Discrete and Computational Geometry manuscript No.
(will be inserted by the editor)

Irrational triangles with polynomial Ehrhart functions

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Abstract While much research has been done on the Ehrhart functions of integral and rational polytopes, little is known in the irrational case. In our main theorem, we determine exactly when the Ehrhart function of a right triangle with legs on the axes and slant edge with irrational slope is a polynomial. We also investigate several other situations where the period of the Ehrhart function of a polytope is less than the denominator of that polytope. For example, we give examples of irrational polytopes with polynomial Ehrhart function in any dimension, and we find triangles with periods dividing any even-index k-Fibonacci number, but with larger denominators.

Keywords Ehrhart function, period collapse

Mathematics Subject Classification (2000) 52B20

The first author would like to thank Martin Gardiner for helpful discussions. The first author was partially supported by NSF grants DMS-1402200 and DMS-1711976.

The second author was supported by the National Science Foundation of China (Grant No. 11601440), the Natural Science Foundation of Chongqing (Grant No. cstc2016jcyjA0245) and Fundamental Research Funds for Central Universities (Grant No. XDJK2018C075)

The third author was partially supported by NSF grant DMS-1068625.

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1 Introduction

1.1 The main theorem

Let $\mathcal{P} \subset \mathbb{R}^d$ be a convex polytope. The counting function

$$I_{\mathcal{P}}(t) := \#(t\mathcal{P} \cap \mathbb{Z}^d)$$

for a positive integer t is called the *Ehrhart function* of $\mathcal{P}$.

The Ehrhart function has been extensively studied in the case where $\mathcal{P}$ is *integral*, meaning its vertices are given by integers, or $\mathcal{P}$ is *rational*, meaning its vertices are given by rational numbers. In particular, recall that a *quasipolynomial* is a function $p : \mathbb{N} \to \mathbb{R}$ satisfying the equation

$$p(t) = c_n(t)t^n + \cdots + c_0(t),$$

where the $c_i(t)$ are periodic functions of t, of integral period. A classical result of Ehrhart [5] asserts that when $\mathcal{P}$ is rational, $I_{\mathcal{P}}(t)$ is a degree d quasipolynomial in t. The minimum common period of the coefficients of $I_{\mathcal{P}}(t)$ is called the *period* of $\mathcal{P}$, while any common period of the coefficients is called a *quasiperiod*.

The main question we are concerned with here is how frequently an *irrational* polytope, namely a polytope that is not rational, has an Ehrhart function that is a quasi-polynomial or a polynomial. An interesting class of examples comes from fixing positive numbers u and v with u/v irrational, and studying the Ehrhart function of the triangle $\mathcal{T}_{u,v} \subset \mathbb{R}^2$ with vertices $(0,0), (1/u, 0)$, and $(0, 1/v)$. It turns out that one can completely determine when the Ehrhart function of such a polytope is a quasipolynomial or a polynomial. To state our result, first define the quantities

$$\alpha := u + v, \qquad \beta := 1/u + 1/v. \tag{1.1}$$

Now recall that any polytope whose Ehrhart function is a polynomial is called *pseudo-integral*. In analogy with this, we will call an (irrational) polytope *pseudo-rational* if its Ehrhart function is a quasipolynomial, and we will define the *period* of this polytope to be the minimal period of this quasipolynomial.

We can now state precisely which triangles $\mathcal{T}_{u,v}$ are pseudo-rational and pseudo-integral. In fact, u and v must be certain special conjugate quadratic irrationalities. Given any rational number x, let $\mathrm{num}(x)$ denote the numerator of x, and $\mathrm{den}(x)$ the denominator of x, when x is written in lowest terms.

Theorem 1 *Let u and v be positive numbers with u/v irrational, and let α and β be as in* (1.1).

(i) The triangle $\mathcal{T}_{u,v}$ is pseudo-rational if and only if $\beta \in \mathbb{Z}$ and $\alpha\beta \in \mathbb{Z}$.
(ii) When $\mathcal{T}_{u,v}$ is pseudo-rational, its period divides $\mathrm{num}(\alpha)$.
(iii) The triangle $\mathcal{T}_{u,v}$ is pseudo-integral if and only if (*i*) *is satisfied and in addition, either* $\mathrm{num}(\alpha) = 1$, *or* $(\mathrm{num}(\alpha), \beta/\mathrm{den}(\alpha)) \in \{(3, 3), (2, 4)\}$.

To simplify the notation, we call a triangle $\mathcal{T}_{u,v}$ such that $\beta \in \mathbb{Z}$ and $\alpha\beta \in \mathbb{Z}$ *admissible*. To get a feel for Theorem 1, the following example, which we prove in §2.1, is illustrative.

Example 1 Let u/v be irrational. The pseudo-integral triangle in the family $\mathcal{T}_{u,v}$ with smallest area corresponds to $(u, v) = (\tau^2, 1/\tau^2)$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio.

Since the Ehrhart functions for pseudo-rational $\mathcal{T}_{u,v}$ are quasipolynomials, one can ask to what degree some of the basic results from Ehrhart theory in the rational case apply. In fact, versions of Ehrhart-Macdonald reciprocity, as well as the nonnegativity theorem and monotonicity theorem of the third author, hold for these triangles; see Proposition 3.

Although our primary interest here is for triangles, we can also give examples of irrational polytopes with quasipolynomial Ehrhart functions in any dimension; see Example 2, Example 3, and Example 4.

1.2 P-recursive sequences

One of the key steps in the proof of the "only if" direction of Theorem 1.(i) involves a slightly stronger statement than what is required, which is of potentially independent interest. Recall that a sequence $f(n)$ is $\mathcal{P}$-*recursive*, of order k, if there are polynomials $p_0, \ldots, p_k$, not all 0, such that the recurrence relation

$$p_k(n+k)f(n+k) + \cdots + p_0(n)f(n) = 0$$

holds for all nonnegative integer n. In general, it can be difficult to show that a sequence is not P-recursive. (For more about P-recursive sequences, see for example [12, §6].) However, natural examples of sequences that are not P-recursive are given by the following.

Theorem 2 *Let u and v be positive numbers with u/v irrational, and define α and β by* (1.1). *Assume that $\alpha \in \mathbb{Q}$ and $\beta \in \mathbb{Q}$, but $\mathcal{T}_{u,v}$ is not pseudo-rational. Then the sequence $f(n) := I_{\mathcal{T}_{u,v}}(n)$ is not P-recursive.*

1.3 Period collapse

For integral and rational polytopes, it is known that the period of $\mathcal{P}$ is bounded from above by the minimum integer $\mathcal{D}$ such that the vertices of $\mathcal{D} \cdot \mathcal{P}$ are integral, called the *denominator* of $\mathcal{P}$. The precise relationship between $\mathcal{P}$ and its period can be quite subtle, however.

For example, in their study of vertices of Gelfand-Tsetlin polytopes, De Leora and McAllister [4] constructed an infinite family of non-integral polytopes for which the Ehrhart function is still a polynomial. Later, McAllister and Woods [8] extended this result to any dimension $d \geq 2$. They showed that, given $\mathcal{D}$ and s such that $s|\mathcal{D}$, there exists a d-dimensional polytope with denominator $\mathcal{D}$ whose Ehrhart quasi-polynomial has period s. Other interesting related work appears in (for example) [2][6][13], and unpublished work of the first author and Kleinman.

