An Analogue of Young's Lattice for Compositions

Anders Björner
Department of Mathematics, Kungl. Tekniska Högskolan
S–100 44 Stockholm, Sweden
`bjorner@math.kth.se`

Richard P. Stanley[1]
Department of Mathematics, Massachusetts Institute of Technology
Cambridge, MA 02139, USA
`rstan@math.mit.edu`

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Abstract

Let $C_n = \{\text{compositions of } n\}$, $C = \cup C_n$. We define a partial order making C into a ranked poset having 1 as its bottom element and C_n as its $(n-1)$-st rank level.

Let $\alpha = a_1 + \cdots + a_k \in C_n$. The interval $[1, \alpha]$ is shown to have the following properties:

- The number of maximal chains in $[1, \alpha]$ equals the number of permutations of $[n]$ with descent set $\{a_1, a_1 + a_2, \ldots\}$.
- The interval $[1, \alpha]$ is CL-shellable.
- The Möbius function satisfies
$$\mu(1, \alpha) = \begin{cases} (-1)^{n-1} & \text{if } \alpha = x22\ldots22y,\ x, y \in \{1, 2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, there is a Pieri-type rule

$$Q_1 Q_\alpha = \sum_{\beta \rhd \alpha} Q_\beta,$$

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for fundamental quasi-symmetric functions Q_α, where the summation runs over all β covering α in the poset. Thus, the poset C plays a role for quasi-symmetric functions analogous to that of Young's lattice for symmetric functions. We also discuss some algebras that may play a role for C analogous to that played by the group algebra of the symmetric group for Young's lattice.

1 Introduction.

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition of $n \geq 0$ (denoted $\lambda \vdash n$), i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $\sum \lambda_i = n$. *Young's lattice* Y is the poset (actually a distributive lattice) of all partitions of all integers $n \geq 0$ ordered by inclusion of their Young diagrams. Thus $\lambda \leq \mu$ in Y if and only if $\lambda_i \leq \mu_i$ for all i. The poset Y has a number of remarkable algebraic and combinatorial properties related to symmetric functions and the symmetric group. These properties include the following. (Unexplained terminology on posets and symmetric functions may be found e.g. in [20][21].)

1. Y is a graded poset, and the rank of a partition $\lambda \vdash n$ is n.

2. The number of saturated chains in Y from $\hat{0}$ (the bottom element of Y, i.e., the partition $\emptyset$ of 0) to a partition λ is the number f^λ of standard Young tableaux of shape λ.

3. The total number of saturated chains from $\hat{0}$ to rank n is the number $t(n)$ of involutions in the symmetric group $\mathfrak{S}_n$.

4. Let s_λ denote a Schur function. Then by Pieri's rule [21, Thm. 7.15.7] we have
$$s_1 s_\lambda = \sum_{\lambda \prec \mu} s_\mu, \qquad (1)$$
where $\lambda \prec \mu$ denotes that μ covers λ in Y.

5. Since Y is a distributive lattice, every interval $[\lambda, \mu]$ is EL-shellable and hence Cohen-Macaulay [2].

6. Y is the Bratteli diagram for the tower of algebras $K\mathfrak{S}_0 \subset K\mathfrak{S}_1 \subset \cdots$, where $K\mathfrak{S}_n$ denotes the group algebra of $\mathfrak{S}_n$ over the field K of characteristic 0. (See Section 5.)

In this paper we define an analogue $\mathcal{C}$ of Y whose elements are the *compositions* α of all integers $n \geq 1$. Thus $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{P}^k$, where $\mathbb{P} = \{1, 2, \dots\}$ and $\sum \alpha_i = n$. Let $\mathrm{Comp}(n)$ denote the set of all compositions of n, so by elementary enumerative combinatorics $\#\mathrm{Comp}(n) = 2^{n-1}$ for $n \geq 1$. For each of the six properties of Y above there is a corresponding property of $\mathcal{C}$.

We take the analogue of property 4, a Pieri rule for fundamental quasisymmetric functions, as our guiding principle. It leads to a combinatorial definition of the partial order of $\mathcal{C}$. Subsequently it turns out that this partial order can also be described in terms of subwords.

Composition analogues of Y have been given previously by Bergeron, Bousquet-Mélou and Dulucq [1], Snellman [17][18], and Sagan and Vatter [16], but our definition is different. In [18] Snellman obtains further properties of $\mathcal{C}$ after learning of this poset from us.

We now define $\mathcal{C}$ in terms of the cover relation $\alpha \prec \beta$. In Section 3 we explain how this definition arises naturally from the theory of quasisymmetric functions. In a poset P, we say that t *covers* s, denoted $s \prec t$, if $s < t$ and no $u \in P$ satisfies $s < u < t$.

