UNIMODALITY OF PARTITIONS WITH DISTINCT PARTS INSIDE FERRERS SHAPES

RICHARD P. STANLEY* AND FABRIZIO ZANELLO**

ABSTRACT. We investigate the rank-generating function F_λ of the poset of partitions contained inside a given shifted Ferrers shape λ. When λ has four parts, we show that F_λ is unimodal when $\lambda = \langle n, n-1, n-2, n-3 \rangle$, for any $n \geq 4$, and that unimodality fails for the doubly-indexed, infinite family of partitions of the form $\lambda = \langle n, n-t, n-2t, n-3t \rangle$, for any given $t \geq 2$ and n large enough with respect to t.

When λ has $b \leq 3$ parts, we show that our rank-generating functions F_λ are all unimodal. However, the situation remains mostly obscure for $b \geq 5$. In general, the type of results that we obtain present some remarkable similarities with those of the 1990 paper of D. Stanton, who considered the case of partitions inside ordinary (straight) Ferrers shapes.

Along the way, we also determine some interesting q-analogs of the binomial coefficients, which in certain instances we conjecture to be unimodal. We state several other conjectures throughout this note, in the hopes to stimulate further work in this area. In particular, one of these will attempt to place into a much broader context the unimodality of the posets $M(n)$ of staircase partitions, for which determining a combinatorial proof remains an outstanding open problem.

1. INTRODUCTION

A classical result in combinatorics is the unimodality of q-binomial coefficient (or Gaussian polynomial) $\binom{n+b}{b}_q$, which is the rank-generating function of the poset of integer partitions having at most b parts and whose largest part is at most n, denoted $L(b, n)$ (see e.g. [12, 14, 15, 19], and of course K. O'Hara's celebrated combinatorial proof [10, 22]). In other words, the coefficients of $\binom{n+b}{b}_q$ are *unimodal*, i.e., they do not increase strictly after a strict decrease.

Recall that a nonincreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_b)$ of positive integers is called a *partition* of N if $\sum_{i=1}^{b} \lambda_i = N$. The λ_i are the *parts* of λ, and the index b is its *length*. A partition λ can be represented geometrically by its *Ferrers diagram*, which is a collection of cells, arranged in left-justified rows, whose ith row contains exactly λ_i cells. With a slight

abuse of notation, we will sometimes also denote by λ the Ferrers diagram of the partition λ.

For some useful introductions and basic results of partition theory, we refer the reader to [2, 3, 11], Section I.1 of [9], and Section 1.8 of [16]. For any other standard combinatorial definition, we refer to [16].

The unimodality of the q-binomial coefficient can be rephrased in terms of Ferrers diagrams, by saying that the rank-generating function of the poset of partitions whose Ferrers diagrams are contained inside a $b \times n$ rectangle, namely G_λ, where $\lambda = (\lambda_1 = n, \lambda_2 = n, \ldots, \lambda_b = n)$, is unimodal. In his 1990 paper [18], D. Stanton studied the rank-generating function G_λ of partitions contained inside other Ferrers shapes λ. Not surprisingly, G_λ can be nonunimodal for certain λ, the smallest of which turned out to be $\lambda = (8, 8, 4, 4)$. Stanton was also able to determine infinitely many nonunimodal partitions λ with $b = 4$ parts, while he proved that unimodality always holds when $b \leq 3$. He also showed that nonunimodal partitions exist for $b = 6$, whereas all examples known to date when $b = 5$ or $b \geq 7$ are unimodal.

A well-known variant of the Ferrers diagram of a partition is the *shifted Ferrers diagram* of a partition λ with distinct parts. Such diagrams have λ_i cells in row i as before, but now each row is indented one cell to the right of the previous row. The goal of this note is to study the rank-generating functions F_λ of the posets of partitions μ contained inside a shifted Ferrers shape λ. Equivalently, μ is a partition *with distinct parts* contained in an ordinary (straight) Ferrers shape λ. We write $\langle \lambda_1, \lambda_2, \ldots, \lambda_n \rangle$ for a partition λ with distinct parts, having at most n parts, regarded as a shifted diagram. Note that 0 is not considered a part, so $\lambda = \langle 4, 2, 1, 0, 0 \rangle$ is a partition with distinct parts. For instance, the partitions contained inside $\lambda = \langle 4, 2, 1 \rangle$ are: $\emptyset$ (the empty partition); $\langle 1 \rangle$ partitioning 1; $\langle 2 \rangle$ partitioning 2, $\langle 3 \rangle$ and $\langle 2, 1 \rangle$ partitioning 3; $\langle 4 \rangle$ and $\langle 3, 1 \rangle$ partitioning 4; $\langle 4, 1 \rangle$ and $\langle 3, 2 \rangle$ partitioning 5; $\langle 4, 2 \rangle$ and $\langle 3, 2, 1 \rangle$ partitioning 6; and $\langle 4, 2, 1 \rangle$ partitioning 7. Thus,

