Some Hecke Algebra Products and Corresponding Random Walks

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September 5, 2008

Abstract. Let $\boldsymbol{i} = 1 + q + \cdots + q^{i-1}$. For certain sequences $(r_1, \ldots, r_l)$ of positive integers, we show that in the Hecke algebra $\mathscr{H}_n(q)$ of the symmetric group $\mathfrak{S}_n$, the product $(1 + \boldsymbol{r_1} T_{r_1}) \cdots (1 + \boldsymbol{r_l} T_{r_l})$ has a simple explicit expansion in terms of the standard basis $\{T_w\}$. An interpretation is given in terms of random walks on $\mathfrak{S}_n$.

Keywords: Hecke Algebra, tight sequence, reduced decompositions.

AMS Classification: 05E99

1 Main Results

Let $\mathfrak{S}_n$ denote the symmetric group of permutations of $\{1, \ldots, n\}$. For $1 \le i \le n-1$ let $s_i = (i, i+1) \in \mathfrak{S}_n$, the *adjacent transposition* interchanging i and $i+1$ and leaving all the other elements fixed. For any $w \in \mathfrak{S}_n$ denote by $\ell(w)$ the *length* of w, i.e., the minimal p such that w can be written as

$$w = s_{r_1} s_{r_2} \cdots s_{r_p}$$

for certain $r_1, r_2, \ldots, r_p$; such a sequence $r = (r_1, \ldots, r_p)$ is called a *reduced decomposition* (or reduced word) provided $p = \ell(w)$.

The *Hecke Algebra* (or *Iwahori-Hecke algebra*) $\mathscr{H}_n(q)$ of the symmetric group $\mathfrak{S}_n$ (e.g., [5, §7.4]) is defined as follows: $\mathscr{H}_n(q)$ is a $\mathbb{R}$-algebra with identity 1 and generators $T_1, T_2, \ldots, T_{n-1}$ which satisfy relations

$$\begin{aligned} (T_i + 1)(T_i - q) &= 0, \\ T_i T_j &= T_j T_i, \quad |i - j| \ge 2, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad 1 \le i \le n-2. \end{aligned} \tag{1.1}$$

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For any $w = s_{r_1}s_{r_2}\cdots s_{r_p} \in \mathfrak{S}_n$ for which $(r_1, r_2, \ldots, r_p)$ is reduced, define $T_w = T_{r_1}T_{r_2}\cdots T_{r_p}$. A basic property of Hecke algebras is that T_w does not depend on the choice of reduced decomposition of w, and for $1 \le k \le n-1$, T_w satisfies

$$T_wT_k = \begin{cases} T_{ws_k}, & \text{if } \ell(ws_k) = \ell(w)+1, \\ qT_{ws_k} + (q-1)T_w, & \text{if } \ell(ws_k) = \ell(w)-1. \end{cases} \tag{1.2}$$

Let $r = (r_1, r_2, \cdots, r_l)$ be any sequence of positive integers (not necessarily reduced). For convenience assume that $\max\{r_1, \ldots, r_l\} = n-1$. Set

$$\boldsymbol{i} = 1 + q + \cdots + q^{i-1}.$$

For any $w \in \mathfrak{S}_n$, define $\alpha_r(w) \in \mathbb{Z}[q]$ by

$$Q(r) := (1 + \boldsymbol{r_1}T_{r_1})(1 + \boldsymbol{r_2}T_{r_2})\cdots(1 + \boldsymbol{r_l}T_{r_l}) = \sum_{w \in \mathfrak{S}_n} \alpha_r(w)T_w.$$

We are primarily concerned with the polynomials $\alpha_r(w)$. In particular, for which r's will $\alpha_r(w)$ have "nice" values for all $w \in \mathfrak{S}_n$?