Definition 1.1. Let $\mathcal{C} = \bigcup_{n\geq 1} \mathrm{Comp}(n)$. Define a partial ordering on $\mathcal{C}$ by letting β cover $\alpha = (\alpha_1, \dots, \alpha_k)$ if β can be obtained from α either by adding 1 to a part, or adding 1 to a part and then splitting this part into two parts. More precisely, for some j we have either

$$\beta = (\alpha_1, \dots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \dots, \alpha_k)$$

or

$$\beta = (\alpha_1, \dots, \alpha_{j-1}, h, \alpha_j + 1 - h, \alpha_{j+1}, \dots, \alpha_k)$$

for some $1 \leq h \leq \alpha_j$.

It is clear that $\mathcal{C}$ is a graded poset for which $\mathrm{Comp}(n)$ is the set of elements of rank $n-1$. The bottom element $\hat{0}$ of $\mathcal{C}$ is the unique

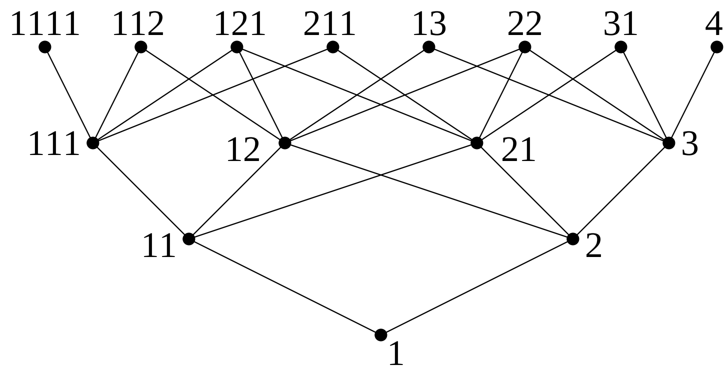

Figure 1: The composition poset $\mathcal{C}$

composition $\alpha = (1)$ of 1. Figure 1 shows the first four levels (i.e., ranks $0, 1, 2, 3$) of $\mathcal{C}$.

In the following sections we develop some combinatorial, topological, and algebraic properties of $\mathcal{C}$. In Section 2 we derive elementary properties of $\mathcal{C}$ that in Section 3 lead to a proof that Definition 1.1 of $\mathcal{C}$ gives the correct Pieri rule. In Section 4 we give the description of $\mathcal{C}$ in terms of subword order on the free monoid on a two-letter alphabet. From this we deduce that intervals in $\mathcal{C}$ are lexicographically shellable, and hence Cohen-Macaulay, and we determine its Möbius function and some related generating functions. Section 5 concerns some speculations on the connection between $\mathcal{C}$ and a class of algebras recently defined by Hivert and Thiéry.

We are grateful to Sergey Fomin for pointing out to us the connection between the composition poset $\mathcal{C}$ and subword order.

2 Descent sets.

Given a permutation $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$, define the *descent set* $D(w)$ by

$$D(w) = \{i \, : \, w_i > w_{i+1}\}.$$

Similarly the *descent composition* $C(w)$ is the composition $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ defined by

$$D(w) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}.$$

Of course $D(w)$ and $C(w)$ contain equivalent information; we will use whichever is more convenient for the situation at hand.

If $\alpha \in \mathrm{Comp}(n)$ then a *saturated chain* from $\hat{0}$ to α, or *saturated α-chain* for short, is a chain

$$\hat{0} = \alpha^1 \prec \alpha^2 \prec \cdots \prec \alpha^n = \alpha,$$

where $\prec$ denotes a covering relation in $\mathcal{C}$. Thus $\alpha^i \in \mathrm{Comp}(i)$. Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n] := \{1, 2, \ldots, n\}$. Given $w \in \mathfrak{S}_n$, write $w[i]$ for the *restriction* of w to $[i]$, i.e., the subsequence of w (regarded as a word $w_1 w_2 \cdots w_n$) consisting of $1, 2, \ldots, i$. For instance, if $w = 5274613$ then $w[4] = 2413$. Define $\mathfrak{m}(w)$ to be the sequence

$$C(w[1]), \ldots,\ C(w[n])$$

of compositions $C(w[i]) \in \mathrm{Comp}(i)$. For instance, if $w = 5274613$, then

$$\mathfrak{m}(w) = (1, 11, 12, 22, 122, 132, 1222).$$

Theorem 2.1. *The map $\mathfrak{m}$ is a bijection from $\mathfrak{S}_n$ to saturated α-chains in $\mathcal{C}$, where α ranges over* $\mathrm{Comp}(n)$.

Proof. Let $w = w_1 \cdots w_n \in \mathfrak{S}_n$, and for $0 \leq i \leq n$ define

$$w_{(i)} = w_1 \cdots w_i \,(n+1)\, w_{i+1} \cdots w_n \in \mathfrak{S}_{n+1}.$$

Thus $w_{(0)}, w_{(1)}, \ldots, w_{(n)}$ are precisely the permutations $u \in \mathfrak{S}_{n+1}$ satisfying $u[n] = w$. It suffices to show that the compositions $C(w_{(i)})$, $1 \leq i \leq n$, are distinct and are precisely the compositions covering $C(w)$ in $\mathcal{C}$.