$$F_{\langle 4,2,1 \rangle}(q) = 1 + q + q^2 + 2q^3 + 2q^4 + 2q^5 + 2q^6 + q^7.$$

While Stanton's work was in part motivated by the interest of the unimodality of rectangular Ferrers shapes, the corresponding prototype of partition in our situation is the "shifted staircase partition" $\lambda = \langle b, b-1, \ldots, 2, 1 \rangle$. The poset of partitions with distinct parts that it generates is often referred to as $M(b)$. It is a standard exercise to show that $F_\lambda(q) = \prod_{i=1}^{b}(1+q^i)$. The unimodality of this polynomial, which was essentially first proved by E.B. Dynkin [6, 7] (see also [12, 15]), is also closely related to the famous Erdös-Moser conjecture, solved by the first author in [15]. Notice, however, that the simplest proof known to date of the unimodality of the staircase partition uses a linear algebra argument [12]; it remains an outstanding open problem in combinatorics to determine a constructive proof.

Though our situation is obviously essentially different from that of Stanton — for instance, it is easy to see that our rank-generating functions F_λ are never symmetric if λ has at least

two parts, with the only exception of the staircase partitions — some of our results for distinct parts will show a remarkable similarity to the case of arbitrary partitions. In this paper we will mostly focus on partitions λ whose parts are in arithmetic progression, even though, similarly to what was done in [18], it is possible to naturally extend some results or conjectures to partitions having distinct parts that lie within certain intervals.

In the next section, we will consider the case when λ has four parts. First, we show that F_λ is unimodal for all "truncated staircases" $\lambda = \langle n, n-1, n-2, n-3 \rangle$. Notice that, unlike in many other instances of nontrivial unimodality results in combinatorics, in this case F_λ is never symmetric (for $n > 4$) nor, as it will be clear from the proof, log-concave.

Our second main result is the existence of a doubly-indexed, infinite family of nonunimodal rank-generating functions F_λ. Namely, we will show that if $\lambda = \langle n, n-t, n-2t, n-3t \rangle$, where $t \geq 2$, then F_λ is always nonunimodal whenever n is large enough with respect to t (the least such n can also be computed effectively). For $t = 2$, as we will see in the subsequent section, the rank-generating function F_λ turns out to be a q-analog of the binomial coefficient $\binom{n+1}{4}$. We will briefly discuss the meaning of these new q-analogs $\binom{a}{b}^q$. Interestingly, even though, unlike the q-binomial coefficient, in general they can be nonunimodal, we will conjecture unimodality for our *central* q-binomial coefficients, $\binom{2n+1}{n}^q$ and $\binom{2n}{n}^q$.

Similarly to Stanton's situation, we will also show that F_λ is unimodal for any partition λ with at most $b = 3$ parts (in fact, we will rely on Stanton's theorem to give a relatively quick proof of our result). Moreover, again like Stanton, we are unaware of the existence of any nonunimodal rank-generating function F_λ when $b = 5$ or $b \geq 7$, and will provide examples of nonunimodal F_λ for $b = 6$ that we have not been able to place into any infinite family.

Next we conjecture the unimodality of all partitions λ having parts in arithmetic progression that begin with the smallest possible positive residue. This conjecture, if true, would place the still little understood unimodality of the staircase partitions into a much broader context. Other conjectures are given throughout this note.

2. Partitions of length four

In this section, we study the rank-generating functions F_λ of partitions λ of length four. We focus in our statements on partitions whose parts are in arithmetic progression, which is the most interesting case; i.e., we consider λ to be of the form $\lambda = \langle n, n-t, n-2t, n-3t \rangle$. Notice, however, that certain arguments could naturally be applied to partitions whose parts lie inside suitable intervals, similarly to some of the cases studied by Stanton [18].

