Counting Conjugacy Classes of Elements of Finite Order in Lie Groups

Tamar Friedmann* and Richard Stanley†

*University of Rochester, Rochester, NY
†Massachusetts Institute of Technology, Cambridge, MA

Abstract

We use combinatorial techniques to derive explicit formulas for the number of conjugacy classes of elements of finite order in unitary, symplectic, and orthogonal Lie groups, as well as the number of such conjugacy classes whose elements have a specified number of distinct eigenvalues.

Keywords: Conjugacy classes, finite order, Lie groups, Chu-Vandermonde Identity, binomial identities

AMS Classification: 05A15 (22E10, 22E40)

1 Introduction

Given a group G of linear transformations and integers m and s, let

$$E(G,m) = \{x \in G \mid x^m = 1\}, \tag{1}$$

and

$$E(G,m,s) = \left\{ x \in E(G,m) \;\middle|\; \begin{array}{c} x \text{ has } s \text{ distinct eigenvalues or} \\ \text{conjugate pairs of eigenvalues} \end{array} \right\}. \tag{2}$$

(For each group G we consider below, we will specify more precisely what s stands for). Also let

$$\begin{aligned} N(G,m) &= \text{number of conjugacy classes of } G \text{ in } E(G,m), \\ N(G,m,s) &= \text{number of conjugacy classes of } G \text{ in } E(G,m,s). \end{aligned}$$

For Γ any finitely generated abelian group and G a Lie group, one can consider the space of homomorphisms $\mathrm{Hom}(\Gamma, G)$ and the space of representations of Γ in G, $\mathrm{Rep}(\Gamma, G) \equiv \mathrm{Hom}(\Gamma, G)/G$

(where G acts by conjugation); using this notation, $E(G,m) = \mathrm{Hom}(\mathbf{Z}/m\mathbf{Z}, G)$ and $N(G,m) = |\mathrm{Rep}(\mathbf{Z}/m\mathbf{Z}, G)|$. For the case $\Gamma = \mathbf{Z}^n$, the spaces $\mathrm{Hom}(\mathbf{Z}^n, G)$ and $\mathrm{Rep}(\mathbf{Z}^n, G)$ have been studied for various Lie groups G in [3, 1, 2, 4] (and references therein), where there has been interest in their number of path-connected components and their cohomology groups.

It is the purpose of this paper to compute $N(G,m) = |\mathrm{Rep}(\mathbf{Z}/m\mathbf{Z}, G)|$ and $N(G,m,s)$ for G a unitary, orthogonal, or symplectic group. Unlike $\mathrm{Rep}(\Gamma, G)$ for $\Gamma = \mathbf{Z}^n$, the representation space $\mathrm{Rep}(\mathbf{Z}/m\mathbf{Z}, G)$ is a finite set, so we can count its number of elements. The results are summarized in Table 1.

The numbers $N(G,m,s)$ have never been studied before in the mathematical literature. What motivated their definition, as well as the definition of $N(G,m)$, was the need to find a formula for the number of certain vacua in the quantum moduli space of M-theory compactifications on manifolds of G_2 holonomy. In that context, the numbers $N(SU(p),q)$ and $N(SU(p),q,s)$, where q and p are relatively prime, were computed in [8]. These numbers are related to symmetry breaking patterns in grand unified theories, with the number $N(SU(p),q,s)$ being particularly significant as s is related to the number of massless fields in the gauge theory that remains after the symmetry breaking. The connections with symmetry breaking patterns arise from the fact that if M is a manifold and $\pi_1(M)$ is its fundamental group, then $\mathrm{Rep}(\pi_1(M), G)$ is the moduli space of isomorphism classes of flat connections on principal G-bundles over M; in grand unified theories arising from string or M-theory, these flat connections (called Wilson lines) serve as a symmetry breaking mechanism. For more on the physical applications and implications of these numbers, see [9].

As for $N(G,m)$, certain cases have been studied previously in the mathematical literature, using different techniques than ours. Two of the quantities we derive, Theorems 2.2 and 3.1, were obtained in [5, 6] using the full machinery of Lie structure theory with a generating function approach; in [16, 7], the case of certain prime power orders is computed; and in [11], Theorem 2.6 is obtained. Our methods are different; they are purely combinatorial and direct, and apply not only to simply connected or adjoint groups as in [5, 6], so we are able to derive formulas for $O(n)$, $SO(n)$, and $U(n)$ alongside those for $SU(n)$ and $Sp(n)$.

Other aspects of elements of finite order in Lie groups have been studied. See for example [10, 13, 12, 14, 15].