The verification of this statement is straightforward. Let $C(w) = (\alpha_1, \ldots, \alpha_k)$. Let $b_j = \alpha_1 + \alpha_2 + \cdots + \alpha_j$. Then

$$C(w_{(b_j)}) = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \ldots, \alpha_k),$$

which for $1 \leq j \leq k$ are distinct compositions covering $C(w)$. On the other hand, suppose that $0 \leq i \leq n$ and i is not of the form $\alpha_1+\alpha_2+\cdots+\alpha_j$. Thus $i = \alpha_1+\cdots+\alpha_j+h$ for some $0 \leq j \leq k-1$ and $1 \leq h < \alpha_{j+1}$. (When $j = 0$ we set $\alpha_1+\cdots+\alpha_j = 0$.) Then $C(w_{(i)})$ is obtained from $C(w)$ by replacing α_{j+1} with the pair $(h, \alpha_{j+1}+1-h)$. These yield all the other (distinct) elements covering α, completing the proof. $\square$

Note. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$. If we replace α_i with the pair $\alpha_i, 1$, then we obtain the same $\beta \succ \alpha$ as when we replace α_{i+1} with $1, \alpha_{i+1}$. Nevertheless, in accordance with the proof of Theorem 2.1, if $C(w) = \alpha$ then there is a unique j for which $C(w_{(j)}) = \beta$, viz., $j = \alpha_1 + \cdots + \alpha_i - 1$.

The following corollaries are an immediate consequence of Theorem 2.1 and its proof.

Corollary 2.2. *The number of saturated α-chains in $\mathcal{C}$ is equal to the number $f_n(\alpha)$ of permutations $w \in \mathfrak{S}_n$ with descent composition α.*

Corollary 2.3. *The total number of saturated chains in $\mathcal{C}$ from $\emptyset$ to rank $n-1$ is given by*

$$\sum_{\alpha \in \mathrm{Comp}(n)} f_n(\alpha) = n!.$$

Corollary 2.4. *If $\alpha \in \mathrm{Comp}(n)$ then α is covered in $\mathcal{C}$ by exactly $n+1$ compositions β.*

A strengthening of Corollary 2.4 is given in Theorem 4.7, part (1) of which can be stated as saying that the number of compositions in $\mathrm{Comp}(p)$ that lie above α equals

$$\sum_{i=0}^{p-n} \binom{p-1}{i}.$$

Corollary 2.4 is the case $p = n+1$.

3 Quasisymmetric functions.

We have given a "naive" definition of the poset $\mathcal{C}$. In this section we give a more motivated definition based on quasisymmetric functions which is completely analogous to the definition (1) of Young's lattice in terms of Schur functions. Let $\sigma = (\sigma_1, \sigma_2, \dots)$, where $\sigma_i \in \mathbb{N} = \{0, 1, 2, \dots\}$ and $\sum \sigma_i < \infty$, and write $x^\sigma = x_1^{\sigma_1} x_2^{\sigma_2} \cdots$. Recall (e.g., [21, §7.19]) that a *quasisymmetric function* (say over $\mathbb{Z}$) is a formal power series $y = \sum_\sigma c_\sigma x^\sigma$ of bounded degree, where $c_\sigma \in \mathbb{Z}$, satisfying the following condition. Let $\tau_1, \dots, \tau_k > 0$ and $i_1 < \cdots < i_k$. Then

$$[x_{i_1}^{\tau_1} \cdots x_{i_k}^{\tau_k}]y = [x_1^{\tau_1} \cdots x_k^{\tau_k}]y,$$

where $[x^\sigma]y$ denotes the coefficient c_σ of x^σ in y.

If $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathrm{Comp}(n)$ then define the *fundamental quasisymmetric function* L_α by

$$L_\alpha = \sum x_{i_1} \cdots x_{i_k},$$

summed over all sequences $1 \le i_1 \le \cdots \le i_k$ such that $i_j < i_{j+1}$ if $j = \alpha_1 + \cdots + \alpha_h$ for some $1 \le h \le k-1$. For instance,

$$L_{212} = \sum_{a \le b < c < d \le e} x_a x_b x_c x_d x_e.$$

It is a standard result [21, Prop. 7.19.1] that $\{L_\alpha : \alpha \in \mathrm{Comp}(n)\}$ is a $\mathbb{Z}$-basis for all quasisymmetric functions that are homogeneous of degree n.

