mathrm{ralt}(\lambda)$ "partially factors." Moreover, there are similar specializations of s_λ which factor nicely, although they don't correspond to values of $\mathrm{alt}(\lambda)$ or $\mathrm{ralt}(\lambda)$, e.g., $s_{\langle p^p \rangle}[E, 0, -E, 0, E, 0, -E, 0, \dots]$ for p even.

5 Cycle type.

A permutation $w \in \mathfrak{S}_n$ has *cycle type* $\rho(w) = (\rho_1, \rho_2, \dots) \vdash n$ if the cycle lengths of w are $\rho_1, \rho_2, \dots$. For instance, the identity permutation has cycle type $\langle 1^n \rangle$. In this section we give an umbral formula for the number of alternating and reverse alternating permutations $w \in \mathfrak{S}_n$ of a fixed cycle type.

Our results are based on a theorem of Gessel-Reutenauer [9], which we now explain. Define a symmetric function

$$L_n = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{n/d}, \tag{11}$$

where μ is the number-theoretic Möbius function. Next define $L_{\langle m^r \rangle} = h_r[L_m]$ (plethysm). Equivalently, if $f(x) = f(x_1, x_2, \cdots)$ then write $f(x^r) = f(x_1^r, x_2^r, \cdots)$. Then for fixed m we have

$$\sum_{r \geq 0} L_{\langle m^r \rangle}(x) t^r = \exp \sum_{r \geq 1} \frac{1}{r} L_m(x^r) t^r. \tag{12}$$

Finally, for any partition $\lambda = \langle 1^{m_1} 2^{m_2} \cdots \rangle$ set

$$L_\lambda = L_{\langle 1^{m_1} \rangle} L_{\langle 2^{m_2} \rangle} \cdots. \tag{13}$$

For some properties of the symmetric functions L_λ see [17, Exer. 7.89].

Theorem 5.1 (Gessel-Reutenauer)**.** *Let $\rho \vdash n$ and $\alpha \in \mathrm{Comp}(n)$. Let $f(\rho, \alpha)$ denote the number of permutations $w \in \mathfrak{S}_n$ satisfying $\rho = \rho(w)$ and $\alpha = \mathrm{co}(w)$. Then*

$$f(\rho, \alpha) = \langle L_\rho, s_{B_\alpha} \rangle.$$

Now for $\rho \vdash n$ let $b(\rho)$ (respectively, $b^*(\rho)$) denote the number of alternating (respectively, reverse alternating) permutations $w \in \mathfrak{S}_n$ of cycle type ρ. The following corollary is then the special cases $B_\alpha = \tau_n$ and $B_\alpha = \tau_n'$ of Theorem 5.1.

Corollary 5.2. *We have $b(\rho) = \langle L_\rho, s_{\tau_n} \rangle$ and $b^*(\rho) = \langle L_\rho, s_{\tau_n'} \rangle$.*

We first consider the case when $\rho = (n)$, i.e., w is an n-cycle. Write $b(n)$ and $b^*(n)$ as short for $b((n))$ and $b^*((n))$. Theorem 5.3 below is actually subsumed by subsequent results (Theorems 5.4 and 5.5), but it seems worthwhile to state it separately.

Theorem 5.3. (a) *If n is odd then*

$$b(n) = b^*(n) = \frac{1}{n}\sum_{d|n} \mu(d)(-1)^{(d-1)/2} E_{n/d}.$$

(b) *If $n = 2^k m$ where $k \geq 1$, m is odd, and $m \geq 3$, then*

$$b(n) = b^*(n) = \frac{1}{n}\sum_{d|m} \mu(d) E_{n/d}.$$

(c) *If $n = 2^k$ and $k \geq 2$ then*

$$b(n) = b^*(n) = \frac{1}{n}(E_n - 1). \tag{14}$$

(d) *Finally, $b(2) = 1$, $b^*(2) = 0$.*

Proof. (a) By Theorem 2.3 and Corollary 5.2 we have for odd n that

$$\begin{aligned} b(n) = b^*(n) &= L_n[E, 0, -E, 0, E, 0, -E, 0, \cdots] \\ &= \frac{1}{n}\sum_{d|n} \mu(d)((-1)^{(d-1)/2} E)^{n/d} \\ &= \frac{1}{n}\sum_{d|n} \mu(d)(-1)^{(d-1)/2} E_{n/d}, \end{aligned}$$

since n/d is odd for each $d \mid n$.

(b) Split the sum (11) into two parts: d odd and d even. Since $\mu(2d) = -\mu(d)$ when d is odd and since $\mu(4d) = 0$ for any d, we obtain

$$\begin{aligned} b(n) &= L_n[E, -1, -E, 1, E, -1, -E, 1, \cdots] \\ &= \frac{1}{n}\left(\sum_{d|m} \mu(d)((-1)^{(d-1)/2} E)^{n/d} - \sum_{d|m} \mu(d)((-1)^d)^{n/2d}\right) \\ &= \frac{1}{n}\left(\sum_{d|m} \mu(d) E_{n/d} - (-1)^{n/2} \sum_{d|m} \mu(d)\right). \end{aligned}$$

The latter sum is 0 since $m > 1$, and we obtain the desired formula for $b(n)$. The argument for $b^*(n)$ is completely analogous; the factor $(-1)^{n/2}$ now becomes $(-1)^{1+\frac{n}{2}}$.

(c) When $n = 2^k$, $k \geq 2$, we have

$$L_n = \frac{1}{n}\left(p_1^n - p_2^{n/2}\right).$$

Substituting $p_1 = E$ and $p_2 = \pm 1$, and using that $n/2$ is even, yields (14).

(d) Trivial. It is curious that only for $n = 2$ do we have $b(n) \neq b^*(n)$. □

Note the special case of Theorem 5.3(a) when $m = p^k$, where p is an odd prime and $k \geq 1$:

$$b(p^k) = \frac{1}{p^k}\left(E_{p^k} - (-1)^{(p-1)/2} E_{p^{k-1}}\right).$$

Is there a simple combinatorial proof, at least when $k = 1$? The same can be asked of equation (14).