Any situation where the period of $\mathcal{P}$ is smaller than its denominator is called *period collapse.*

We can view Theorem 1 as a particularly extreme example of period collapse. When u and v are rational, the period collapse question for $\mathcal{T}_{u,v}$ is less well understood than in the irrational case. Nevertheless, we find many new examples of rational triangles of this form exhibiting significant period collapse. The key is the following criterion.

Theorem 3 *Let $u = q/p$ and $v = s/r$ in lowest terms. Then q is a quasiperiod of the Ehrhart quasipolynomial for $\mathcal{T}_{u,v}$ if*

$$s|p, \quad p|(rq+1), \quad \text{and} \quad \gcd\left(\frac{rq+1}{p}, s\right) = 1. \tag{1.2}$$

For example, if $q = 1$, then one obtains the McAllister and Woods example of period collapse mentioned above as a corollary of Theorem 3. Indeed, the theorem implies that the triangle with vertices $(0,0)$, $(p,0)$ and $\left(0, \frac{p-1}{p}\right)$ is a pseudo-integral triangle with denominator p. This triangle is unimodularly equivalent to the pseudo-integral triangle found by McAllister and Woods [8, Theorem 2.2], which has vertices $(0,0)$, $(p,0)$ and $\left(1, \frac{p-1}{p}\right)$, via the map

$$\varphi(x) = x \begin{pmatrix} -1 & 0 \\ -p & 1 \end{pmatrix} + (p, 0).$$

Theorem 3 can be used to construct other pseudo-integral triangles, via the following result.

Corollary 1 *Let $u = q/p, v = s/r$ in lowest terms. The triangle $\mathcal{T}_{u,v}$ is pseudo-integral if*

$$s|p, \quad p|(rq+1), \quad \gcd\left(\frac{rq+1}{p}, s\right) = 1 \tag{1.3}$$

and

$$q|r, \quad r|(sp+1), \quad \gcd\left(\frac{sp+1}{r}, q\right) = 1. \tag{1.4}$$

The criteria of (1.2) also have a nice relationship with the k-Fibonacci numbers. Specifically, we can use Theorem 3 to construct triangles with period dividing any even-index k-Fibonacci number, and high denominator (Theorem 5). If $s = p$ and $r = q$, then the condition (1.2) is also necessary for q to be a quasiperiod, which we also show (Theorem 4).

1.4 Relationship with symplectic geometry

We briefly remark that the triangles $\mathcal{T}_{u,v}$ with u/v irrational seem to have interesting relationships with symplectic geometry. For example, the triangle from Example 1 is closely related to a foundational result of McDuff and Schlenk [9] about symplectic embedding problems. Some of the other pseudo-rational triangles in the family $\mathcal{T}_{u,v}$ also seem to be relevant in the context of symplectic embeddings. This is further explored in work in progress between the first author and Holm, Mandini, and Pires. See Schlenk [10] for a survey.

1.5 Acknowledgements

We thank Bjorn Poonen for his help with Lemma 2. We also thank the anonymous referees for many extremely helpful comments and suggestions.

2 Irrational triangles with Ehrhart quasipolynomials

2.1 Proof of the main theorem

The purpose of this section is to prove Theorem 1. The proof will follow from several lemmas and propositions, which will also imply Theorem 2. We will first state and prove these results, and then finish the section by proving the theorem. Throughout, we assume that α and β are defined by (1.1). At the end of this section, we will also prove Theorem 2, which follows from similar arguments.

We begin with the following simple calculation:

Lemma 1 *For any pair* (u, v),

$$I_{\mathcal{T}_{u,v}}(t) = \sum_{m=0}^{\lfloor \frac{t}{\alpha} \rfloor} \left(1 + \left\lfloor \frac{t - m\alpha}{v} \right\rfloor + \left\lfloor \frac{t - m\alpha}{u} \right\rfloor\right). \tag{2.1}$$

Proof We know that $I_{\mathcal{T}_{u,v}}(t)$ is given by the number of nonnegative integer solutions (x, y) to

$$ux + vy \le t. \tag{2.2}$$

Let $0 \le m \le \lfloor t/\alpha \rfloor$ be an integer. By (1.1), the number of solutions to (2.2) with $x = m$ and $y \ge m$ is $1 + \lfloor \frac{t-m\alpha}{v} \rfloor$. Similarly, the number of solutions to (2.2) with $y = m$ and $x > m$ is $\lfloor \frac{t-m\alpha}{u} \rfloor$. The identity (2.1) now follows.

We can now prove the following proposition, which will be key in proving Theorem 1.(ii), the “if” direction of Theorem 1.(i), and the “if” direction of Theorem 1.(iii).

Proposition 1 *Let u/v be irrational, and assume that $\beta \in \mathbb{Z}$ and $\alpha\beta \in \mathbb{Z}$. Then the triangle $\mathcal{T}_{u,v}$ is pseudo-rational, with period dividing* $\mathrm{num}(\alpha)$.

Proof We will use Lemma 1.

By (1.1), we know that

$$\left\lfloor \frac{t - m\alpha}{v} \right\rfloor = \lfloor (t - m\alpha)(\beta - 1/u) \rfloor$$
$$= \left\lfloor (t\beta - m\alpha\beta) - \frac{(t - m\alpha)}{u} \right\rfloor$$
$$= (t\beta - m\alpha\beta) + \left\lfloor -\frac{(t - m\alpha)}{u} \right\rfloor.$$

So, we can rewrite the right hand side of (2.1) as

$$\sigma(t) + \sum_{m=0}^{\lfloor \frac{t}{\alpha} \rfloor} (t\beta - m\alpha\beta), \tag{2.3}$$

where

$$\sigma(t) := \begin{cases} 1 & \text{if } t \text{ is divisible by } \mathrm{num}(\alpha), \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Now, $\sigma(t)$ is a quasipolynomial, of period $\mathrm{num}(\alpha)$, the quantity

$$\sum_{m=0}^{\lfloor \frac{t}{\alpha} \rfloor} (1 + t\beta - m\alpha\beta)$$

is the Ehrhart function of the rational triangle

$$\mathrm{Conv}\{(0,0), (0, \alpha^{-1}), (\beta, 0)\},$$

and so is a quasipolynomial of period dividing $\mathrm{num}(\alpha)$, and $\lfloor \frac{t}{\alpha} \rfloor$ is also a quasipolynomial, of period $\mathrm{num}(\alpha)$. Thus, (2.3) is a quasipolynomial of period dividing $\mathrm{num}(\alpha)$, and so by (2.1), $I_{\mathcal{T}_{u,v}}(t)$ is as well.