In addition to the quantities $N(G,m)$ and $N(G,m,s)$, which count conjugacy classes of elements of any order dividing m, we consider also conjugacy classes of elements of exact order m in G: let

$$F(G,m) = \{x \in G \mid x^m = 1, x^n \neq 1 \text{ for all } n < m\},$$

and

$$F(G,m,s) = \left\{ x \in F(G,m) \;\middle|\; \begin{array}{c} x \text{ has } s \text{ distinct eigenvalues or} \\ \text{conjugate pairs of eigenvalues} \end{array} \right\}.$$

Also let

$$\begin{aligned} K(G,m) &= \text{number of conjugacy classes of } G \text{ in } F(G,m), \\ K(G,m,s) &= \text{number of conjugacy classes of } G \text{ in } F(G,m,s). \end{aligned}$$

Since

$$\begin{aligned} N(G,m) &= \sum_{d|m} K(G,d), \\ N(G,m,s) &= \sum_{d|m} K(G,d,s), \end{aligned}$$

we have, by the Mobius inversion formula,

$$K(G,m) = \sum_{d|m} \mu(d) N(G, \frac{m}{d}), \tag{3}$$

$$K(G,m,s) = \sum_{d|m} \mu(d) N(G, \frac{m}{d}, s), \tag{4}$$

where $\mu(d)$ is the Mobius function.

The reader is invited to obtain $K(G,m)$ and $K(G,m,s)$ from Table 1 and equations (3) and (4) above.

Table 1: Number of conjugacy classes of elements of finite order in Lie groups

G	m	$N(G,m)$	$N(G,m,s)$
$U(n)$	any	$\binom{n+m-1}{m-1}$	$\frac{s}{n}\binom{n}{s}\binom{m}{s}$
$SU(n)$	$(n,m)=1$	$\frac{1}{m}\binom{n+m-1}{n}$	$\frac{s}{nm}\binom{n}{s}\binom{m}{s}$
	any	$\frac{1}{m}\sum_{d\mid(n,m)}\phi(d)\binom{(n+m-d)/d}{n/d}$	$\frac{1}{m}\sum_{d\mid(n,m)}\sum_{j\geq 0}\phi(d)\binom{(n+m-jd-d)/d}{(n-jd)/d}\binom{m/d}{j}\binom{jd}{s}(-1)^{j+s}$
$Sp(n)$	any	$\binom{n+[\frac{m}{2}]}{n}$	$\frac{s}{n}\binom{n}{s}\binom{[\frac{m}{2}]+1}{s}$
$SO(2n+1)$	any	$\binom{n+[\frac{m}{2}]}{n}$	$\frac{s}{n}\binom{n}{s}\binom{\left[\frac{m}{2}\right]+1}{s}$
$O(2n+1)$	$2k+1$	$\binom{n+[\frac{m}{2}]}{n}$	$\frac{s}{n}\binom{n}{s}\binom{\left[\frac{m}{2}\right]+1}{s}$
$O(2n)$	$2k+1$	$\binom{n+[\frac{m}{2}]}{n}$	$\frac{s}{n}\binom{n}{s}\binom{\left[\frac{m}{2}\right]+1}{s}$
$SO(2n)$	$2k+1$	$\binom{n+[\frac{m}{2}]-1}{n-1}\frac{n+m-1}{n}$	$\frac{s}{n}\binom{n}{s}\binom{\left[\frac{m}{2}\right]}{s}\frac{m+1-s}{\left[\frac{m}{2}\right]+1-s}$
$O(2n+1)$	$2k$	$2\binom{n+\frac{m}{2}}{n}$	$\frac{2s}{n}\binom{n}{s}\binom{\frac{m}{2}+1}{s}$
$O(2n)$	$2k$	$\binom{n+\frac{m}{2}-1}{n-1}\frac{4n+m}{2n}$	$\frac{2n-s-1}{n-s}\binom{n-2}{s-1}\binom{\frac{m}{2}+1}{s}$
$SO(2n)$	$2k$	$\binom{n+\frac{m}{2}}{n}+\binom{n+\frac{m}{2}-2}{n}$	$\frac{s}{n}\binom{n}{s}\left[2\binom{\frac{m}{2}}{s}+\binom{\frac{m}{2}-1}{s-2}\right]$

2 Counting conjugacy classes in unitary groups

We begin with $N(U(n),m)$, with no conditions on the integers m and n. Since every element of $U(n)$ is diagonalizable, every conjugacy class has diagonal elements. The diagonal entries are m^{th} roots of unity, $e^{2\pi i k_j/m}$, $k_j = 0, \ldots$, and $m-1$, $j = 1, \ldots, n$. In each conjugacy class there is a unique diagonal element for which the diagonal entries are ordered so that the k_j are nondecreasing with j. Therefore, $N(U(n),m)$ is the number of such diagonal matrices with nondecreasing k_j.