We next turn to the case $\lambda = \langle m^r \rangle$, i.e., all cycles of w have length m. Write $b(m^r)$ as short for $b(\langle m^r \rangle)$, and similarly for $b^*(m^r)$. Set

$$\begin{aligned} F_m(t) &= \sum_{r \geq 0} b(m^r) t^r \\ F_m^*(t) &= \sum_{r \geq 0} b^*(m^r) t^r. \end{aligned}$$

First we consider the case when m is odd.

Theorem 5.4. (a) *Let m be odd and $m \geq 3$. Then*

$$F_m(t) = F_m^*(t) = \exp\left[\frac{1}{m}\left(\sum_{d|m} \mu(d)(-1)^{(d-1)/2} E^{m/d}\right)(\tan^{-1} t)\right].$$

(b) *We have*

$$\begin{aligned} F_1(t) &= \sinh(E \tan^{-1} t) + \frac{1}{\sqrt{1+t^2}} \cosh(E \tan^{-1} t) \\ F_1^*(t) &= \sinh(E \tan^{-1} t) + \sqrt{1+t^2} \cosh(E \tan^{-1} t). \end{aligned}$$

Proof. (a) By equations (3) and (12) we have that the terms of $F_m(t)$ and $F_m^*(t)$ of odd degree (in t) are given by

$$\begin{aligned}
\frac{1}{2}(F_m(t) - F_m(-t)) &= \frac{1}{2}(F_m^*(t) - F_m^*(-t)) \\
&= \left(\sinh \sum_{r\,\mathrm{odd}} \frac{1}{r} L(x^r) t^r\right) \\
&\qquad \left(\exp \sum_{r\,\mathrm{even}} \frac{1}{r} L(x^r) t^r\right) [E, 0, -E, 0, \ldots] \\
&= \left(\sinh \sum_{r\,\mathrm{odd}} \frac{t^r}{mr} \sum_{d|m} \mu(d) p_{rd}^{m/d}\right) [E, 0, -E, 0, \ldots] \\
&= \sinh \sum_{r\,\mathrm{odd}} \frac{t^r}{mr} \sum_{d|m} \mu(d) (-1)^{(rd-1)/2} E^{m/d} \\
&= \sinh \frac{1}{m} \sum_{d|m} \mu(d) (-1)^{(d-1)/2} E^{m/d} \left(t - \frac{t^3}{3} + \frac{t^5}{5} - \cdots\right) \\
&= \sinh \frac{1}{m} \left(\sum_{d|m} \mu(d) (-1)^{(d-1)/2} E^{m/d}\right) (\tan^{-1} t). \qquad (15)
\end{aligned}$$

Similarly the terms of $F_m(t)$ of even degree are given by

$$\begin{aligned}
\frac{1}{2}(F_m(t)+F_m(-t)) &= \left(\cosh \sum_{r\,\mathrm{odd}} \frac{1}{r}L(x^r)t^r\right) \\
&\quad \cdot \left(\exp \sum_{r\,\mathrm{even}} \frac{1}{r}L(x^r)t^r\right)[E,-1,-E,1,\dots] \\
&= \left(\cosh \sum_{r\,\mathrm{odd}} \frac{t^r}{mr}\sum_{d|m}\mu(d)p_{rd}^{m/d}\right) \\
&\quad \cdot \left(\exp \sum_{r\,\mathrm{even}} \frac{t^r}{mr}\sum_{d|m}\mu(d)p_{rd}^{m/d}\right)[E,-1,-E,1,\dots] \\
&= \left(\cosh \sum_{r\,\mathrm{odd}} \frac{t^r}{mr}\sum_{d|m}\mu(d)((-1)^{(rd-1)/2})^{m/d}E^{m/d}\right) \\
&\quad \cdot \left(\exp \sum_{r\,\mathrm{even}} \frac{t^r}{mr}\sum_{d|m}\mu(d)((-1)^{rd/2})^{m/d}\right) \\
&= \left(\cosh \frac{1}{m}\sum_{d|m}\mu(d)(-1)^{(d-1)/2}E^{m/d}\tan^{-1}t\right) \\
&\quad \cdot \left(\exp \sum_{r\,\mathrm{even}} \frac{t^r}{mr}(-1)^{r/2}\sum_{d|m}\mu(d)\right)
\end{aligned}$$

$$= \cosh \frac{1}{m}\left(\sum_{d|m}\mu(d)(-1)^{(d-1)/2}E^{m/d}\right)(\tan^{-1}t). \qquad (16)$$

Adding equations (15) and (16) yields (a) for $F_m(t)$.

The computation for $F_m^*(x)$ is identical, except that the factor $(-1)^{rd/2}$ is replaced by $(-1)^{1+rd/2}$. This alteration does not affect the final answer.

(b) The computation of the odd part of $F_1(t)$ and $F_1^*(t)$ is the same as in (a), yielding

$$\begin{aligned}\frac{1}{2}(F_1(t) - F_1(-t)) &= \frac{1}{2}(F_1^*(t) - F_1^*(-t)) \\ &= \sinh(E\tan^{-1} t).\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{1}{2}(F_1(t) + F_1(-t)) &= \cosh(E\tan^{-1} t) \cdot \left(\exp \sum_{r\,\text{even}} \frac{t^r}{r}\mu(1)(-1)^{r/2}\right) \\ &= \frac{\cosh(E\tan^{-1} t)}{\sqrt{1+t^2}},\end{aligned}$$

and the proof for $F_1(t)$ follows. For $F_1^*(t)$ the factor $(-1)^{r/2}$ becomes $(-1)^{1+r/2}$, so the factor $\sqrt{1+t^2}$ moves from the denominator to the numerator. $\square$

Clearly the only alternating permutation of cycle type $\langle 1^r\rangle$ is 1 (when $r = 1$). Hence from Theorem 5.4(b) we obtain the umbral identity

$$\sinh(E\tan^{-1} t) + \frac{1}{\sqrt{1+t^2}}\cosh(E\tan^{-1} t) = 1 + t. \tag{17}$$

One may wonder what is the point of Theorem 5.4(b) since $b(1^r)$ is trivial to compute directly. Its usefulness will be seen below (Theorem 5.6), when we consider "mixed" cycle types, i.e., not all cycle lengths are equal.