To prove the "only if" direction of Theorem 1.(i) and the "only if" direction of Theorem 1.(iii), the following lemma, which was explained to us by Bjorn Poonen, will be helpful:

Lemma 2 *Let u/v be irrational. Then $I_{\mathcal{T}_{u,v}}(t) = \frac{1}{2uv}t^2 + \frac{1}{2}\left(\frac{1}{u} + \frac{1}{v}\right)t + o(t)$ for $t \in \mathbb{R}_{>0}$.*

Proof By scaling, we may assume that $u = 1$. By counting in each vertical line, we see that

$$I_{\mathcal{T}_{1,1/v}}(t) = \sum_{m=0}^{\lfloor t \rfloor} (\lfloor t/v - m/v \rfloor + 1). \tag{2.5}$$

It is convenient to define a function $f(x)$ by

$$\lfloor x \rfloor + 1 = x + f(x).$$

We can then rewrite (2.5) as

$$I_{\mathcal{T}_{1,v}}(t) = \sum_{m=0}^{\lfloor t \rfloor} (t/v - m/v) + \sum_{m=0}^{\lfloor t \rfloor} f(t/v - m/v). \tag{2.6}$$

Now note that the mth term in the first sum computes the area of the trapezoid defined by $m - 1/2 \leq x \leq m + 1/2, 0 \leq y \leq t/v - x/v$. The first sum is therefore within $O(1)$ of the area of the triangle defined by $-1/2 \leq x \leq t, 0 \leq y \leq t/v - x/v$, and so we have

$$\sum_{m=0}^{\lfloor t \rfloor} (t/v - m/v) = (t^2 + t)/2v + O(1). \tag{2.7}$$

The second sum is a sum of values of a bounded, integrable, periodic function f at points that become equidistributed mod 1 as $t \to \infty$ (by Weyl's criterion for uniform distribution), and so we find that

$$\sum_{m=0}^{\lfloor t \rfloor} f(t/v - m/v) = t \int_0^1 f(x)dx + o(t) = t/2 + o(t). \tag{2.8}$$

Lemma 2 now follows by combining (2.6), (2.7), and (2.8).

Corollary 2 *Let u/v be irrational. If $\mathcal{T}_{u,v}$ is pseudo-rational, then $\alpha \in \mathbb{Q}$ and $\beta \in \mathbb{Q}$.*

Proof The assumptions of the corollary imply that $I_{\mathcal{T}_{u,v}}(t)$ is a quasipolynomial in the positive integer t, of period C. Write $I_{\mathcal{T}_{u,v}}(Ct) = A(C^2t^2) + B(Ct) + D$. We know that the number $I_{\mathcal{T}_{u,v}}(t)$ must always be an integer. Hence, the numbers A and B here must be rational. Lemma 2 now implies that $u + v$ and $1/u + 1/v$ must be rational as well.

We can now prove the key:

Proposition 2 *Assume that $\alpha \in \mathbb{Q}$, $\beta \in \mathbb{Q}$, and u/v is irrational. If the Ehrhart function of $\mathcal{T}_{u,v}$ is P-recursive, then $\beta \in \mathbb{Z}$ and $\alpha\beta \in \mathbb{Z}$.*

Proof By Lemma 1, we can write, $I_{\mathcal{T}_{u,v}}(t)$ as

$$\sum_{m=0}^{\lfloor \frac{t}{\alpha} \rfloor} \left(1 + \left\lfloor \frac{t - m\alpha}{v} \right\rfloor + \left\lfloor \frac{t - m\alpha}{u} \right\rfloor \right).$$

We know that $\left\lfloor \frac{t-m\alpha}{v} \right\rfloor = \left\lfloor \beta(t - m\alpha) - \frac{t-m\alpha}{u} \right\rfloor$. To simplify the notation for what will follow, define[1]

$$\sigma(t) := \# \{ m \in \mathbb{Z} \mid 0 \leq m \leq \lfloor t/\alpha \rfloor \text{ and } \{\beta(t - m\alpha)\} \geq \{(t - m\alpha)/u\} \}, \tag{2.9}$$

[1] This might seem like an abuse of notation, since we also defined a function $\sigma(t)$ in (2.4). Notice, however, that under the assumptions of Proposition 1, in particular under the assumption that α and $\alpha\beta$ are both integers, the functions in (2.4) and (2.9) agree.

where $\{\cdot\}$ denotes the fractional part function. Also define

$$q(t) := \sum_{m=0}^{\lfloor t/\alpha \rfloor} \lfloor t\beta - m\alpha\beta \rfloor. \tag{2.10}$$

Then,

$$I_{\mathcal{T}_{u,v}}(t) = \sigma(t) + q(t). \tag{2.11}$$

Now assume that $I_{\mathcal{T}_{u,v}}(t)$ is P-recursive, and write the recurrence as

$$p_s(t)\Big(\sigma(t) + q(t)\Big) + \cdots + p_0(t)\Big(\sigma(t-s) + q(t-s)\Big) = 0. \tag{2.12}$$

The function q is a quasipolynomial in t, since it is the Ehrhart function of the rational triangle $\mathrm{Conv}\{(0,0),(0,\alpha^{-1}),(\beta,0)\}$.

Write $\alpha = \frac{c}{d}$ in lowest terms, and define the function

$$\mathrm{head}(t) = \#\left\{m \in \mathbb{Z} \mid 0 \le m \le (d-1) \text{ and } \{\beta(t-m\alpha)\} > \{(t-m\alpha)/u\}\right\}.$$

Also introduce the two rotations f_1, f_2 from $[0,1]$ (mod 1) to itself given by:

$$f_1(x) = \{x - \alpha\beta\}, \qquad f_2(x) = \{x - \alpha/u\}.$$

Then, $f_1^d(x) = \{x - c\beta\}$ and $f_2^d(x) = \{x - c/u\}$, so $\sigma(t) - \sigma(t-c) = \mathrm{head}(t)$ for all $t \ge c$. Now for $t \ge c+s$ apply (2.12) twice to get

$$\begin{aligned} &p_s(t)\,\mathrm{head}(t) + \cdots + p_0(t)\,\mathrm{head}(t-s) + \\ &p_s(t)q(t) - p_s(t-c)q(t-c) + \cdots + p_0(t)q(t-s) - p_0(t-c)q(t-c-s) \\ &+ \sigma(t-c)(p_s(t) - p_s(t-c)) + \cdots + \sigma(t-s-c)(p_0(t) - p_0(t-c)) = 0. \end{aligned} \tag{2.13}$$

Now assume that either β or $\alpha\beta$ or both are not integers. We will derive a contradiction.

Write $\beta = k/l$ in lowest terms, and let C be the period of the quasipolynomial q. Introduce the set $S = \{1 + iCl \,|\, i \in \mathbb{Z}_{\ge 0}\}$. Then, if β is not an integer, then for any $t \in S$, $\{t\beta\}$ is some fixed nonzero number a_0 independent of t. If β is an integer, then $\alpha\beta$ is not, and in particular $\{-\alpha\beta\}$ is some nonzero number b_0. To complete the proof of Proposition 2, we will need the following lemma.

Lemma 3 *Under the assumptions of Proposition 2:*

(a) If β is not an integer, there exists $\varepsilon > 0$ such that if $\{t/u\} \in (a_0 - \varepsilon, a_0 + \varepsilon)$, then:

- *(i)* $\mathrm{head}(t)$ *is determined by whether or not $a_0 > \{t/u\}$, and* $\mathrm{head}(t)$ *will differ depending on whether or not $a_0 > \{t/u\}$*
- *(ii) If, in addition, $t \in S$ and $t \ge s + c$,*

$$\mathrm{head}(t \pm 1), \ldots, \mathrm{head}(t \pm s)$$

do not depend on t.

(b) If β is an integer, there exists $\varepsilon > 0$ such that if $\{(t-\alpha)/u\} \in (b_0 - \varepsilon, b_0 + \varepsilon)$, then:

(i) head(t) *is determined by whether or not* $b_0 > \{(t-\alpha)/u\}$*, and* head(t) *will differ depending on whether or not* $b_0 > \{(t-\alpha)/u\}$.