Our first main result of this section is that if $\lambda = \langle n, n-1, n-2, n-3 \rangle$, then F_λ is unimodal, for any $n \geq 4$. In contrast, our second result will show that the doubly-indexed, infinite family of partitions $\lambda = \langle n, n-t, n-2t, n-3t \rangle$ are nonunimodal, for any given $t \geq 2$ and n large enough with respect to t. The proofs of both results will be mostly

combinatorial, and rely in part on the following elegant properties of the coefficients of the q-binomial coefficients $\binom{a+4}{4}_q$, which are of independent interest.

Lemma 2.1. *Let* $\binom{a+4}{4}_q = \sum_{i=0}^{4a} d_{a,i} q^i$. *Define* $f(a,c) = d_{a,2a-c} - d_{a,2a-c-1}$ *for* $c \geq 0$, *and* $f(a,c) = 0$ *for* $c < 0$. *(Hence,* $f(a,c) = 0$ *for* $c > 2a$.) *We have:*

(a)

$$\sum_{a,c} f(a,c) q^a t^c = \frac{1}{(1-q^2)(1-q^3)(1-qt^2)} + \frac{q^2 t^2}{(1-q^2)(1-q^3)(1-qt^2)(1-qt)};$$

(b) $f(a,c) \geq 0$ *for all* a *and* c. *Moreover, if* $a \geq 2$ *and* $0 \leq c \leq 2a$, *then equality holds if and only if* $c = 1$, $c = 2a - 1$, *or* $(a,c) = (4,3)$;

(c) $f(a,0) = \lfloor (a+3\delta)/6 \rfloor$, *where* $\delta = 1$ *if* a *is odd, and* $\delta = 2$ *if* a *is even. In particular,* $f(a,0)$ *goes to infinity when* a *goes to infinity.*

Proof. (a) See Theorem 2.2 of [17].

(b) That $f(a,c) \geq 0$ for all a and c is obvious from part (a) or also from the unimodality of the q-binomial coefficient $\binom{a+4}{4}_q$. As for the second part of the statement, the fact that the coefficients of $q^a t$, $q^a t^{2a-1}$ and $q^4 t^3$ are 0 for all a is easy to check directly. Proving the converse implication requires some careful but entirely standard analysis, so we will omit the details.

(c) From part (a), we immediately have that the generating function for $f(a,0)$ is

$$\sum_{a \geq 0} f(a,0) q^a = \frac{1}{(1-q^2)(1-q^3)}.$$

In other words, $f(a,0)$ counts the number of partitions of a whose parts can only assume the values 2 and 3. That their number now is the one in the statement is a simple exercise that we leave to the reader. This completes the proof of the lemma. □

Remark 2.2. It is possible to give an entirely combinatorial proof of parts (b) and (c) of Lemma 2.1, using D. West's symmetric chain decomposition for the poset $L(4,a)$, whose rank-generating function is of course $\binom{a+4}{4}_q$ (see [20] for all details). However, the argument would be less elegant and require significantly more work than using part (a) of the lemma.

On the other hand, the portions of the statement that will later suffice to show the nonunimodality of $\lambda = \langle n, n-t, n-2t, n-3t \rangle$ for $t \geq 2$ and n large — namely that $f(a,1) = 0$ for all a, and $f(a,0)$ goes to infinity when a goes to infinity — are easy and interesting to show using West's result. Indeed, it is clear that in his decomposition of $L(4,a)$ there exist no symmetric chains of cardinality three (one should only check that the cardinality of the chains $D_{i,j}$ defined at the middle of page 13 of [20] cannot equal 3, by how the indexes i and j are defined for the new chains on page 7). This immediately gives that $f(a,1) = 0$, since, clearly, $f(a,c) = 0$ if and only if in West's construction there exist no symmetric chains of cardinality $2c + 1$.

In order to show that $f(a, 0)$ goes to infinity, notice that West's proof implies that $f(a, 0)$ is nondecreasing, since in the inductive step he makes an injection between the chains constructed for $a - 1$ and those for a. Thus, in the formula for the cardinality of $c_{i,j}$ at the middle of page 13 of [20], one can for instance choose a to be a multiple of 6, $i = a/3$, and $j = 0$. This easily shows the existence of an extra chain of cardinality 1 for those values of a which does not come from $a - 1$, and thus it suffices to make $f(a, 0)$ go to infinity. (In fact, a little more work proves in this fashion all of part (3) of Lemma 2.1.)