Theorem 5.4 can be restated "non-umbrally" analogously to Theorem 3.1. For instance, if $m = p^k$ where p is prime and $p \equiv 3 \pmod 4$, then

$$F_m(t) = F_m^*(t) = \sum_{i,j\geq 0} E_{(m/p)i+mj} \frac{\left(\frac{1}{p}\tan^{-1} t\right)^{i+j}}{i!\, j!},$$

while if $p \equiv 1 \pmod 4$, then

$$F_m(t) = F_m^*(t) = \sum_{i,j\geq 0} (-1)^i E_{(m/p)i+mj} \frac{\left(\frac{1}{p}\tan^{-1} t\right)^{i+j}}{i!\, j!}.$$

For general odd m, $F_m(t)$ will be expressed as a $2^{\nu(m)}$-fold sum, where $\nu(m)$ is the number of distinct prime divisors of m.

Theorem 5.5. (a) *Let $m = 2^k h$, where $k \geq 1$, $h \geq 3$, and h is odd. Then*

$$F_m(t) = F_m^*(t) = \left(\frac{1+t}{1-t}\right)^{\frac{1}{2m}\sum_{d|h}\mu(d)E^{m/d}}.$$

(b) *Let $m = 2^k$ where $k \geq 2$. Then*

$$F_m(t) = F_m^*(t) = \left(\frac{1+t}{1-t}\right)^{\frac{1}{2m}(E^m-1)}.$$

(c) *Let $m = 2$. Then*

$$\begin{aligned} F_2(t) &= \left(\frac{1+t}{1-t}\right)^{(E^2+1)/4} \\ F_2^*(t) &= \frac{F_2(t)}{1+t} \quad \text{(compare (7)).} \end{aligned}$$

Proof. (a) The argument is analogous to the proof of Theorem 5.4.

We have

$$
\begin{aligned}
F_m(t) &= \left(\exp \sum_{r\geq 1} \frac{1}{r} L(x^r) t^r\right) [E, -1, -E, 1, \ldots] \\
&= \left(\exp \sum_{r\geq 1} \frac{t^r}{mr} \sum_{d|m} \mu(d) p_{rd}^{m/d}\right) [E, -1, -E, 1, \ldots] \\
&= \exp \left(\sum_{r\,\mathrm{odd}} \frac{t^r}{rm} \sum_{d|h} ((-1)^{(rd-1)/2})^{m/d} \mu(d) E^{m/d} \right. \\
&\qquad \left. + \sum_{r\,\mathrm{even}} \frac{t^r}{rm} \sum_{d|h} ((-1)^{rd/2})^{m/d} - \sum_{r} \frac{t^r}{rm} \sum_{d|h} ((-1)^{rd/2})^{m/d} \right) \\
&= \exp \sum_{r\,\mathrm{odd}} \frac{t^r}{rm} \sum_{d|h} \mu(d) E^{m/d} \\
&= \exp \left(\frac{1}{m} \sum_{d|h} \mu(d) E^{m/d} \right) \frac{1}{2} \log \frac{1+t}{1-t},
\end{aligned}
$$

and the proof follows for $F_m(t)$. The same argument holds for $F_m^*(t)$ since -1 was always raised to an even power or was multiplied by a factor $\sum_{d|h} \mu(d) = 0$ in the proof.

(b) We now have

$$
\begin{aligned}
F_m(t) &= \exp \sum_{r\geq 1} \frac{t^r}{rm} \left(p_r^m - p_{2r}^{m/2} \right) [E, -1, -E, 1, \ldots] \\
&= \exp \frac{1}{m} \left(\sum_{r\,\mathrm{odd}} \frac{t^r}{r} \left(((-1)^{(r-1)/2} E)^m - (-1)^{rm/2} \right) \right. \\
&\qquad \left. + \sum_{r\,\mathrm{even}} \frac{t^r}{r} \left((-1)^{rm/2} - (-1)^{rm/2} \right) \right) \\
&= \left(\frac{1-t}{1+t} \right)^{1/2m} \exp \frac{E^m}{2m} \log \frac{1+t}{1-t},
\end{aligned}
$$

etc. Again the computation for $F_m^*(t)$ is the same.

(c) We have

$$\begin{aligned}
F_2(t) &= \exp \frac{1}{2} \sum_{r \geq 1} \left(p_r^2 - p_{2r}\right) \frac{t^r}{r} [E, -1, -E, 1, \ldots] \\
&= \exp \frac{1}{2} \left[\sum_{r \text{ odd}} \left(\left((-1)^{(r-1)/2} E\right)^2 - (-1)^r \right) \frac{t^r}{r} \right. \\
&\qquad \left. + \sum_{r \text{ even}} \left((-1)^r - (-1)^r\right) \right] \\
&= \exp \frac{1}{2} \sum_{r \text{ odd}} (E^2 + 1) \frac{t^r}{r}.
\end{aligned}$$

etc. We leave the case $F_2^*(t)$ to the reader. $\square$

The expansion of $F_2(t)$ begins

$$F_2(t) = 1 + t + t^2 + 2t^3 + 5t^4 + 17t^5 + 72t^6 + 367t^7 + 2179t^8 + \cdots. \quad (18)$$

Ramanujan asserts in Entry 16 of his second notebook (see [3, p. 545]) that as t tends to $0+$,

$$2 \sum_{n \geq 0} (-1)^n \left(\frac{1-t}{1+t} \right)^{n(n+1)} \sim 1 + t + t^2 + 2t^3 + 5t^4 + 17t^5 + \cdots. \quad (19)$$

Berndt [3, (16.6)] obtains a formula for the complete asymptotic expansion of $2 \sum_{n \geq 0} (-1)^n \left(\frac{1-t}{1+t}\right)^{n(n+1)}$ as $t \to 0+$. It is easy to see that Berndt's formula can be written as $\left(\frac{1+t}{1-t}\right)^{(E^2+1)/4}$ and is thus equal to $F_2(t)$. Theorem 5.5(c) therefore answers a question of Galway [7, p. 111], who asks for a combinatorial interpretation of the coefficients in Ramanujan's asymptotic expansion.