(ii) If, in addition, $t \in S$ *and* $t \geq s + c$*, then:*

$$\text{head}(t \pm 1), \ldots, \text{head}(t \pm s)$$

do not depend on t.

Proof To prove Lemma 3.(a).(i), note that since $\alpha\beta$ is rational, while α/u is irrational, for rational x we can never have $f_1^m(x) = f_2^m(x)$ for any positive integer m. With this in mind, consider $f_1(a_0), \ldots, f_1^{d-1}(a_0)$. If we take ε sufficiently small, we can guarantee that for any $y \in (a_0-\varepsilon, a_0+\varepsilon)$, $f_2^m(y) \neq f_1^m(a_0)$ for any $1 \leq m \leq d-1$. By shrinking ε if necessary, we can also conclude that for any y in this interval, $f_2^i(y) \neq 0$, hence Lemma 3.(a)(i) is proved.

We can use the same argument to prove Lemma 3.(b).(i).

To prove Lemma 3.(a).(ii), let $a_1^{\pm}, \ldots, a_s^{\pm}$ be the rational numbers $\{a_0 \pm i\beta\}$. For any $1 \leq i \leq s$, we can never have $\{a_0 \pm \frac{i}{u}\} = a_i^{\pm}$, since $\{a_0 \pm \frac{i}{u}\}$ is irrational. Consider then $a_i^{\pm}$ and $z_i^{\pm} = \{a_0 \pm \frac{i}{u}\}$. By the argument in the previous paragraph, if ε is sufficiently small and $x_i^{\pm}$ is some irrational number in $(z_i^{\pm} - \varepsilon, z_i^{\pm} + \varepsilon)$, then

$$\#\left\{m \in \mathbb{Z} \ \mid \ 0 \leq m \leq (q-1) \ \text{ and } \ \{a_i^{\pm} - m\alpha\beta\} > \{x_i^{\pm} - \frac{m\alpha}{u}\}\right\}$$

does not depend on $x_i^{\pm}$. Since we can make $\{\frac{t \pm i}{u}\}$ arbitrarily close to $z_i^{\pm}$ by making $\{\frac{t}{u}\}$ sufficiently close to a_0, Lemma 3.(a).(ii) now follows. As above, the same argument proves Lemma 3.(b).(ii).

We can now complete the proof of Proposition 2.

Let $p_{s-\tilde{s}}$ be one of the polynomials p_j, with the property that no other p_j has higher degree. Let t_i be any infinite sequence with the property that $t_i - \tilde{s} \in S$. Then, since we are fixing t, mod C,

$$\text{head}(t_i - \tilde{s}) = a(t_i)/p_{s-\tilde{s}}(t_i) - R(t_i)/p_{s-\tilde{s}}(t_i), \tag{2.14}$$

where a is some fixed polynomial of t_i, whose coefficients do not depend on i. Meanwhile, the term $R(t)$ is given by

$$R(t_i) = \sum_{j=0}^{s} \sigma(t_i - c - j)(p_{s-j}(t_i) - p_{s-j}(t_i - c)).$$

We know that if $x_r - \tilde{s} \in S$ is any sequence of points tending to $+\infty$, then there exists some constant M such that

$$\lim_{r\to\infty} R(x_r)/p_{s-\tilde{s}}(x_r) = M, \tag{2.15}$$

because the degree of $p_{s-\tilde{s}}$ is strictly greater than the degree of any of the polynomials $p_{s-j}(t) - p_{s-j}(t-p)$, while $\sigma(t) = \ell(t) + o(t)$, for some linear polynomial $\ell(t)$ by combining (2.11) and Lemma 2.

Now note that as t ranges over S, $\{t/u\}$ is dense in $(0, 1)$.

If β is not an integer, we produce a contradiction as follows. Take an infinite sequence t_i such that $\{(t_i - \tilde{s})/u\} \in (a_0, a_0 + \varepsilon)$ and $t_i - \tilde{s} \in S$. Since $\{\frac{t_i - \tilde{s}}{u}\} \in (a_0, a_0 + \varepsilon)$, by Lemma 3.(ii), (2.15), and (2.14), the rational function $a(t_i)/p_{s-\tilde{s}}(t_i)$ must have a horizontal asymptote as $t_i \to \infty$. Now choose some collection of $\hat{t}_i - \tilde{s} \in S$ such that $\{(\hat{t}_i - \tilde{s})/u\} \in (a_0 - \varepsilon, a_0)$ and $p_{s-\tilde{s}}(\hat{t}_i) \neq 0$. By again invoking Lemma 3.(ii), (2.15), and (2.14), the rational function $a(\hat{t}_i)/p_{s-\tilde{s}}(\hat{t}_i)$ has a horizontal asymptote as $\hat{t}_i$ goes to $+\infty$. Since $a_0 > \{\frac{\hat{t}_i - \tilde{s}}{u}\}$ while $a_0 < \{\frac{t_i - \tilde{s}}{u}\}$, these two asymptotes are different, by Lemma 3.(ii); this can not happen for a rational function.

If β is an integer, then the argument in the previous paragraph still produces a contradiction, by choosing $t - \tilde{s}$ ranging over S with $\{\frac{t - \tilde{s} - \tilde{\alpha}}{u}\}$ in $(b_0 - \varepsilon, b_0 + \varepsilon)$ instead.

We can now finally give the proof of Theorem 1.

Proof The "if" direction of Theorem 1.(i) follows directly from Proposition 1. To prove the "only if" direction of Theorem 1.(i), first note that if the Ehrhart function of $\mathcal{T}_{u,v}$ is pseudo-rational, then by Corollary 2 we can assume that $\alpha \in \mathbb{Q}$ and $\beta \in \mathbb{Q}$. Moreover, if $\mathcal{T}_{u,v}$ is pseudo-rational, then the Ehrhart function must be P-recursive. The "only if" direction now follows from Proposition 2.

Theorem 1.(ii) follows directly from Proposition 1.

To prove Theorem 1.(iii), we can assume that β and $\alpha\beta$ are integers. We can therefore apply the argument in Proposition 1 to conclude that $I_{\mathcal{T}_{u,v}}(t)$ must be given by (2.3). Now rewrite (2.3) as

$$\sigma(t) + (\lfloor t/\alpha \rfloor + 1)\left(t\beta - \frac{\alpha\beta\lfloor t/\alpha \rfloor}{2}\right). \tag{2.16}$$

Write $k := \beta/\operatorname{den}(\alpha)$.