For each $w = w_1w_2\cdots w_n \in \mathfrak{S}_n$, we write $w \preceq r$ if $w = s_{c_1}\cdots s_{c_k}$ for some subsequence $c_1, \ldots, c_k$ of $r = (r_1, \ldots, r_l)$. It follows from equation (1.2) that $\alpha_r(w) = 0$ unless $w \preceq r$. Let $a_r(i)$ denote the number of i's in r, and let $\mathrm{inv}(w) = (\mathrm{inv}_w(1), \mathrm{inv}_w(2), \ldots, \mathrm{inv}_w(n-1))$ denote the *inversion sequence* of w, i.e., for any $1 \le i \le n-1$, $\mathrm{inv}_w(i)$ is the number of j's such that $w_j < w_i$ and $j > i$. It is easy to see that if $w \preceq r$ then $\mathrm{inv}_w(i) \le a_r(i)$ for all $1 \le i \le n-1$, and these two conditions are equivalent when r is reduced.

We call $r = (r_1, \ldots, r_l)$ a *tight sequence* if it satisfies the following:

1. For each prefix $r[i] := (r_1, \ldots, r_i)$ of r, $1 \le i \le l$, we have
$$a_{r[i]}(1) \ge a_{r[i]}(2) \ge a_{r[i]}(3) \ge \cdots.$$
2. For all $2 \le i \le l$ and $r_i \ge 2$, if $a_{r[i]}(r_i) \le \max\{r_1, \ldots, r_i\} + 1 - r_i$, then $a_{r[i]}(r_i - 1) = a_{r[i]}(r_i)$.

Notice that any prefix of the sequences $(1, 2, 1, 3, 2, 1, \ldots)$ or $(1, 2, \ldots, n, 1, 2, \ldots, n-1, \ldots, 1, 2, 1)$ is a tight sequence.

The main result of this paper is the following.

Theorem 1.1. *Let r be a tight sequence with $\max\{r\} = n-1$. Then for any $w \in \mathfrak{S}_n$ and $w \preceq r$, we have*

$$\alpha_r(w) = \prod_{i=2}^{n-1} \boldsymbol{i}^{\max\{a_r(i-1)-1, \mathrm{inv}_w(i)\}}. \tag{1.3}$$

Example 1.2. (a) Define the *standard tight sequence* ρ_n of degree n by

$$\rho_n = (1, 2, 1, 3, 2, 1, 4, 3, 2, 1, \ldots, n-1, n-2, \ldots, 1).$$

It is easy to see that ρ_n is not only a tight sequence but also a reduced decomposition of the element $w_0 = n, n-1 \ldots, 1 \in \mathfrak{S}_n$. Theorem 1.1 becomes

$$\alpha_{\rho_n}(w) = \mathbf{1}^{n-1}\mathbf{2}^{n-2} \cdots (\boldsymbol{n-1}), \tag{1.4}$$

independent of $w \in \mathfrak{S}_n$.

(b) Let $r = (1, 2, 1, 1, 3, 1)$. Then we have

$$\alpha_r(w) = \mathbf{2}^3, \quad \forall w \in \{1234, 1324, 2134, 2314, 3124, 3214\}$$

and

$$\alpha_r(w) = \mathbf{2}^3\mathbf{3}, \quad \forall w \in \{1243, 1342, 2143, 2341, 3142, 3241\}.$$

Otherwise we have $\alpha_r(w) = 0$.

Alexander Molev has pointed out (private communication dated September 1, 2008) that Theorem 1.1 in the case of the standard tight sequence is connected to the "fusion procedure" for the Hecke algebra, which goes back to Cherednik [1][2].

2 Proof of the Main Theorem

For the proof of Theorem 1.1 we need the following lemma.