Let $\{n_k\} = (n_0, \ldots, n_{m-1})$, $\sum_{k=0}^{m-1} n_k = n$ with $n_k \geq 0$. Such a sequence is a weak m-composition of n, and it is well-known that there are $\binom{n+m-1}{m-1}$ such sequences [17]. There is a bijective map between such sequences and diagonal matrices in $U(n)$ with ordered entries: $\{n_k\}$ corresponds to the diagonal $U(n)$ matrix with n_k repetitions of the eigenvalue $e^{2\pi i k/m}$:

$$\mathrm{diag}(\underbrace{1,1,\cdots,1}_{n_0},\underbrace{e^{2\pi i/m},\cdots,e^{2\pi i/m}}_{n_1},\cdots,\underbrace{e^{2(m-1)\pi i/m},\cdots,e^{2(m-1)\pi i/m}}_{n_{m-1}}) \ . \tag{5}$$

Thus $N(U(n),m)$ is the number of weak m-compositions of n, so we obtain the following formula.

Theorem 2.1 *For any positive integers n and m,*

$$N(U(n),m) = \binom{n+m-1}{m-1} \tag{6}$$

Note that $N(U(n),m)$ is also the number of inequivalent unitary representations of $\mathbf{Z}/m\mathbf{Z}$ of dimension n.

Now we turn to the special unitary group $SU(p)$, and calculate $N(SU(p),q)$ where $(p,q)=1$. Given a sequence $\{n_k\}$, $k = 0, \ldots q-1$ with $\sum_{k=0}^{q-1} n_k = p$, $n_k \geq 0$ (i.e., a weak q-composition of p), the determinant of the corresponding matrix x is $\exp \frac{2\pi i}{q}\left(\sum_{k=0}^{q-1} k n_k\right)$, so the condition $\det x = 1$ requires $\sum_k k n_k \equiv 0 \bmod q$. Thus for a weak q-composition of p to determine a matrix in $SU(p)$, we need $\sum_k k n_k \equiv 0 \bmod q$.

We now show the family of weak q-compositions of p are partitioned into sets of size q where in each such set there is exactly one such composition with $\sum k n_k \equiv 0$. Consider the q distinct sequences

$$\{n_k^{(j)}\} = \{n_{k+j}\} \qquad j = 0, 1, \ldots, q-1 \ , \text{ indices are understood mod q.} \tag{7}$$

(The only way for the sequences not to be distinct is if all n_k were equal, which would imply $q n_k = p$, impossible when $(p,q)=1$). The determinant of the matrix x_j corresponding to the

j^{th} sequence is $\exp \frac{2\pi i}{q}\left(\sum_{k=0}^{q-1} kn_{k+j}\right)$. Since $(p,q)=1$ and

$$\sum_{k=0}^{q-1} kn_{k+j} - \sum_{k=0}^{q-1} kn_{k+j+1} \equiv p \bmod q\ , \tag{8}$$

exactly one of the q values of j gives the sum $\sum_k kn_{k+j} \equiv 0 \bmod q$, so $\det(x_j) = 1$ for that value of j. We conclude:

Theorem 2.2 *For* $(p,q)=1$,

$$N(SU(p),q) = \frac{1}{q}\binom{p+q-1}{q-1} = \frac{(p+q-1)!}{p!\,q!}\ . \tag{9}$$

Now we turn to counting conjugacy classes whose elements have a given number s of distinct eigenvalues. We begin with $N(U(n),m,s)$. A $U(n)$ matrix with s distinct eigenvalues (which has centralizer of the form $\Pi_{i=1}^{s} U(n_i)$) corresponds to a sequence $\{n_a\} = (n_1,\dots n_s)$, $\sum_{a=1}^{s} n_a = n$, $n_a \geq 1$. Such a sequence is an s-composition of n and there are $\binom{n-1}{s-1}$ such sequences [17]). There are also $\binom{m}{s}$ ways to choose the s eigenvalues themselves. We therefore obtain the following formula.

Theorem 2.3 *For any positive integers n and m,*

$$N(U(n),m,s) = \binom{n-1}{s-1}\binom{m}{s} = \frac{s}{n}\binom{n}{s}\binom{m}{s}\ . \tag{10}$$

For the special unitary group, again we impose $(p,q)=1$. Given an s-composition of p, $\{n_a\} = (n_1,\dots n_s)$, $\sum_{a=1}^{s} n_a = p$, $n_a > 0$, consider $\{\lambda_a\} = (\lambda_1,\dots,\lambda_s)$ where $\lambda_a \in \{0,\dots,q-1\}$ determine the eigenvalues $e^{\frac{2\pi i \lambda_a}{q}}$ with multiplicity n_a of the corresponding matrix. Arrange the $\binom{q}{s}s!$ possibilities for $\{\lambda_a\}$ in sets of size q given by

$$\{\lambda_a^{(j)}\} = (\lambda_1 + j,\dots,\lambda_s + j),\ \ j = 0,\dots,q-1 \quad \text{(all numbers are understood mod } q). \tag{11}$$