Let $z := t\operatorname{den}(\alpha) \pmod{\operatorname{num}(\alpha)}$. Then $\lfloor \frac{t}{\alpha} \rfloor = \frac{\operatorname{den}(\alpha)t - z}{\operatorname{num}(\alpha)}$. Hence we can rewrite (2.16) as

$$\sigma(t) + \frac{(t\operatorname{den}(\alpha) - (z - \operatorname{num}(\alpha))(tk\operatorname{den}(\alpha) + zk)}{2\operatorname{num}(\alpha)}$$

$$= \frac{t^2 k \operatorname{den}(\alpha)^2 + t\operatorname{num}(\alpha)\operatorname{den}(\alpha)k + zk(\operatorname{num}(\alpha) - z) + 2\operatorname{num}(\alpha)\sigma(t)}{2\operatorname{num}(\alpha)}. \tag{2.17}$$

The coefficients of t^2 and t in (2.17) do not depend on t. So $\mathcal{T}_{u,v}$ is pseudo-integral if and only if

$$\frac{zk(\operatorname{num}(\alpha) - z) + 2\operatorname{num}(\alpha)\sigma(t)}{2\operatorname{num}(\alpha)} = 1 \tag{2.18}$$

for all equivalence classes z of t modulo $\operatorname{num}(\alpha)$. To prove the "only if" part of Theorem 1.1 (iii), we assume that $\mathcal{T}_{u,v}$ is pseudo-integral. Now choose t such that $t\operatorname{den}(\alpha) \equiv 1 \pmod{\operatorname{num}(\alpha)}$. By (2.18), this gives

$$\frac{k(\operatorname{num}(\alpha) - 1)}{2\operatorname{num}(\alpha)} = 1, \tag{2.19}$$

so $k = 2\,\mathrm{num}(\alpha)/(\mathrm{num}(\alpha)-1)$. Now observe that β and $\alpha\beta$ are integers if and only if β and k are. The only solutions to (2.19) with $\mathrm{num}(\alpha) > 1$ and k an integer are $(3,3)$ and $(2,4)$. Conversely, if $\mathrm{num}(\alpha) = 1$, or $(\mathrm{num}(\alpha), k) \in \{(3,3),(2,4)\}$, then (2.18) holds for all equivalence classes of z.

We can also now give the proof that was owed for Example 1.

Proof Let (u, v) be such that $\mathcal{T}_{u,v}$ is pseudo-integral. Then $uv = \frac{\alpha}{\beta}$. To minimize the area of $\mathcal{T}_{u,v}$, we would like to maximize $\frac{\alpha}{\beta}$; the maximum occurs when $\mathrm{den}(\alpha) = 1$. By Theorem 1, if $\mathrm{num}(\alpha) > 1$, then $(\mathrm{num}(\alpha), \beta/\mathrm{den}(\alpha)) \in \{(3,3),(2,4)\}$. The largest possible value in these two cases is $\frac{\alpha}{\beta} = 1$, which is uniquely obtained by the "golden mean" triangle.

For $\alpha = 1$, the only possible value of β that could give an equally large $\frac{\alpha}{\beta}$ is when $\beta = 1$. There are no real numbers satisfying $u+v = 1, 1/u+1/v = 1$, however.

Remark 1 Similar arguments can be used to give another characterization of the "golden mean" triangle: it is the only pseudo-integral triangle of the form $\mathcal{T}_{u,v}$ where u and v are quadratic irrational algebraic integers. We omit the proof for brevity.

Finally, we can prove Theorem 2.

Proof We argue by contradiction. Assume that the sequence $f(n)$ is P-recursive. Then, we can apply Proposition 2 to conclude that $\beta \in \mathbb{Z}$ and $\alpha\beta \in \mathbb{Z}$. We can then apply the "if" direction of Theorem 1.(i) to conclude that $\mathcal{T}_{u,v}$ is pseudo-rational, contradicting one of the hypotheses of the Theorem.

2.2 Properties of admissible irrational triangles

We now prove some properties of the Ehrhart functions of admissible triangles that mirror properties from the rational case; compare [1, §3], [3]. For simplicity, we state some of the results for pseudo-integral triangles, although we expect they should hold more generally. Along those lines, recall from [1, Lem. 3.9] that if $\mathcal{T}_{u,v}$ is pseudo-integral, then we can write

$$\sum_{t\geq 0} I_{\mathcal{T}_{u,v}}(t) z^t = \frac{g_{u,v}(z)}{(1-z)^3},$$

where $g_{u,v}(z)$ is a polynomial of degree at most 2.

Proposition 3 *Let $\mathcal{T}_{u,v}$ be admissible. Then the Ehrhart function of $\mathcal{T}_{u,v}$ satisfies*

(i) (Reciprocity) If t is positive, then

$$I_{\mathcal{T}_{u,v}}(-t) = \{\#(t\mathcal{T}^o_{u,v} \cap \mathbb{Z}^2)\} + \sigma(t),$$

where the superscript ^{o} denotes the interior.

(ii) (Nonnegativity) If $\mathcal{T}_{u,v}$ is pseudo-integral, then each coefficient of $g_{u,v}$ is nonnegative.
(iii) (Monotonicity) If $\mathcal{T}_{u,v}$ and $\mathcal{T}_{u',v'}$ are both pseudo-integral, and $\mathcal{T}_{u,v} \subset \mathcal{T}_{u',v'}$, then each coefficient of $g_{u,v}$ is less than or equal to the corresponding coefficient of $g_{u',v'}$.

Proof To prove Proposition 3.(i), note that the number of lattice points on the boundary of the triangle with vertices $(t/u, 0), (0, t/v)$ and $(0, 0)$ is $\lfloor t/u \rfloor + \lfloor t/v \rfloor + 1$, plus the number of points on the slant edge. We know that

$$\lfloor t/u \rfloor + \lfloor t/v \rfloor = t\beta - 1,$$

and we know that $ux + vy = t$ only if $x = y$. Proposition 3.(i) now follows by subtracting the number of lattice points on the boundary from the formula in (2.17).

To prove Proposition 3.(ii) and Proposition 3.(iii), note that if $g_{u,v}(z) = a_0 + a_1 z + a_2 z^2$, then $a_0 = 1$, $a_1 = I_{\mathcal{T}_{u,v}}(1) - 3$, and $a_2 = 3 - 3I_{\mathcal{T}_{u,v}}(1) + I_{\mathcal{T}_{u,v}}(2)$ (here, we are implicitly using the fact that $I_{\mathcal{T}_{u,v}}(0) = 1$, as can be seen by (2.18).) To show Proposition 3.(ii), we therefore first have to show that $I_{\mathcal{T}_{u,v}}(1) \geq 3$, or equivalently, by (2.17), that $\beta/\alpha + \beta \geq 4$. If $\left(\mathrm{num}(\alpha), \frac{\beta}{\mathrm{den}(\alpha)}\right) \in \{(3,3),(2,4)\}$, then this holds, so by Theorem 1 we can assume that $\mathrm{num}(\alpha) = 1$. It suffices to show that $\beta\,\mathrm{den}(\alpha) + \beta \geq 4$, which follows immediately since we can never have $(\alpha, \beta) \in \{(1,1), (\frac{1}{2}, 1)\}$. So nonnegativity for a_1 follows. For nonnegativity of a_2, we need to show that $(\beta - \alpha\beta)/\alpha \geq -2$. If $\mathrm{num}(\alpha) = 1$, then this is automatic; if $\left(\mathrm{num}(\alpha), \frac{\beta}{\mathrm{den}(\alpha)}\right) \in \{(3,3),(2,4)\}$, then it holds as well, which proves Proposition 3.(ii).

For Proposition 3.(iii), we are given that $1/u \leq 1/u'$ and $1/v \leq 1/v'$. That $a_0 \leq a_0'$ and $a_1 \leq a_1'$ are immediate; to see that $a_2 \leq a_2'$, we need to show that $\beta(\frac{1}{\alpha} - 1) \leq \beta'(\frac{1}{\alpha'} - 1)$. This follows from $1/u \leq 1/u', 1/v \leq 1/v'$.