Let $w \in \mathfrak{S}_n$, and let v be a permutation of $\{n+1, n+2, \cdots, n+m\}$. Let $C(w) = \alpha$ and $C(v) = \beta$. Another basic result on quasisymmetric functions [21, Exer. 7.93] asserts that

$$L_\alpha L_\beta = \sum_u L_{C(u)},$$

where u runs over all *shuffles* of w and v, i.e., all permutations $u \in \mathfrak{S}_{n+m}$ such that $u[n] = w$ and the restriction of u to $\{n+1, \dots, n+m\}$ is v. Apply this result to the case $m = 1$. The shuffles of w and v are precisely the permutations $w_{(i)}$ appearing in the proof of Theorem 2.1. Hence we obtain the formula

$$L_1 L_\alpha = \sum_{\alpha \prec \beta} L_\beta. \tag{2}$$

Equation (2) could therefore be taken as the *definition* of $\mathcal{C}$ (defined by its cover relations $\alpha \prec \beta$). Note that Corollary 2.2 is equivalent to the quasisymmetric function identity

$$L_1^n = \sum_{\alpha \in \mathrm{Comp}(n)} f_n(\alpha) L_\alpha.$$

This identity can be proved directly in a number of ways. It is analogous to the symmetric function identity [21, Cor. 7.12.5]

$$s_1^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda.$$

4 Subword order: shellability and Möbius function.

Let A^* denote the set of all words in the two-letter alphabet $A = \{a, b\}$. Similarly, let F^{**} denote the set of all words in the two-letter alphabet $F = \{1, +\}$, beginning and ending with "1" and without consecutive plusses. The substitutions

$$a \overset{\varphi}{\mapsto} +1 \tag{3}$$
$$b \overset{\varphi}{\mapsto} 1 \tag{4}$$

induce a bijection

$$A^* \xrightarrow{\varphi} F^{**} \tag{5}$$
$$a_1 \ldots a_p \mapsto 1\varphi(a_1) \ldots \varphi(a_p) \tag{6}$$

There is also a bijection

$$F^{**} \xrightarrow{\psi} \mathcal{C} \tag{7}$$

obtained by replacing maximal strings of ones by their lengths. A couple of examples should make the idea of the composite bijection $A^* \xrightarrow{\psi\circ\varphi} \mathcal{C}$ clear:

$$\begin{array}{llll} aaabbab & \overset{\varphi}{\to} & 1+1+1+1\;1\;1+1\;1 & \overset{\psi}{\to} & (1,1,1,3,2) \\ bbbaaba & \overset{\varphi}{\to} & 1\;1\;1\;1+1+1\;1+1 & \overset{\psi}{\to} & (4,1,2,1) \end{array}$$

We say that a word u is a *subword* of a word $w = a_1 \cdots a_p$ if $u = a_{i_1} \cdots a_{i_k}$ for some string $1 \le i_1 < \cdots < i_k \le p$. The subword relation $u \le w$ introduces a structure of a partial order on the set A^*.

Theorem 4.1. *The map $\psi \circ \varphi$ is an isomorphism of A^* and $\mathcal{C}$ as partially ordered sets.*

Proof. We have seen that $\psi \circ \varphi$ gives a bijection between words of length $n-1$ and $\mathrm{Comp}(n)$ for all $n \ge 1$. Thus, it remains only to check that the covering relations agree.

Suppose that $\psi \circ \varphi\,(u) = (\alpha_1, \ldots, \alpha_k)$, and that the word w is obtained from u by inserting somewhere a single letter. If that letter is b then

$$\psi \circ \varphi\,(w) = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \ldots, \alpha_k)$$

for some j. If the letter is a then

$$\psi \circ \varphi\,(w) = (\alpha_1, \ldots, \alpha_{j-1}, h, \alpha_j + 1 - h, \alpha_{j+1}, \ldots, \alpha_k)$$

for some j and some $1 \le h \le \alpha_j$. Thus

$$u \prec w \quad \Longrightarrow \quad \psi \circ \varphi\,(u) \prec \psi \circ \varphi\,(w).$$

Conversely, every covering relation $\alpha \prec \beta$ in $\mathcal{C}$ is obtained this way, as is easily seen from Definition 1.1 and the construction. $\square$

Subword order (on alphabets of arbitrary size) has been previously studied, see [3, 5] and the references given there. By Theorem 4.1 we can transfer known results from A^* to $\mathcal{C}$. A basic such result is the following.

Theorem 4.2. *Intervals in the composition poset $\mathcal{C}$ are dual CL-shellable, and hence Cohen-Macaulay.*

Proof. This is [3, Thm. 3] transferred to $\mathcal{C}$. A direct proof is given in Section 6, where an explicit dual CL-labeling of intervals in $\mathcal{C}$ is constructed. This labeling is combinatorially equivalent to the one obtained via transfer from [3]. However, it is described directly in terms of compositions rather than words. $\square$

Next we determine the Möbius function of A^*. The *length* of a word $w = a_1 \cdots a_p$ is $\ell(w) = p$, and its *repetition set* is $\mathcal{R}(w) = \{i : a_{i-1} = a_i\}$. An *embedding* of a subword u in w is a sequence $1 \leq i_1 < \cdots < i_k \leq p$ such that $u = a_{i_1} \cdots a_{i_k}$. It is a *normal embedding* if also $\mathcal{R}(w) \subseteq \{i_1, \ldots, i_k\}$.