The determinant of the matrix x_j corresponding to the j^{th} choice is

$$\exp \frac{2\pi i}{q}\left(\sum_{a=1}^{s} n_a(\lambda_a + j)\right).$$

Since $(p,q)=1$ and

$$\sum_a n_a(\lambda_a + j) - \sum_a n_a(\lambda + j + 1) = p,$$

exactly one of the q matrices has determinant 1. Since so far neither the λ_a's nor the n_a's have been ordered, once we arrange the eigenvalues to have increasing λ_a's, each matrix would appear $s!$ times. Dividing by $s!q$, we obtain

Theorem 2.4 *For* $(p, q) = 1$,

$$N(SU(p), q, s) = \frac{1}{q}\binom{p-1}{s-1}\binom{q}{s} = \frac{s}{pq}\binom{p}{s}\binom{q}{s} . \tag{12}$$

From Theorems 2.2 and 2.4, we deduce an intriguing symmetry between p and q:

Corollary 2.1 *For* $(p, q) = 1$,

$$\begin{aligned} N(SU(p), q) &= N(SU(q), p); \\ N(SU(p), q, s) &= N(SU(q), p, s). \end{aligned}$$

This symmetry has implications involving dualities of gauge theories; see [9].

It is clear that for any G and m, we must have

$$\sum_s N(G, m, s) = N(G, m). \tag{13}$$

Since $N(G, m, s) = 0$ when $s > m$, the sum is finite. Applying equation (13) to $G = U(n)$ gives

$$\sum_s \binom{n-1}{s-1}\binom{m}{s} = \binom{n+m-1}{m-1}, \tag{14}$$

which is a special case of the Chu-Vandermonde identity [17].

We may also obtain both $N(SU(n), m)$ and $N(SU(n), m, s)$ without requiring $(n, m) = 1$ via a generating function approach. Let

$$F(x, t, u) = \prod_{k=0}^{m-1}\left(1 + u\sum_{a=1}^{\infty}(t^k x)^a\right).$$

A typical term in $F(x, t, u)$ is

$$x^{\sum n_k} t^{\sum k n_k} u^s,$$

where n_k, $k = 0, \ldots, m-1$ are nonnegative integers and s is the number of k's for which $n_k \neq 0$. If $\sum n_k = n$ and $\sum k n_k \equiv 0 \bmod m$ then the sequence $\{n_k\}$ corresponds to a diagonal $SU(n)$ matrix of order m with s distinct eigenvalues. To pick out the terms in $F(x, t, u)$ for which $\sum k n_k \equiv 0 \bmod m$, let $\zeta = \exp 2\pi i/m$ and recall

$$\frac{1}{m}\sum_{j=0}^{m-1}\zeta^{jb} = \begin{cases} 1, & \text{if } m|b \\ 0, & \text{else} \end{cases},$$

so

$$G(x, u) = \frac{1}{m}\sum_{j=0}^{m-1} F(x, \zeta^j, u) = \sum_{n,s} N(SU(n), m, s) x^n u^s. \tag{15}$$

Rewriting

$$\left(1+u\sum_{a=1}^{\infty}(t^kx)^a\right)=(1-u)+\frac{u}{1-t^kx}=\frac{1-t^k(1-u)x}{1-t^kx},$$

we have

$$G(x,u)=\frac{1}{m}\sum_{j=0}^{m-1}\prod_{k=0}^{m-1}\frac{1-\zeta^{kj}(1-u)x}{1-\zeta^{kj}x}. \tag{16}$$

For ζ^j a primitive d^{th} root of unity, we have the factorization $(1-x^d)=\prod_{l=0}^{d-1}(1-\zeta^{jl}x)$. Since ζ^j, $j=0,\ldots,m-1$ is a primitive d^{th} root of unity $\phi(d)$ times, where $\phi(d)$ is Euler's function, we have

$$G(x,u)=\frac{1}{m}\sum_{d|m}\phi(d)\frac{\left[1-(1-u)^dx^d\right]^{m/d}}{(1-x^d)^{m/d}}. \tag{17}$$

Expanding in binomial series gives

$$G(x,u)=\frac{1}{m}\sum_{d|m}\phi(d)\sum_{k,j,l\geq 0}\binom{k+m/d-1}{k}\binom{m/d}{j}\binom{jd}{l}(-1)^{j+l}\,x^{d(k+j)}u^l.$$

Setting $d(k+j)=n$ and $l=s$, we have

Theorem 2.5 *For any positive integers $n,m,$ and s,*

$$N(SU(n),m,s)=\frac{1}{m}\sum_{d|(n,m)}\sum_{j\geq 0}\phi(d)\binom{n/d+m/d-j-1}{n/d-j}\binom{m/d}{j}\binom{jd}{s}(-1)^{j+s}. \tag{18}$$

We may deduce from Theorems 2.5 and 2.4 that for $(p,q)=1$,

$$\frac{1}{q}\sum_{j\geq 0}\binom{p+q-j-1}{p-j}\binom{q}{j}\binom{j}{s}(-1)^{j+s}=\frac{s}{pq}\binom{p}{s}\binom{q}{s}. \tag{19}$$

For $N(SU(n),m)$ we apply equation (13), or equivalently set $u=1$ in $G(x,u)$, and obtain (see also [11]) the next result.