2.3 Examples in other dimensions

Here we briefly mention some examples of polytopes in other dimensions that are not rational, but nevertheless have Ehrhart functions that are polynomials.

In dimension 1, such examples are easy to come by:

Example 2 Let $\mathcal{P} = [u, v] \subset \mathbb{R}$, where u and v are irrational numbers with $v - u = m$ and m is an integer. Then for positive integer t, $I_{\mathcal{P}}(t) = tm$.

Proof We know that $I_{\mathcal{P}}(t) = \lfloor tv \rfloor - \lfloor tu \rfloor = tm$.

In higher dimensions, we do not currently know many examples of truly different character. However, one has:

Example 3 Suppose that $n \geq 2$ and let $\mathcal{P}_n$ denote the polytope in $\mathbb{R}^n$ with the following $n+1$ vertices:

$$(0,0,\ldots,0),(1,0,\ldots,0),(0,1,\ldots,0),$$

$$\left(0,0,1,\ldots,0),\ldots,(0,\ldots,\frac{1}{u},0\right),\left(0,\ldots,0,\frac{1}{v}\right),$$

where (u,v) satisfies the conditions in Theorem 1.1 (iii). Then $I_{\mathcal{P}}(t)$ is a polynomial in t.

To see this, note that we have:

$$\mathcal{P}_n = \{(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n \mid x_i \geq 0, x_1+x_2+\cdots+x_{n-2}+ux_{n-1}+vx_n \leq 1\}$$

and

$$\begin{aligned} I_{\mathcal{P}_n}(t) &= \#\{(x_1,x_2,\ldots,x_n) \in \mathbb{Z}^n \mid x_i \geq 0, x_1+x_2+\cdots+x_{n-2}+ux_{n-1}+vx_n \leq t\} \\ &= \sum_{i=0}^{t} \#\{(x_2,\ldots,x_n) \in \mathbb{Z}^{n-1} \mid x_i \geq 0, x_2+\cdots+x_{n-2}+ux_{n-1}+vx_n \leq t-i\} \\ &= \sum_{i=0}^{t} I_{\mathcal{P}_{n-1}}(t-i). \end{aligned}$$

So, the proof is immediate by induction on n.

A more interesting example is given in dimension three:

Example 4 The polytope with vertices $(0,0,0)$, $(\frac{1}{2},0,0)$, $(0,2+\sqrt{2},0)$ and $(0,0,2-\sqrt{2})$ has an Ehrhart function which is a polynomial.

The proof of Example 4 is deferred to §3.3.

3 Rational examples

We now give the proof of Theorem 3.

Proof Firstly, it is easy to see that $\gcd(rq, ps) = 1$. Let ξ_m denote a primitive mth root of unity. By [1, Theorem 2.10], we have

$$\begin{aligned}
I_{\mathcal{T}_{q/p,s/r}}(t) &= \frac{1}{2\cdot rq\cdot ps}(t\cdot pr)^2 + \frac{1}{2}(t\cdot pr)\left(\frac{1}{rq}+\frac{1}{ps}+\frac{1}{rq\cdot ps}\right)\\
&+\frac{1}{4}\left(1+\frac{1}{rq}+\frac{1}{ps}\right)+\frac{1}{12}\left(\frac{rq}{ps}+\frac{ps}{rq}+\frac{1}{rq\cdot ps}\right)\\
&+\frac{1}{rq}\sum_{j=1}^{rq-1}\frac{\xi_{rq}^{-jt\cdot pr}}{(1-\xi_{rq}^{j\cdot ps})(1-\xi_{rq}^{j})}+\frac{1}{ps}\sum_{l=1}^{ps-1}\frac{\xi_{ps}^{-lt\cdot pr}}{(1-\xi_{ps}^{l\cdot rq})(1-\xi_{ps}^{l})}\\
&=\frac{pr}{2qs}\cdot t^2+\frac{1}{2}\left(\frac{p}{q}+\frac{r}{s}+\frac{1}{qs}\right)t+\frac{1}{4}\left(1+\frac{1}{rq}+\frac{1}{ps}\right)\\
&\quad+\frac{1}{12}\left(\frac{rq}{ps}+\frac{ps}{rq}+\frac{1}{rq\cdot ps}\right)\\
&+\frac{1}{rq}\sum_{j=1}^{rq-1}\frac{\xi_{q}^{-jtp}}{(1-\xi_{rq}^{j\cdot ps})(1-\xi_{rq}^{j})}+\frac{1}{ps}\sum_{l=1}^{ps-1}\frac{\xi_{s}^{-ltr}}{(1-\xi_{ps}^{l\cdot rq})(1-\xi_{ps}^{l})}. \qquad (3.1)
\end{aligned}$$

Then it suffices to show that

$$\frac{1}{ps}\sum_{l=1}^{ps-1}\frac{\xi_{s}^{-ltr}}{(1-\xi_{ps}^{l\cdot rq})(1-\xi_{ps}^{l})}$$

is a constant function in t. In fact, writing $l = is + u : 0 \le i < p, 0 \le u < s$ and using the fact that $rq \equiv -1 \pmod{p}$, we have

$$\begin{aligned}
&\frac{1}{ps}\sum_{l=1}^{ps-1}\frac{\xi_{s}^{-ltr}}{(1-\xi_{ps}^{l\cdot rq})(1-\xi_{ps}^{l})}\\
&=\frac{1}{ps}\sum_{i=1}^{p-1}\frac{1}{(1-\xi_{p}^{irq})(1-\xi_{p}^{i})}+\frac{1}{ps}\sum_{u=1}^{s-1}\xi_s^{-utr}\sum_{i=0}^{p-1}\frac{1}{(1-\xi_{ps}^{(is+u)rq})(1-\xi_{ps}^{is+u})}\\
&=\frac{1}{ps}\sum_{i=1}^{p-1}\frac{1}{(1-\xi_{p}^{irq})(1-\xi_{p}^{i})}+\frac{1}{ps}\sum_{u=1}^{s-1}\xi_s^{-utr}\sum_{i=0}^{p-1}\frac{1}{(1-\xi_{ps}^{urq-is})(1-\xi_{ps}^{u+is})} \qquad (3.2)
\end{aligned}$$

Keeping in mind that $s|(rq+1)$ and $\gcd\left(\frac{rq+1}{p}, s\right) = 1$, we find that

$$ps \nmid u(rq+1), \quad \text{and} \quad ps|u(rq+1)p$$

for any $1 \le u \le s-1$. By Lem. 2.1 in arXiv:1307.5493v1, we deduce that

$$\sum_{i=0}^{p-1}\frac{1}{(1-\xi_{ps}^{urq-is})(1-\xi_{ps}^{u+is})}=0$$

for any $1 \le u \le s-1$. Therefore, we have

$$\frac{1}{ps}\sum_{l=1}^{ps-1}\frac{\xi_{s}^{-ltr}}{(1-\xi_{ps}^{l\cdot rq})(1-\xi_{ps}^{l})}=\frac{1}{ps}\sum_{i=1}^{p-1}\frac{1}{(1-\xi_{p}^{irq})(1-\xi_{p}^{i})},$$

which is a constant function in t. It follows from (3.1) that q is a quasiperiod of $I_{\mathcal{T}_{q/p,s/r}}(t)$.

Next we prove Corollary 1.