Theorem 4.3 ([3], Theorem 1)**.**

$$\mu_{A^*}(u, w) = (-1)^{\ell(w)-\ell(u)} \cdot \textit{number of normal embeddings of } u \textit{ in } w$$

This result determines the Möbius function of $\mathcal{C}$ via the isomorphism $\psi \circ \varphi$. For example,

$$\begin{aligned} \mu_{\mathcal{C}}(5, (3,3,3)) &= \mu_{A^*}(bbbb, bbabbabb) \\ &= (-1)^{8-4} \cdot \text{number of normal embeddings} \\ &= 3. \end{aligned}$$

Corollary 4.4. *The Möbius function of lower intervals in $\mathcal{C}$ is*

$$\mu_{\mathcal{C}}(1, \alpha) = \begin{cases} (-1)^{|\alpha|-1} & \textit{if } \alpha = (x, 2, 2, \ldots, 2, 2, y),\ x, y \in \{1, 2\}, \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, for given $n > 1$ there are exactly two compositions $\alpha \in \mathrm{Comp}(n)$ such that $\mu(1, \alpha) \neq 0$, namely $22 \cdots 22$ and $122 \cdots 221$ if n is even, and $122 \cdots 22$ and $22 \cdots 221$ if n is odd.

Proof. Let w be the word corresponding to α. A normal embedding of the empty word into w exists if and only if the repetition set of w is empty, and the embedding is then unique. Therefore, $\mu_{A^*}(\emptyset, w) \neq 0 \Leftrightarrow \mathcal{R}(w) = \emptyset \Leftrightarrow w = ababa \cdots$ or $w = babab \cdots$.

Transferring this information to the poset $\mathcal{C}$, we deduce that $\mu_{\mathcal{C}}(1,\alpha) \neq 0$ if and only if α is of the stated type, and that its value is then ± 1. $\square$

Let $\widetilde{\mathcal{C}}_n = (\bigcup_{1\le i\le n} \mathrm{Comp}(i)) \bigcup \{\omega\}$, where ω is a new top element and the order relation is otherwise as before. Thus, $\widetilde{\mathcal{C}}_n$ is graded and of length n.

Corollary 4.5. *The poset $\widetilde{\mathcal{C}}_n$ is shellable. Its Möbius function satisfies*

$$\mu(1,\omega) = (-1)^n.$$

Proof. Every composition $(\alpha_1,\dots,\alpha_k) \in \mathrm{Comp}(n)$ lies below the composition $(2,\dots,2) \in \mathrm{Comp}(2n)$ in $\mathcal{C}$. To see this, first group the parts $(2+\cdots+2)+\cdots+(2+\cdots+2)$ so that the i-th parenthesis contains α_i 2's, then reduce the i-th group to α_i. Thus, $\widetilde{\mathcal{C}}_n$ is obtainable via rank-selection from the interval $[1,(2,\dots,2)]$ of $\mathcal{C}$, so shellability follows by [2].

For the Möbius function we get, using the previously computed expressions from Corollary 4.4:

$$\mu(1,\omega) = -\sum_{|\alpha|\le n} \mu(1,\alpha) = -(1-2+2-2+\cdots) = (-1)^n. \qquad \square$$

In the following we write $|\alpha| = n$ to mean $\alpha \in \mathrm{Comp}(n)$.

Theorem 4.6. (1) *Let $\beta \in \mathrm{Comp}(k)$. Then*

$$\sum_{\alpha\in\mathcal{C}} \mu_{\mathcal{C}}(\beta,\alpha) t^{|\alpha|-k} = \frac{1-t}{(1+t)^k}.$$

(2)

$$\sum_{\alpha,\beta\in\mathcal{C}} \mu_{\mathcal{C}}(\beta,\alpha) t^{|\alpha|-1} q^{|\beta|-1} = \frac{1-t}{1-(2q-1)t}.$$

Proof. This is [3, Theorem 2] transferred to $\mathcal{C}$. $\square$

There are similar rational expressions for the zeta function of $\mathcal{C}$.

Theorem 4.7. (1) *Let* $\beta \in \mathrm{Comp}(k)$*. Then*

$$\sum_{\alpha \in \mathcal{C}} \zeta_{\mathcal{C}}(\beta, \alpha) t^{|\alpha|-k} = \frac{1}{(1-2k)(1-t)^{k-1}}.$$

(2)

$$\sum_{\alpha, \beta \in \mathcal{C}} \zeta_{\mathcal{C}}(\beta, \alpha) t^{|\alpha|-1} q^{|\beta|-1} = \frac{1-t}{(1-2t)(1-(2q+1)t)}.$$

Proof. This is [3, Remark 3] transferred to $\mathcal{C}$. □

An interesting feature of the series in part (1) of these theorems is that the right-hand side does not depend on the combinatorial structure of the composition β, only on its rank $k-1$.