Theorem 2.6 *For any positive integers n and m,*

$$N(SU(n),m)=\frac{1}{m}\sum_{d|(n,m)}\phi(d)\binom{n/d+m/d-1}{n/d}. \tag{20}$$

3 Counting conjugacy classes in symplectic groups

The diagonal elements of $U(n)$ and $SU(p)$ that we counted in the previous section belonged to the maximal tori of those groups. For $\mathrm{Sp}(n) \equiv \mathrm{Sp}(n, \mathbf{C}) \cap U(2n)$, the maximal torus is

$$T_{\mathrm{Sp}(n)} = \left\{ (e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}, e^{-2\pi i\theta_1}, \ldots, e^{-2\pi i\theta_n}) \right\}. \tag{21}$$

Since $\mathrm{Sp}(n)$ is compact and connected, we have $\mathrm{Sp}(n) = \bigcup_{x\in G} \, xT_{\mathrm{Sp}(n)}x^{-1}$. Hence, every element $x \in G$ can be conjugated into the torus, so every conjugacy class has elements in $T_{\mathrm{Sp}(n)}$. Any two elements x and x' of $T_{\mathrm{Sp}(n)}$ that differ only by $\theta'_l = -\theta_l$ for some l's are in the same conjugacy class; the symplectic matrix $E_{l,n+l} - E_{n+l,l}$, where $(E_{ab})_{cd} = \delta_{ac}\delta_{bd}$, conjugates them. So a conjugacy class is fully determined by n values of θ_l restricted to $[0, 1/2]$.

Conjugacy classes of elements of order m have a unique element in $T_{\mathrm{Sp}(n)}$ such that $\theta_l \in \frac{1}{m}(0, 1, \ldots, [\frac{m}{2}])$ and the θ_l are nondecreasing as i runs from 1 to n. Following the arguments leading to Theorem 2.1, and noting that here we have weak $([\frac{m}{2}]+1)$-compositions of n, rather than weak m-compositions of n, we obtain our next theorem.

Theorem 3.1 *For any positive integers n and m,*

$$N(\mathrm{Sp}(n), m) = \binom{n + [\frac{m}{2}]}{[\frac{m}{2}]}. \tag{22}$$

We now consider $N(\mathrm{Sp}(n), m, s)$ where s denotes the number of complex conjugate pairs of eigenvalues. Following the arguments leading to Theorem 2.3, but replacing m by $([\frac{m}{2}] + 1)$, we have

Theorem 3.2 *For any positive integers n, m, and s,*

$$N(\mathrm{Sp}(n), m, s) = \binom{n-1}{s-1}\binom{[\frac{m}{2}]+1}{s}. \tag{23}$$

4 Counting conjugacy classes in orthogonal groups

The maximal tori of the different orthogonal groups depend on the parity of l in $SO(l)$ or $O(l)$ and also on whether the orthogonal group is special or not:

$$\begin{aligned} T_{SO(2n)} &= \{\operatorname{diag}(A(\theta_1), A(\theta_2), \ldots, A(\theta_n))\}\,, && (24)\\ T_{SO(2n+1)} &= \{\operatorname{diag}(A(\theta_1), A(\theta_2), \ldots, A(\theta_n), 1)\}\,, && (25)\\ T_{O(2n)} &= \left\{ \begin{array}{l} T_{1,even} = \operatorname{diag}(A(\theta_1), A(\theta_2), \ldots, A(\theta_n)) \\ T_{2,even} = \operatorname{diag}(A(\theta_1), A(\theta_2), \ldots, A(\theta_{n-1}), B) \end{array} \right\}, && (26)\\ T_{O(2n+1)} &= \left\{ \begin{array}{l} T_{1,\text{odd}} = \operatorname{diag}(A(\theta_1), A(\theta_2), \ldots, A(\theta_n), 1) \\ T_{2,\text{odd}} = \operatorname{diag}(A(\theta_1), A(\theta_2), \ldots, A(\theta_n), -1) \end{array} \right\}, && (27) \end{aligned}$$

where

$$A(\theta) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix} \; ; \; B = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \tag{28}$$

The identity

$$BA(\theta)B^{-1} = A(-\theta) \tag{29}$$

will become useful below.

With the maximal tori defined as above, every element of the orthogonal group can be conjugated to the torus, so each conjugacy class has a nonempty intersection with the group's maximal torus.

The counting of conjugacy classes depends on the parity of the order m of the elements, so we treat the odd and even cases separately.