Note. The following formula for $F_2(t)$ follows from equation (19) and an identity of Ramanujan proved by Andrews [2, $(6.3)_{\mathrm{R}}$]:

$$F_2(t) = 2 \sum_{n \geq 0} q^n \frac{\prod_{j=1}^{n} (1 - q^{2j-1})}{\prod_{j=1}^{2n+1} (1 + q^j)},$$

where $q = \left(\frac{1-t}{1+t}\right)^{2/3}$. It is not hard to see that this is a *formal* identity, unlike the asymptotic identity (19).

NOTE. We can put Theorems 5.4(a) into a form more similar to Theorem 5.5(a) by noting the identity

$$\exp(\tan^{-1} t) = \left(\frac{1-it}{1+it}\right)^{i/2}.$$

Hence when m is odd and $m \geq 3$ we have

$$F_m(t) = F_m^*(t) = \left(\frac{1-it}{1+it}\right)^{\frac{i}{2m}\sum_{d|m}\mu(d)(-1)^{(d-1)/2}E^{m/d}}.$$

The multiplicativity property (13) of L_λ allows us write down a generating function for the number $b(\lambda)$ (respectively, $b^*(\lambda)$) of alternating (respectively, reverse alternating) permutations of any cycle type λ. For this purpose, let $t_1, t_2, \ldots$ and t be indeterminates and set $\deg(t_i) = i$, $\deg(t) = 1$. If $F(t_1, t_2, \ldots)$ is a power series in $t_1, t_2, \ldots$ or $F(t)$ is a power series in t, then write $\mathcal{O}F$ (respectively, $\mathcal{E}F$) for those terms of F whose total degree is odd (respectively, even). For instance,

$$\mathcal{O}F(t_1, t_2, \ldots) = \frac{1}{2}\left(F(t_1, t_2, t_3, t_4 \ldots) - F(-t_1, t_2, -t_3, t_4, \ldots)\right).$$

Define the "cycle indicators"

$$\begin{aligned} Z(t_1, t_2, \ldots) &= \sum_{\lambda = \langle 1^{m_1} 2^{m_2} \cdots \rangle} b(\lambda) t_1^{m_1} t_2^{m_2} \cdots \\ Z^*(t_1, t_2, \ldots) &= \sum_{\lambda = \langle 1^{m_1} 2^{m_2} \cdots \rangle} b^*(\lambda) t_1^{m_1} t_2^{m_2} \cdots, \end{aligned}$$

where both sums range over all partitions λ of all integers $n \geq 0$.

Theorem 5.6. *We have*

$$\begin{aligned} &\mathcal{O}Z(t_1, t_2, \ldots) = \mathcal{O}Z^*(t_1, t_2, \ldots) \\ &\quad = \mathcal{O}\exp(E\tan^{-1} t_1) \cdot \left(\frac{1+t_2}{1-t_2}\right)^{E^2/4} F_3(t_3)F_4(t_4)\cdots \qquad (20) \end{aligned}$$

$$\begin{aligned}
\mathcal{E}Z(t_1,t_2,\dots) &= \mathcal{E}\frac{\exp(E\tan^{-1}t_1)}{\sqrt{1+t_1^2}}\left(\frac{1+t_2}{1-t_2}\right)^{(E^2+1)/4} F_3(t_3)F_4(t_4)\cdots \\
\mathcal{E}Z^*(t_1,t_2,\dots) &= \mathcal{E}\sqrt{1+t_1^2}\,\exp(E\tan^{-1}t_1)\cdot\frac{1}{1+t_2}\left(\frac{1+t_2}{1-t_2}\right)^{(E^2+1)/4} \\
&\qquad \cdot F_3(t_3)F_4(t_4)\cdots.
\end{aligned}$$

It is understood that in these formulas $F_j(t_j)$ is to be written in the umbral form given by Theorems 5.4 and 5.5.

Proof. Let

$$G_m(t) = \exp\sum_{r\geq 1}\frac{1}{r}L_m(x^r)t^r. \tag{21}$$

It follows from equations (12) and (13) that

$$\mathcal{O}Z(t_1,t_2,\dots) = \mathcal{O}\prod_{m\geq 1}G_m(t_m)[E,0,-E,0,\dots].$$

The proofs of Theorems 5.4(a) and 5.5(a,b) show that for $m\geq 3$,

$$G_m(t)[E,0,-E,0,\dots] = G_m(t)[E,-1,-E,1,\dots].$$

Hence we obtain the factors $F_3(t_3)F_4(t_4)\cdots$ in equation (20). It is straightforward to compute $G_m(t_m)[E,0,-E,0,\dots]$ (and is implicit in the proofs of Theorems 5.4(b) and 5.5(c)) for $m=1,2$. For instance,

$$\begin{aligned}
G_1(t_1)[E,0,-E,0\dots] &= \exp\left(\sum_{r\geq 1}\frac{1}{r}p_r t_1^r\right)[E,0,-E,0,\dots] \\
&= \exp\sum_{r\,\mathrm{odd}}\frac{1}{r}(t_1-\frac{1}{3}t_1^3+\cdots) \\
&= \exp(\tan^{-1}t_1). \qquad (22)
\end{aligned}$$

Thus we obtain the remaining factors in equation (20). The remaining formulas are proved analogously. □

We mentioned in Section 1 that Ehrenborg and Readdy raised the question of counting alternating involutions $w\in S_n$. An answer to this question is a simple consequence of Theorem 5.6.

Corollary 5.7. *Let $c(n)$ (repectively, $c^*(n)$) denote the number of alternating (respectively, reverse alternating) involutions $w \in \mathfrak{S}_n$. Then*

$$\begin{aligned}
\sum_{n\geq 0} c(2n+1)t^{2n+1} &= \sinh(E\tan^{-1} t)\cdot\left(\frac{1+t^2}{1-t^2}\right)^{E^2/4} \\
\sum_{n\geq 0} c(2n)t^{2n} &= \frac{1}{\sqrt[4]{1-t^4}}\cosh(E\tan^{-1} t)\cdot\left(\frac{1+t^2}{1-t^2}\right)^{E^2/4} \\
c^*(n) &= c(n).
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\sum_{n\geq 0} c(2n+1)t^{2n+1} &= \sum_{i,j\geq 0}\frac{E_{2i+2j+1}}{(2i+1)!\,j!\,4^j}\tan^{-1}(t)^{2i+1}\left(\log\frac{1+t^2}{1-t^2}\right)^j \\
\sum_{n\geq 0} c(2n)t^{2n} &= \frac{1}{\sqrt[4]{1-t^4}}\sum_{i,j\geq 0}\frac{E_{2i+2j}}{(2i)!\,j!\,4^j}\tan^{-1}(t)^{2i}\left(\log\frac{1+t^2}{1-t^2}\right)^j.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\sum_{n\geq 0} c(2n+1)t^{2n+1} &= \mathcal{O}Z(t,t^2,0,0,\dots) \\
\sum_{n\geq 0} c(2n)t^{2n} &= \mathcal{E}Z(t,t^2,0,0,\dots),
\end{aligned}$$

and similarly for $c^*(n)$. The result is thus a special case of Theorem 5.6. $\square$