Theorem 2.3. *Let* $\lambda = \langle n, n - 1, n - 2, n - 3 \rangle$, *where* $n \geq 4$. *Then the rank-generating function* F_λ *is unimodal.*

Proof. In the case when the partitions lying inside $\lambda = \langle n, n - 1, n - 2, n - 3 \rangle$ have three or four parts, by removing the staircase $\langle 3, 2, 1 \rangle$ from λ, it is easy to see that such partitions are in bijection with the *arbitrary* partitions contained inside a $4 \times (n - 3)$ rectangle, whose rank-generating function is $\binom{n+1}{4}_q$. In a similar fashion, when the partitions lying inside $\lambda = \langle n, n - 1, n - 2, n - 3 \rangle$ have at most two parts, by removing the staircase $\langle 1 \rangle$ we can see that such partitions are enumerated by $\binom{n+1}{2}_q$.

From this, we immediately have that F_λ decomposes as:

$$(1) \qquad F_\lambda(q) = 1 + q\binom{n + 1}{2}_q + q^6 \binom{n + 1}{4}_q.$$

We can assume for simplicity that $n \geq 8$, since the result is easy to check (e.g., using Maple) for $n \leq 7$. Let c_i be the coefficient of degree i of $q^6 \binom{n+1}{4}_q$. Hence, $c_i \neq 0$ if and only if $6 \leq i \leq 4n - 6$, and because of the unimodality of the q-binomial coefficient $\binom{n+1}{4}_q$, the c_i are also unimodal with a peak at c_{2n}. Further, it easily follows from Lemma 2.1 that $c_i > c_{i-1}$ for all $n + 1 \leq i \leq 2n$, with the exception of $i = 2n - 1$, which gives $c_{2n-1} = c_{2n-2}$.

On the other hand, notice that the q-binomial coefficient $q\binom{n+1}{2}_q$ is a unimodal function; its coefficients d_i are nonzero for $1 \leq i \leq 2n - 1$, and they assume a peak at d_n. Also, it is a simple exercise to check that when $\binom{n+1}{2}_q$ decreases (or by symmetry, it increases), it does so by at most 1. Finally, notice that $d_{2n-1} = d_{2n-2}$ (they are both equal to 1), which implies that the coefficients of F_λ in degree $2n - 1$ and $2n - 2$ are also equal.

Putting all of the above together, since by equation (1) F_λ can be written as

$$F_\lambda(q) = 1 + \sum_{i=1}^{4n-6} (c_i + d_i)q^i,$$

it is easy to check that F_λ is unimodal. (In fact, we have shown that it has a peak in degree $2n$.) $\qquad\square$

Theorem 2.4. *Let* $\lambda = \langle n, n-t, n-2t, n-3t \rangle$, *where* $t \geq 2$ *is fixed. Then the rank-generating function* F_λ *is nonunimodal for all integers* n *large enough with respect to* t.

Proof. Let $F_\lambda(q) = \sum_{i=0}^{4n-6t} c_i^t q^i$. We will prove that F_λ is nonunimodal for any $t \geq 2$ and n large enough with respect to t, by showing that

$$c_{2n}^t > c_{2n-1}^t < c_{2n-2}^t.$$

We assume from now on that n is large enough. It is easy to see from Lemma 2.1 and the proof of Theorem 2.3 that $c_{2n-1}^1 = c_{2n-2}^1$, and that $c_{2n}^1 - c_{2n-1}^1$ goes to infinity. Indeed, we have shown in that proof that, in degree $2n$, the rank-generating function of $\langle n, n-1, n-2, n-3 \rangle$ is the same as that of $\binom{n+1}{4}_q$, whereas in degrees $2n-1$ and $2n-2$ it is exactly one more than that of $\binom{n+1}{4}_q$, because of the extra contributions coming from $q\binom{n+1}{2}_q$.

We begin by showing that $c_{2n}^2 > c_{2n-1}^2 < c_{2n-2}^2$. Notice that the only partition of $2n$ that lies inside $\langle n, n-1, n-2, n-3 \rangle$ but not inside $\langle n, n-2, n-4, n-6 \rangle$ is $\langle n, n-1, 1 \rangle$. Thus, $c_{2n}^2 = c_{2n}^1 - 1$. Similarly, the only partition of $2n-1$ that lies inside $\langle n, n-1, n-2, n-3 \rangle$ but not $\langle n, n-2, n-4, n-6 \rangle$ is $\langle n, n-1 \rangle$, and therefore, $c_{2n-1}^2 = c_{2n-1}^1 - 1$.