Lemma 2.3. *Let $r = (r_1, \ldots, r_l) \in \mathbb{P}^l$ with $\max\{r\} = n-1$, where $\mathbb{P} = \{1, 2, \ldots\}$. Set $r' = (r, k)$ (the concatenation of r and k), $1 \le k \le n-1$. Then for any $w \in \mathfrak{S}_n$, we have*

1. *If $w \not\preceq r$, then $\alpha_{r'}(w) = \alpha_r(ws_k) \cdot \boldsymbol{k}$,*
2. *If $w \preceq r, ws_k \not\preceq r$, then $\alpha_{r'}(w) = \alpha_r(w)$,*
3. *If $w, ws_k \preceq r$, and $\ell(ws_k) = \ell(w) + 1$, then $\alpha_{r'}(w) = \alpha_r(w) + \alpha_r(ws_k) \cdot \boldsymbol{k}q$,*
4. *If $w, ws_k \preceq r$, and $\ell(ws_k) = \ell(w) - 1$, then $\alpha_{r'}(w) = \alpha_r(w) \cdot q^k + \alpha_r(ws_k) \cdot \boldsymbol{k}$.*

Proof. We have

$$Q(r') = Q(r)(1 + \boldsymbol{k}T_k) = \sum_{w \preceq r} \alpha_r(w) T_w + \sum_{u \preceq r} \alpha_r(u) T_u \cdot \boldsymbol{k}T_k. \tag{2.1}$$

We will prove the desired result by applying (1.2), and comparing the coefficients of T_w on both sides of (2.1).

1. If $w \not\preceq r$, then there is $u \preceq r$ such that $u \cdot s_k = w$. In this case T_w can only be obtained by $T_u \cdot \boldsymbol{k}T_k$, so we have $\alpha_{r'}(w) = \alpha_r(u) \cdot \boldsymbol{k} = \alpha_r(ws_k) \cdot \boldsymbol{k}$.
2. If $w \preceq r$ and $ws_k \not\preceq r$, then there is no $u \preceq r$ such that $us_k = w$. Hence T_w can only be obtained by $T_w \cdot 1$, so we have $\alpha_{r'}(w) = \alpha_r(w)$.

3. If $w, ws_k \preceq r$ and $\ell(ws_k) = \ell(w) + 1$, then $T_w \cdot \boldsymbol{k}T_k = \boldsymbol{k}T_{ws_k}$, and there is $u = w \cdot s_k \preceq r$ such that $T_u \cdot \boldsymbol{k}T_k = \boldsymbol{k}((q-1)T_u + qT_{u \cdot s_k}) = \boldsymbol{k}((q-1)T_{ws_k} + qT_w)$. Therefore we have $\alpha_{r'}(w) = \alpha_r(w) + \alpha_r(u) \cdot \boldsymbol{k}q = \alpha_r(w) + \alpha_r(ws_k) \cdot \boldsymbol{k}q$.

4. If $w, ws_k \preceq r$ and $\ell(ws_k) = \ell(w) - 1$, then $T_w \cdot \boldsymbol{k}T_k = \boldsymbol{k}((q-1)T_w + qT_{ws_k})$, and there is $u = w \cdot s_k \preceq r$ such that $T_u \cdot \boldsymbol{k}T_k = \boldsymbol{k}T_{u \cdot s_k} = \boldsymbol{k}T_w$. Therefore we have
$$\begin{aligned} \alpha_{r'}(w) &= \alpha_r(w) + \alpha_r(u) \cdot \boldsymbol{k} + \alpha_r(w) \cdot \boldsymbol{k}(q-1) \\ &= \alpha_r(w) \cdot q^k + \alpha_r(ws_k) \cdot \boldsymbol{k}. \end{aligned}$$

∎

We also want to list the following result related to $\mathrm{inv}(w)$ and $\mathrm{inv}(ws_k)$, which is frequently used in the proof of Theorem 1.1. The proof of this result is quite straightforward and is omitted here.