The identity $c(n) = c^*(n)$ does not seem obvious. It can also be obtained using properties of the RSK algorithm, analogous to the argument after the proof of Theorem 3.1 Namely, $w \in \mathfrak{S}_n$ is an alternating (respectively, reverse alternating) involution if and only if $w \stackrel{\text{rsk}}{\rightarrow} (P,P)$, where P is an alternating (respectively, reverse alternating) SYT. Hence if $w' \stackrel{\text{rsk}}{\rightarrow} (P^t,P^t)$, then the map $w \mapsto w'$ interchanges alternating involutions $w \in \mathfrak{S}_n$ with reverse alternating involutions $w' \in \mathfrak{S}_n$.

6 Fixed points.

P. Diaconis (private communication) raised the question of enumerating alternating permutations by their number of fixed points. It is easy to answer this question using Theorem 5.6. Write $d_k(n)$ (respectively, $d_k^*(n)$) for the number of alternating (respectively, reverse alternating) permutations in $\mathfrak{S}_n$ with k fixed points. Write $\mathcal{O}_t$ and $\mathcal{E}_t$ for the odd and even part of a power series with respect to t (ignoring other variables), i.e., $\mathcal{O}_t F(t) = \frac{1}{2}(F(t) - F(-t))$ and $\mathcal{E}_t F(t) = \frac{1}{2}(F(t) + F(-t))$.

Proposition 6.1. *We have*

$$\begin{aligned}
\sum_{k,n\geq 0} d_k(2n+1)q^k t^{2n+1} &= \mathcal{O}_t \frac{\exp(E(\tan^{-1} qt - \tan^{-1} t))}{1-Et} && (23)\\
d_k^*(2n+1) &= d_k(2n+1) && (24)\\
\sum_{k,n\geq 0} d_k(2n)q^k t^{2n} &= \mathcal{E}_t \sqrt{\frac{1+t^2}{1+q^2t^2}} \frac{\exp(E(\tan^{-1} qt - \tan^{-1} t))}{1-Et}\\
\sum_{k,n\geq 0} d_k^*(2n)q^k t^{2n} &= \mathcal{E}_t \sqrt{\frac{1+q^2t^2}{1+t^2}} \frac{\exp(E(\tan^{-1} qt - \tan^{-1} t))}{1-Et}.
\end{aligned}$$

Equivalently, we have the nonumbral formulas

$$\begin{aligned}
\sum_{k,n\geq 0} d_k(2n+1)q^k t^{2n+1} &= \sum_{\substack{i,j\geq 0\\ i\not\equiv j \pmod 2}} \frac{E_{i+j}}{j!} t^i (\tan^{-1} qt - \tan^{-1} t)^j\\
\sum_{k,n\geq 0} d_k(2n)q^k t^{2n} &= \sqrt{\frac{1+t^2}{1+q^2t^2}} \sum_{\substack{i,j\geq 0\\ i\equiv j \pmod 2}} \frac{E_{i+j}}{j!} t^i (\tan^{-1} qt - \tan^{-1} t)^j\\
\sum_{k,n\geq 0} d_k^*(2n)q^k t^{2n} &= \sqrt{\frac{1+q^2t^2}{1+t^2}} \sum_{\substack{i,j\geq 0\\ i\equiv j \pmod 2}} \frac{E_{i+j}}{j!} t^i (\tan^{-1} qt - \tan^{-1} t)^j.
\end{aligned}$$

Proof. It is not hard to see (e.g., [16, (1)]) that

$$\sum_{\lambda\vdash n} L_\lambda = p_1^n,$$

where $p_1 = x_1 + x_2 + \cdots$. It follows from equations (12), (13) and (21) that

$$G_1(t)G_2(t)\cdots = \sum_{n\geq 0} p_1^n t^n = \frac{1}{1-p_1 t}.$$

Hence by equation (20) we have

$$\begin{aligned}
\sum_{k,n\geq 0} d_k(2n+1)q^k t^{2n+1} &= \sum_{k,n\geq 0} d_k^*(2n+1)q^k t^{2n+1} \\
&= \mathcal{O}_t \exp(E\tan^{-1} qt)\left(\frac{1+t}{1-t}\right)^{E^2/4} F_3(t)F_4(t)\cdots \\
&= \mathcal{O}_t \frac{\exp(E\tan^{-1} qt)}{\exp(E\tan^{-1} t)\cdot(1-Et)},
\end{aligned}$$

proving (23) and (24). The proof for n even is analogous. $\square$

Corollary 6.2. *For $n > 1$ we have $d_0(n) = d_1(n)$ and $d_0^*(n) = d_1^*(n)$.*

Proof. Let

$$M(q,t) = \mathcal{O}_t \frac{\exp E(\tan^{-1} qt - \tan^{-1} t)}{1-Et}.$$

By equation (24) it follows that

$$\begin{aligned}
\sum_{n\,\mathrm{odd}} d_0(n)t^n &= M(0,t) \\
\sum_{n\,\mathrm{odd}} d_1(n)t^n &= \left.\frac{\partial}{\partial q}M(q,t)\right|_{q=0}.
\end{aligned}$$

It is straightforward to compute that

$$\left.\frac{\partial}{\partial q}M(q,t)\right|_{q=0} - M(0,t) = \sinh(E\tan^{-1} t).$$

By equation (17) we have $\sinh(E\tan^{-1} t) = t$, and the proof follows for n odd. The proof for n even is completely analogous. $\square$

We have a conjecture about certain values of $d_k(n)$ and $d_k^*(n)$. It is not hard to see that

$$\begin{aligned}
\max\{k : d_k(n) \neq 0\} &= \lceil n/2 \rceil, \quad n \geq 4 \\
\max\{k : d_k^*(n) \neq 0\} &= \lceil (n+1)/2 \rceil, \quad n \geq 5.
\end{aligned}$$

Conjecture 6.3. *Let D_n denote the number of derangements (permutations without fixed points) in $\mathfrak{S}_n$. Then*

$$\begin{aligned}
d_{\lceil n/2 \rceil}(n) &= D_{\lfloor n/2 \rfloor}, \quad n \geq 4 \\
d^*_{\lceil (n+1)/2 \rceil}(n) &= D_{\lfloor (n-1)/2 \rfloor}, \quad n \geq 5.
\end{aligned}$$

It is also possible to obtain asymptotic information from Proposition 6.1. The next result considers alternating or reverse alternating derangements (permutations without fixed points).