Similar expressions for the generating functions for powers $\mu^d_{\mathcal{C}}(\beta, \alpha)$ and $\zeta^d_{\mathcal{C}}(\beta, \alpha)$ can be transferred from the results of [5, Section 4]. Here "powers" are to be understood in the sense of the incidence algebra of $\mathcal{C}$.

A further consequence of Theorem 4.1 was pointed out to us by Sergey Fomin. The poset A^* (and hence $\mathcal{C}$), together with the complete binary tree $\mathcal{T}$, form a pair of *dual graded graphs* [9, Examples 2.3.6, 2.4.1]. This means that all the algebraic and combinatorial machinery of [9] and [10] can be applied to $\mathcal{C}$. In particular, let $\mathcal{C}'$ denote $\mathcal{C}$ with a new bottom element $\emptyset$ (the empty composition of 0) adjoined, and consider walks on the vertices of $\mathcal{C}'$ with steps as follows: (a) from α we can step to $\beta \succ \alpha$, and (b) from α we can step to the composition obtained by subtracting one from the last part of α. There are then explicit formulas for the cardinality of many classes of such walks. We mention two such formulas to convey the flavor of this topic.

- The number of walks from $\emptyset$ to $\emptyset$ in $2n$ steps beginning with n steps of type (a) (and therefore ending with n steps of type (b)) is $n!$. This result is equivalent to Corollary 2.3 since the steps of type (b) are uniquely determined.

- The total number of walks from $\emptyset$ to $\emptyset$ in $2n$ steps is $1 \cdot 3 \cdot 5 \cdots (2n-1)$.

5 Representation theory.

Let $K = A_0 \subset A_1 \subset \cdots$ be a tower $\mathcal{T}$ of finite-dimensional semisimple algebras over a field K. Let $\mathrm{irr}(A_n)$ denote the set of irreducible representations of A_n. Define a poset $P_\mathcal{T}$, with each cover relation $x \prec y$ weighted by a positive integer $\omega(x, y)$, on the set $\bigcup_{n\geq 0} \mathrm{irr}(A_n)$ as follows. Let $y \in \mathrm{irr}(A_n)$. Let $y{\downarrow}_{n-1}$ denote the restriction of y to A_{n-1}. If $x \in A_{n-1}$ appears with multiplicity m in $y{\downarrow}_{n-1}$, then define $x \prec y$ and $\omega(x, y) = m$.

The weighted poset $(P_\mathcal{T}, \omega)$ is called the *Bratteli diagram* of $\mathcal{T}$ [12, §2.3]. If all the multiplicities m are equal to 1, then the Bratteli diagram becomes an ordinary (unweighted) poset. Since $A_0 = K$, the Bratteli diagram has a unique minimal element $\hat{0}$. Let $f(x)$ denote the weighted number of saturated chains from $\hat{0}$ to x, where each chain is weighted by the product of the weights of its cover relations. A fundamental property of the Bratteli diagram is that $f(x) = \dim(x)$, the dimension of the representation x. In particular, by elementary representation theory we have

$$\sum_{x\in \mathrm{irr}(A_n)} f(x)^2 = \dim A_n.$$

The prototypical example of a tower $\mathcal{T}$ of algebras is

$$K \subset K\mathfrak{S}_1 \subset K\mathfrak{S}_2 \subset \cdots,$$

where K is a field of characteristic 0, $K\mathfrak{S}_n$ denotes the group algebra of the symmetric group $\mathfrak{S}_n$ over K and the embedding $K\mathfrak{S}_n \subset K\mathfrak{S}_{n+1}$ is induced by the "obvious" embedding $\mathfrak{S}_n \subset \mathfrak{S}_{n+1}$ obtained by identifying $\mathfrak{S}_n$ with those $w \in \mathfrak{S}_{n+1}$ that fix $n + 1$. In this case it is well-known that when $\mathrm{char}(K) = 0$ the Bratteli diagram of $\mathcal{T}$ is just Young's lattice Y.

In view of the above remarks it is natural to ask whether the composition poset $\mathcal{C}$ is the Bratteli diagram of a "nice" tower $\mathcal{T}$ of algebras. (Every graded weighted poset with $\hat{0}$ and with finitely many elements at each rank is the Bratteli diagram of *some* tower $\mathcal{T}$ of algebras A_n, but $\mathcal{T}$ may not have any desirable properties such as simple generators and relations or a direct combinatorial description.) A necessary condition on a candidate tower $\mathcal{T}$ is that the irreducible (or perhaps indecomposable) representations of A_n are indexed by compositions α of n and have dimension $f_n(\alpha)$, the number of $w \in \mathfrak{S}_n$ with descent composition α. We will point out three towers $\mathcal{T}$ with this property as a direction for further investigation. We are grateful to Arun Ram for explaining to us the first two towers and their close relationship. We are also grateful to Florent Hivert for explaining the third tower, which he is currently investigating with Nicolas Thiéry. It seems plausible that the algebras in the third tower are a quotient of the ones in the first tower $\mathcal{T}_1$, but this remains open. None of the three towers are semisimple, so there is more than one way to define what should be their Bratteli diagram. We will not discuss here the possible definitions of Bratteli diagrams of non-semisimple towers.