4.1 Odd m

We begin with $N(SO(2n+1), m)$. The block-diagonal matrix $\operatorname{diag}(B, I_{2n-2}, -1)$ is an element of $SO(2n+1)$ and equation (29) shows that conjugation by it takes $x \in T_{SO(2n+1)}$ to $x' \in T_{SO(2n+1)}$ where $\theta'_1 = -\theta_1$ and the other θ_i remain the same. Similarly, two elements x and x' of $T_{SO(2n+1)}$ that differ by $\theta'_l = -\theta_l$ for any $l = 1, \ldots, n$ belong to the same conjugacy class. We therefore consider only elements of $T_{SO(2n+1)}$ with $\theta_l \in [0, 1/2]$ as we did for the symplectic case. As before, we order the θ_l to be nondecreasing with l.

For elements of order m, we have $\theta_l \in \frac{1}{m}(0, 1, \ldots, [\frac{m}{2}])$. So $N(SO(2n+1), m)$ is the number of weak $([\frac{m}{2}] + 1)$-compositions of n:

Theorem 4.1 *For any positive integer n and any odd integer $m = 2k+1$,*

$$N(SO(2n+1), m) = \binom{n + \left[\frac{m}{2}\right]}{\left[\frac{m}{2}\right]}. \tag{30}$$

For $O(2n+1)$, there are two conjugacy classes of maximal tori, i.e., $T_{SO(2n+1)}$, and $T_{2,\mathrm{odd}}$ in equation (27). However, all elements of $T_{2,\mathrm{odd}}$ have even order, so none has order $m = 2k+1$. Therefore, the number of conjugacy classes of elements of odd order in $O(2n+1)$ is the same as that for $SO(2n+1)$, so we get the following result.

Theorem 4.2 *For any positive integer n and any odd integer $m = 2k+1$,*

$$N(O(2n+1), m) = \left(\begin{array}{c} n + \left[\frac{m}{2}\right] \\ \left[\frac{m}{2}\right] \end{array} \right). \tag{31}$$

For $O(2n)$, again $T_{2,even} \in T_{O(2n)}$ does not play a role when m is odd. Also, the block diagonal matrix $\mathrm{diag}(B, I_{2n-2})$ is an element of $O(2n)$, so the results for $O(2n+1)$ and $O(2n)$ are the same.

Theorem 4.3 *For any positive integer n and any odd integer $m = 2k+1$,*

$$N(O(2n), m) = \left(\begin{array}{c} n + \left[\frac{m}{2}\right] \\ \left[\frac{m}{2}\right] \end{array} \right). \tag{32}$$

Things become more subtle for $SO(2n)$: $\mathrm{diag}(B, I_{2n-2})$ has determinant -1 so it is not an element of $SO(2n)$. Therefore, it is no longer the case that if $x, x' \in T_{SO(2n)}$ differ only by $\theta_i' = -\theta_i$ for some i's then x and x' are necessarily in the same conjugacy class. However, the block diagonal matrix $\mathrm{diag}(B, B, I_{2n-4})$ *is* in $SO(2n)$, so if $\theta_l' = -\theta_l$ for an even number of l's, so x and x' are in the same conjugacy class.

There are two cases to consider: $\theta_1' = \theta_1 = 0$ and $\theta_l \neq 0$ for all l. In the first case, $A(\theta_1) = A(\theta_1') = I_2$, and if $\theta_l' = -\theta_l$ for any additional $l \geq 2$ (not necessarily an even number of times), then x and x' are in the same conjugacy class. The number of conjugacy classes that are represented by elements of $T_{SO(2n)}$ with $\theta_1 = 0$ is the number of weak $\left(\left[\frac{m}{2}\right] + 1\right)$-compositions of $n-1$. In the second case $\theta_l \neq 0$ for all l, the number of classes is the number of weak $\left[\frac{m}{2}\right]$-compositions of n; since here, flipping the sign of one θ_l, say $\theta_1' = -\theta_1$ and leaving the others fixed lands in a different conjugacy class, we multiply the number by two to include all the classes. This leads to the following theorem.

Theorem 4.4 *For any positive integer n and any odd integer $m = 2k+1$,*

$$N(SO(2n), m) = \binom{n + \left[\frac{m}{2}\right] - 1}{\left[\frac{m}{2}\right]} + 2\binom{n + \left[\frac{m}{2}\right] - 1}{\left[\frac{m}{2}\right] - 1} = \binom{n + \left[\frac{m}{2}\right] - 1}{\left[\frac{m}{2}\right]} \frac{n + m - 1}{n} . \tag{33}$$

We now turn to $N(SO(2n+1), m, s)$, where as for the symplectic groups, s denotes the number of distinct conjugate pairs of eigenvalues of the elements. For all the orthogonal groups, there are n θ_l's and $\binom{n-1}{s-1} = \frac{s}{n}\binom{n}{s}$ ways to partition them into s nonzero parts. There are $\left[\frac{m}{2}\right]+1$ possible values for the θ_i. The same is true for $O(2n+1)$, and $O(2n)$, yielding the next result.