Lemma 2.4. *For any permutation $w \in \mathfrak{S}_n$ and adjacent transposition $s_k, 1 \le k \le n-1$, we have the following properties of the statistic inv_w.*

1. *If $\ell(ws_k) = \ell(w) - 1$, then*
$$\mathrm{inv}_w(k) > \mathrm{inv}_w(k+1), \quad \mathrm{inv}_{ws_k}(k) = \mathrm{inv}_w(k+1), \quad \textit{and} \quad \mathrm{inv}_{ws_k}(k+1) = \mathrm{inv}_w(k) - 1.$$

2. *If $\ell(ws_k) = \ell(w) + 1$ then*
$$\mathrm{inv}_w(k) \le \mathrm{inv}_w(k+1), \quad \mathrm{inv}_{ws_k}(k) = \mathrm{inv}_w(k+1) + 1, \quad \textit{and} \quad \mathrm{inv}_{ws_k}(k+1) = \mathrm{inv}_w(k).$$

Now we are ready to prove the main theorem.

Proof of Theorem 1.1. The proof is by induction on l, the length of the sequence r. It is trivial to check that (1.3) holds for $r = (i)$ for any positive integer i. Suppose that (1.3) holds for some tight sequence r, and $r' = (r, k)$ is also a tight sequence. We want to prove that (1.3) also holds for r'. The case when $k > \max\{r\}$ is trivial, so from now on we will assume that $\max\{r\} = \max\{r'\} = n - 1$.

For any $i \neq k, k+1$ $(1 \le i \le n-1)$, we have $a_{r'}(i) = a_r(i)$, and $\mathrm{inv}_w(i) = \mathrm{inv}_{ws_k}(i)$. Therefore
$$\max\{a_{r'}(i-1) - 1, \mathrm{inv}_w(i)\} = \max\{a_r(i-1) - 1, \mathrm{inv}_w(i)\} = \max\{a_r(i-1) - 1, \mathrm{inv}_{ws_k}(i)\}$$
holds for any $i \neq k, k+1$ $(2 \le i \le n-1)$. Hence we only need to concentrate on the values of $\max\{a_{r'}(i-1)-1, \mathrm{inv}_w(i)\}$ for $i = k, k+1$. (When $k = 1$, we only consider $\max\{a_{r'}(1), \mathrm{inv}_w(2)$.)

Next we will prove that $\alpha_{r'}(w) = \prod_{i=2}^{n-1} \boldsymbol{i}^{\max\{a_{r'}(i-1)-1, \mathrm{inv}_w(i)\}}$ according to the four cases in Lemma 2.3, and we will frequently use Lemma 2.4.

1. Let $w \not\preceq r$. In this case $\mathrm{inv}_w(k) = a_r(k) + 1 = a_{r'}(k) \le n - k$. Since r, r' are both tight sequences we have $a_{r'}(k-1) = a_r(k-1) = a_r(k) + 1$. Moreover, since $\mathrm{inv}_{ws_k}(k) = \mathrm{inv}_w(k+1) < \mathrm{inv}_w(k) = a_r(k) + 1$, we have
$$\max\{a_{r'}(k-1) - 1, \mathrm{inv}_w(k)\} = a_r(k) + 1 = \max\{a_r(k-1) - 1, \mathrm{inv}_{ws_k}(k)\} + 1.$$

Since $\mathrm{inv}_{ws_k}(k+1) = \mathrm{inv}_w(k) - 1 = a_r(k)$ and $a_{r'}(k) = a_r(k)+1$, we have

$$\max\{a_{r'}(k)-1, \mathrm{inv}_w(k+1)\} = \max\{a_r(k)-1, \mathrm{inv}_{ws_k}(k+1)\} = a_r(k).$$

Hence we conclude that

$$\alpha_{r'}(w) = \alpha_r(ws_k)\cdot \boldsymbol{k} = \prod_{i=2}^{n-1} \boldsymbol{i}^{\max\{a_r(i-1)-1, \mathrm{inv}_{ws_k}(i)\}} \cdot \boldsymbol{k} = \prod_{i=2}^{n-1} \boldsymbol{i}^{\max\{a_{r'}(i-1)-1, \mathrm{inv}_w(i)\}}.$$