Proof First we note that $\gcd(q,s) = 1$. In fact, suppose that d is an arbitrary common divisor of q and s, then we obviously have $d|rq$. On the other hand, the assumption $d|s$ and the condition (1.3) imply that $d|(rq+1)$. It follows that $d|1$ and so $\gcd(q,s) = 1$. Next we show that both q and s are quasiperiods of $I_{\mathcal{T}_{q/p,s/r}}(t)$.

By Theorem 3, condition (1.3) implies that q is a quasiperiod of $I_{\mathcal{T}_{q/p,s/r}}(t)$. On the other hand, it is obvious that

$$(x,y) \mapsto (y,x)$$

is a bijection between lattice points in triangles $\mathcal{T}_{q/p,s/r}$ and $\mathcal{T}_{s/r,q/p}$. So we have

$$I_{\mathcal{T}_{q/p,s/r}}(t) = I_{\mathcal{T}_{s/r,q/p}}(t).$$

By Theorem 3 again, condition (1.4) means that s is also a quasiperiod of $I_{\mathcal{T}_{q/p,s/r}}(t)$. It follows that 1 is a quasiperiod of $I_{\mathcal{T}_{q/p,s/r}}(t)$. Therefore, the triangle $I_{\mathcal{T}_{q/p,s/r}}(t)$ is a pseudo-integral triangle.

3.1 The case where $u = 1/v$

As mentioned in the introduction, for when $s = p$ and $r = q$, one can also give a necessary condition for a version of Theorem 3. Specifically, we have:

Theorem 4 *Suppose that p, q are relatively prime positive integers. Then q is a quasiperiod of $I_{\mathcal{T}_{q/p,p/q}}(t)$ if and only if*

$$p|(q^2+1) \qquad \text{and} \qquad \gcd\left(\frac{q^2+1}{p}, p\right) = 1. \tag{3.3}$$

Proof Clearly, the "if " part follows from the proof of Theorem 3. We now proceed to the proof of the "only if" part.

Suppose that q is a quasiperiod of $I_{\mathcal{T}_{q/p,p/q}}(t)$. By (3.1), we deduce that

$$f_{p,q}(t) := \frac{1}{p^2} \sum_{l=1}^{p^2-1} \frac{\xi_p^{-ltq}}{(1-\xi_{p^2}^{lq^2})(1-\xi_{p^2}^{l})}$$

is a periodic function of t with period q. Clearly, p is also a period of $f_{p,q}(t)$. Since $(p,q) = 1$, we deduce that $f_{p,q}(t)$ is a constant function of t. It follows from (3.2) that

$$\frac{1}{p^2} \sum_{u=1}^{p-1} \xi_p^{-utq} \sum_{i=0}^{p-1} \frac{1}{(1-\xi_{p^2}^{uq^2-ip})(1-\xi_{p^2}^{u+ip})} = C$$

for some constant C. Keeping in mind the fact that $\gcd(p,q)=1$, we have

$$\{\xi_p^{-utq} : 1 \le u \le p-1\} = \{\xi_p^{jt} : 1 \le j \le p-1\}.$$

So

$$\{1, \xi_p^{-tq}, \xi_p^{-2tq}, \dots, \xi_p^{-(p-1)tq}\} = \{1, \xi_p^{t}, \xi_p^{2t}, \dots, \xi_p^{(p-1)t}\},$$

which consists of p linearly independent functions from $\mathbb{N}$ to $\mathbb{C}$. Hence we have

$$\sum_{i=0}^{p-1} \frac{1}{(1-\xi_{p^2}^{uq^2-ip})(1-\xi_{p^2}^{u+ip})} = 0$$

for any $1 \le u \le p-1$. By applying Lem. 2.1 in arXiv:1307.5493v1, we deduce that

$$p | u(q^2+1) \qquad \text{and} \qquad p^2 \nmid u(q^2+1)$$

for each $1 \le u \le p-1$. Now we can conclude immediately that

$$p | (q^2+1) \qquad \text{and} \qquad \gcd\left(p, \frac{q^2+1}{p}\right) = 1.$$

3.2 The k-Fibonacci numbers

In Thm. 1.2 of arXiv:1307.5493v1, it was shown that $\mathcal{T}_{q/p,p/q}$ is pseudo-integral if and only if $p=q=1$ or $\{p,q\} = \{F_{2k-1}, F_{2k+1}\}$ for some $k \ge 1$, where p and q are relatively prime positive integers, F_m denotes the mth Fibonacci number. We now further study the relationship between the period collapse problem and recursive sequences, by proving a similar result, involving two consecutive terms in the sequence of generalized Fibonacci numbers.

Recall first for any integer $k \ge 1$, the k-Fibonacci sequence $\{F_n(k)\}$, defined recursively as follows:

$$F_0(k) = 0, F_1(k) = 1,\ F_n(k) = kF_{n-1}(k) + F_{n-2}(k) \quad (n \ge 2).$$

Clearly, when $k=1$, we get the Fibonacci sequence. For notational simplicity, for any $k \ge 1, n \ge 2$, we let

$$I_{k,n}(t) := I_{\mathcal{T}_{F_n(k)/F_{n-1}(k),F_{n-1}(k)/F_n(k)}}(t).$$

In the following we shall consider period collapse in $I_{k,n}(t)$. To this end, we need the following immediate facts:

Fact 1: For any $k,n \ge 1$, $\gcd(F_n(k), F_{n-1}(k)) = 1$ and $\gcd(F_n(k), k) = 1$.

Fact 2: For any $k,n \ge 1$, we have

$$F_n(k)^2 - kF_{n-1}(k)F_n(k) - F_{n-1}(k)^2 + (-1)^n = 0.$$

Both Fact 1 and Fact 2 can be verified immediately by induction on n. We only give the proof of Fact 2 here. Clearly, the fact holds for $n = 1$. Assume $n \geq 2$ and Fact 2 is true for $n - 1$ and then we have

$$\begin{aligned}
&F_n(k)^2 - kF_{n-1}(k)F_n(k) - F_{n-1}(k)^2 + (-1)^n \\
&\quad = (kF_{n-1}(k) + F_{n-2}(k))^2 - kF_{n-1}(k)(kF_{n-1}(k) + F_{n-2}(k)) - F_{n-1}(k)^2 + (-1)^n \\
&\quad = kF_{n-1}(k)F_{n-2}(k) + F_{n-2}(k)^2 - F_{n-1}(k)^2 + (-1)^n \\
&\quad = -(F_{n-1}(k)^2 - kF_{n-1}(k)F_{n-2}(k) - F_{n-2}(k)^2 + (-1)^{n-1}) \\
&\quad = 0.
\end{aligned}$$

It follows from Fact 1 and Fact 2 that, when n is even, both $(p, q) = (F_{n-1}(k), F_n(k))$ and $(p, q) = (F_{n+1}(k), F_n(k))$ satisfy condition (3.3). We therefore get:

Theorem 5 *For any $k \geq 1$ and even integer $n \geq 2$, $F_n(k)$ is a common quasiperiod of $I_{k,n}(t)$ and $I_{k,n+1}(t)$.*

3.3 Tetrahedra

We now give a few higher dimensional examples of period collapse.