Finally, $c_{2n-2}^2 = c_{2n-2}^1$, since all partitions with distinct parts of $2n-2$ that lie inside $\langle n, n-1, n-2, n-3 \rangle$ clearly cannot have $n-1$ as their second largest part. It follows that the difference between $(c_{2n}^1, c_{2n-1}^1, c_{2n-2}^1)$ and $(c_{2n}^2, c_{2n-1}^2, c_{2n-2}^2)$ is $(1, 1, 0)$, which immediately proves that $c_{2n}^2 > c_{2n-1}^2 < c_{2n-2}^2$, i.e., the theorem for $t = 2$.

In a similar fashion, it is easy to check that, in passing from $\langle n, n-2, n-4, n-6 \rangle$ to $\langle n, n-3, n-6, n-9 \rangle$, the difference between $(c_{2n}^2, c_{2n-1}^2, c_{2n-2}^2)$ and $(c_{2n}^3, c_{2n-1}^3, c_{2n-2}^3)$ is $(3, 2, 2)$, showing the result for $t = 3$.

In general, if n_i is the number of partitions of i into two distinct parts, employing the same idea as above easily gives us that in passing from $\langle n, n-(t-1), n-2(t-1), n-3(t-1) \rangle$ to $\langle n, n-t, n-2t, n-3t \rangle$, the difference $c_{2n}^t - c_{2n-1}^t$ decreases by $n_{2t-3} - n_{t-2}$ with respect to $c_{2n}^{t-1} - c_{2n-1}^{t-1}$.

Notice that

$$n_{2t-3} - n_{t-2} = \lceil (2t-3)/2 \rceil - \lceil (t-2)/2 \rceil = \lfloor t/2 \rfloor,$$

if as usual we denote by $\lceil x \rceil$ and $\lfloor x \rfloor$ the smallest integer $\geq x$ and the largest $\leq x$, respectively.

Therefore, the difference between $c_{2n}^1 - c_{2n-1}^1$ and $c_{2n}^t - c_{2n-1}^t$ amounts to $\sum_{i=1}^t \lfloor i/2 \rfloor$, which has order of magnitude $t^2/4$.

Since $c_{2n}^1 - c_{2n-1}^1$ goes to infinity, this completes the proof that $c_{2n}^t > c_{2n-1}^t$ for all $t \geq 2$ and n large enough with respect to t.

Notice that $c_{2n-1}^2 < c_{2n-2}^2$. Hence, in order to complete the proof of the theorem it now suffices to show that, in passing from $\langle n, n-(t-1), n-2(t-1), n-3(t-1) \rangle$ to $\langle n, n-t, n-2t, n-3t \rangle$, c_{2n-1}^t decreases at least as much as c_{2n-2}^t does, for any $t \geq 3$. Thus, since n is large enough with respect to t, one moment's thought gives that it is enough to show that there are at least as many partitions of $2n-1$ than there are of $2n-2$, which are contained inside $\langle n, n-(t-1), n-2(t-1), n-3(t-1) \rangle$ and have $n-(t-1)$ as their second largest part.

But if $\mu = \langle \mu_1, \mu_2 = n-(t-1), \mu_3, \mu_4 \rangle$ is such a partition of $2n-2$, notice that $\mu_1 \geq n-t+2$, and therefore

$$\mu_3 + \mu_4 \leq 2n - 2 - (n - (t-1)) - (n - t + 2) = 2t - 5,$$

which is smaller than $\mu_2 = n - (t-1)$ by at least 2, since n is large.