Corollary 6.4. (a) We have for n odd the asymptotic expansion

$$\begin{aligned}
d_0(n) &\sim \frac{1}{e}\left(E_n + a_1 E_{n-2} + a_2 E_{n-4} + \cdots\right) \qquad (25) \\
&= \frac{1}{e}\left(E_n + \frac{1}{3}E_{n-2} - \frac{13}{90}E_{n-4} + \frac{467}{5760}E_{n-6} + \cdots\right),
\end{aligned}$$

where

$$\sum_{k \geq 0} a_k x^{2k} = \exp\left(1 - \frac{1}{x}\tan^{-1} x\right).$$

(b) We have for n even the asymptotic expansion

$$\begin{aligned}
d_0(n) &\sim \frac{1}{e}\left(E_n + b_1 E_{n-2} + b_2 E_{n-4} + \cdots\right) \qquad (26) \\
&= \frac{1}{e}\left(E_n + \frac{5}{6}E_{n-2} - \frac{37}{360}E_{n-4} + \frac{281}{9072}E_{n-6} + \cdots\right),
\end{aligned}$$

where

$$\sum_{k \geq 0} b_k x^{2k} = \sqrt{1+x^2}\exp\left(1 - \frac{1}{x}\tan^{-1} x\right).$$

(c) We have for n even the asymptotic expansion

$$d_0^*(n) \sim \frac{1}{e}\left(E_n + c_1 E_{n-2} + c_2 E_{n-4} + \cdots\right) \tag{27}$$
$$= \frac{1}{e}\left(E_n - \frac{1}{6}E_{n-2} + \frac{23}{360}E_{n-4} - \frac{1493}{45360}E_{n-6} + \cdots\right),$$

where

$$\sum_{k\geq 0} c_k x^{2k+1} = \frac{1}{\sqrt{1+x^2}} \exp\left(1 - \frac{1}{x}\tan^{-1} x\right).$$

NOTE. Equations (25), (26), and (27) are genuine asymptotic expansions since $E_m \sim 2(2/\pi)^{m+1} m!$, so for fixed k,

$$E_{n-k} \sim 2\left(\frac{\pi}{2}\right)^k \frac{1}{n^k} E_n$$

as $n \to \infty$. In fact, since

$$E_m = 2\left(\frac{2}{\pi}\right)^{m+1} m!(1 + O(3^{-m})),$$

we can rewrite (25) (and similarly (26) and (27)) as

$$d_0(n) \sim \frac{E_n}{e}\left(1 + a_1\left(\frac{\pi}{2}\right)^2 \frac{1}{(n)_2} + a_2\left(\frac{\pi}{2}\right)^4 \frac{1}{(n)_4} + \cdots\right),$$

where $(n)_j = n(n-1)\cdots(n-j+1)$.

Proof of Corollary 6.4. (a) It follows from equation (23) that

$$\sum_{n\,\mathrm{odd}} d_0(n) t^n = \mathcal{O}_t \frac{\exp(-E\tan^{-1} t)}{1 - Et}.$$

This series has the form

$$\sum_{n\,\mathrm{odd}} t^n (a_{n0}E^n + a_{n1}E^{n-2} + a_{n2}E^{n-4} + \cdots).$$

If we replace t with Et and E with $1/E$ we therefore obtain

$$\mathcal{O}_t \frac{\exp(-E^{-1}\tan^{-1} tE))}{1-t} = \sum_{n\,\mathrm{odd}} t^n (a_{n0} + a_{n1}E^2 + a_{n2}E^4 + \cdots). \tag{28}$$

We claim that for fixed j the coefficients a_{nj} rapidly approach (finite) limits as $n \to \infty$. If we expand the left-hand side of (28) as a power series in E, it is not hard to see that the coefficient of E^{2j} has the form $Q_j(t)/(1-t^2)$, where $Q_j(t)$ is a polynomial in t, e^t and e^{-t}. Hence the coefficient of t^{2n+1} in $Q_j(t)$ has the form $p_j(n)/(2n+1)!$ for some polynomial $p_j(n)$. It follows that

$$a_{nj} = Q_j(1) + o(n^{-r})$$

for all $r > 0$. Now

$$\mathcal{O}_t \frac{\exp(-E^{-1}\tan^{-1} tE))}{1-t} = \frac{(1+t)e^{-\frac{1}{E}\tan^{-1} tE} - (1-t)e^{\frac{1}{E}\tan^{-1} tE}}{2(1-t^2)}.$$

Multiplying by $1-t^2$ and setting $t=1$ gives $e^{-\frac{1}{E}\tan^{-1} E}$, and the proof follows. The argument for (b) and (c) is analogous. □

7 Multisets.

In this section we give simple umbral formulas for the number of alternating and reverse alternating permutations of a multiset of positive integers, with various interpretations of the meaning of "alternating." There has been some previous work on alternating multiset permutations. Goulden and Jackson [11, Exer. 4.2.2(b); solution, pp. 459–460] obtain a formula for the number of alternating permutations of the multiset with one occurrence of i for $1 \leq i \leq m$ and two occurrences of i for $m+1 \leq i \leq m+n$. Gessel [8, pp. 265–266] extends this result to multisets with one, two, or three multiplicities of each part, or with one or four multiplicities of each part. Upon being told about the results in this section, Gessel (private communication) was able to extend his argument to arbitrary multisets, obtaining a result equivalent to the case $A = \emptyset$ of Theorem 7.3. Zeng [19] obtains an even more general result concerning the case $A = \emptyset$.