Recall first the sequence given by $a_1 = 2, a_2 = 3, a_3 = 10, a_4 = 17$ and

$$a_n = 6a_{n-2} - a_{n-4}. \tag{3.4}$$

It follows from Thm. 1.6(i) in arXiv:1307.5493v1 that for each $n \geq 1$, the triangle with vertices $(0, 0)$, $\left(\frac{a_{2n+1}}{a_{2n}}, 0\right)$ and $\left(0, \frac{2a_{2n}}{a_{2n+1}}\right)$ is a pseudo-integral triangle with Ehrhart polynomial

$$I_n(t) = (t + 1)^2.$$

Using this we can show:

Theorem 6 *Let $\{a_n\}$ be the sequence defined by (3.4). Then for any $n \geq 1$, the tetrahedron T_n with vertices $(0, 0, 0)$, $\left(\frac{1}{2}, 0, 0\right)$, $\left(0, \frac{a_{2n+1}}{a_{2n}}, 0\right)$ and $\left(0, 0, \frac{2a_{2n}}{a_{2n+1}}\right)$ is a pseudo-integral tetrahedron.*

Proof Let $f_n(t)$ denote the Ehrhart function for T_n. Then for any positive integer t, we have

$$\begin{aligned} f_n(t) &= \#\{(x,y,z) \in \mathbb{Z}^3 \mid 2x + \frac{a_{2n}}{a_{2n+1}} y + \frac{a_{2n+1}}{2a_{2n}} z \leq t,\ x,y,z \geq 0\} \\ &= \sum_{x=0}^{\lfloor \frac{t}{2} \rfloor} \#\{(y,z) \in \mathbb{Z}^2 \mid \frac{a_{2n}}{a_{2n+1}} y + \frac{a_{2n+1}}{2a_{2n}} z \leq t - 2x,\ y,z \geq 0\} \\ &= \sum_{x=0}^{\lfloor \frac{t}{2} \rfloor} I_n(t-2x) = \sum_{x=0}^{\lfloor \frac{t}{2} \rfloor} (t-2x+1)^2 \\ &= \frac{1}{6} t^3 + t^2 + \frac{11}{6} t + 1, \end{aligned}$$

where $I_n(t)$ denotes the Ehrhart function of the triangle with vertices $(0,0)$, $\left(\frac{a_{2n+1}}{a_{2n}}, 0\right)$ and $\left(0, \frac{2a_{2n}}{a_{2n+1}}\right)$.

We now give the proof of Example 4. Note that Example 4 is natural to consider, in view of Theorem 6. It is easy to show that

$$\lim_{n\to\infty} \frac{a_{2n+1}}{a_{2n}} = 2 + \sqrt{2}, \quad \text{and} \quad \lim_{n\to\infty} \frac{2a_{2n}}{a_{2n+1}} = 2 - \sqrt{2}.$$

Theorem 6 states that for any $n \geq 1$, the tetrahedron with vertices

$$(0,0,0), \left(\frac{1}{2}, 0, 0\right), \left(0, \frac{a_{2n+1}}{a_{2n}}, 0\right) \quad \text{and,} \quad \left(0, 0, \frac{2a_{2n}}{a_{2n+1}}\right)$$

is a pseudo-integral tetrahedron with the Ehrhart polynomial

$$f(t) = \frac{1}{6} t^3 + t^2 + \frac{11}{6} t + 1,$$

which is independent of n. Thus, it is reasonable to expect that the Ehrhart function of the limiting tetrahedron T with vertices $(0,0,0)$, $\left(\frac{1}{2}, 0, 0\right)$, $\left(0, 2+\sqrt{2}, 0\right)$ and $\left(0, 0, 2-\sqrt{2}\right)$ is also equal to the polynomial $f(t)$, and one can indeed show this by using Theorem 6 plus a limiting argument. We instead give a more direct proof that does not require Theorem 6.

Proposition 4 *The Ehrhart function of the irrational tetrahedron T with vertices $(0,0,0)$, $\left(\frac{1}{2}, 0, 0\right)$, $\left(0, 2+\sqrt{2}, 0\right)$ and $\left(0, 0, 2-\sqrt{2}\right)$ is the polynomial $f(t) = \frac{1}{6} t^3 + t^2 + \frac{11}{6} t + 1$.*

Proof Let $g(t)$ denote the Ehrhart function of T. Then we have

$$\begin{aligned}
&g(t) = \#(tT \cap \mathbb{Z}^3)\\
=&\#\{(x,y,z) \in \mathbb{Z}^3 \,|\, 2x + \left(1 - \frac{\sqrt{2}}{2}\right) y + \left(1 + \frac{\sqrt{2}}{2}\right) z \le t, x, y, z \ge 0\}\\
=&\sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \#\{(i,y,z) \,|\, (i,y,z) \in tT \cap \mathbb{Z}^3\}\\
=&\sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \#\{(i,y,z) \,|\, (i,y,z) \in tT \cap \mathbb{Z}^3, y \ge z\}\\
&\quad + \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \#\{(i,y,z) \,|\, (i,y,z) \in tT \cap \mathbb{Z}^3, y < z\}\\
=&\sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{t-2i}{2} \rfloor} \#\{(i,y,j) \,|\, (i,y,j) \in tT \cap \mathbb{Z}^3, y \ge j\}\\
&\quad + \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{t-2i-1+\frac{\sqrt{2}}{2}}{2} \rfloor} \#\{(i,k,z) \,|\, (i,k,z) \in tT \cap \mathbb{Z}^3, k < z\}\\
=&\sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{t-2i}{2} \rfloor} \left(\left\lfloor \frac{t-2i-2j}{1-\frac{\sqrt{2}}{2}} \right\rfloor + 1 \right) + \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \sum_{k=0}^{\left\lfloor \frac{t-2i-1+\frac{\sqrt{2}}{2}}{2} \right\rfloor} \left\lfloor \frac{t-2k-2i}{1+\frac{\sqrt{2}}{2}} \right\rfloor\\
=&\sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{t-2i}{2} \rfloor} \left(\left\lfloor \frac{t-2i-2j}{1-\frac{\sqrt{2}}{2}} \right\rfloor + 1 + \left\lfloor \frac{t-2i-2j}{1+\frac{\sqrt{2}}{2}} \right\rfloor \right)
\end{aligned}$$

Keeping in mind that

$$\frac{1}{1-\frac{\sqrt{2}}{2}} = 2 + \sqrt{2}, \quad \frac{1}{1+\frac{\sqrt{2}}{2}} = 2 - \sqrt{2},$$

we deduce that

$$\begin{aligned}
\left\lfloor \frac{t-2i-2j}{1-\frac{\sqrt{2}}{2}} \right\rfloor &= \lfloor (t-2i-2j)(2+\sqrt{2}) \rfloor\\
&= \lfloor (t-2i-2j)(4-(2-\sqrt{2})) \rfloor\\
&= 4t - 8i - 8j + \left\lfloor -\frac{t-2i-2j}{1+\frac{\sqrt{2}}{2}} \right\rfloor.
\end{aligned}$$

So we have

$$g(t) = \begin{cases} \frac{t}{2} + 1 + \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{t-2i}{2} \rfloor} (4t - 8i - 8j), \; t \text{ is even;} \\ \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{t-2i}{2} \rfloor} (4t - 8i - 8j), \qquad t \text{ is odd} \end{cases}$$

$$= \frac{1}{6}t^3 + t^2 + \frac{11}{6}t + 1.$$

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