1. Let $\tilde{H}_n$ denote the affine Hecke algebra of type GL_n. The center of $\tilde{H}_n$ is the ring of symmetric functions $\Lambda_n = \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{\mathfrak{S}_n}$, and on any finite-dimensional simple $\tilde{H}_n$-module the center will act by a central character $\gamma : \Lambda_n \to \mathbb{C}$. Given a central character γ, the quotient

$$\tilde{H}_n^{[\gamma]} = \tilde{H}_n/(f - \gamma(f) \,:\, f \in \Lambda_n)$$

is a finite-dimensional algebra of dimension $n!^2$ [14, (1.9), Thm. 1.13, Thm. 1.17].

Let γ be the central character defined by

$$\gamma(f) = f(1, q, q^2, \ldots, q^{n-1}). \tag{8}$$

The irreducible reprentations of $\tilde{H}_n$ are indexed by $\alpha \in \mathrm{Comp}(n)$ and have dimension $f_n(\alpha)$ (a consequence of [15, Thm. 4.1] or [14, Thm. 3.5, Thm. 5.9]), so the first tower $\mathcal{T}_1$ is given by

$$\tilde{H}_0^{[\gamma]} \subset \tilde{H}_1^{[\gamma]} \subset \cdots,$$

with an obvious embedding $\tilde{H}_n^{[\gamma]} \subset \tilde{H}_{n+1}^{[\gamma]}$ analogous to the embedding $K\mathfrak{S}_n \subset K\mathfrak{S}_{n+1}$.

2. Let $H_n(0)$ denote the 0-Hecke algebra (of type A or GL_n). The simple $H_n(0)$-modules L_α are indexed by compositions $\alpha \in \mathrm{Comp}(n)$ and are all 1-dimensional. The projective indecomposable $H_n(0)$-modules P_α are therefore also indexed by $\alpha \in \mathrm{Comp}(n)$, but now $\dim P_\alpha = f_n(\alpha)$ [6]. Hence we can define a second tower $\mathcal{T}_2$ by

$$H_0(0) \subset H_1(0) \subset \cdots,$$

again with an obvious embedding.

There is a close connection between the representation theory of $\tilde{H}_n^{[\gamma]}$ (with γ given by (8)) and $H_n(0)$. Let $M^{[\gamma]}$ be the principal series module for $\tilde{H}_n^{[\gamma]}$; we have $\dim M^{[\gamma]} = n!$. By [15, Cor. 6.3] the simple $\tilde{H}_n$-modules $\tilde{H}_n^\alpha$ which appear as composition factors of $M^{[\gamma]}$ are indexed by compositions $\alpha \in \mathrm{Comp}(n)$ and have dimension $f_n(\alpha)$. In fact, these simple $\tilde{H}_n$-modules are precisely the projective indecomposable $H_n(0)$-modules. The action of $\tilde{H}_n(0)$ on $M^{[\gamma]}$ can be produced using the τ-operators of [14, Prop. 2.14] or [15, Prop. 3.2].

3. The third tower $\mathcal{T}_3$ is a consequence of recent work of Hivert and Thiéry [13], as mentioned above. Let V be the vector space over a field K of characteristic 0 with basis $\mathfrak{S}_n$. Define Γ_n to be the algebra generated by the following two classes of operators: (1) ordinary right multiplication by $w \in \mathfrak{S}_n$, and (2) the "sorting operators" [11, §3] π_i defined by

$$w\pi_i = \begin{cases} ws_i, & \text{if } w(i) > w(i+1) \\ w, & \text{if } w(i) < w(i+1). \end{cases}$$

where ws_i is the ordinary product of w with the adjacent transposition $s_i = (i, i+1)$. Hivert and Thiéry show that $\dim \Gamma_n$ is the number $d(n)$ of pairs $(u, v) \in \mathfrak{S}_n \times \mathfrak{S}_n$ such that $D(u) \cap D(v) = \emptyset$. It was shown by Carlitz, Scoville, and Vaughan [7][8] (see also [19, (28)]) that

$$\sum_{n\geq 0} d(n) \frac{x^n}{n!^2} = \left(\sum_{n\geq 0} (-1)^n \frac{x^n}{n!^2} \right)^{-1}.$$

Hivert and Thiéry further show that the irreducible representations I_α of Γ_n can be indexed by compositions $\alpha \in \mathrm{Comp}(n)$ such that $\dim I_\alpha = f_n(\alpha)$. Hence the third tower $\mathcal{T}_3$ is given by

$$\Gamma_0 \subset \Gamma_1 \subset \cdots ,$$

once again with an obvious embedding.

6 Appendix: a CL-labeling

We refer to [4] for definitions and further details about the concepts used here.