2. Let $w \preceq r$ and $ws_k \not\preceq r$. In this case we have $\ell(ws_k) = \ell(w)+1$ and $\mathrm{inv}_w(k) = a_r(k)$. Since $\mathrm{inv}_w(k+1) \geq \mathrm{inv}_w(k) = a_r(k)$ and $a_{r'}(k) = a_r(k)+1$, we have

$$\max\{a_r(k)-1, \mathrm{inv}_w(k+1)\} = \max\{a_{r'}(k)-1, \mathrm{inv}_w(k+1)\}.$$

It follows that $\alpha_{r'}(w) = \alpha_r(w) = \prod_{i=2}^{n-1} \boldsymbol{i}^{\max\{a_{r'}(i-1)-1, \mathrm{inv}_w(i)\}}$.

3. Let $w, ws_k \preceq r$ and $\ell(ws_k) = \ell(w)+1$. Since $\mathrm{inv}_w(k) < a_r(k)$, $\mathrm{inv}_{ws_k}(k) \leq a_r(k)$ and $a_r(k-1)-1 \geq a_r(k)$, we have

$$\max\{a_r(k-1)-1, \mathrm{inv}_w(k)\} = \max\{a_r(k-1)-1, \mathrm{inv}_{ws_k}(k)\} = a_r(k-1)-1.$$

Since $\mathrm{inv}_w(k+1) = \mathrm{inv}_{ws_k}(k) - 1 \leq a_r(k)-1$, and $\mathrm{inv}_{ws_k}(k+1) = \mathrm{inv}_w(k) < a_r(k)$, we have

$$\max\{a_r(k)-1, \mathrm{inv}_w(k+1)\} = \max\{a_r(k)-1, \mathrm{inv}_{ws_k}(k+1)\} = a_r(k)-1.$$

Hence $\alpha_r(w) = \alpha_r(ws_k)$. Therefore we have

$$\alpha_{r'}(w) = \alpha_r(w) + \alpha_r(ws_k)\cdot \boldsymbol{k}q = \alpha_r(w)(\boldsymbol{k+1}) = \prod_{i=2}^{n-1} \boldsymbol{i}^{\max\{a_r(i-1)-1, \mathrm{inv}_w(i)\}} \cdot (\boldsymbol{k+1}).$$

Moreover, since $\max\{a_{r'}(k)-1, \mathrm{inv}_w(k+1)\} = a_r(k) = \max\{a_r(k)-1, \mathrm{inv}_w(k+1)\}+1$, we have $\alpha_{r'}(w) = \prod_{i=2}^{n-1} \boldsymbol{i}^{\max\{a_{r'}(i-1)-1, \mathrm{inv}_w(i)\}}$.

4. Let $w, ws_k \preceq r$ and $\ell(ws_k) = \ell(w)-1$. In this case $\mathrm{inv}_w(k) \leq a_r(k)$. Since $\mathrm{inv}_{ws_k}(k) = \mathrm{inv}_w(k+1) < \mathrm{inv}_w(k) \leq a_r(k)$ and $a_r(k-1)-1 \geq a_r(k)$, we have

$$\max\{a_r(k-1)-1, \mathrm{inv}_w(k)\} = \max\{a_r(k-1)-1, \mathrm{inv}_{ws_k}(k)\} = a_r(k-1)-1.$$

Since $\mathrm{inv}_{ws_k}(k+1) = \mathrm{inv}_w(k)-1$, we have

$$\max\{a_r(k)-1, \mathrm{inv}_w(k+1)\} = \max\{a_r(k)-1, \mathrm{inv}_{ws_k}(k+1)\} = a_r(k)-1.$$