Therefore, we can define an injection between the above partitions of $2n - 2$ and those of $2n - 1$ by mapping μ to $\theta = \mu + (0, 0, 1, 0)$. This shows that there are at least as many of the above partitions of $2n - 1$ as there are of $2n - 2$, which completes the proof of the theorem. $\qquad\square$

Remark 2.5. The same idea of the proof of Theorem 2.4 can prove the nonunimodality of F_λ also for other partitions $\lambda = \langle \lambda_1, \lambda_2, \lambda_3, \lambda_4 \rangle$; namely, those that are obtained by "perturbing" $\langle n, n-t, n-2t, n-3t \rangle$ in a way that the λ_i remain within suitable intervals. This fact, which is quite natural, is also consistent with the results of Stanton [18] in the case of arbitrary partitions with parts lying within certain intervals. We only remark here that in general, however, an actual "interval property" (see e.g. [21]) does not hold in this context. In fact, it is easy to check, e.g. using Maple, that the rank-generating function F_λ is nonunimodal when $\lambda = \langle 19, 16, 11, 8 \rangle$ and $\lambda = \langle 19, 16, 9, 8 \rangle$, while it is unimodal for $\lambda = \langle 19, 16, 10, 8 \rangle$.

3. Other shapes

We begin by presenting a new q-analog of the binomial coefficients, that we call $\binom{a}{b}^q$, for any integers a and b such that $1 \leq b \leq a/2$. We will show that this q-analog is the rank-generating function F_λ of partitions with distinct parts contained inside $\lambda = \langle a-1, a-3, \ldots, a-(2b-1) \rangle$. After discovering an independent proof of this fact, we found out that it can also be easily deduced from a theorem of R. Proctor concerning shifted plane partitions (see [13, Theorem 1]). However, since our argument, unlike Proctor's, is combinatorial, we include a sketch of it below for completeness. We use the following lemma without proof, since it is equivalent to the well-known fact that the number of standard Young tableaux with a boxes, at most two rows, and at most b boxes in the second row is $\binom{a}{b}$.

Lemma 3.1. *Fix integers a and b such that $1 \leq b \leq a/2$. Then the number of binary sequences of length a containing at most b 1's, and such that no initial string contains more 1's than 0's, is $\binom{a}{b}$.*

Proposition 3.2. *Fix integers a and b such that $1 \leq b \leq a/2$, and let $\lambda = \langle a - 1, a - 3, \ldots, a - (2b-1) \rangle$. Then the number of partitions μ with distinct parts lying inside λ is $\binom{a}{b}$.*

Proof. Let $\mu = \langle \mu_1, \ldots, \mu_t \rangle$ be a partition into distinct parts contained inside $\lambda = \langle a - 1, \ldots, a - (2b - 1) \rangle$. In particular, $a - 1 \geq \mu_1 > \cdots > \mu_t \geq 1$, where $t \leq b$.

We associate to μ a binary sequence of length a, say $W_\mu = w_a w_{a-1} \cdots w_1$, such that $w_i = 1$ if i is a part of μ, and $w_i = 0$ otherwise. (Notice that w_a is always 0, since $a > \mu_1$ for all partitions μ.)

By definition, the number of 1's in W_μ is $t \le b$, and since the parts of λ differ by exactly 2, it is a standard exercise to check that the above correspondence is indeed a bijection between our partitions μ and those binary sequences W_μ where no initial string of W_μ contains more 1's than 0's. Thus, by Lemma 3.1, the number of partitions μ is $\binom{a}{b}$, as desired. $\square$

Therefore, from Proposition 3.2 we immediately have the following result.

Corollary 3.3. *If* $\lambda = \langle a - 1, a - 3, \ldots, a - (2b - 1) \rangle$, *then* $F_\lambda(q) = \binom{a}{b}^q$.

For example,

$$F_{\langle 4,2 \rangle} = \binom{5}{2}^q = 1 + q + q^2 + 2q^3 + 2q^4 + 2q^5 + q^6$$

$$F_{\langle 5,3 \rangle} = \binom{6}{2}^q = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 2q^6 + 2q^7 + q^8$$

$$F_{\langle 8,6,4,2 \rangle} = \binom{9}{4}^q = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + 5q^7 + 6q^8 + 7q^9 + 8q^{10} +$$

$$9q^{11} + 10q^{12} + 11q^{13} + 12q^{14} + 11q^{15} + 11q^{16} + 10q^{17} + 7q^{18} + 4q^{19} + q^{20}.$$