To define the chain labeling it is useful to first restate the definition of the partial order of $\mathcal{C}$. This will be done by describing the elements covered by $\alpha = \alpha_1 + \cdots + \alpha_k \in \mathrm{Comp}(n)$.

Equivalent definition. Say that a part α_j is *legal* if either $j = 1$, or $j > 1$ and $\alpha_j \geq 2$. The elements covered by $\alpha = \alpha_1 + \cdots + \alpha_k$ in $\mathcal{C}$ are, for legal α_j (zero parts are suppressed)

- $\alpha_1 + \cdots + \alpha_{j-1} + (\alpha_j - 1) + \alpha_{j+1} + \cdots + \alpha_k$,
- $\alpha_1 + \cdots + \alpha_{j-1} + (\alpha_j + \alpha_{j+1} - 1) + \alpha_{j+1} + \cdots + \alpha_k$.

Chain labeling. Given $\alpha = \alpha_1 + \cdots + \alpha_k \in \mathrm{Comp}(n)$ we now define a labeling of the downward maximal chains in the interval $[1, \alpha]$. The ordered set of labels is $1 < 1' < 2 < 2' < \cdots < (n-1)' < n$. We model the combinatorics of moving down a maximal chain by a process of removing balls from urns. The starting position consists of a sequence of urns $U_1, \ldots, U_k$, ordered from left to right, with α_j balls in urn U_j. There are two types of moves, each receiving a label by the following rule. At each step of the procedure, say that an urn is *legal* if either it is the first nonempty urn (left-to-right), or it contains at least two balls.

- Move of type 1: Remove one ball from a legal urn U_j. Label this move by j.

- Move of type 2: If U_j is a legal urn with at least two balls and U_i, $i > j$, is the first nonempty urn to its right, then move all balls from U_i over into U_j, then remove one ball from U_j. Label this move by j'.

It is clear that sequences of moves model downward maximal chains in the interval $[1, \alpha]$, and thus their associated label sequences induce a chain labeling, let us call it λ.

Theorem 6.1. *The labeling λ is a dual CL-labeling.*

Proof. The induced labeling on rooted intervals in $[1, \alpha]$ is of the same kind. Thus it suffices to consider an interval $[\beta, \alpha]$ and check that the labeling has the required properties there.

1. *The lexicographically first chain m in $[\beta, \alpha]$ has a weakly increasing label.*

Note first that all edges down from an element in the poset $\mathcal{C}$ receive distinct labels, so the lex-first chain m is well-defined.

Suppose that $\lambda(m)$ has a descent. Then somewhere there is an occurrence in consecutive positions in $\lambda(m)$ of one of the following six patterns:

(i) $\lambda(m) = (\ldots, j, i, \ldots)$, $i < j$,

(ii) $\lambda(m) = (\ldots, j, i', \ldots)$, $i < j$,

(iii) $\lambda(m) = (\ldots, j', i, \ldots)$, $i < j$,

(iv) $\lambda(m) = (\ldots, j', i', \ldots)$, $i < j - 1$,

(v) $\lambda(m) = (\ldots, j', (j-1)', \ldots)$,

(vi) $\lambda(m) = (\ldots, j', j, \ldots)$.

Considering the urn model of the combinatorial process it is in the first five cases easy to see that, in each case, there exists a chain m' in $[\beta, \alpha]$ such that, respectively,

(i) $\lambda(m') = (\ldots, i, j \ldots)$,

(ii) $\lambda(m') = (\ldots, i', j, \ldots)$, or $\lambda(m') = (\ldots, i', i, \ldots)$,

(iii) $\lambda(m') = (\ldots, i, j', \ldots)$,

(iv) $\lambda(m') = (\ldots, i', j', \ldots)$,

(v) $\lambda(m') = (\ldots, (j-1)', (j-1)', \ldots)$.

The sixth case requires a little more care, depending on whether urn U_j has 2 balls, or more than 2 balls, at the moment of the j'-labeled move.
Case (vi-1): $|U_j| > 2$, or U_j is the first non-empty urn.
Case (vi-2): $|U_j| = 2$ and there is a non-empty urn to its left. Let U_c be the right-most such having more than one ball, if such an urn exists; otherwise U_c is the first non-empty urn.
Then there exists m' such that

(vi-1) $\lambda(m') = (\ldots, j, j', \ldots)$,

(vi-2) $\lambda(m') = (\ldots, j, c', \ldots)$.

Thus, in all six cases there is a chain m' in $[\beta, \alpha]$ with $\lambda(m') <_{lex} \lambda(m)$, contradicting our assumption.

2. *No other chain has weakly increasing label.*

Say that the first move along m is to remove a ball from U_j. If not taken, all moves with strictly greater labels will weakly increase the number of balls in U_j, and will leave all urns to the left of U_j untouched. Thus, we must eventually return to a move with label j (or less) in order to reach a correct final distribution.

For a move of type 2 the reasoning is similar. Thus, in both cases any deviation from the chain m will later be punished with a descent in the label. □

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