Hence $\alpha_r(w) = \alpha_r(ws_k)$. Therefore we have

$$\alpha_{r'}(w) = \alpha_r(w)\cdot q^k + \alpha_r(ws_k)\cdot \boldsymbol{k} = \alpha_r(w)\cdot(\boldsymbol{k+1}) = \prod_{i=2}^{n-1} \boldsymbol{i}^{\max\{a_r(i-1)-1, \mathrm{inv}_w(i)\}} \cdot (\boldsymbol{k+1}).$$

Moreover, since $\max\{a_{r'}(k)-1, \mathrm{inv}_w(k+1)\} = a_r(k) = \max\{a_r(k)-1, \mathrm{inv}_w(k+1)\}+1$, we have $\alpha_{r'}(w) = \prod_{i=2}^{n-1} \boldsymbol{i}^{\max\{a_{r'}(i-1)-1, \mathrm{inv}_w(i)\}}$.

Hence the proof is complete. ∎

We can use Theorem 1.1 and its proof to compute $\alpha_r(w)$ for certain sequences r that are not tight sequences.

Corollary 2.5. *Let r be a sequence of adjacent transpositions, and* $\max\{r\} = n-1$. *If r has the prefix $\rho_n = (1,2,1,3,2,1\ldots,n,n-1,\ldots,1)$, then we have*

$$\alpha_r(w) = \prod_{i=2}^{n-1} \boldsymbol{i}^{\max\{a_r(i-1)-1,\mathrm{inv}_w(i)\}}. \tag{2.2}$$

Proof. We will prove equation (2.2) by induction on the length of r. Since ρ_n is a tight sequence, from Theorem 1.1 we know that the result holds for $r=\rho_n$. Next assume the result for r and let $r'=(r,k)$ with $1\le k\le n-1$. We do an induction similar to what we did in the proof of Theorem 1.1. Since r has the prefix ρ_n, it follows that for any $w\in\mathfrak{S}_n$, $w, ws_k \preceq r$. Therefore only cases 3 and 4 will occur. Moreover, since $a_r(k-1)\ge n-(k-1)$, $a_r(k)\ge n-k$ and $a_{r'}(k)=a_r(k)+1$, we have

$$\max\{a_r(k-1)-1,\mathrm{inv}_w(k)\} = \max\{a_r(k-1)-1,\mathrm{inv}_{ws_k}(k)\} = a_r(k-1)-1,$$

$$\max\{a_r(k)-1,\mathrm{inv}_w(k+1)\} = \max\{a_r(k)-1,\mathrm{inv}_{ws_k}(k+1)\} = a_r(k)-1,$$

and

$$\max\{a_{r'}(k)-1,\mathrm{inv}_w(k+1)\} = a_r(k) = \max\{a_r(k)-1,\mathrm{inv}_w(k+1)\}+1.$$

Hence for both case 3 and 4 we have $\alpha_{r'}(w)=\prod_{i=2}^{n-1}\boldsymbol{i}^{\max\{a_{r'}(i-1)-1,\mathrm{inv}_w(i)\}}$. ∎.

Note that r is a reduced decomposition of $w\in\mathfrak{S}_n$ if and only if the reverse of r is a reduced decomposition of w^{-1}. Thus we have the following result.

Corollary 2.6. *Let r be a sequence of adjacent transpositions, and* $\max\{r\}=n-1$. *If*

1. *r is the reverse of a tight sequence, or*
2. *r has suffix $\rho_n=(1,2,1,3,2,1\ldots,n,n-1,\ldots,1)$,*

then for any $w\in\mathfrak{S}_n$ and $w\preceq r$, we have

$$\alpha_r(w) = \prod_{i=2}^{n-1} \boldsymbol{i}^{\max\{a_r(i-1)-1,\mathrm{inv}_{w^{-1}}(i)\}}. \tag{2.3}$$

Note. If a sequence r' is obtained from r by transposing two adjacent terms that differ by at least 2, then $Q(r)=Q(r')$, so $\alpha_w(r)=\alpha_w(r')$. Thus our results extend to sequences that can be obtained from those of Theorem 1.1, Corollary 2.5, and Corollary 2.6 by applying such "commuting transpositions" to r.