Notice that, for $b > 1$, our q-analog $\binom{a}{b}^q$ of the binomial coefficient $\binom{a}{b}$ is always different from the Gaussian polynomial $\binom{a}{b}_q$ (in fact, it is easy to show that $\binom{a}{b}^q$ is never symmetric for $b > 1$). While $\binom{a}{b}_q$ is well known to be unimodal, the case $t = 2$ of Theorem 2.4 proves that, in general, unimodality may fail quite badly for $\binom{a}{b}^q$, even when $b = 4$. The smallest such nonunimodal example is

$$F_{\langle 9,7,5,3 \rangle} = \binom{10}{4}^q = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + 5q^7 + 6q^8 + 8q^9 + 9q^{10} + 10q^{11} + 12q^{12} + 13q^{13}$$

$$+ 15q^{14} + 16q^{15} + \mathbf{17}q^{16} + \mathbf{16}q^{17} + \mathbf{17}q^{18} + 15q^{19} + 14q^{20} + 11q^{21} + 7q^{22} + 4q^{23} + q^{24}.$$

However, we conjecture that the following fact is true, which is a special case of a conjecture that we will state later.

Conjecture 3.4. *Our q-analogs of the* central *binomial coefficients are all unimodal. That is,* $\binom{2n+1}{n}^q$ *and* $\binom{2n}{n}^q$ *are both unimodal, for any* $n \ge 1$.

For partitions with three parts, all rank-generating functions are unimodal. We will provide a bijective proof of this result, assuming the corresponding theorem of Stanton for arbitrary partitions ([18, Theorem 7]).

Lemma 3.5 (Stanton). *If λ is any arbitrary partition of length $b \le 3$, then the rank-generating function G_λ is unimodal.*

Lemma 3.6. *Consider the partition $\lambda = (p, r, s)$, and let $G_{(p,r,s)}(q) = \sum_{i=0}^{p+r+s} a_i q^i$. We have:*
  (1) *If $2 \le p \le 2r + s$, then $a_{p-1} < a_p$:*
  (2) *If $p = 1$ or $p \ge 2r + s + 1$, then $a_{p-1} = a_p$ and $a_i \ge a_{i+1}$ for all $i \ge p - 1$.*

Proof. Since the largest part of λ is p, notice that there is a natural injection ϕ between the set A_{p-1} of partitions μ of $p-1$ contained inside the Ferrers diagram of λ and the set of partitions θ in A_p, where $\theta = \phi(\mu) = \mu + (1, 0, 0)$.

Thus, for any $\lambda = (p, r, s)$, we have $a_{p-1} \leq a_p$. Clearly, equality holds if and only if there exists no partition $\theta = (\theta_1, \theta_2, \theta_3)$ in A_p such that $\theta_1 = \theta_2$, since these are the only partitions not in the image of the map ϕ.

It is a standard exercise now to show that if $p \leq 2r+s$, then there always exists a partition $\theta \in A_p$ such that $\theta_1 = \theta_2$. Indeed, if $p = 2r + s$, then we can pick $\theta = (r, r, s)$; for $p < 2r + s$, one can for instance first decrease the value of s until it reaches 0 (i.e., until p is down to $2r$), and then consider the partitions $\theta = (d, d, \epsilon)$, where d decreases by 1 at the time and ϵ is either 0 or 1, depending on the parity of p. This proves part (1).

In order to prove (2), notice that the case $p = 1$ is trivial, since here $\lambda = (p, r, s) = (1, 1, 1)$. Thus let $p \geq 2r+s+1$. Then we have that no partition θ of p inside $\lambda = (p, r, s)$ can satisfy $\theta_1 = \theta_2$, since $\theta_3 \leq s$, $\theta_2 \leq r$, and therefore $\theta_1 \geq r + 1$. Thus, this is exactly the case where $a_{p-1} = a_p$, and in order to finish the proof of the lemma, now it suffices to show that $a_i \geq a_{i+1}$ for all $i \geq p - 1$.

But this can be done in a symmetric fashion to the above argument, by defining a map ψ from A_{i+1} to A_i such that $\beta = \psi(\alpha) = \alpha - (1, 0, 0)$. Since $p \geq 2r + s + 1$, it is easy to see that ψ is well defined and injective. Thus, $a_i \geq a_{i+1}$ for all $i \geq p - 1$, as we wanted to show. $\qquad \square$

Theorem 3.7. *If λ is any partition with length $b \leq 3$, then the rank-generating function F_λ is unimodal.*

Proof. When $b = 1$ the result is obvious, and when $b = 2$ it is also easy to check. Indeed, this can be done directly, or by observing that if $\lambda = \langle p + 1, r \rangle$, for some $p \geq r \geq 1$, then one promptly obtains that

$$F_{\langle p+1, r \rangle}(q) = 1 + qG_{(p,r)}(q).$$

Thus F_λ is unimodal, since $G_{(p,r)}(q)$ is unimodal by Lemma 3.5.