Our basic tool, in addition to Theorem 2.3, is the following extension of Theorem 4.1 to skew shapes λ/μ. We define the descent composition of an SYT T of shape λ/μ exactly as for ordinary shapes, viz., T has descent composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ if

$\{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\cdots+\alpha_{k-1}\}$ is the set of those i for which $i+1$ appears in T in a lower row than i.

Lemma 7.1. *Let λ/μ be a skew partition of size n, with corresponding skew Schur function $s_{\lambda/\mu}$ [17, Def. 7.10.1], and let $\alpha \in \mathrm{Comp}(n)$. Then $\langle s_{\lambda/\mu}, s_{B_\alpha}\rangle$ is equal to the number of SYT of shape λ/μ and descent composition α.*

Proof. Let $s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu$. Let T be an SYT of shape λ/μ, and apply jeu de taquin [17, §A1.2] to T to obtain an SYT T' of some ordinary shape ν. Two fundamental properties of jeu de taquin assert the following:

- As T runs over all SYT of shape λ/μ, we obtain by jeu de taquin each SYT T' of shape ν exactly $c^\lambda_{\mu\nu}$ times.
- We have $\mathrm{co}(T) = \mathrm{co}(T')$.

The first item above appears e.g. in [17, Thm. A1.3.1], while the second item is easily proved by showing that the descent composition is preserved by a single jeu de taquin slide. The proof of the lemma follows immediately from the two items above. □

We can define $\mathrm{alt}(\lambda/\mu)$ and $\mathrm{ralt}(\lambda/\mu)$ for skew shapes λ/μ exactly as we did for ordinary shapes λ. The following corollary is then immediate from Theorem 2.3 and Lemma 7.1.

Corollary 7.2. *Let λ/μ be a skew shape of odd size $|\lambda/\mu|$. Then*

$$\mathrm{alt}(\lambda/\mu) = \mathrm{ralt}(\lambda/\mu) = s_{\lambda/\mu}[E, 0, -E, 0, E, 0, -E, 0, \ldots].$$

If $|\lambda/\mu|$ is even then

$$\begin{aligned} \mathrm{alt}(\lambda/\mu) &= s_{\lambda/\mu}[E, -1, -E, 1, E, -1, -E, 1, \ldots] \\ \mathrm{ralt}(\lambda/\mu) &= s_{\lambda/\mu}[E, 1, -E, -1, E, 1, -E, -1, \ldots]. \end{aligned}$$

We are now ready to enumerate alternating permutations of a multiset. If two equal elements i in a permutation appear consecutively, then we need to decide whether they form an ascent or a descent. We can make this decision separately for each i. Let $k \geq 1$, and let A, B be complementary subsets of $[k]$, i.e., $A \cup B = [k]$, $A \cap B = \emptyset$. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of some $n \geq 1$ into k parts. An *α-permutation* of $[k]$ is a permutation of the multiset $M = \{1^{\alpha_1}, \ldots, k^{\alpha_k}\}$, i.e., a sequence $a_1 a_2 \cdots a_n$ with α_i occurrences of i, for $1 \leq i \leq k$. An α-permutation is said to be *(A, B)-alternating* if

$$a_1 > a_2 < a_3 > a_4 < \cdots a_n,$$

where we define $j > j$ if $j \in A$ and $j < j$ if $j \in B$. For instance, if $A = \{1, 3\}$, $B = \{2, 4\}$, and $\alpha = (3, 2, 2, 3)$, then the α-permutation $w = 1142214343$ is (A, B)-alternating since

$$1 > 1 < 4 > 2 < 2 > 1 < 4 > 3 < 4 > 3$$

according to our definition. Similarly we define *reverse (A, B)-alternating*. For example, 2213341414 is a reverse (A, B) α-permutation (with α, A, B as before), since

$$2 < 2 > 1 < 3 > 3 < 4 > 1 < 4 > 1 < 4.$$

Let $N(\alpha, A, B)$ (respectively, $N^*(\alpha, A, B)$) denote the number of (A, B)-alternating (respectively, reverse (A, B)-alternating) α-permutations. Write e_i and h_i for the elementary and complete symmetric functions of degree i.

Theorem 7.3. *Let $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathrm{Comp}(n)$, and let A, B be complementary subsets of $[k]$.*

(a) *If n is odd, then*

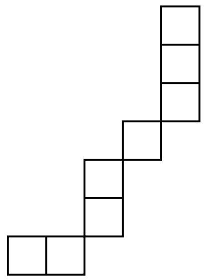
$$\begin{aligned} N(\alpha, A, B) &= N^*(\alpha, A, B) \\ &= \prod_{i \in A} e_{\alpha_i} \cdot \prod_{j \in B} h_{\alpha_j}[E, 0, -E, 0, E, 0, -E, 0, \ldots]. \end{aligned}$$

(b) *If n is even, then*

$$N(\alpha, A, B) = \prod_{i \in A} e_{\alpha_i} \cdot \prod_{j \in B} h_{\alpha_j}[E, -1, -E, 1, E, -1, -E, 1, \ldots]$$

$$N^*(\alpha, A, B) = \prod_{i \in A} e_{\alpha_i} \cdot \prod_{j \in B} h_{\alpha_j}[E, 1, -E, -1, E, 1, -E, -1, \ldots]$$

Proof. Let $\sigma = \sigma(\alpha, A, B)$ be the skew shape consisting of a disjoint union of single rows and columns, as follows. There are k connected components, of sizes $\alpha_1, \dots, \alpha_k$ from top to bottom. If $i \in A$ then the ith component is a single row, and otherwise a single column. For instance, $\sigma((3,1,2,2), \{2,4\}, \{1,3\})$ and $\sigma((3,1,2,2), \{4\}, \{1,2,3\})$ both have the following diagram:



Suppose that n is odd. By Corollary 7.2 we have

$$\mathrm{alt}(\sigma) = \mathrm{ralt}(\sigma) = s_\sigma[E, 0, -E, 0, \dots].$$

Given an alternating or reverse alternationg SYT T of shape σ, define an α-permutation $w = a_1 \cdots a_n$ by the condition that $a_i = j$ if a_i appears in the jth component of σ. For instance, if

$$\sigma = \begin{array}{cccccccc} &&&& 4 & 8 & 10 & 12 \\ &&& 2 &&&& \\ &&& 3 &&&& \\ &&& 11 &&&& \\ & 5 & 6 & 9 &&&& \\ 1 &&&&&&& \\ 7 &&&&&&& \end{array},$$

then $w = 422133413121$. This construction sets up a bijection between alternating (respectively, reverse alternating) SYT of shape σ and (A, B)-alternating (respectively, reverse alternating) α-permutations, so the proof follows for n odd. Exactly the same argument works for n even. $\square$

Some values of the relevant specializations of e_i and h_i are as follows:

$$\begin{array}{rcl} e_1[E,0,-E,0,\dots] &=& h_1[E,0,-E,0,\dots] &=& E \\ e_2[E,0,-E,0,\dots] &=& h_2[E,0,-E,0,\dots] &=& \frac{1}{2}E^2 \\ e_3[E,0,-E,0,\dots] &=& h_3[E,0,-E,0,\dots] &=& \frac{1}{6}(E^3-2E) \\ e_4[E,0,-E,0,\dots] &=& h_4[E,0,-E,0,\dots] &=& \frac{1}{24}(E^4-8E^2) \\ e_5[E,0,-E,0,\dots] &=& h_5[E,0,-E,0,\dots] &=& \frac{1}{120}(E^5-20E^3+24E) \end{array}$$

$$\begin{aligned} &e_1[E,-1,-E,1,\dots] = e_1[E,1,-E,-1,\dots] \\ &\quad = h_1[E,-1,-E,1,\dots] = h_1[E,1,-E,-1,\dots] = E \end{aligned}$$

$$\begin{array}{rcl} e_2[E,-1,-E,1,\dots] &=& h_2[E,1,-E,-1,\dots] &=& \frac{1}{2}(E^2+1) \\ e_2[E,1,-E,-1,\dots] &=& h_2[E,-1,-E,1,\dots] &=& \frac{1}{2}(E^2-1) \\ e_3[E,-1,-E,1,\dots] &=& h_3[E,1,-E,-1,\dots] &=& \frac{1}{6}(E^3+E) \\ e_3[E,1,-E,-1,\dots] &=& h_3[E,-1,-E,1,\dots] &=& \frac{1}{6}(E^3-5E) \\ e_4[E,-1,-E,1,\dots] &=& h_4[E,1,-E,-1,\dots] &=& \frac{1}{24}(E^4-2E^2-3) \\ e_4[E,1,-E,-1,\dots] &=& h_4[E,-1,-E,-1,\dots] &=& \frac{1}{24}(E^4-7E^2+9) \\ e_5[E,-1,-E,1,\dots] &=& h_5[E,1,-E,-1,\dots] &=& \frac{1}{120}(E^5-10E^3-11E) \\ e_5[E,1,-E,-1,\dots] &=& h_5[E,-1,-E,1,\dots] &=& \frac{1}{120}(E^5-30E^3+89E). \end{array}$$

It is easy to see (see equations (29), (30), (31) below) that for all i we have

$$\begin{array}{rcl} e_i[E,0,-E,0,\dots] &=& h_i[E,0,-E,0,\dots] \\ e_i[E,-1,-E,1,\dots] &=& h_i[E,1,-E,-1,\dots] \\ e_i[E,1,-E,-1,\dots] &=& h_i[E,-1,-E,1,\dots] \end{array}$$

These formulas, together with Theorem 7.3 and the commutativity of the ring of symmetric functions, yield some results about the equality of certain values of $N(\alpha,A,B)$. For instance, if n is odd, then $N(\alpha,A,B)$ depends only on the multiset of parts of α, not on their order, and also not on A and B. If n is even, then $N(\alpha,A,B)$ depends only on the multiset of parts of α and on which submultiset of these parts index the elements of A and B.

The specialization of e_i and h_i for small i lead to some nonumbral formulas for certain values of $N(\alpha, A, B)$. For instance, let k be odd, $\alpha = (3^k)$ (i.e., k parts equal to 3), $A = \emptyset$, so that $N((3^k), \emptyset, [k])$ is the number of alternating permutations $a_1 > a_2 \leq a_3 > a_4 \leq a_5 > \cdots \leq a_{3k}$ (where $>$ and $\leq$ have their usual meaning) of the multiset $\{1^3, 2^3, \ldots, k^3\}$. Then

$$\begin{aligned} N((3^k), \emptyset, [k]) &= h_3^k[E, 0, -E, 0, \ldots] \\ &= \frac{1}{6^k} E^k (E^2 - 2)^k \\ &= \frac{1}{6^k} \sum_{j=0}^{k} \binom{k}{j} (-2)^{k-j} E_{2j+k}. \end{aligned}$$

In the same way we obtain the formulas in [8, pp. 265–266].

It is easy to find generating functions for the specializations of e_n and h_n that we are considering, using the identities

$$\begin{aligned} \sum_{n \geq 0} e_n t^n &= \exp \sum_{j \geq 1} (-1)^{j-1} \frac{p_j}{j} \\ \sum_{n \geq 0} h_n t^n &= \exp \sum_{j \geq 1} \frac{p_j}{j}. \end{aligned}$$

Namely,

$$\begin{aligned} \sum_{n \geq 0} e_n[E, 0, -E, 0, \ldots] t^n &= \sum_{n \geq 0} h_n[E, 0, -E, 0, \ldots] t^n \\ &= \exp E \tan^{-1} t \qquad (29) \\ \sum_{n \geq 0} e_n[E, 1, -E, -1, \ldots] t^n &= \sum_{n \geq 0} h_n[E, -1, -E, 1, \ldots] t^n \\ &= \frac{1}{\sqrt{1+t^2}} \exp E \tan^{-1} t \qquad (30) \\ \sum_{n \geq 0} e_n[E, -1, -E, 1, \ldots] t^n &= \sum_{n \geq 0} h_n[E, 1, -E, -1, \ldots] t^n \\ &= \sqrt{1+t^2} \exp E \tan^{-1} t \qquad (31) \end{aligned}$$

Equation (29) in fact is a restatement of (22).

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