Theorem 4.5 *For any positive integers n and s, and any odd integer $m = 2k+1$,*

$$N(SO(2n+1), m, s) = N(O(2n+1), m, s) = N(O(2n), m, s) = \frac{s}{n}\binom{n}{s}\binom{\left[\frac{m}{2}\right] + 1}{s}. \tag{34}$$

The above derivation does not apply to $SO(2n)$ because as before, some classes need to be counted twice due to the absence of (B, I_{2n-2}) in $SO(2n)$. First, we divide the n eigenvalue pairs into s nonzero parts (s-compositions of n). In choosing the s eigenvalues out of the $\left[\frac{m}{2}\right]+1$ possibilities, we differentiate the cases where $\theta_1 = 0$, which we count once, from the cases where $\theta_1 \neq 0$, which we need to count twice to account for $\theta_1' = -\theta_1$, $\theta_l' = \theta_l$, $l > 1$ which is in a distinct conjugacy class. We get the following formula.

Theorem 4.6 *For any positive integers n and s and any odd integer $m = 2k+1$,*

$$\begin{aligned} N(SO(2n), m, s) &= \binom{n-1}{s-1}\left[\binom{\left[\frac{m}{2}\right]}{s-1} + 2\binom{\left[\frac{m}{2}\right]}{s}\right] \\ &= \frac{s}{n}\binom{n}{s}\binom{\left[\frac{m}{2}\right]}{s}\frac{m+1-s}{\left[\frac{m}{2}\right]+1-s} . \end{aligned}$$

4.2 Even m

Unlike the case for odd m, here we will have to consider T_2 in both $O(2n)$ and $O(2n+1)$. There will also be changes from the odd m case due to the fact that $\theta_l = 1/2$, corresponding to $A(\theta_l) = -I_2$, can appear.

For $SO(2n+1)$, we have essentially the same as we did for odd m, i.e., weak $\left(\frac{m}{2}+1\right)$-compositions of n.

Theorem 4.7 *For any positive integer n and any even integer $m = 2k$,*

$$N(SO(2n+1), m) = \begin{pmatrix} n + \frac{m}{2} \\ \frac{m}{2} \end{pmatrix}. \tag{35}$$

For $O(2n+1)$, we have to consider conjugacy classes with elements in $T_{2,\text{odd}}$ of $T_{O(2n+1)}$. But the counting is exactly the same as in $T_{1,\text{odd}}$, so the next theorem follows.

Theorem 4.8 *For any positive integer n and any even integer $m = 2k$,*

$$N(O(2n+1), m) = 2 \begin{pmatrix} n + \frac{m}{2} \\ \frac{m}{2} \end{pmatrix}. \tag{36}$$

Turning to $O(2n)$, we note that elements in $T_{2,even}$ have only $n-1$ θ_l's. Other than that, the counting is the same as before, and we have

Theorem 4.9 *For any positive integers n and any even integer $m = 2k$,*

$$\begin{aligned} N(O(2n), m) &= \begin{pmatrix} n + \frac{m}{2} \\ \frac{m}{2} \end{pmatrix} + \begin{pmatrix} n + \frac{m}{2} - 1 \\ \frac{m}{2} \end{pmatrix} \\ &= \begin{pmatrix} n + \frac{m}{2} - 1 \\ \frac{m}{2} \end{pmatrix} \frac{4n+m}{2n}. \end{aligned}$$

For $SO(2n)$, again we need to be careful since $\theta_l' = \pm\theta_l$ does not always mean x and x' are in the same conjugacy class. Only when at least one of the θ_i is 0 or 1/2, so that $A(\theta_l) = \pm I_2$ for that l, which commutes with B, does $\theta_l' = \pm\theta_l$ mean x and x' are in the same conjugacy class. If no θ_l is 0 or 1/2 then if say $\theta_1' = -\theta_1$ and $\theta_l' = \theta_l$, $l > 1$, we have a different conjugacy class for x and x'. The number of conjugacy classes such that at least one θ_l is 0 or 1/2 is the number of weak $\left(\frac{m}{2}+1\right)$-compositions of $n-1$ (where we have fixed $\theta_1 = 0$) plus the number of weak $\left(\frac{m}{2}\right)$-compositions of $n-1$ (where we do not allow $\theta_l = 0$ and we require $\theta_l = 1/2$ for some l). The number of conjugacy classes where no θ_l is 0 or 1/2 is twice the number of weak $\left(\frac{m}{2}-1\right)$-compositions of n. After some algebra we obtain the next result.

Theorem 4.10 *For any positive integer n and any even integer $m = 2k$,*

$$N(SO(2n), m) = \begin{pmatrix} n + \frac{m}{2} \\ \frac{m}{2} \end{pmatrix} + \begin{pmatrix} n + \frac{m}{2} - 2 \\ \frac{m}{2} - 2 \end{pmatrix}. \tag{37}$$

For $N(SO(2n+1), m, s)$, we have the same calculation as for odd m, and for $N(O(2n+1), m, s)$, we simply double the result to account for the elements in $T_{2,\text{odd}}$, giving

Theorem 4.11 *For any positive integers n and s and any even integer $m = 2k$,*

$$N(SO(2n+1), m, s) = \frac{s}{n}\binom{n}{s}\binom{\frac{m}{2}+1}{s};$$

$$N(O(2n+1), m, s) = \frac{2s}{n}\binom{n}{s}\binom{\frac{m}{2}+1}{s}.$$

Next is $O(2n)$, where $T_{2,even}$ has only $n-1$ θ_l's, so the contribution from $T_{2,even}$ differs from that from $T_{1,even}$ by replacing n with $n-1$. After some algebra we get the following theorem.