Hence, let $b = 3$, and set $\lambda = \langle p + 2, r + 1, s \rangle$, where $p \geq r \geq s \geq 1$. Clearly, any partition μ contained inside λ has at most three parts, and it is easy to see that those with at least two parts are in bijective correspondence with arbitrary partitions contained inside (p, r, s), by removing the staircase $\langle 2, 1 \rangle$.

From this, it follows that

$$F_{\langle p+2, r+1, s \rangle}(q) = q^3 G_{(p,r,s)}(q) + (1 + q + q^2 + \cdots + q^{p+2}).$$

Therefore, since by Lemma 3.5, $G_{(p,r,s)}(q)$ is unimodal, we have that in order to prove the unimodality of $F_{\langle p+2, r+1, s \rangle}(q)$, it suffices to show that if the coefficients of $G_{(p,r,s)}(q)$ coincide in degree $p - 1$ and p, then they are nonincreasing from degree $p - 1$ on. But this follows from Lemma 3.6, thus completing the proof of the theorem. $\qquad \square$

For partitions λ with $b \geq 5$ parts, the scenario becomes more and more unclear, and it again bears several similarities with Stanton's situation for arbitrary partitions. For instance, when $b = 5$, all examples we have computed are unimodal, and for $b = 6$, while it is possible to construct nonunimodal partitions, we have not been able to place them into any infinite family.

In particular, even the "truncated staircases" $\lambda = \langle n, n-1, \ldots, n-(b-1) \rangle$ in general need not be unimodal when $b < n$. For instance, the rank-generating function F_λ is nonunimodal for $\lambda = \langle 15, 14, 13, 12, 11, 10 \rangle$, $\lambda = \langle 17, 16, 15, 14, 13, 12 \rangle$, and $\lambda = \langle 19, 18, 17, 16, 15, 14 \rangle$, though this sequence does not continue in the obvious way. In fact, L. Alpoge [1] has recently proved, by means of a nice analytic argument, that the truncated staircases are all unimodal for n sufficiently large with respect to b, a fact that was conjectured in a previous version of our paper.

Theorem 3.8 ([1]). *Let $\lambda = \langle n, n-1, \ldots, n-(b-1) \rangle$. Then the rank-generating function F_λ is always unimodal for n large enough with respect to b.*

As we mentioned earlier, recall that for $b = n$, i.e., for the staircase partitions $\lambda = \langle b, b-1, \ldots, 2, 1 \rangle$, the unimodality of F_λ has already been established, though no combinatorial proof is known to date. The following conjecture attempts to place this result into a much broader context.

Conjecture 3.9. *The rank-generating function F_λ is unimodal for all partitions $\lambda = \langle a, a - t, \ldots, a - (b-1)t \rangle$ such that $t \geq a/b$. In other words, unimodality holds for all partitions with parts in arithmetic progression that begin with the smallest possible positive integer.*

Notice that, again similarly to Stanton's situation of arbitrary partitions, all examples we have constructed of nonunimodal rank-generating functions F_λ have exactly two peaks. However, it seems reasonable to expect that nonunimodality may occur with an arbitrary number of peaks, though showing this fact will probably require a significantly new idea.

Conjecture 3.10. *For any integer $N \geq 2$, there exists a partition λ whose rank-generating function F_λ is nonunimodal with (exactly?) N peaks.*

Finally, recall that a sequence $(a_0, a_1, \ldots, a_N)$ is *flawless* if $a_i \leq a_{N-i}$ for all $i \leq N/2$. Though this property is not as well studied as symmetry or unimodality, several natural and important sequences in algebra and combinatorics happen to be flawless (see e.g. [4, 5, 8]). We conclude this section by stating the following intriguing conjecture.

Conjecture 3.11. *For any partition λ, the rank-generating function F_λ is flawless.*

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DEPARTMENT OF MATHEMATICS, MIT, CAMBRIDGE, MA 02139-4307
E-mail address: `rstan@math.mit.edu`

DEPARTMENT OF MATHEMATICAL SCIENCES, MICHIGAN TECH, HOUGHTON, MI 49931-1295
E-mail address: `zanello@mtu.edu`