3 A Connection with Random Walks on $\mathfrak{S}_n$

There is a huge literature on random walks on $\mathfrak{S}_n$, e.g., [3]. Our results can be interpreted in this context. First consider the case $q = 1$. In this case the Hecke algebra $\mathscr{H}_n(q)$ reduces to the group algebra $\mathbb{R}\mathfrak{S}_n$ of $\mathfrak{S}_n$, and the generator T_i becomes the adjacent transposition s_i. Thus

$$Q(r)_{q=1} = (1 + r_1 s_{r_1})(1 + r_2 s_{r_2}) \cdots (1 + r_l s_{r_l}).$$

We normalize this expression by dividing each factor $1 + r_i s_i$ by $1 + r_i$. Write

$$D_j = (1 + js_j)/(1 + j),$$

and set

$$\widetilde{Q}(r) = D_{r_1} D_{r_2} \cdots D_{r_l}.$$

If P is a probability distribution on $\mathfrak{S}_n$, then let $\sigma_P = \sum_{w \in \mathfrak{S}_n} P(w)w \in \mathbb{R}\mathfrak{S}_n$. If P' is another probability distribution on $\mathfrak{S}_n$, then $\sigma_P \sigma_{P'} = \sigma_{P*P'}$ for some probability distribution $P * P'$, the *convolution* of P and P'. It follows that $\widetilde{Q}(r) = \sigma_{P_r}$ for some probability distribution P_r on $\mathfrak{S}_n$. Theorem 1.1 gives (after setting $q = 1$ and normalizing) an explicit formula for the distribution P_r, i.e., the values $P_r(w)$ for all $w \in \mathfrak{S}_n$. Note in particular that if r is the standard tight sequence $\rho_n = (1, 2, 1, 3, 2, 1, 4, 3, 2, 1, \ldots, n-1, n-2, \ldots, 1)$, then from equation (1.4) we get

$$\widetilde{Q}(\rho_n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} w = \sigma_U,$$

where U is the uniform distribution on $\mathfrak{S}_n$. (We have been informed by Alexander Molev that an equivalent result was given by Jucys [6] in 1966. We have also been informed by Persi Diaconis that this result, and similar results for some other groups, were known by him and Colin Mallows twenty years ago.) It is not hard to see directly why we obtain the uniform distribution. Namely, start with any permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$. Do nothing with probability 1/2 or apply s_1 (i.e., interchange w_1 and w_2) with probability 1/2, obtaining $y_1 y_2 w_3 \cdots w_n$. Thus y_2 is equally likely to be w_1 or w_2. Now either do nothing with probability 1/3 or apply s_2 with probability 2/3, obtaining $y_1 z_2 z_3 w_4 \cdots w_n$. Then z_3 is equally likely to be w_1, w_2 or w_3. Continue in this way, applying $s_3, \ldots, s_{n-1}$ at each step or doing nothing, with probability $1/(i + 1)$ of doing nothing at the ith step, obtaining $d_1 \cdots d_n$. Then d_n is equally likely to be any of $1, 2, \ldots, n$. Now apply $s_1, s_2, \ldots, s_{n-2}$ or do nothing as before, obtaining $e_1 \cdots e_n$. The last element e_n has never switched, so $e_n = d_n$, and now e_{n-1} is equally likely to be any element of $\{1, 2, \ldots, n\} - \{d_n\}$. Continue as before with $s_1, \ldots, s_{n-3}$, then $s_1, \ldots, s_{n-4}$, etc., ending in s_1, s_2, s_1, at which point we obtain a uniformly distributed random permutation.