Theorem 4.12 *For any positive integers n and s and any even integer $m = 2k$,*

$$N(O(2n), m, s) = \frac{2n-s-1}{n-s}\binom{n-2}{s-1}\binom{\frac{m}{2}+1}{s}. \qquad (38)$$

For $SO(2n)$, for each s-composition of n, the number of conjugacy classes of $T_{SO(2n)}$ with $\theta_l \neq 0, 1/2$ for all l is $\binom{\frac{m}{2}-1}{s}$, and the number of conjugacy classes with at least one $\theta_l = 0, 1/2$ is the sum of $\binom{\frac{m}{2}}{s-1}$, which gives the number of conjugacy classes with $\theta_1 = 0$, and $\binom{\frac{m}{2}-1}{s-1}$ which gives the number of conjugacy classes with $\theta_l \neq 0$ $\forall l$ and $\theta_l = 1/2$ for some l. As before, we multiply the number for $\theta_l \neq 0, 1/2$ by 2, and add the rest. After some algebra, we have our final result.

Theorem 4.13 *For any positive integers n and s and any even integer $m = 2k$,*

$$N(SO(2n), m, s) = \binom{n-1}{s-1}\left[\binom{\frac{m}{2}+1}{s} + \binom{\frac{m}{2}-1}{s}\right]. \qquad (39)$$

Acknowledgments

It is a pleasure to thank Jonathan Pakianathan, Steve Gonek, Ben Green, and Dragomir Djokovic for discussions. It is also a pleasure to thank Jonathan Pakianathan for comments on an earlier draft. The work of the first author was supported in part by US DOE Grant number DE-FG02-91ER40685, and of the second author in part by NSF Grant number DMS-1068625.

References

[1] A. Adem, F. R. Cohen, "Commuting elements and spaces of homomorphisms," Math. Ann. **338** (2007), no. 3, 587-626; Erratum.

[2] A. Adem, F. R. Cohen, and J. M. Gomez, "Commuting elements in central products of special unitary groups," arXiv:0905.2895.

[3] A. Borel, R. Friedman, and J. W. Morgan, "Almost commuting elements in compact Lie groups," Mem. Amer. Math. Soc. **157** (2002) no. 747.

[4] T. Baird, L. C. Jeffrey, and P. Selick, "The space of commuting n-tuples in $SU(2)$," Illinois J. Math, to appear, arXiv:0911.4953.

[5] D. Z. Djokovic, "On conjugacy classes of elements of finite order in compact or complex semisimple Lie groups," Proc. Amer. Math. Soc. **80** (1980), no. 1, 181–184.

[6] D. Z. Djokovic, "On conjugacy classes of elements of finite order in complex semisimple Lie groups," J. Pure Appl. Algebra **35** (1985), no. 1, 1–13.

[7] F. Destrempes and A. Pianzola, "Elements of compact connected simple Lie groups with prime power order and given field of characters," Geometriae Dedicata **45**, 225 (1993).

[8] T. Friedmann, "On the Quantum Moduli Space of M–Theory Compactifications," Nucl. Phys. B **635**: 384-394, 2002.

[9] T. Friedmann, "The string landscape: on formulas for counting vacua," preprint.

[10] V. Kac, "Automorphisms of finite order of semisimple Lie algebras," Func. Anal. Appl. **3** (1969) 252.

[11] O. P. Lossers, Solution to problem 288, Canad. Math. Bull. **26** 503 (1983).

[12] M. de Montigny, "Elements of finite order in Lie groups and discrete gauge symmetries," Nucl. Phys. B **439**, 665 (1995) [hep-th/9406064].

[13] R. V. Moody and J. Patera, "Characters of elements of finite order in Lie groups," SIAM J. Alg. Disc. Meth. **5** (1984) no. 3, 359.

[14] A. Pianzola, "On the arithmetic of the representation ring and elements of finite order in Lie groups," J. Algebra **108** (1987), 1.

[15] A. Pianzola, "On the rationality and regularity of certain elements of finite order in Lie groups," J. reine Angew. Math. **377** (1987), 40.

[16] A. Pianzola and A. Weiss, "The rationality of elements of prime order in compact connected simple Lie groups," J. Alg. **144**, 510 (1991).

[17] R. Stanley, *Enumerative Combinatorics, Vol. 1*, Cambridge University Press, Cambridge, 2012.