Now consider the situation for $\mathscr{H}_n(q)$. If P is a probability distribution on $\mathfrak{S}_n$ then write $\tau_P = \sum_{w \in \mathfrak{S}_n} P(w) T_w \in \mathscr{H}_n(q)$. If P' is another probability distribution on $\mathfrak{S}_n$, then in general it is not true that $\tau_P \tau'_P = \tau_R$ for some probability distribution R. A probabilistic interpretation of Theorem 1.1 requires the use of a Markov chain. Let $0 < q < 1$. Note that from equation (1.2) we have

$$T_w(1 + \boldsymbol{k} T_k) = \begin{cases} T_w + \boldsymbol{k} T_{ws_k}, & \ell(ws_k) = \ell(w) + 1 \\ q^k T_w + q\boldsymbol{k} T_{ws_k}, & \ell(ws_k) = \ell(w) - 1. \end{cases}$$

Divide each side by $1 + \boldsymbol{k}$. Let $w = w_1 \cdots w_n$. We can then interpret multiplication of T_w by $(1 + \boldsymbol{k}T_w)/(1 + \boldsymbol{k})$ as follows. If $w_k < w_{k+1}$ then transpose w_k and w_{k+1} with probability $\boldsymbol{k}/(1 + \boldsymbol{k})$, or do nothing with probability $1/(1 + \boldsymbol{k})$. If $w_k > w_{k+1}$, then transpose w_k and w_{k+1} with probability $q\boldsymbol{k}/(1 + \boldsymbol{k})$, or do nothing with probability $q^k/(1 + \boldsymbol{k})$. Since

$$\frac{q\boldsymbol{k}}{1+\boldsymbol{k}} + \frac{q^k}{1+\boldsymbol{k}} < 1,$$

we have a "leftover" probability of $(1 - (q\boldsymbol{k} + q^k)/(1 + \boldsymbol{k}))$. In this case the process has failed and we should start it all over. Let us call this procedure a *k-step*.

If $r = (r_1, \dots, r_l)$ is a tight sequence, then begin with the identity permutation and apply an r_1-step, r_2-step, etc. If we need to start over, then we again begin with the identity permutation and apply an r_1-step, r_2-step, etc. Eventually (with probability 1) we will apply r_i-steps for all $1 \leq i \leq l$, ending with a random permutation v. In this case, Theorem 1.1 tells us the distribution of v, namely, the probability of v is

$$P(v) = \frac{\alpha_r(v)}{\prod_{i=1}^{l}(1 + \boldsymbol{r_i})}.$$

In particular, if r is the standard tight sequence ρ_n, then v is uniformly distributed.

Example 3.7. Start with the permutation 123 and $r = \rho_3 = (1, 2, 1)$. Let us calculate by "brute force" the probability $P = P(123)$ that $v = 123$. There are three ways to achieve $v = 123$.

(a) Apply a 1-step, a 2-step, and a 1-step, doing nothing each time. This has probability $(1/2)(1/(2 + q))(1/2) = 1/4(2 + q)$.

(b) Apply a 1-step and switch. Apply a 2-step and do nothing. Apply a 1-step and switch. This has probability $q/4(2 + q)$.

(c) Apply a 1-step and switch. Apply a 2-step and do nothing. Try to apply a 1-step but go back to the beginning, after which we continue the process until ending up with 123. This has probability

$$\frac{1}{2}\frac{1}{2+q}(1-q)P = \frac{P(1-q)}{2(2+q)}.$$

Hence

$$P = \frac{1}{4(2+q)} + \frac{q}{4(2+q)} + \frac{P(1-q)}{2(2+q)}.$$

Solving for P gives (somewhat miraculously!) $P = 1/6$. Similarly for all other $w \in \mathfrak{S}_3$ we get $P(w) = 1/6$.

Note. A probabilistic interpretation of certain Hecke algebra products different from ours appears in a paper by Diaconis and Ram [4].

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